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# Notes on the Diffraction of Sound

*by*

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# NOTES ON THE DIFFRACTION OF SOUND

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SUMMARY

Two problems are considered, both involving a semi-infinite plane with adjacent source:

(1) the system at rest (Part A);

(2) the system in motion through the fluid with either a stationary or moving observer (Part B).

Asymptotic forms are given for large wave number and for the observer in the far field.

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† Replaces RAE Technical Report 69283 - ARC 32130.

## CONTENTS

## INTRODUCTION

Diffraction of sound by a semi-infinite plane from an adjacent point source is considered in the present paper, both when the source and plane are at rest and moving. Macdonald's solution[1] for the source at rest is used, and some asymptotic forms are derived for the sound pressure in the far field and at large values of the wave number. Since this part of the analysis is self-contained it is given first as Part A of the present paper.

Part B concerns the moving source and plane, and in it a general transformation is first derived and then applied in two examples, both involving the diffraction past a semi-infinite plane and using the results of Part A. In the first of these examples the plane is moving normal to its edge whereas in the second example it moves in an inclined direction. In each example the results are given both for an observer at rest relative to the fluid and moving with the source and plane, conditions which apply respectively to an aircraft in flight past a stationary observer, and to a model aircraft in a wind-tunnel with a stationary observer.

## PART A

## SOURCE NEAR TO A SEMI-INFINITE PLANE

### 1 PRELIMINARIES

We take the plane as $y = 0,\ x > 0$ with the z axis along the edge, and take cylindrical polar coordinates $(r_o, \phi_o, 0)$ for the source S and $(r, \phi, z)$ for the receiving point P. We let

$$PS = R = \{r^2 + r_o^2 - 2r\,r_o \cos(\phi - \phi_o) + z^2\}^{\frac{1}{2}} \tag{1}$$

and denote the projection of PS on the plane $z = 0$ by R' where

$$R' = \{r^2 + r_o^2 - 2r\,r_o \cos(\phi - \phi_o)\}^{\frac{1}{2}} \tag{2}$$

*(see Fig.1).*

We also consider the 'image' source $\bar{S}$ whose coordinates are $(r_o, -\phi_o, 0)$ and write

$$\bar{R} = P\bar{S} = \{r^2 + r_o^2 - 2r\,r_o \cos(\phi + \phi_o) + z^2\}^{\frac{1}{2}} \tag{3}$$

$$\bar{R}' = \{r^2 + r_o^2 - 2r\,r_o \cos(\phi + \phi_o)\}^{\frac{1}{2}} \ . \tag{4}$$

For future reference we also note that from equations (2) and (4) we can show that

$$2\sqrt{r\,r_o}\ \left|\cos \tfrac{1}{2}(\phi - \phi_o)\right| = \{(r + r_o)^2 - R'^2\}^{\frac{1}{2}} \tag{5}$$

$$2\sqrt{r\,r_o}\ \left|\cos \tfrac{1}{2}(\phi + \phi_o)\right| = \{(r + r_o)^2 - \bar{R}'^2\}^{\frac{1}{2}} \ . \tag{6}$$

### 2 GENERAL FORMULA FOR SOUND PRESSURE

If the sound pressure from an undisturbed source is written

$$p = \frac{A\,e^{ik(ct-R)}}{R} \ ,$$

where c is the velocity of sound and k the wave number then Macdonald[1] has shown that in the presence of a semi-infinite plane

$$p = A\,e^{ikct}\,U$$

where

$$U = \frac{i\,k}{\pi}\int_{-\infty}^{\xi_o} K_1\,(i\,k\,R\cosh\xi)\,d\xi + \frac{i\,k}{\pi}\int_{-\infty}^{\bar{\xi}_o} K_1\,(i\,k\,\bar{R}\cosh\xi)\,d\xi \quad (7)$$

$$= V + \bar{V}\ \text{(say)}\ .$$

Here

$$\sinh\xi_o = \frac{2(r\,r_o)^{\frac{1}{2}}}{R}\cos\tfrac{1}{2}(\phi - \phi_o) = \omega_o\ \text{(say)}$$

$$\sinh\bar{\xi}_o = \frac{2(r\,r_o)^{\frac{1}{2}}}{\bar{R}}\cos\tfrac{1}{2}(\phi + \phi_o) = \bar{\omega}_o\ \text{(say)}\ .$$

(Note: Macdonald's definition of $r$ is different to that used here. The result is in our present notation.)

