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Parameter Estimation for the Log-Normal Parent Population of Fatigue Failures from a Sample Containing both Failed and Non-Failed Members

by

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PARAMETER ESTIMATION FOR THE LOG-NORMAL PARENT POPULATION OF FATIGUE FAILURES FROM A SAMPLE CONTAINING BOTH FAILED AND NON-FAILED MEMBERS

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SUMMARY

A Maximum Likelihood technique is applied to provide estimates of the mean and standard deviation of the parent (log-normal) population of a sample of fatigue test results, for the case when the sample consists of some specimens that have not broken as well as specimens that have failed. The estimates produced by this method of analysis are compared with those given by the suitable application of a technique developed by Gupta and with those resulting from a graphical procedure suggested by Weibull and Johnson. The samples used for these comparisons were fictitious, being obtained from an assumed parent population by a Monte Carlo technique, and, although limited in number and scope, they indicate that the Maximum Likelihood technique gives reasonable approximations to the population parameters.

Use of the most suitable of the mentioned methods of analysis to correlate early service failures with a test failure should enable a check to be made on the validity of the fatigue monitoring process being applied to the service aircraft.

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*Replaces RAE Technical Report 70145 - ARC 32594.

CONTENTS

1 INTRODUCTION

Economic servicing of a group of structures which are subject to fatigue failures depends upon an efficient system of stocking spares (i.e. neither too many nor too few at any time) and, when the failures are of a fail-safe nature, upon an efficient inspection system (i.e. neither too seldom nor too often). Overstocking results in the wastage of space, whilst understocking leads to aircraft being unserviceable for too long. Too many inspections mean a large bill for the man hours spent inspecting needlessly, whilst too few inspections will create a hazard to the safety of the aircraft. Similarly the economic utilization of such a group of structures requires efficient planning so that too many structures are not unserviceable at any one time, thus keeping to a minimum the number of structures needed to cover the usage requirements.

Clearly a knowledge of the exact times to occurrence of cracks in each part of the structure on every structure would provide the ideal basis both for economic servicing and for economic utilization of a group of structures. However, fatigue strength is, like all macroscopic material properties, subject to inherent scatter and so the nearest approach to this ideal situation that can be attained in practice is a knowledge of the probabilities that the various parts of each structure should have failed by any time in the life of the structure.

The estimation of these probabilities of failure of the individual components of a structure is based either on the various times to failure of the corresponding individual components, as derived from the fatigue test of the complete structure, or on calculations of the times to failure of the various components. In general the two factors, expense and time, restrict the number of fatigue tests of the complete structure, i.e. full scale tests, that are made on any one design of aircraft. For example it is unusual for more than one full scale aircraft specimen to be tested under fatigue loading, whilst in some cases no full scale tests at all have been carried out. Thus it is important to obtain any extra information that can add confidence either to the deductions made from the full scale test or to the results of the calculations, when no tests have been carried out, for this added confidence would be carried through to the estimation of the probabilities of failure and thence would lead to more efficient fleet planning and servicing[1].

In the particular case of the fatigue test of a complete airframe, cracks of minor importance (in fail-safe structures they may even be of major importance) will usually occur in various positions throughout the test airframe structure before any catastrophic failure takes place. Provided that the loading distribution applied to the test airframe is at least reasonably representative of service conditions and that the specimen is reasonably representative of service aircraft, cracks will occur in service aircraft in some at least of the various positions indicated by the full scale test. If the test loading is not representative of service conditions cracks may occur in service in positions other than those indicated by the full scale test. By correlating the times to occurrence of cracks in some of the service aircraft, additional confidence can be derived for the estimates of the times to occurrence of cracks in the other aircraft of the fleet. Clearly the benefit gained by this type of correlation would be minimal if nearly all the aircraft of the fleet have to be cracked before the correlation can be carried out. Thus a method of analysis is required for a fleet of aircraft in which some members have already cracked but in which the majority of members remain uncracked.

In this Report the Principle of Maximum Likelihood[2] is applied to the solution of this problem and the parameter estimates thus obtained are compared with estimates given by three other modes of analysis.

2 PRESENT METHODS OF ANALYSIS

The most common graphical procedure for the analysis of a sample containing both failed and unfailed members is an adaptation of the procedure used for analysing samples consisting only of failed specimens (see Appendix). The introduction of non-failures produces problems in the allocation of a mean order number to be associated with each member of the sample. The 'usual' method of overcoming this difficulty is to treat each member, whether failure or non-failure, as a failed item and to obtain the corresponding mean or median ranks accordingly. Then, when these ranks are plotted against the value of the property considered on probability paper, only those members of the sample which have actually failed are used. In this way the unfailed members of the sample influence the ranks associated with the failed members but are not themselves explicitly involved in the final graphical analysis.

A further adaptation of this 'usual' technique is proposed by both Johnson[3] and Weibull[4] for estimating the parameters of the parent Weibull distribution for a sample containing both failed and unfailed members. Their proposed method of analysis follows exactly the same lines as the 'usual' procedure outlined above but differs in the values of the mean order numbers allocated to the failed members of the sample (see Appendix). Although in the original references this method of analysis was proposed for application to samples from Weibull distributions and the estimation of the two parameters appropriate to that form of distribution, the technique is equally valid for the estimation of the mean and standard deviation when the parent population is log-normal.

The problem of parameter estimation for a sample that is censored in such a way 'that the (n-k) smallest or greatest observations out of a sample of size n are censored' was studied theoretically by Gupta[5]. The type of sample to which Gupta's method of analysis is suitable is produced by testing n specimens simultaneously and stopping the test when k specimens have failed, so that all the other (n-k) specimens are known to have values greater than the kth failure, the individual values being known. This is not quite the situation achieved in aircraft usage, where the aircraft do not all accumulate fatigue damage at the same rate and in fact do not all enter service at the same time. This results in the non-failures often being interspersed amongst the failures. However, by suitable approximations, involving the loss of a certain amount of data, Gupta's technique can be applied to the present problem to give an estimate of the population parameters.

In an effort to avoid the loss of any relevant data that is provided by the non-failures in a sample the Principle of Maximum Likelihood, as used by Gupta, is applied in this Report to the more general problem when the failures and non-failures are interspersed. As in Gupta's work the parent distribution is assumed to be normal, the variate being the logarithm of the time to failure under fatigue loading, and the mean and standard deviation of this parent population are estimated by the analysis. However, whereas the situation Gupta studied enabled him to standardize the process to some degree, thus simplifying the analysis and allowing the presentation of tables of certain standardized variables to help in the solution of his type of problem, the present, more general situation leads to no such standardization. Each situation must be analysed from fundamental principles as an entirely new situation.

3 MAXIMUM LIKELIHOOD ANALYSIS. GENERAL CASE

The method of analysis proposed in this Report for the analysis of a sample containing both failures and non-failures depends on the Principle of Maximum Likelihood[2], whereby the likelihood of observing the sample in existence is maximized for variations in the position and extent, but not shape, of the parent population which is assumed to be normal. The values of the estimated parameters that give this Maximum Likelihood are then taken as the best estimates of the population parameters. The normal distribution has two parameters μ and σ and so the likelihood of observing the sample is maximized for both of these parameters simultaneously.

Taking m and s as the postulated population parameters of the distribution of $y = \log x$ where x is the time to crack occurrence, the probability density function of $y = \log x$ will be defined by

$$p(y)\,dy = \frac{1}{s\sqrt{2\pi}} e^{-(y-m)^2/2s^2}\,dy \quad . \tag{1}$$

Consider a fleet of aircraft of which n members have already cracked at a particular station after lives $x_1, x_2, \ldots x_n$ and of which c members remain uncracked after lives $x_{n+1}, x_{n+2}, \ldots x_{n+c}$. No stipulations are made about the relative magnitudes of the x's.

The probability that aircraft $(n + i)$, $i = 1,2,\ldots c$ should not have failed by the time x_{n+i} (the life it has achieved to date) will be given by the area under the distribution defined by equation (1) to the right of the ordinate at $y_{n+i} = \log (x_{n+i})$ (Fig.1) and so

$$P_{(n+i)} = \frac{1}{s\sqrt{2\pi}} \int_{y_{(n+i)}}^{+\infty} \exp\left\{-\frac{(y-m)^2}{2s^2}\right\} dy \qquad i = 1,2,\ldots c\ . \tag{2}$$

Also the probability that aircraft i $(i = 1,2,\ldots n)$ should have failed at a life between $x_i - \frac{\delta x}{2}$ and $x_i + \frac{\delta x}{2}$ will be given by (Fig.2)

$$P_i = \frac{1}{s\sqrt{2\pi}} \exp\left\{-\frac{(y-m)^2}{2s^2}\right\} \delta y \qquad i = 1,2,\ldots n \tag{3}$$

where $\delta y = \log\left(x_i + \frac{\delta x}{2}\right) - \log\left(x_i - \frac{\delta x}{2}\right)$, and
δx is very small.

Thus the total probability P that, with parent population parameters postulated as m and s, the present situation of the fleet should be observed will be given by the product of the individual probabilities for each aircraft, for these are all independent of one another. Then

$$P = \prod_{i=1}^{i=c} \left\{ \frac{1}{s\sqrt{2\pi}} \int_{y_{n+i}}^{+\infty} \exp - \tfrac{1}{2}\left(\frac{y-m}{s}\right)^2 dy \right\} \prod_{i=1}^{i=n} \left\{ \frac{1}{s\sqrt{2\pi}} \exp - \tfrac{1}{2}\left(\frac{y_i - m}{s}\right)^2 \delta y \right\}$$

which reduces to

$$P = \frac{1}{(s\sqrt{2\pi})^n} \prod_{i=1}^{i=c} \left\{ \frac{1}{\sqrt{2\pi}} \int_{(y_{n+i}-m/s)}^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt \right\} \prod_{i=1}^{i=n} \left\{ \exp - \tfrac{1}{2}\left(\frac{y_i - m}{s}\right)^2 \delta y \right\} . \quad \ldots.(4)$$

If the best estimates of the mean and standard deviation derived from this estimator P are denoted by $\hat{m}$ and $\hat{s}$ respectively, the conditions required for the calculation of these values are

$$\frac{\partial P}{\partial s} = 0 \quad \text{and} \quad \frac{\partial P}{\partial m} = 0 \qquad \text{at} \quad m = \hat{m} \quad \text{and} \quad s = \hat{s}$$

and also

$$\frac{\partial^2 P}{\partial s^2} < 0 \quad \text{and} \quad \frac{\partial^2 P}{\partial m^2} < 0 \ .$$

These conditions are equivalent to

$$\left.\begin{aligned} &\frac{1}{P}\frac{\partial P}{\partial s} = \frac{\partial}{\partial s}(\log P) = 0 \quad \text{and} \quad \frac{1}{P}\frac{\partial P}{\partial m} = \frac{\partial}{\partial m}(\log P) = 0 \\ &\text{with} \\ &\frac{\partial^2 P}{\partial s^2} < 0 \quad \text{and} \quad \frac{\partial^2 P}{\partial m^2} < 0 \ . \end{aligned}\right\} \quad (5)$$

Taking equations (4) and (5) together then gives the values of the best estimates $\hat{m}$ and $\hat{s}$.