Nowadays one would use the function $H$ instead of $K$, when the argument is imaginary. We have in fact

$$K_1\,(i\,z) = -\tfrac{1}{2}\pi\,H_1^{(2)}\,(z)\ .$$

It is easy to see that when $P$ is in the geometric shadow of $S$ then both $\xi_o$ and $\bar{\xi}_o$ are negative. We shall confine our attention to this case.

## 3 ASYMPTOTIC VALUES FOR LARGE $R/r_o$ AND $k\,r_o$

Consider the first integral in equation (7), when $P$ is in the geometric shadow of $S$, so that $\xi_o$ is negative. It may be written

$$V = -\tfrac{1}{2}i\,k\int_{-\infty}^{-|\xi_o|} H_1^{(2)}\,(k\,R\cosh\xi)\,d\xi\ .$$

Putting $\sinh\xi = t$ we have

$$V = -\tfrac{1}{2}i\,k\int_{-\infty}^{-|\omega_o|} \frac{H_1^{(2)}\,\{k\,R\sqrt{1 + t^2}\}}{\sqrt{1 + t^2}}\,dt$$

and the integral $\int_{-\infty}^{-|\omega_o|}$ may be written $\int_{-\infty}^{o} + \int_{o}^{-|\omega_o|}$. The first of these can be evaluated by the use of equation 7.14(49) of Erdelyi[2] and in the second $t$ is replaced by $-t$ to give

$$V = \frac{e^{-ikR}}{2R} + \tfrac{1}{2} i\, k \int_{o}^{|\omega_o|} \frac{H_1^{(2)}\{k\, R\sqrt{1 + t^2}\}}{\sqrt{1 + t^2}}\, dt \,. \qquad (8)$$

Now $k\,R$ is large and so we may use the asymptotic series for $H_1^{(2)}$, and evaluate the integrals term by term.

The result is (see Appendix)

$$V = \frac{e^{-ikR}}{R}\left\{\tfrac{1}{2} - \frac{1+i}{2}\left[C(\beta) - i\,S(\beta)\right] - \sqrt{\frac{2}{\pi\, k\, R}}\; e^{-2ikRy'^2 + \frac{3i\pi}{4}}\left[-\frac{3y'}{8} + \text{terms of order } y'^3 \text{ and } (kR)^{-1}\right]\right\}. \qquad \ldots\ldots (9)$$

Here

$$y'^2 = \tfrac{1}{2}\{\sqrt{1 + \omega_o^2} - 1\}\,,$$

$$\beta = 2\sqrt{\frac{k\,R}{\pi}}\, y'\,,$$

$$\omega_o = 2\frac{\sqrt{r\, r_o}}{R}\cos\tfrac{1}{2}(\phi - \phi')\,.$$

$C(\beta)$ and $S(\beta)$ are Fresnel integrals

$$\begin{matrix} C(\beta) \\ S(\beta) \end{matrix} = \int_o^\beta \begin{matrix} \cos \\ \sin \end{matrix}\left(\frac{\pi\, t^2}{2}\right) dt\,.$$

This expression can be rearranged by the use of equation (5). We find

$$\omega_o^2 = \left(\frac{r + r_o}{R}\right)^2 - \frac{R'^2}{R^2}$$

and since $R'^2 = R^2 - z^2$ this leads to

$$2k\,R\,y'^2 = k\,\{\sqrt{z^2 + (r + r_o)^2} - R\}$$

$$\beta^2 = \frac{2k}{\pi}\,\{\sqrt{z^2 + (r + r_o)^2} - R\}\ .$$

Now $w_o$ is small and so is $y'$ but $2k\,R\,y'^2$ and $\beta$ may be large, though $\beta^2/k\,R$ is small. We may write finally

$$V = \frac{e^{-ikR}}{R}\left\{\tfrac{1}{2} - \frac{1+i}{2}\left[C(\beta) - i\,S(\beta)\right] + (-1 + i)\,e^{-\frac{i\pi\beta^2}{2}}\,(3\beta/16k\,R + \ldots)\right\} \qquad \ldots\ldots (10)$$

with

$$\beta^2 = \frac{2k}{\pi}\,\{\sqrt{z^2 + (r + r_o)^2} - R\}\ . \qquad (11)$$

There is a similar form for $\overline{V}$ with $R$ replaced by $\overline{R}$ in equations (10) and (11).