4 'ALL FAILED' VERSION OF GENERAL CASE

It is instructive to consider the case of 'all failed', i.e. $c = 0$. Equation (4) reduces to

$$P = \frac{1}{(s\sqrt{2\pi})^n} \exp - \frac{1}{2s^2} \sum_{i=1}^{i=n} (y_i - m)^2 \, \delta y \quad . \qquad (6)$$

Now δy is an arbitrarily chosen small interval and plays no part in determining the relative magnitudes of P for different combinations of m and s and thus will not affect the position of the maximum, although it will change its magnitude. Thus without loss of generality δy can be put equal to 1 for all $i = 1,2,\ldots n$. Then $\log P = \text{constant} - n \log s - \frac{1}{2s^2} \sum_{i=1}^{i=n} (y_i - m)^2$

$$\frac{\partial \log P}{\partial m} = \frac{1}{s^2} \sum_{i=1}^{i=n} (y_i - m) \quad ; \qquad \frac{\partial \log P}{\partial s} = -\frac{n}{s} + \sum_{i=1}^{i=n} \frac{(y_i - m)^2}{s^3} \quad . \qquad (7)$$

Also

$$\frac{\partial^2 P}{\partial s^2} = \frac{\partial}{\partial s}\left(P \frac{\partial \log P}{\partial s}\right) = \frac{\partial P}{\partial s} \frac{\partial \log P}{\partial s} + P \frac{\partial^2 \log P}{\partial s^2} \quad .$$

Therefore at $\hat{m}$ and $\hat{s}$ as $\frac{\partial \log P}{\partial s} = 0$, $\quad \frac{\partial^2 P}{\partial s^2} = P \frac{\partial^2 \log P}{\partial s^2}$

Similarly at $\hat{m}$ and $\hat{s}$ $\quad \frac{\partial^2 P}{\partial m^2} = P \frac{\partial^2 \log P}{\partial m^2}$. $\qquad$ (8)

From equation (8)

$$\frac{\partial^2 P}{\partial m^2} = P \frac{1}{s^2}(-n) \quad ; \qquad \frac{\partial^2 P}{\partial s^2} = P \left[+ \frac{n}{s^2} - 3 \sum_{i=1}^{i=n} \frac{(y_i - m)^2}{s^4} \right] \qquad (9)$$

equations (5) and (7) give $\frac{1}{\hat{s}^2} \sum_{i=1}^{i=n} (y_i - \hat{m}) = 0$ and $-\frac{n}{\hat{s}} + \sum_{i=1}^{i=n} \frac{(y_i - \hat{m})^2}{\hat{s}^3} = 0$

or
$$\hat{m} = \sum_{i=1}^{i=n} \frac{y_i}{n}$$

and
$$\hat{s}^2 = \sum_{i=1}^{i=n} \frac{(y_i - \hat{m})^2}{n} \quad . \qquad (10)$$

Substitution of these values for $\hat{m}$ and s^2 in equation (9) makes both $\frac{\partial^2 P}{\partial m^2}$ and $\frac{\partial^2 P}{\partial s^2}$ less than 0, for P is positive, and so the conditions (5) are satisfied.

Comparison of these results, equation (10), with the standard 'best' estimates, m_1 and s_1, for the parent mean μ and standard deviation σ respectively, where

$$m_1 = \sum_{i=1}^{i=n} \frac{\log x_i}{n} \qquad (11)$$

$$s_1^2 = \sum_{i=1}^{i=n} \frac{(\log x_i - m_1)^2}{(n - 1)} \qquad (12)$$

show that the Maximum Likelihood estimation $\hat{m}$ for the parent mean μ is identical to the standard 'best' estimation m_1 for the parent mean μ when all the members of the sample have failed. On the other hand the Maximum Likelihood estimator $\hat{s}$ for the parent standard deviation σ is biassed, whilst the standard 'best' estimator s_1 has been rendered unbiassed. This point is discussed fully by Kendall, who, however, only considers the 'all failed' case where the estimate of the mean can be made without any reference to the standard deviation.

For in our terminology

$$\frac{\partial^2 P}{\partial m\, \partial s} = \frac{\partial}{\partial m}\left\{P\, \frac{\partial \log P}{\partial s}\right\} = \frac{\partial P}{\partial m}\, \frac{\partial \log P}{\partial s} + P \frac{\partial^2 \log P}{\partial m\, \partial s} \quad .$$

But when $m = \hat{m}$ and $s = \hat{s}$, $\frac{\partial \log P}{\partial s} = 0$

and so

$$\frac{\partial^2 P}{\partial m\, \partial s} = P\, \frac{\partial^2 \log P}{\partial m\, \partial s} = P \left\{\sum_{i=1}^{i=n} (y_i - m)\right\}\left\{-\frac{2}{s^3}\right\} .$$

Thus $\frac{\partial^2 P}{\partial m \, \partial s}$ has zero expectation when $m = \hat{m}$ and so $\hat{m}$ and $\hat{s}$ are independent of one another.

In the more general case considered in this Report the two parameters will no longer be independent as can be seen from the form of equation (4) and thus they must be estimated simultaneously. Attempts at an analytical solution to the general equations (4) and (5) were unsuccessful and so numerical solutions to particular problems were obtained from a computer programme written in Mercury Autocode to be run on the Manchester University Atlas computer. The programme entails a simple search through values of P calculated for paired values of postulated m and s and picks that pairing which maximises the calculated P. The result of the calculations and search is a two-way table of $\log P$ with regard to m and s and centred on $\hat{m}$ and $\hat{s}$, the estimates that give the maximum value to P (or $\log P$).

5 CONFIDENCE IN ESTIMATES. GENERAL CASE

An approximate evaluation of the standard errors of the estimates of the mean and standard deviation resulting from the above analysis can also be made from the values of $\log P$ in the two-way table mentioned above. From Ref.2 the asymptotic variance-covariance matrix is given by the reciprocal of the matrix

$$\left(\frac{\partial \log P}{\partial \theta_r} \cdot \frac{\partial \log P}{\partial \theta_s}\right)_{\theta=\hat{\theta}} \tag{13}$$

where θ in the present case defines the two parameters m and s, $\hat{\theta}$ is the value of θ at the Maximum Likelihood condition, i.e. $\hat{\theta}$ defines $\hat{m}$ and $\hat{s}$, and θ_1 and θ_2 are m and s respectively. The same reference also shows that

$$\left(\frac{\partial \log P}{\partial \theta_r} \cdot \frac{\partial \log P}{\partial \theta_s}\right)_{\theta=\hat{\theta}} = -\left(\frac{\partial^2 \log P}{\partial \theta_r \, \partial \theta_s}\right)_{\hat{\theta}=\theta} = -E\left(\frac{\partial^2 \log P}{\partial \theta_r \, \partial \theta_s}\right) \tag{14}$$

where E denotes the expectation.

Now for the case considered in this Report the expectations of the second order partial differentials, equation (14), are not known and so are replaced by the estimates of these differentials at the Maximum Likelihood condition. Thus equations (13) and (14) combine to give

$$\begin{pmatrix} \text{var}(\hat{m}) & \text{covar}(\hat{m},\hat{s}) \\ \text{covar}(\hat{m},\hat{s}) & \text{var}(\hat{s}) \end{pmatrix} \simeq \begin{pmatrix} -\left\{\dfrac{\partial^2 \log P}{\partial m^2}\right\}_{\hat{m},\hat{s}} & -\left\{\dfrac{\partial^2 \log P}{\partial m\ \partial s}\right\}_{\hat{m},\hat{s}} \\ -\left\{\dfrac{\partial^2 \log P}{\partial s\ \partial m}\right\}_{\hat{m},\hat{s}} & -\left\{\dfrac{\partial^2 \log P}{\partial s^2}\right\}_{\hat{m},\hat{s}} \end{pmatrix} . \quad (15)$$

The estimates of $\left(\dfrac{\partial^2 \log P}{\partial m^2}\right)_{\hat{m},\hat{s}}$ and $\left(\dfrac{\partial^2 \log P}{\partial s^2}\right)_{\hat{m},\hat{s}}$ at the Maximum Likelihood point $(\hat{m},\hat{s})$ are made using the approximate equation

$$\frac{d^2y}{dx^2} \simeq \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

and taking the values of $\log P$ from the two-way table produced in assessing $\hat{m}$ and $\hat{s}$. The validity of this approximation is checked by taking three different values of h and comparing the values thus obtained for $\left(\dfrac{\partial^2 \log P}{\partial m^2}\right)_{\hat{m},\hat{s}}$ and $\left(\dfrac{\partial^2 \log P}{\partial s^2}\right)_{\hat{m},\hat{s}}$. Similarly $\left(\dfrac{\partial^2 \log P}{\partial m\ \partial s}\right)_{\hat{m},\hat{s}}$ is estimated using the approximate form

$$\left(\frac{\partial^2 y}{\partial z\ \partial x}\right)_{z,x} \simeq \frac{y(x+h, z+d) + y(x-h, z-d) - y(x+h, z-d) - y(x-h, z+d)}{4hd}$$

and this approximation is also checked by comparing the values derived for three different sets of values of h and d. Equation (15) results in the three equations below, where the differentials are estimates at the Maximum Likelihood point

$$\left.\begin{aligned} \text{var}(\hat{m}) &\simeq -\frac{\partial^2 \log P}{\partial s^2} \Bigg/ \frac{\partial^2 \log P}{\partial m^2} \cdot \frac{\partial^2 \log P}{\partial s^2} - \left(\frac{\partial^2 \log P}{\partial m\ \partial s}\right)^2 \\ \text{var}(\hat{s}) &\simeq -\frac{\partial^2 \log P}{\partial m^2} \Bigg/ \frac{\partial^2 \log P}{\partial m^2} \cdot \frac{\partial^2 \log P}{\partial s^2} - \left(\frac{\partial^2 \log P}{\partial m\ \partial s}\right)^2 \\ \text{and}\quad \text{covar}(\hat{m},\hat{s}) &\simeq \frac{\partial^2 \log P}{\partial m\ \partial s} \Bigg/ \frac{\partial^2 \log P}{\partial m^2} \cdot \frac{\partial^2 \log P}{\partial s^2} - \left(\frac{\partial^2 \log P}{\partial m\ \partial s}\right)^2 . \end{aligned}\right\} \quad (16)$$

Thus ρ the coefficient of correlation between m and s is given by

$$\rho = \frac{\text{covar}\,(\hat{m},\hat{s})}{\sqrt{\text{var}\,(\hat{m})\;\text{var}\,(\hat{s})}} \approx \left(\frac{\partial^2 \log P}{\partial m\,\partial s}\right) \Big/ \left(\frac{\partial^2 \log P}{\partial m^2}\right)^{\frac{1}{2}} \left(\frac{\partial^2 \log P}{\partial s^2}\right)^{\frac{1}{2}} .$$

The computer programme written to estimate $\hat{m}$ and $\hat{s}$ also includes instructions to estimate var $(\hat{m})$, var $(\hat{s})$, covar $(\hat{m},\hat{s})$ and the coefficient of correlation, according to the above equations. Using these values an idea can be gained of the confidence that can be placed on the estimates $\hat{m}$ and $\hat{s}$.

6 CONFIDENCE IN ESTIMATES. 'ALL FAILED' CASE

The 'all failed' case can be treated analytically as an example of the above procedure; however the equalities will be exact and not approximate. From equations (18) and (19)

$$\left.\begin{aligned} E\left\{\frac{\partial^2 \log P}{\partial m^2}\right\} &= -\frac{n}{s^2} \\ E\left\{\frac{\partial^2 \log P}{\partial s^2}\right\} &= -\frac{2n}{s^2} \\ E\left\{\frac{\partial^2 \log P}{\partial m\,\partial s}\right\} &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{as } E\sum_{i=1}^{i=n}(y_i - m)^2 = ns^2 \qquad (17) \\ &\text{as } E\sum_{i=1}^{i=n}(y_i - m) = 0 \;. \end{aligned}$$

Equations (16) and (17) then give

$$\text{var}\,(\hat{m}) = \frac{(\hat{s})^2}{n}$$

$$\text{var}\,(\hat{s}) = \frac{(\hat{s})^2}{2n}$$

$$\text{covar}\,(\hat{m},\hat{s}) = 0$$

which are the usual results and show that m and s, the Maximum Likelihood estimates of the mean and standard deviation, are independent of one another.

7 NUMERICAL EXAMPLES

Practical verification of the suggested method of parameter estimation for the distribution of a particular fatigue failure in a fleet of aircraft would require an accurate knowledge of the fatigue state of a large fleet throughout its entire life - the fatigue state of a fleet at any time being defined not only in terms of the fatigue damage accumulated by each aircraft up to that time but also in terms of the presence by that time of fatigue failures in specific aircraft which have cracked after known amounts of fatigue damage have been accumulated. Only when all this information is available, for the period from the time the first aircraft flew until all the aircraft in the fleet have suffered the particular failure being considered, could any estimate of the validity of the proposed method be made using practical data, for a knowledge of the true population parameters of the failure distribution is a necessary part of any such estimation.

In the past this ideal situation, when all the members of a fleet are allowed to fail, has seldom, if ever, been achieved. Generally, if areas on the aircraft are known to be subject to early fatigue cracking, modifications are applied before all the aircraft have cracked in that area or, if the cracking does not occur very early in the life of the aircraft, some members of the fleet will not have experienced the considered failure before being retired from service. However the advent of the present method of treatment of some fatigue failures as 'fail-safe' has increased the possibility that at some time in the future the ideal situation could be achieved and that a verification of the proposed method of parameter estimation using practical data could then take place.

7.1 Derivation of two model fleets

Meanwhile, in an effort to overcome this lack of suitable physical data, a Monte Carlo technique has been applied to the problem, whereby a fictitious model fleet of aircraft has been developed and investigated. In setting up this fictitious fleet, two main assumptions were made:-

(1) A fleet of 30 aircraft was to be considered. This number was chosen as being a reasonable size for a fleet of aircraft, sufficiently large to keep small the sampling errors caused by the lack of an infinite population but small enough to keep the computer time short.

(2) The probability distribution of times to failure was taken to be logarithmically normal, if time is measured in terms of the fatigue index units (FIU)* accumulated. The population, of which the fleet of 30 aircraft was a sample, was assumed to have a mean μ of 100 FIU (i.e. a $\log_{10}$ mean of 2) and a standard deviation σ $(\log_{10})$ of 0.17 FIU. The first stage in the derivation of the model fleet, the creation of the 'all failed' situation, was achieved by generating a string of 30 pseudo random numbers between 0 and 1000 (see Table 1, column 1). Each of these numbers, divided by 1000, was then interpreted as a cumulative probability and the corresponding normal deviate, that gave the required probability, was calculated, Fig.3. These 30 deviates were then analysed in terms of a log-normal distribution of mean 100 FIU and standard deviation 0.17 FIU and so a set of 30 failure times was derived for a fleet obeying the original two assumptions, Table 1. Analysis of this sample of 30 failures using equations (11) and (12) gave estimates of 107.2 FIU and 0.182 FIU for the mean and standard deviation respectively of the parent population.

The further derivation of the annual fatigue state of the fleet of aircraft required the definition of an average rate of accumulation of fatigue index units throughout the fleet and the assumption of some form of distribution of the individual rates of fatigue index unit accumulation for the separate aircraft within the fleet. This distribution was taken to be normal in form, with a standard deviation of 5 FIU per year and with a mean, that is the fleet average rate of consumption, of 15 FIU per year. Applying the same procedure as for the 'all failed' situation strings of 30 random numbers were generated, each number being interpreted as a cumulative probability which was converted into a normal deviate and then reduced to the individual rate of consumption of fatigue index for one particular aircraft for one year. Each string of 30 numbers thus represented the consumption of fatigue index units by each of the 30 members of the fleet during one year and the successive addition of the corresponding members of these strings produced a year by year tally of the fatigue damage accumulated by the 30 individual aircraft in the fleet. Comparison of this progress table with the list of times to occurrence of the crack

* The number of fatigue index units accumulated by an aircraft is a measure, in terms of an arbitrary linear scale, of the proportion of the fatigue life of of the aircraft that has been used up. For each aircraft it is calculated by substitution of the readings, recorded at each acceleration level on a Fatigue Load Meter, into a formula derived as in Ref.6.

in the individual members of the fleet derived earlier showed which aircraft had experienced that crack during the previous year and so the fatigue state of the aircraft at the end of each year was known.

Two points that arose in this derivation of the fatigue state of the model fleet must be noted, viz:-

(1) During the construction of the tally of the consumption of fatigue index units, the interpretation of the random number sometimes led to a negative normal deviate of magnitude greater than 3 standard deviations, which if strictly applied would have led to a negative fatigue consumption during that year for that particular aircraft (μ = 15 FIU; σ = 5 FIU). When this situation occurred, the fatigue damage for that aircraft for that year was arbitrarily put equal to 0 FIU.

(2) Once an aircraft had experienced the failure considered, the accumulation of fatigue index units by that aircraft was artificially frozen at the exact value at which the failure occurred.

The fatigue state of the fleet derived using the above method and assumptions is presented in Table 2, which is a year-by-year statement of the situation, and Fig.4, which provides a diagrammatic representation of the same data.

A survey of Fig.4 shows that, after the first four years of service, the majority of the uncracked members of the fleet, had accumulated a greater number of fatigue index units than any of the cracked aircraft. This situation, brought about by the assumption of too small a standard deviation of the distribution of rates of fatigue index consumption, was felt to be rather unrealistic, as in practice it is usually those aircraft that have flown most that crack first, and so a second fictitious model fleet was derived. The same failure times for the individual aircraft as in the first fleet were used but the standard deviation of the distribution of rates of consumption of fatigue index units within the fleet was changed from 5 FIU to 10 FIU per year, keeping the average rate of consumption at 15 FIU per year. The fatigue state of this fleet was evolved in the same manner as that of the first fleet, but using the altered assumptions, and the result is presented in Table 3 and Fig.5, which clearly shows that the occurrence of the failures in this fleet is not so ordered as in the first fleet and generally appears more realistic.

7.2 Results of the Maximum Likelihood analysis of the model fleets

It was decided to analyse both the model fleets to see how the Maximum Likelihood estimates would be affected by the unrealistic situations provided by the first fleet. The results produced by the programme for these two fleets are presented in Tables 4 and 5. The accuracy of the estimates of the mean and standard deviation given by the programme is easily assessed by comparing the estimates, columns 4 and 6 of Tables 4 and 5, with the assumed population parameters of 2.0 and 0.17. It can be readily seen that even in the most unfavourable circumstances the greatest error in the mean estimate is no more than 10%, but that of the standard deviation is as high as 76%. It is interesting to note that the largest discrepancies in the estimate of the mean occur when the estimate of the standard deviation is also considerably in error.

7.3 Results of the analysis of the model fleets using Gupta's technique

The only state of the first fleet to which Gupta's method (see section 2) can be applied immediately is the 'after seven years' state in which nine failures have occured (Fig.4e) and all the uncracked members of the fleet have accumulated more damage than the failed aircraft. By considering all these 21 unfailed aircraft to have censored totals of fatigue index units equal to that of the greatest failure, thereby loosing a certain amount of information, a state suitable for the application of Gupta's technique is obtained. With rather more drastic changes, i.e. the treatment of some failures as non-failures, five of the other fleet fatigue states can also be altered to be amenable to Gupta's treatment. The resulting states are given in Table 6, whilst the analysis of these states is presented in Table 7. The remaining three fatigue states not shown in Table 6 cannot be converted to a suitable form because the aircraft with the lowest consumption of fatigue index units has not failed.

The failures and non-failures in the second model fleet (Fig.5) were so intermingled that a large amount of information would have been lost in the conversion of this fleet to a form suitable for analysis by Gupta's technique. So the second fleet was not analysed by this method.

7.4 Results of two graphical analyses of the model fleets

Comparisons of the Weibull/Johnson plotting technique (see section 2) with the standard plotting procedure, for the fatigue states of the first model fleet, are presented in Figs.6 and 7 which show that in the cases illustrated Johnson's method gives at least as good, and usually better, estimates of the parameters than the standard graphical technique which will always give a pessimistic estimate. The plots obtained for the other fatigue states of the fleet not included in Figs.6 and 7 are omitted, as the median ranks for the vast majority of the failures derived by the two separate methods of analysis are identical, the only difference between the two occurring for failures which have accumulated more damage than some non-failures. Thus the plots for the remaining fatigue states produced by the two techniques are almost identical and are just progressively more complete versions of the plot for the total fleet (Fig.7).

The second model fleet which could not be sensibly analysed by Gupta's technique (section 7.3) was suitable for analysis by the Weibull/Johnson and the standard plotting procedures in all those cases in which more than one failure has occurred. The plots for four of those fleet fatigue states are presented in Figs.8 and 9. As for the first fleet the plots for the remainder of the fleet states have been omitted for the sake of brevity as they are just progressively more complete versions of the plot for the 'all failed' state (Fig.7) and would thus provide much the same estimates as that plot.

7.5 Comparison and discussion of the results given by the four methods of analysis

If the fourth and fifth cases of Table 4 are omitted temporarily, the largest error in the Maximum Likelihood estimate of the mean for the first fleet is about 3½%, the errors when there were only 1, 2 and 3 failures respectively being 2.7%, 1.0% and 1.2%. A glance at Figs.4d and 4e shows that, in the two situations that produce the largest errors, i.e. the fourth and fifth cases, the failures that have occurred are the failures from the low tail of the distribution of failures and that the great majority of the non-failures have accumulated more fatigue damage than any of the failed aircraft. These are clearly unlikely situations which, if they did arise in practice, are clearly suitable for analysis by Gupta's technique, which (Table 7) gives estimates much closer to the population values. Table 7 shows that, although a certain amount of information is lost in the alteration of a fatigue state to a suitable form, Gupta's technique of analysis gives better estimates than the Maximum Likelihood method for the type of distribution of failures and non-failures provided by the first model fleet, and that the standard errors of the estimates provided by Gupta's method are similar to those given by the Maximum Likelihood technique in most cases but smaller in the two cases when the number of failures is relatively small.

The estimates of the mean and standard deviation given by the Maximum Likelihood method for the second fleet are in excellent agreement with the parent population, the largest error being 3% on the mean and 20% on the standard deviation, omitting temporarily the fleet states when only one failure had occurred. Consideration of these latter two cases, with only one failure, shows that, when the failure is in one of the aircraft that has accumulated the most damage, the estimates of both the mean and standard deviation are low but when the failure occurs in one of the aircraft in the middle of the range of fatigue damaged aircraft then both estimates are high. This latter state compares with the 'after three years' state of the first fleet at which time only one aircraft had failed (that aircraft which was twelfth in a decreasing list of percentage life expired) but for which the parameters were reasonable, showing the dependence of the estimates in this situation on the relative magnitudes of the failures and non-failures. As the second model fleet was not analysed by Gupta's method (section 7.3) no comparison of the Maximum Likelihood method of analysis with Gupta's method is possible for this fleet.