When $\beta$ is large we may write

$$C(\beta) - i\,S(\beta) = \frac{1-i}{2} + \frac{i}{\pi\,\beta}\,e^{-\frac{i\pi\beta^2}{2}} + O(\beta^{-3})$$

and then

$$V = \frac{e^{-ikR}}{R\,\sqrt{2}}\,e^{-\frac{i\pi\beta^2}{2} - i\frac{\pi}{4}}\left[\frac{1}{\pi\,\beta} - \frac{3\beta}{8k\,R}\right]$$

$$= \frac{e^{-ik\sqrt{z^2+(r+r_o)^2} - i\frac{\pi}{4}}}{R\,\sqrt{2}}\left[\frac{1}{\pi\,\beta} - \frac{3\beta}{8k\,R}\right],$$

noting that

$$i\,k\,R + \frac{i\,\pi\,\beta^2}{2} = i\,k\sqrt{z^2 + (r + r_o)^2}\ .$$

Combining the two terms $V$ and $\overline{V}$ we obtain

$$p = \frac{A\, e^{ik[ct-\sqrt{z^2+(r+r_o)^2}]-i\frac{\pi}{4}}}{R\sqrt{2}} \left[\frac{1}{\pi\,\beta} - \frac{3\beta}{8k\,R} + \frac{R}{\overline{R}}\left\{\frac{1}{\pi\,\overline{\beta}} - \frac{3\overline{\beta}}{8k\,\overline{R}}\right\}\right] \qquad (12)$$

with $\beta$ given by (11), and $\overline{\beta}$ by (11) with $R$ replaced by $\overline{R}$.

It may sometimes be sufficient to ignore the terms of the form $3\beta/8k\,R$ compared with $1/\pi\,\beta$. In one example $\beta$ was about 25, whilst $R$ and $k$ were each 100 so that the ratio of the second to the first terms was about 0.06.

## 4 ATTENUATION

The sound intensity is proportional to $|p|^2$ and so without the screen the intensity is proportional to $A^2/R^2$.

If we take the attenuation to be the inverse of the ratio of the actual sound intensity to what it would have been without the screen then we find that the attenuation in the case $\beta$ large is

$$2\left[\frac{1}{\pi\,\beta} - \frac{3}{8k\,R} + \frac{R}{\overline{R}}\left(\frac{1}{\pi\,\overline{\beta}} - \frac{3\overline{\beta}}{8k\,\overline{R}}\right)\right]^{-2}$$

and this can be considerable, as numerical examples indicate.

However, there may be circumstances in which $\beta$ is not large and then equation (12) is not applicable. For instance if $R$, $r$ and $k$ are large, with $r_o$ of order unity we find from the results of section 1, that

$$\sqrt{z^2 + (r + r_o)^2} - R \simeq \frac{r\, r_o\, [1 + \cos\,(\phi - \phi_o)]}{\sqrt{z^2 + r^2}} \quad .$$

Hence

$$\left.\begin{matrix}\beta^2 \\ \overline{\beta}^2\end{matrix}\right\} \simeq \frac{2k\, r\, r_o}{\pi\sqrt{z^2 + r^2}}\,[1 + \cos\,(\phi \mp \phi_o)] \ ,$$

and so if $z = O(1)$ we have $\beta^2 = O(2k\,r_o)$, $\overline{\beta}^2 = O(2k\,r_o)$ unless $\cos\,(\phi \mp \phi_o)$ is near to $-1$. Both of these possibilities are excluded when the receiver is in the shadow but not too near the edge of the shadow. Thus, excluding this situation, $\beta$ and $\overline{\beta}$ will be large unless $r_o$ is very small; that is, unless the source is so close to the edge that $k\,r_o$ is $O(1)$ in spite of $k$ being large. However, if $z$ is large, then $\beta$ could be small without $r_o$ being small. In such a case equation (12) is not applicable but equation (10) is, and can be written approximately

$$V = \frac{e^{-ikR}}{R}\left\{\frac{1}{2} - \frac{1+i}{2}\left[C(\beta) - i\,S(\beta)\right]\right\} . \qquad (13)$$

In this case the attenuation will not be so great, [$C(\beta)$ and $S(\beta)$ are small if $\beta$ is small]. This implies (see Fig.2) that if P is in the shadow of S and is in the position N the attenuation will be large, but if it moves parallel to the edge to a position such as $P_1$ the attenuation will not be so great. (Remember that in computing the attenuation we are comparing the intensity at $P_1$ with what it would have been without the screen.)