Figs.6, 7, 8 and 9 show that the Weibull/Johnson plotting technique gives estimates of the parameters which are at least as good as, and usually better than, the estimates given by the standard graphical technique. Comparison of the estimates of the mean and standard deviation from Fig.6 with those from the Maximum Likelihood technique in Table 4 shows that the latter produces by far the better estimates for the fleet states in which only a few members have failed. However the graphical procedure gives good and occasionally better estimates of the parameters for those fatigue states in which an appreciable number of failures have occurred and in which there are no non-failures intermingled with the failures to distort the plot of the lower end of the distribution. Unfortunately the Weibull/Johnson method does not provide any procedure for obtaining the standard errors of the graphical estimates and so the confidence that can be placed in these estimates is unknown.

Besides showing the improvement gained by adopting the Weibull-Johnson plotting technique rather than the standard plotting technique, the results from Figs.8 and 9 indicate that the Maximum Likelihood method of analysis gives better estimates than the graphical procedures for the case in which only three items have failed. When there is only one failure, the graphical procedures clearly cannot cope at all and do not yield any results.

8 CONCLUSIONS

Various fictitious states of a fleet of aircraft involving both failed and non-failed members have been analysed, where possible, by each of four methods to give estimates of the mean and standard deviation of the parent population. The true parameters of the parent population, assumed to be normal, from which the fictitious fleets were derived by a Monte Carlo technique were given values representative of aircraft fleet data. The four methods of analysis used were

(a) Maximum Likelihood method,

(b) Gupta's technique[5],

(c) Weibull/Johnson plotting procedure[3,4],

(d) A standard plotting procedure (Appendix).

Comparisons of the estimates of the population mean and standard deviation given by the two graphical methods of analysis indicate that the Weibull/Johnson graphical method always gives estimates which are as good as, and often better than, the estimates given by the standard plotting procedures.

For the rather limited set of conditions provided by the examples in this Report it appears that in general the estimates given by the Maximum Likelihood technique are in better agreement with the true values than are the estimates given by Gupta's method, especially when only a few members of the fleet have failed. However, in those cases in which the application of Gupta's technique requires only a small approximation the best results are usually given by this method. The Weibull/Johnson graphical procedure is clearly a great improvement over the standard graphical procedure and in some cases gives excellent results, but when the number of failures in the fleet is small the agreement of the estimates given by this graphical procedure with the true parameters is not as good as that given by the other two methods of analysis.

It appears then that a suitable choice, between Gupta's method of analysis for a censored sample[5], the Weibull/Johnson plotting procedure[3,4] and the Maximum Likelihood technique proposed in this Report, should enable reasonable estimates of the mean and standard deviation of the parent normal distribution to be made from samples containing both failed and unfailed members for all situations but that of only one failure. In this latter case the Maximum Likelihood technique will provide estimates (the other methods will not) but these estimates will be sensitive to the relative magnitudes of the percentage life expired of the one failure and the non-failures in the sample. It seems likely, therefore, that valuable information can be provided by a correlation of service failures with the test failure using the most suitable of the methods compared in this Report.

Appendix

The simple standard graphical procedure for the analysis of a sample of n failed items consists of five separate steps, namely

(1) Collection of the n items in order of ascending magnitude of the property considered.

(2) Numbering the n items in ascending order from 1 to n (the general item being the jth). These numbers are the mean order numbers.

(3) Corresponding to each mean order number there is a mean and a median rank. Gumbel[7] proposes the use of $\frac{j}{n + 1}$ (a mean rank) as the plotting position for the jth item and this is the most commonly accepted procedure. However Johnson[8] uses the median rank given by

$$\frac{j - (1 - \log_e 2) - (2 \log_e 2 - 1)\left(\frac{j - 1}{n - 1}\right)}{n} .$$

(4) Having assigned either a median or a mean rank to each failure, the values are plotted on some form of probability paper.

(5) A straight line is fitted to the points thus produced, either by eye or by some form of least squares method[7] and this straight line then provides the 'best' estimates of the population parameters.

When non-failures are present, the value of the increment between successive failures, the (j - 1)th and the jth failures in the ordered list, (where there are r non-failures between these two failures) is often changed from the $\frac{1}{n + 1}$, given by the Gumbel form for the 'all failed' case, to $\frac{1 + r}{(n + 1)}$. This is the 'usual' method adopted in Figs.6 to 9, where the non-failures are not plotted. Johnson however proposed that the increment between the mean order numbers for the jth and (j - 1)th failure should be

$$\frac{(n + 1) - (\text{previous mean order number})}{1 + (\text{number of items above the present set of non-failures in the ascending order of items})} .$$

He then assigned median ranks to each of the mean order numbers derived for the failures in the sample. Otherwise the steps in the analysis are as for the 'usual' procedure. Clearly as in the 'usual' procedure the actual values of the non-failures are not used, their only effect being in the spacing of the failures in the order of ascending magnitudes.

Table 1

DERIVATION OF THE FAILURE TIMES (IN FIU) OF THE MODEL FLEET

Random number	Corresponding deviate. (A) {see Fig.3}	$\mu + A$ where $\mu = 2.0$ and $\sigma = 0.17$	Corresponding failures {antilog. column 3}	Ordered list of failures (FIU)
636	$+0.35\sigma$	2.0595	114.7	44.6
77	-1.43σ	1.7569	57.1	47.9
634	$+0.34\sigma$	2.0578	114.2	56.2
987	$+2.23\sigma$	2.3791	239.4	57.1
901	$+1.29\sigma$	2.2193	165.7	63.3
368	-0.34σ	1.9422	87.5	74.8
829	$+0.95\sigma$	2.1615	145.1	76.3
470	-0.08σ	1.9864	96.9	87.5
620	$+0.31\sigma$	2.0527	112.9	88.9
528	$+0.07\sigma$	2.0119	102.8	96.9
989	$+2.29\sigma$	2.3893	245.1	101.5
770	$+0.74\sigma$	2.1258	133.6	102.4
30	-1.88σ	1.6804	47.9	102.8
543	$+0.11\sigma$	2.0187	104.4	104.4
121	-1.17σ	1.8011	63.3	107.7
384	-0.30σ	1.9490	88.9	112.0
883	$+1.19\sigma$	2.2023	159.3	112.9
20	-2.06σ	1.6498	44.6	114.2
677	$+0.46\sigma$	2.0782	119.7	114.7
791	$+0.81\sigma$	2.1377	137.3	119.7
614	$+0.29\sigma$	2.0493	112.0	122.6
244	-0.69σ	1.8827	76.3	133.6
71	-1.47σ	1.7501	56.2	137.3
517	$+0.04\sigma$	2.0068	101.5	145.1
576	$+0.19\sigma$	2.0323	107.7	159.3
978	$+2.01\sigma$	2.3417	219.6	165.7
231	-0.74σ	1.8742	74.8	172.3
524	$+0.06\sigma$	2.0102	102.4	219.6
918	$+1.39\sigma$	2.2363	172.3	239.4
701	$+0.52\sigma$	2.0884	122.6	245.1

Table 2

PROGRESS OF THE FATIGUE STATE OF THE FIRST MODEL FLEET

Time in service	3 years	3½ years	4 years	6 years	7 years	8 years	9 years	10 years	'All failed' state
Total number of failures	1	2	3	6	9	16	22	25	30
Total number of non-failures	29	28	27	24	21	14	8	5	0
Individual aircraft damage accumulations {FIU}	44.6	44.6	44.6	44.6	44.6	44.6	44.6	44.6	44.6
	30.7	47.9	47.9	47.9	47.9	47.9	47.9	47.9	47.9
	31.8	37.0	57.1	56.2	56.2	56.2	56.2	56.2	56.2
	31.9	42.1	42.2	57.1	57.1	57.1	57.1	57.1	57.1
	32.9	43.0	45.9	63.3	63.3	63.3	63.3	63.3	63.3
	36.3	43.2	47.9	76.3	74.8	74.8	74.8	74.8	74.8
	36.6	43.3	49.2	68.0	76.3	76.3	76.3	76.3	76.3
	37.7	44.5	49.6	72.1	87.5	87.5	87.5	87.5	87.5
	38.1	44.7	49.8	76.2	88.9	88.9	88.9	88.9	88.9
	38.2	45.1	52.0	77.3	89.6	96.9	96.9	96.9	96.9
	38.5	45.4	53.1	78.0	89.9	101.5	101.5	101.5	101.5
	38.7	45.8	53.4	79.6	94.6	102.4	102.4	102.4	102.4
	39.4	47.3	54.2	80.5	95.2	102.8	102.8	102.8	102.8
	39.7	48.3	55.8	80.9	95.3	104.4	104.4	104.4	104.4
	40.2	48.5	56.2	82.2	97.0	107.7	107.7	107.7	107.7
	41.9	48.5	56.8	84.2	97.2	112.0	112.0	112.0	112.0
	42.3	49.6	56.9	85.7	98.2	104.4	112.9	112.9	112.9
	42.7	50.5	57.1	85.8	99.3	104.8	114.2	114.2	114.2
	43.7	52.0	57.2	86.5	99.8	111.3	119.7	114.7	114.7
	44.8	53.0	58.9	88.3	102.4	113.0	122.6	119.7	119.7
	46.3	53.1	61.4	88.6	103.2	117.4	133.6	122.6	122.6
	47.9	53.7	61.7	90.6	103.8	119.8	137.3	133.6	133.6
	49.0	54.5	64.7	92.5	106.8	122.2	113.8	137.3	137.3
	50.0	57.4	67.5	93.2	107.9	123.4	135.8	145.1	145.1
	51.1	58.3	68.3	95.9	111.7	128.1	138.1	159.3	159.3
	52.0	59.8	68.7	98.0	114.6	128.9	139.2	146.6	165.7
	52.2	61.5	71.8	100.8	115.3	128.9	141.8	154.1	172.3
	54.6	63.3	71.9	100.9	118.3	136.4	150.2	158.4	219.6
	56.7	65.7	74.7	101.6	121.5	136.8	154.3	170.8	239.4
	70.8	77.7	84.6	102.1	124.3	139.3	161.2	171.7	245.1

Values above the stepped line indicate failures; those below indicate non-failures.

For the sake of brevity the fatigue states of the fleet in the less interesting periods have been omitted.