## 5 THE VELOCITY POTENTIAL

The disturbance potential $\phi$ satisfies the same equation and the same boundary conditions as the disturbance pressure p. Hence the solution given above applies equally well for the velocity potential as it does for the pressure. If the undisturbed density and velocity are $\rho_o$ and $\underline{V}$ and the pressure and velocity increments are p and $\underline{u}$ which are supposed to be small then the linearized form of Bernoulli's equation is

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho_o} + \underline{u}\,\underline{V} = 0$$

and so for zero undisturbed velocity, as in the above work we have

$$p = -\rho_o \frac{\partial\phi}{\partial t} ,$$

which gives the relation between the pressure and the velocity potential.

## 6 SOUND INTENSITY

The intensity vector $\underline{I}$ is defined[3,4] to be the time average of the flux of acoustic energy across unit area and is equal to the time average of p $\underline{u}$, where $\underline{u}$ is the velocity (or the velocity increment if the fluid is moving). Hence we have

$$\underline{I} = -\rho_o \frac{\partial\phi}{\partial t} \nabla\phi .$$

For a source we have

$$\phi = \frac{A\,e^{i\omega\left(t-\frac{R}{c}\right)}}{R}, \qquad |\nabla\phi| = -\frac{A\,i\,\omega}{c\,R}\,e^{i\omega\left(t-\frac{R}{c}\right)} + 0\left(\frac{1}{R^2}\right)$$

$$p = -\rho_o \frac{\partial\phi}{\partial t} = -\rho_o \frac{A\,i\,\omega}{R}\,e^{i\omega\left(t-\frac{R}{c}\right)} .$$

Hence $|\nabla\phi|$ means the magnitude of the vector $\nabla\phi$ .

When the complex notation is used as above we must be careful when dealing with second order quantities which the products are. We are really taking the product of the real parts of complex quantities and then averaging over time. Hence, for the above source, the magnitude of the intensity vector is

$$\frac{\rho_o\,A^2\,\omega^2}{2c\,R^2} ,$$

and this may be written

$$\frac{|p|^2}{2c\,\rho_o} .$$

This is the result we have used in section 4 above.

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## Appendix

## THE ASYMPTOTIC EXPANSION

We have to evaluate

$$V = \frac{e^{-ikR}}{2R} + W ,$$

where

$$W = \tfrac{1}{2} i k \int_0^{|\omega_o|} \frac{H_1^{(2)} \{k R \sqrt{1 + t^2}\}}{\sqrt{1 + t^2}} dt .$$

We write $\sqrt{1 + t^2} = 1 + 2y^2$ and find

$$W = i k \int_0^{y'} \frac{H_1^{(2)} \{k R (1 + 2y^2)\}}{\sqrt{1 + y^2}} dy ,$$

where

$$y'^2 = \tfrac{1}{2} \{\sqrt{1 + \omega_o^2} - 1\} ,$$

and $y'$ is small if $R$ is large compared with $r_o$.

Now, the asymptotic expansion of the Bessel function $H$ is[2]

$$H_1^{(2)}(z) = e^{\frac{3i\pi}{4}} \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[1 - \frac{3i}{8z} + \frac{15}{128z^2} - \dots\right] e^{-iz}$$

and hence we have

$$W = \frac{i k}{(k R)^{\frac{1}{2}}} \sqrt{\frac{2}{\pi}} e^{\frac{3i\pi}{4}} e^{-ikR} \int_0^{y'} \frac{e^{-2ikRy^2}}{(1 + y^2)^{\frac{1}{2}} (1 + 2y^2)^{\frac{1}{2}}} \left\{1 - \frac{3i}{8k R (1 + 2y^2)} + \frac{15}{128k^2 R^2 (1 + 2y^2)^2} \dots\right\} dy$$

$$= \frac{i k}{(k R)^{\frac{1}{2}}} \sqrt{\frac{2}{\pi}} e^{\frac{3i\pi}{4}} e^{-ikR} \int_0^{y'} e^{-2ikRy^2} \left(1 - \frac{3y^2}{2} + \frac{19y^4}{8} - \frac{63y^6}{16} + \dots\right) \times$$

$$\left(1 - \frac{3i}{8k R} + \frac{15}{128k^2 R^2} + \frac{3i y^2}{4k R} - \frac{3i y^4}{2k R} + \dots\right) dy \quad (14)$$

If

$$\int_0^{y'} e^{-2ikRy^2}\,dy = D$$

then

$$\int_0^{y'} y^2\, e^{-2ikRy^2}\,dy = \frac{i}{4k\,R}\,(y'\,E - D)$$

where

$$E = e^{-2ikRy'^2} ,$$

on integrating by parts.