Table 3

PROGRESS OF THE FATIGUE STATE OF THE SECOND MODEL FLEET

Time in service	2 years	3 years	$3\frac{1}{2}$ years	4 years	5 years	6 years	8 years	9 years	'All failed' state
Total number of failures	1	1	3	5	8	12	18	21	30
Total number of non-failures	29	29	27	25	22	18	12	9	0
Individual aircraft damage accumulations {FIU}	47.9	47.9	47.9	47.9	44.6	44.6	44.6	44.6	44.6
	12.3	21.7	56.2	56.2	47.9	47.9	47.9	47.9	47.9
	13.9	29.6	57.1	57.1	56.2	56.2	56.2	56.2	56.2
	15.9	29.9	24.1	87.5	57.1	57.1	57.1	57.1	57.1
	17.4	31.5	30.8	107.7	76.3	63.3	63.3	63.3	63.3
	20.8	32.4	31.2	28.2	87.5	76.3	74.8	74.8	74.8
	21.1	33.1	36.2	31.7	101.5	87.5	76.3	76.3	76.3
	22.5	38.5	42.8	33.2	107.7	101.5	87.5	87.5	87.5
	23.4	39.5	43.1	36.5	43.2	102.4	88.9	88.9	88.9
	23.9	40.4	43.9	42.8	46.7	102.8	96.9	96.9	96.9
	24.6	40.6	44.3	43.4	49.8	107.7	101.5	101.5	101.5
	25.1	40.8	44.8	45.3	52.5	112.0	102.4	102.4	102.4
	25.3	40.8	48.1	46.5	53.9	53.1	102.8	102.8	102.8
	25.7	40.9	48.9	48.7	55.4	54.9	107.7	104.4	104.4
	27.1	42.1	49.8	50.6	56.5	56.9	112.0	107.7	107.7
	28.0	42.9	51.0	50.9	60.4	70.2	112.9	112.0	112.0
	28.5	43.3	52.3	53.3	63.7	72.5	114.2	112.9	112.9
	29.3	46.2	53.1	54.5	65.5	74.9	119.7	114.2	114.2
	29.7	46.3	53.4	55.3	67.4	77.0	81.8	114.7	114.7
	32.6	46.9	55.9	55.8	71.3	77.5	97.9	119.7	119.7
	35.0	47.5	60.0	58.0	71.4	77.6	99.1	137.3	122.6
	35.4	48.4	62.4	61.0	73.3	79.5	105.4	98.5	133.6
	35.5	50.9	68.1	66.4	73.4	80.7	109.7	112.4	137.3
	35.5	52.3	68.7	69.5	77.3	83.5	110.1	115.1	145.1
	36.0	59.3	70.8	71.9	82.0	84.4	111.7	127.3	159.3
	37.6	64.9	83.9	74.7	93.0	86.3	116.4	128.5	165.7
	38.0	67.6	85.4	77.5	93.7	98.5	119.9	129.1	172.3
	45.1	75.6	86.9	90.7	101.4	99.3	121.2	130.7	219.6
	47.0	81.5	89.4	91.3	102.4	111.8	126.5	136.3	239.4
	66.1	97.6	104.0	92.1	104.1	115.7	133.8	139.2	245.1

Values above the stepped line indicate failures; those below indicate non-failures.

For the sake of brevity the fatigue states of the fleet in the less interesting periods have been omitted.

Table 4

RESULTS OF THE 'MAXIMUM LIKELIHOOD' PARAMETER ESTIMATION FOR THE FIRST MODEL FLEET

Time in service (years)	Number of failures	Number of non-failures	Estimated mean $\{\log_{10}\}$	Standard error of estimated mean $\{\log_{10}\}$	Estimated standard deviation $\{\log_{10}\}$	Standard error of estimated standard deviation $\{\log_{10}\}$	Estimated covariance	Estimated coefficient of correlation	Estimated mean minus one standard error {FIU}	Estimated mean {FIU}	Estimated mean plus one standard error {FIU}
3	1	29	1.947	0.133	0.147	0.104	6.83×10^{-2}	0.933	65.1	88.4	120.0
3½	2	28	1.981	0.076	0.169	0.071	2.46×10^{-2}	0.861	80.4	95.7	114.0
4	3	27	2.024	0.132	0.191	0.087	5.45×10^{-2}	0.893	77.9	105.6	143.1
6	6	24	2.195	0.143	0.304	0.110	7.10×10^{-2}	0.847	112.7	156.7	218.1
7	9	21	2.157	0.092	0.291	0.081	2.80×10^{-2}	0.706	116.2	143.7	177.6
8	16	14	2.068	0.048	0.221	0.044	4.58×10^{-3}	0.408	104.5	116.7	130.5
9	22	8	2.035	0.037	0.191	0.030	9.95×10^{-4}	0.169	99.3	108.3	117.9
10	25	5	2.030	0.034	0.182	0.026	5.11×10^{-4}	0.107	99.1	107.2	116.0
'All failed' state	30	0	2.030	0.033	0.182	0.023	-2.35×10^{-5}	-0.0059	99.3	107.2	115.7

Table 5

RESULTS OF THE 'MAXIMUM LIKELIHOOD' PARAMETER ESTIMATION FOR THE SECOND MODEL FLEET

Time in service (years)	Number of failures	Number of non-failures	Estimated mean $\{\log_{10}\}$	Standard error of estimated mean $\{\log_{10}\}$	Estimated standard deviation $\{\log_{10}\}$	Standard error of estimated standard deviation $\{\log_{10}\}$	Estimated covariance	Estimated coefficient of correlation	Estimated mean minus one standard error {FIU}	Estimated mean {FIU}	Estimated mean plus one standard error {FIU}
2	1	29	1.813	0.099	0.104	0.065	2.57×10^{-2}	0.751	51.7	65.0	81.7
3	1	29	2.174	0.306	0.248	0.166	2.48×10^{-1}	0.924	73.8	149.1	301.6
3½	3	27	2.060	0.126	0.208	0.085	4.78×10^{-2}	0.842	85.7	114.7	153.1
4	5	25	1.990	0.067	0.156	0.048	1.18×10^{-2}	0.706	83.8	97.7	113.9
5	8	22	2.016	0.061	0.182	0.047	1.03×10^{-2}	0.671	90.1	103.8	119.4
6	12	18	1.999	0.042	0.165	0.034	3.56×10^{-3}	0.478	90.6	99.7	109.1
8	18	12	2.026	0.038	0.182	0.033	2.18×10^{-3}	0.325	97.3	106.2	115.9
9	21	9	2.022	0.034	0.174	0.028	1.01×10^{-3}	0.204	97.2	105.1	113.7
'All failed' state	30	0	2.030	0.033	0.182	0.023	-2.35×10^{-5}	-0.0059	99.3	107.2	115.7

Table 6

PROGRESS OF THE FATIGUE STATE OF THE FIRST MODEL FLEET ADAPTED TO PERMIT THE APPLICATION OF GUPTA'S TECHNIQUE

Time in service	6 years	7 years	8 years	9 years	10 years	All failed' state
Total number of failures	5	9	14	17	24	30
Total number of non-failures	25	21	16	13	6	0
Individual aircraft damage accumulations {FIU}	44.6	44.6	44.6	44.6	44.6	44.6
	47.9	47.9	47.9	47.9	47.9	47.9
	56.2	56.2	56.2	56.2	56.2	56.2
	57.1	57.1	57.1	57.1	57.1	57.1
	63.3	63.3	63.3	63.3	63.3	63.3
	63.3	74.8	74.8	74.8	74.8	74.8
	63.3	76.3	76.3	76.3	76.3	76.3
	63.3	87.5	87.5	87.5	87.5	87.5
	63.3	88.9	88.9	88.9	88.9	88.9
	63.3	88.9	96.9	96.9	96.9	96.9
	63.3	88.9	101.5	101.5	101.5	101.5
	63.3	88.9	102.4	102.4	102.4	102.4
	63.3	88.9	102.8	102.8	102.8	102.8
	63.3	88.9	104.4	104.4	104.4	104.4
	63.3	88.9	104.4	107.7	107.7	107.7
	63.3	88.9	104.4	112.0	112.0	112.0
	63.3	88.9	104.4	112.9	112.9	112.9
	63.3	88.9	104.4	112.9	114.2	114.2
	63.3	88.9	104.4	112.9	114.7	114.7
	63.3	88.9	104.4	112.9	119.7	119.7
	63.3	88.9	104.4	112.9	122.6	122.6
	63.3	88.9	104.4	112.9	133.6	133.6
	63.3	88.9	104.4	112.9	137.3	137.3
	63.3	88.9	104.4	112.9	145.1	145.1
	63.3	88.9	104.4	112.9	145.1	159.3
	63.3	88.9	104.4	112.9	145.1	165.7
	63.3	88.9	104.4	112.9	145.1	172.3
	63.3	88.9	104.4	112.9	145.1	219.6
	63.3	88.9	104.4	112.9	145.1	239.4
	63.3	88.9	104.4	112.9	145.1	245.1

Values above the stepped line indicate failures; those below non-failures.

Table 7

COMPARISON BETWEEN THE RESULTS OF TWO ANALYSES OF THE FIRST MODEL FLEET; ONE BY GUPTA'S TECHNIQUE, THE OTHER BY THE 'MAXIMUM LIKELIHOOD' METHOD

Time in service	Number of failures of: original	Number of failures of: changed	Number of non-failures of: original	Number of non-failures of: changed	RESULTS FOR ANALYSIS BY GUPTA'S TECHNIQUE: Estimated mean {$\log_{10}$}	Standard error of estimated mean {$\log_{10}$}	Estimated standard deviation {$\log_{10}$}	Standard error of estimated standard deviation {$\log_{10}$}	Estimated covariance	RESULTS FOR 'MAXIMUM LIKELIHOOD' ANALYSIS: Estimated mean {$\log_{10}$}	Standard error of estimated mean {$\log_{10}$}	Estimated standard deviation {$\log_{10}$}	Standard error of estimated standard deviation {$\log_{10}$}	Estimated covariance
6 years	6	5	24	25	1.932	0.070	0.135	0.052	3.12×10^{-3}	2.195	0.143	0.304	0.110	7.10×10^{-2}
7 years	9	9	21	21	2.016	0.068	0.212	0.058	2.77×10^{-3}	2.157	0.092	0.291	0.081	2.80×10^{-2}
8 years	16	14	14	16	2.042	0.046	0.196	0.042	9.24×10^{-4}	2.068	0.048	0.221	0.044	4.58×10^{-3}
9 years	22	17	8	13	2.032	0.039	0.186	0.035	4.99×10^{-4}	2.035	0.037	0.191	0.030	9.95×10^{-4}
10 years	25	24	5	6	2.025	0.034	0.178	0.027	1.13×10^{-4}	2.030	0.034	0.182	0.026	5.11×10^{-4}
'All failed' state	30	30	0	0	2.031	0.034	0.186	0.024	0.00×10^{0}	2.030	0.033	0.182	0.023	-2.35×10^{-5}

SYMBOLS

c	total number of uncracked aircraft in a fleet of size $n + c$
d	increment of z used in estimating the quantity $\frac{\partial^2 y}{\partial z\, \partial x}$
E	denotes the expectation of a variate
h	increment of x used in estimating the quantities $\frac{d^2 y}{dx^2}$ and $\frac{\partial^2 y}{\partial z\, \partial x}$
$p(y)$	probability density function of y
P	probability that, with postulated parameters m and s, the present situation should be observed
P_i	$(i = 1,2...n)$ probability that the ith aircraft should have failed at a life between $x_i - \frac{\delta x}{2}$ and $x_i + \frac{\delta x}{2}$
$P_{(n+i)}$	$(i = 1,2...c)$ probability that the $(n + i)$th aircraft $(i = 1,2...c)$ should not have failed by time $x_{(n+i)}$
m	postulated mean of the parent normal population
m_1	estimated mean of the parent normal population from a sample of all failed members
$\hat{m}$	best estimate of the mean of the parent population derived by the Maximum Likelihood technique
s	postulated standard deviation of the parent normal population
s_1	estimated standard deviation of the parent normal population from a sample of all failed members
$\hat{s}$	best estimate of the standard deviation of the parent population derived by the Maximum Likelihood technique
t	Student's 't' distribution
x	the time to occurrence of a crack on a general aircraft
x_i	the time to occurrence of a crack at the considered position on the ith aircraft $(i = 1,2...n)$
$x_{(n+i)}$	the length of time for which an uncracked aircraft has flown $(i = 1,2...c)$
y	$\log x = \log$ (time to occurrence of a crack on the general aircraft)
y_i	$\log x_i \quad (i = 1,2...n)$
$y_{(n+i)}$	$\log (x_{n+i}) \; (i = 1,2...c)$
θ	defines the two parameters m and s
$\hat{\theta}$	defines the two values $\hat{m}$ and $\hat{s}$
μ	true mean of the parent normal population
σ	true standard deviation of the parent normal population
χ^2	'chi-squared' distribution

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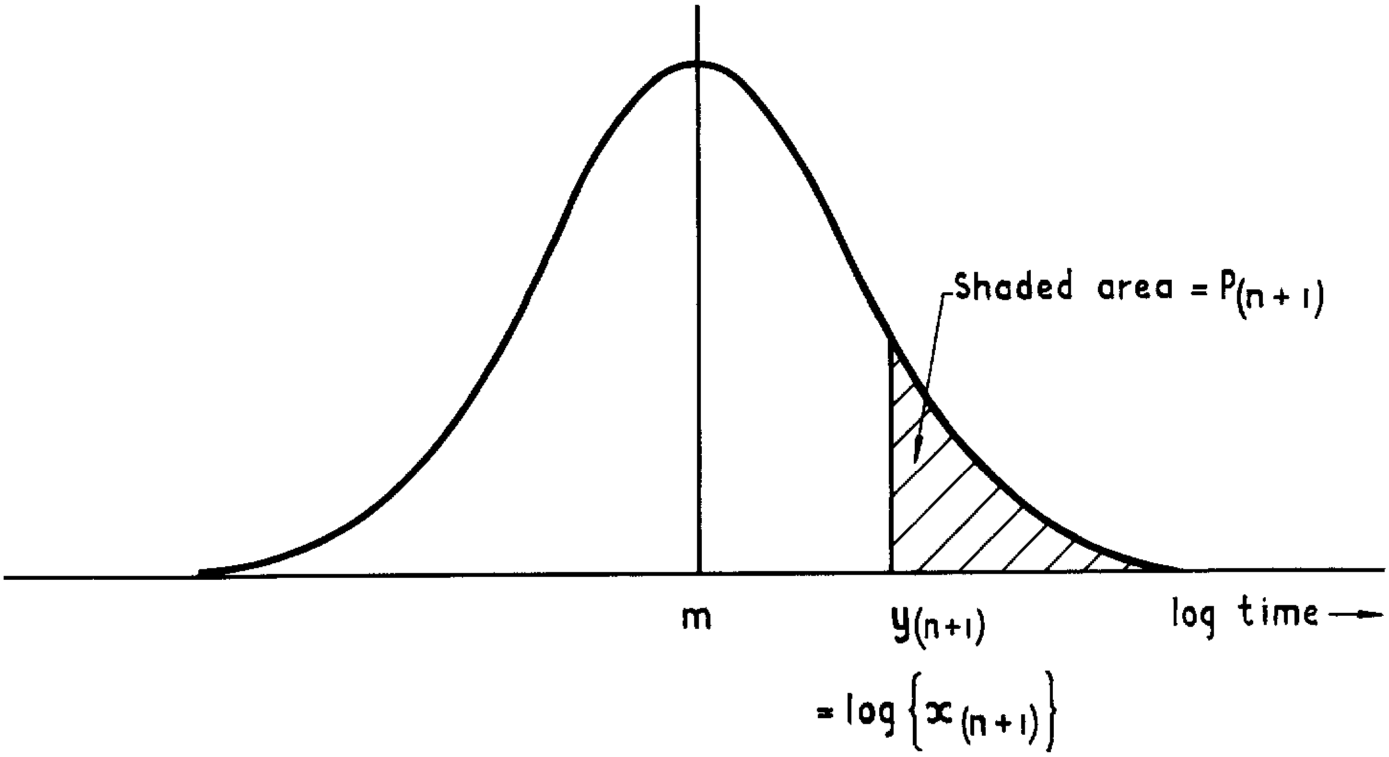

Normal distribution with mean m and standard deviation S

Fig 1 $P_{(n+1)}$. The probability of aircraft (n+1) remaining uncracked until time $x_{(n+1)}$

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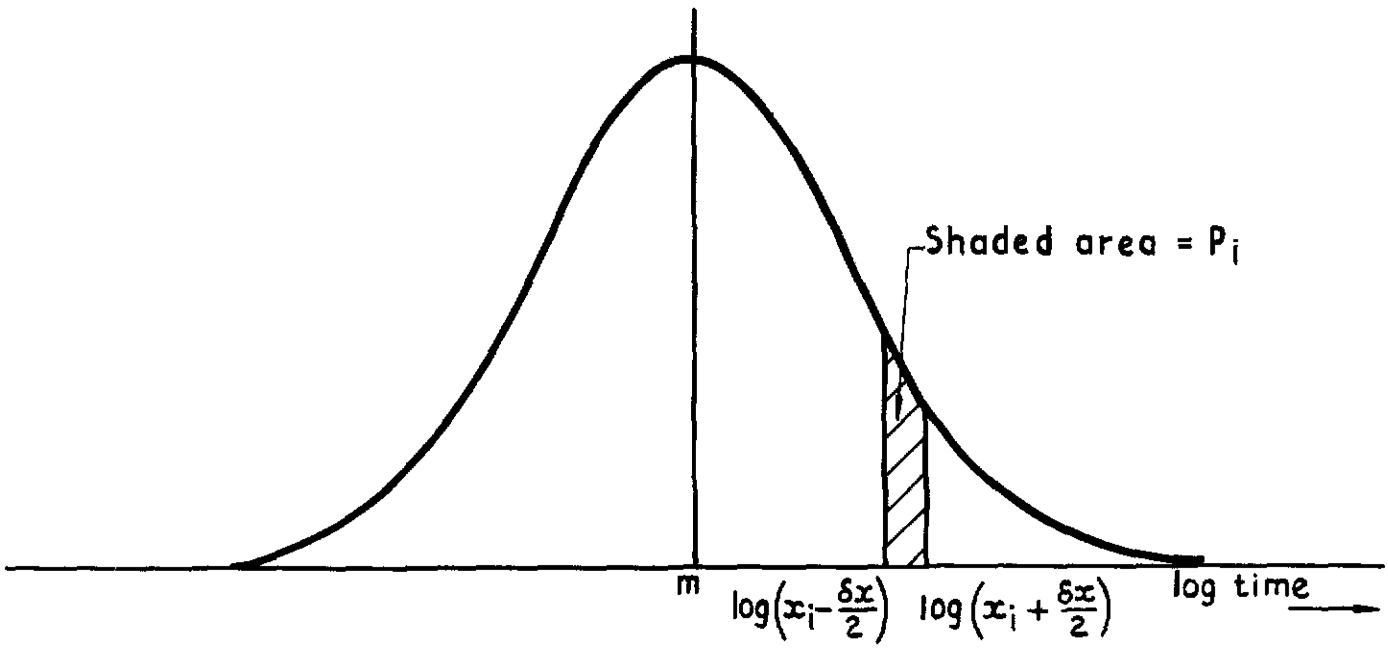
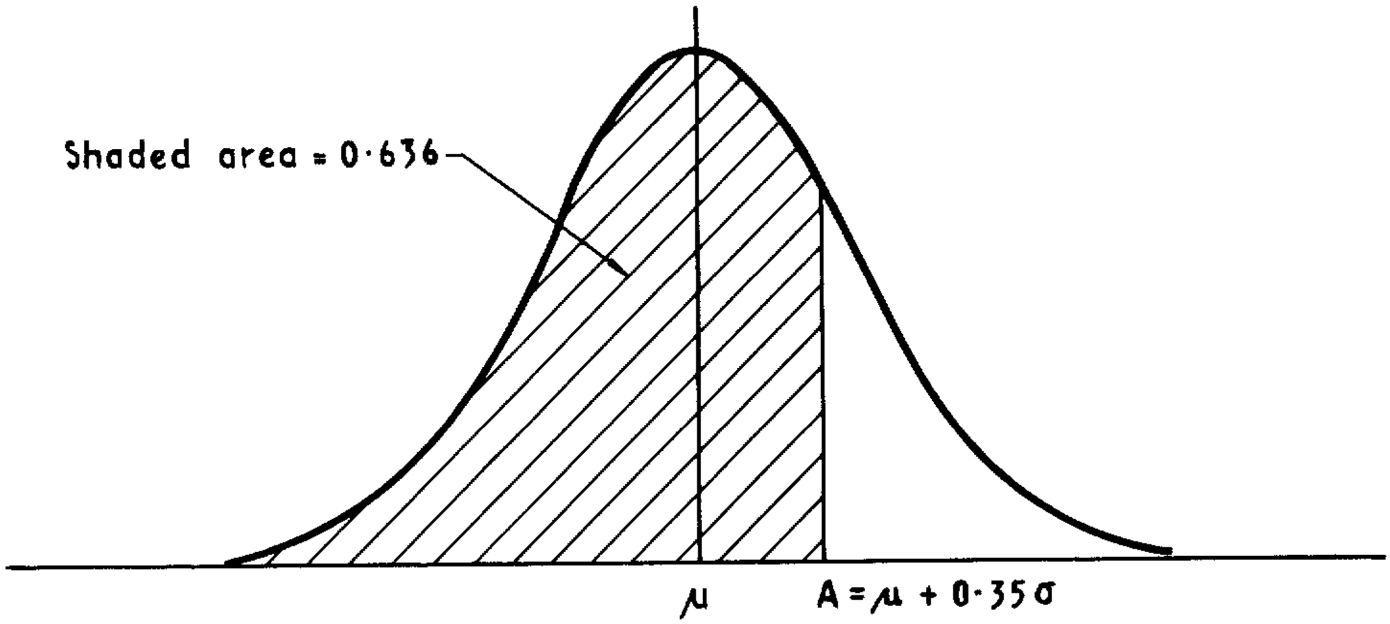


Normal distribution with mean m and standard deviation S

Fig.2 P_i. The probability of aircraft i cracking between times $x_i - \frac{\delta x}{2}$ and $x_i + \frac{\delta x}{2}$

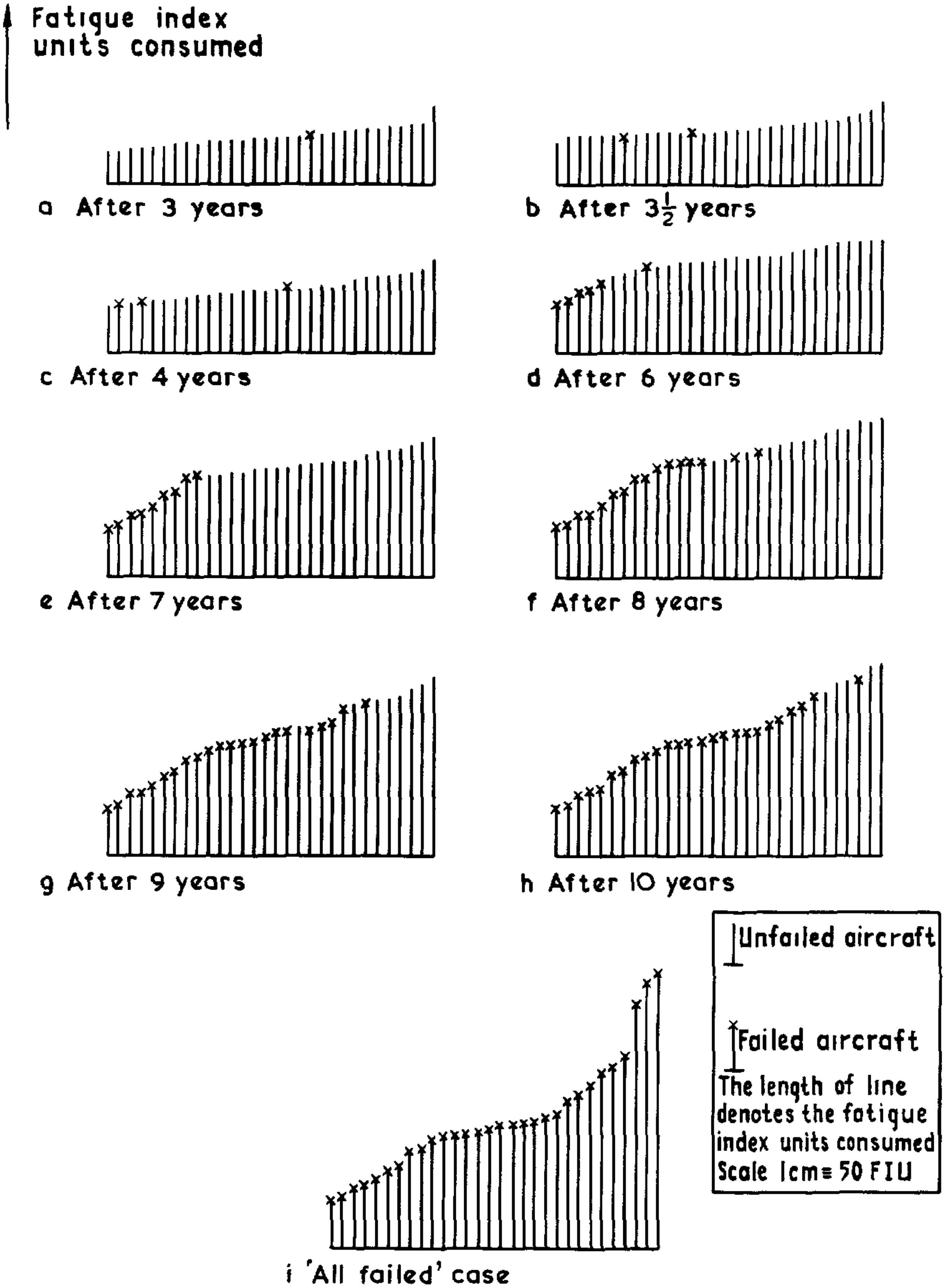


1st random number = 636 ie a cumulative probability of 0·636
Area under normal distribution up to, to deviate +0·35σ = 0·636
Thus if the population mean is 2 and the population standard deviation is 0·17,

$$A = \mu + 0\cdot35\,\sigma = 2\cdot0595$$

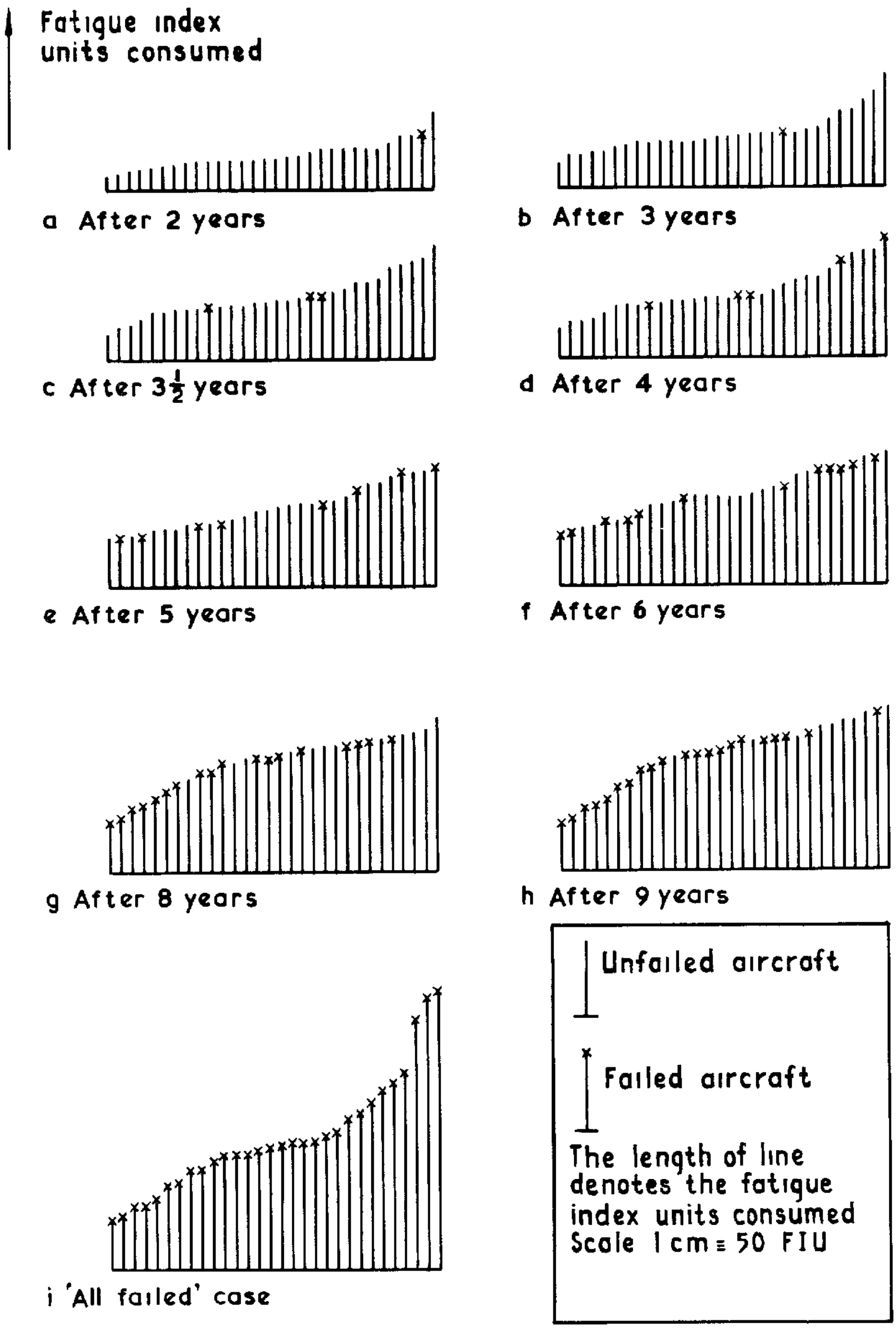
Hence the random number 636 gives a failure at antilog_{10} 2·0595 = 114·7

Fig. 3 Derivation of the individual failure times for the model fleet



Data from Table 2

Fig.4a-i Progress of the fatigue state of the first model fleet



Data from Table 3

Fig.5 a-i Progress of the fatigue state of the second model fleet

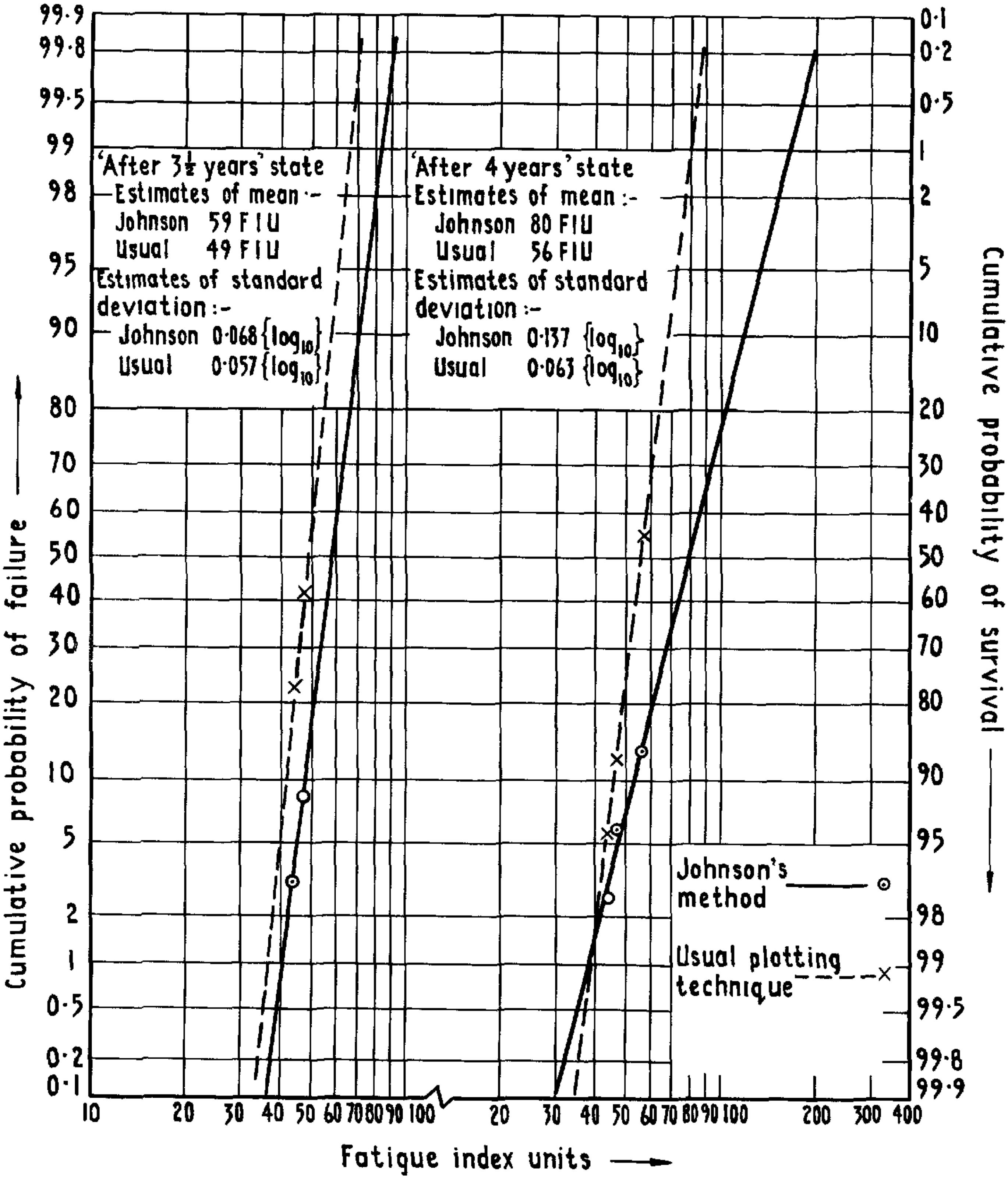


Fig. 6 A comparison of two methods of graphical analysis of the first model fleet

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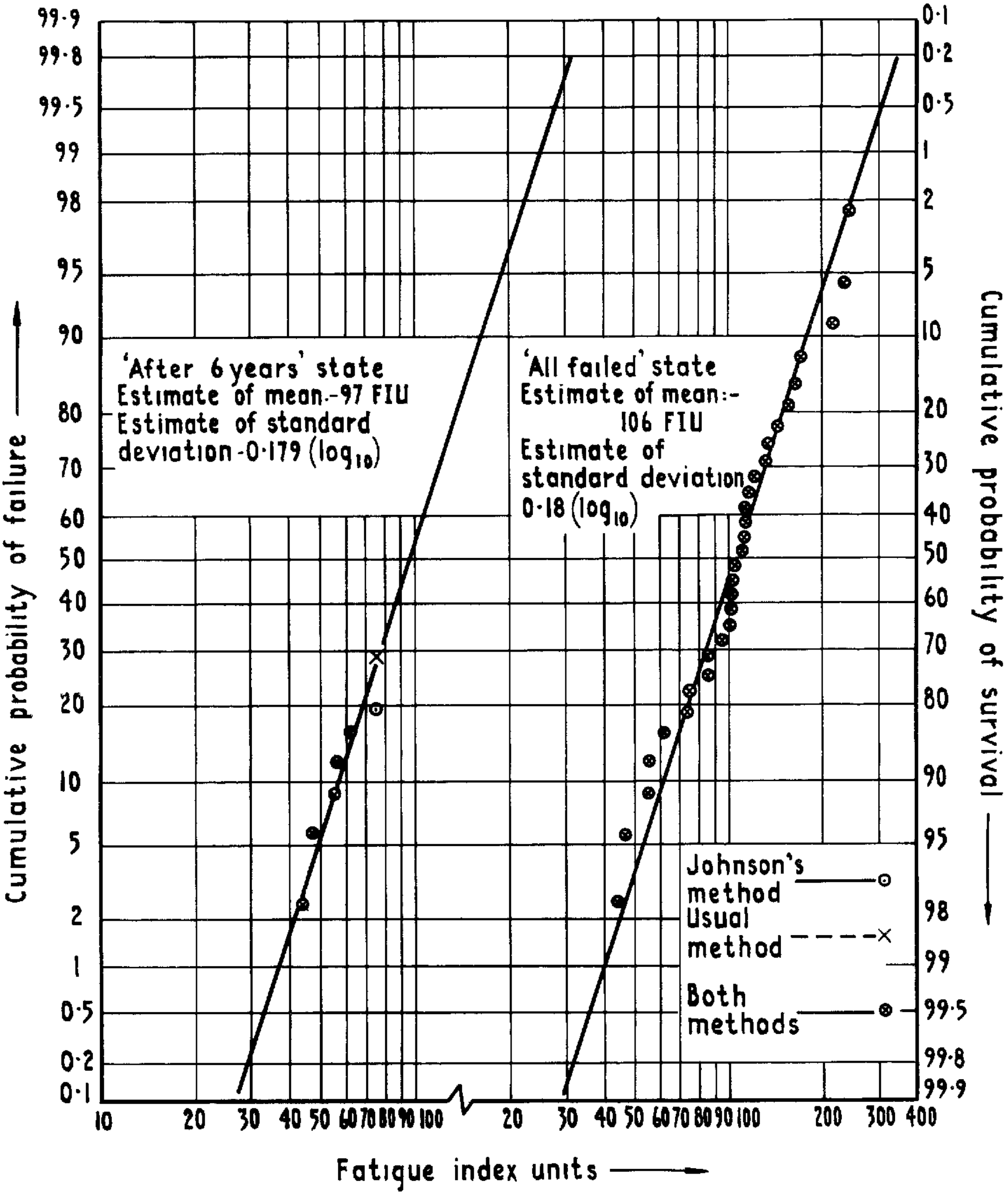

Fig.7 A comparison of two methods of graphical analysis of the first model fleet

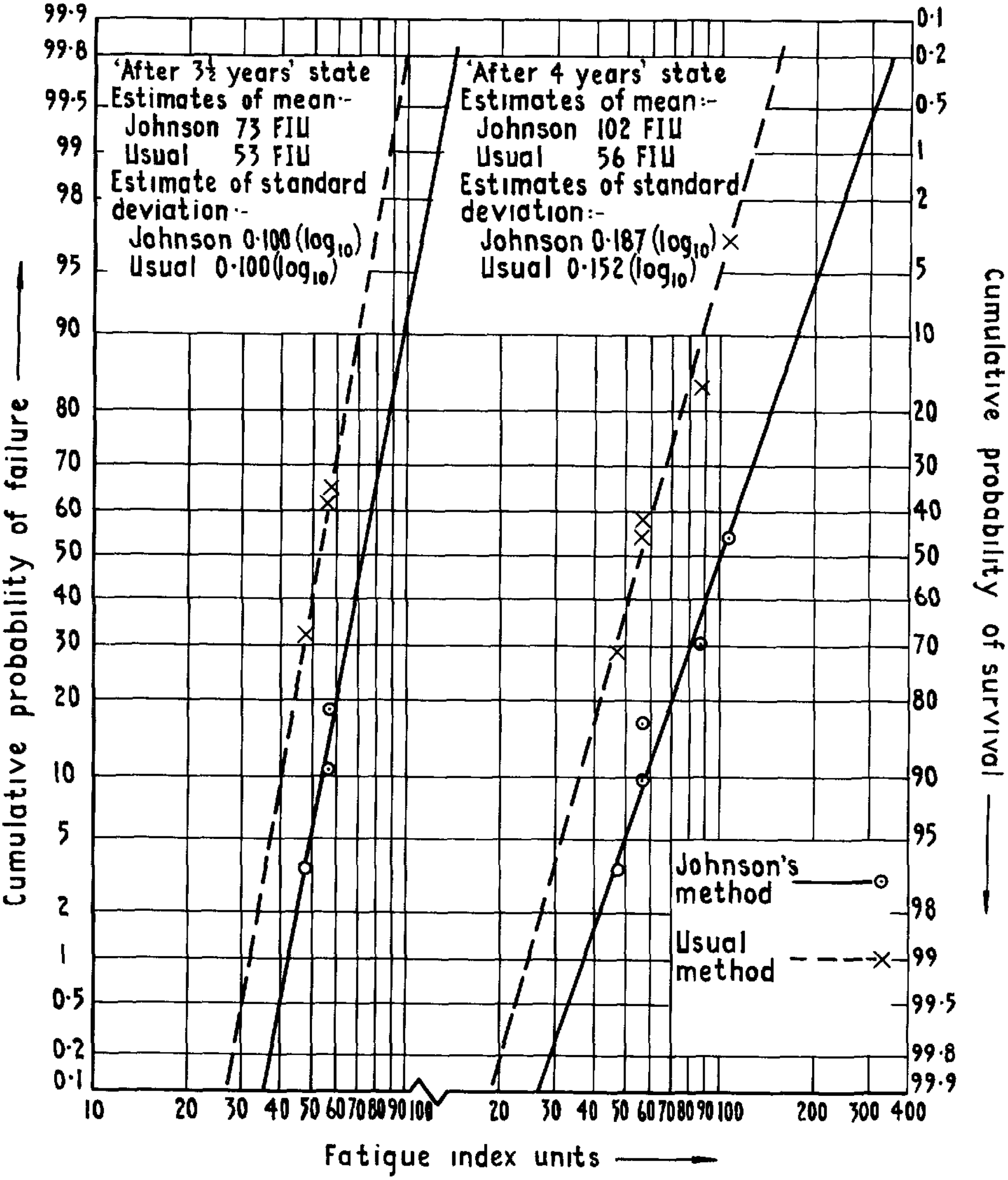

Fig.8 A comparison of two methods of graphical analysis of the second model fleet

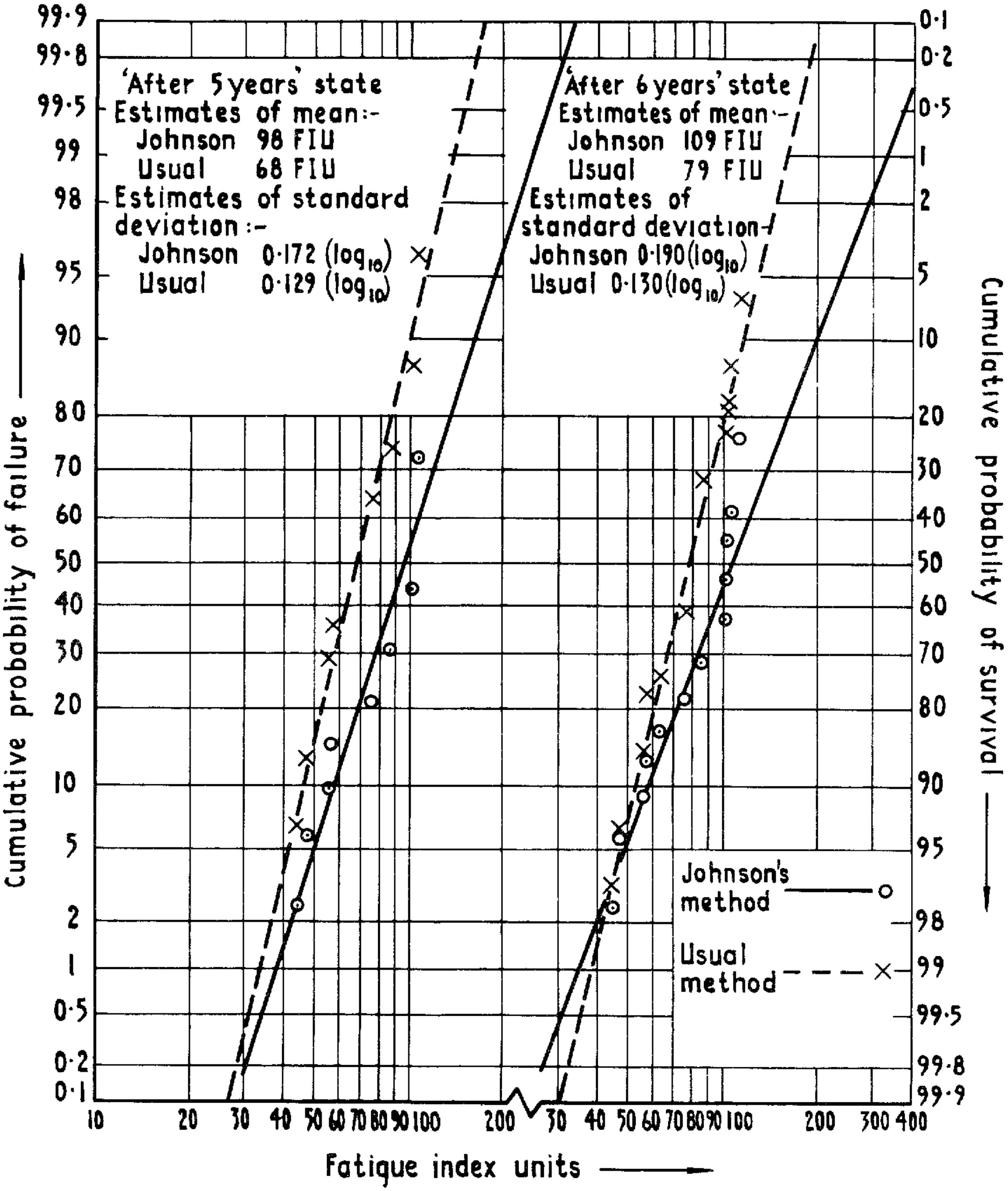

Fig.9 A comparison of two methods of graphical analysis of the second model fleet

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Use of the most suitable of the mentioned methods of analysis to correlate early service failures with a test failure should enable a check to be made on the validity of the fatigue monitoring process being applied to the service aircraft.

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