Further products by even powers of $y$ can be integrated by parts in succession.

The algebra is extremely tedious, and we shall omit the details. We go as far as terms in $y'^3$ and terms in $1/k^2\,R^2$, since $y'$ is small and $k\,R$ is large. We find that the integral in equation (14), excluding its multiplier, is

$$D\,[1 + O(k^{-3}\,R^{-3})] + e^{-2ikRy'^2}\left\{\frac{1}{k\,R}\left[-\frac{3i\,y'}{8} + \frac{19i}{32}\,y'^3 + O(y'^5)\right] + \frac{1}{k^2\,R^2}\left[\frac{15y'}{128} - \frac{45y'^3}{128} + O(y'^5)\right]\right\}$$

and

$$D = \int_0^{y'} e^{-2ikRy^2}\,dy$$

$$= \tfrac{1}{2}\sqrt{\frac{\pi}{k\,R}}\,[C(\beta) - i\,S(\beta)] ,$$

with

$$\beta = 2\sqrt{\frac{k\,R}{\pi}}\,y' ,$$

$C(\beta)$ and $S(\beta)$ being Fresnel integrals.

Now multiply by the factor outside the integral in equation (14) and simplify. We finally obtain

$$V = \frac{e^{-ikR}}{2R} + W$$

$$= \frac{e^{-ikR}}{R}\left\{\frac{1}{2} - \frac{1+i}{2}\left[C(\beta) - i\,S(\beta)\right] - \sqrt{\frac{2}{\pi\,k\,R}}\, e^{-2ikRy'^2 + \frac{3i\pi}{4}} \left[-\frac{3y'}{8} + \frac{19y'^3}{32} - \frac{1}{k\,R}\left(\frac{15y'}{128} - \frac{45y'^3}{128}\right)\right]\right\}.$$

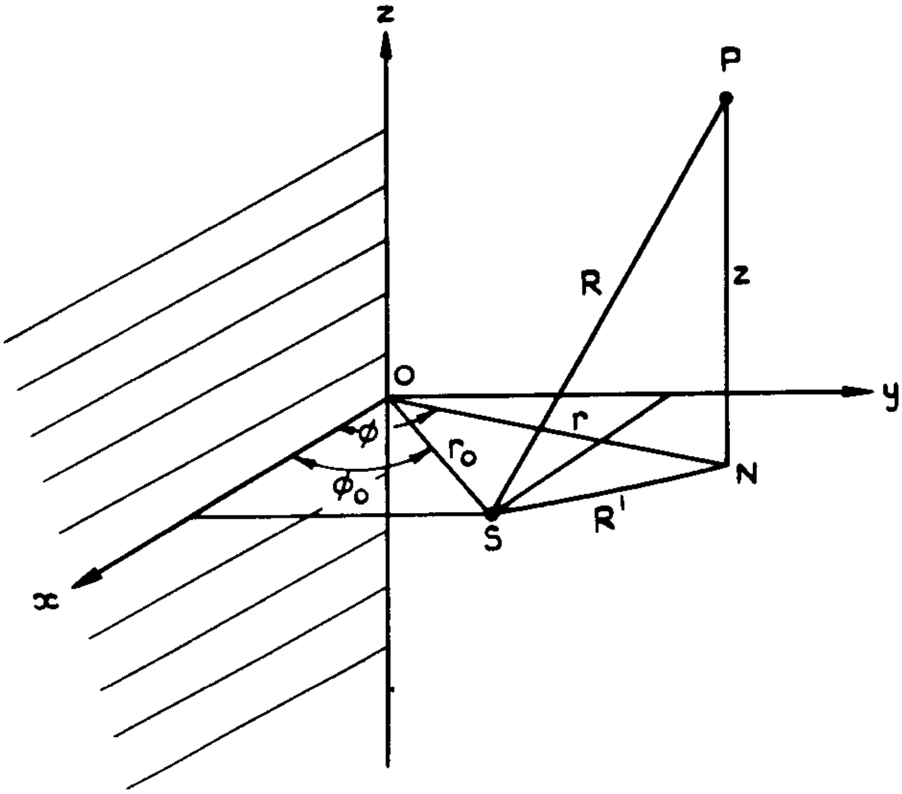
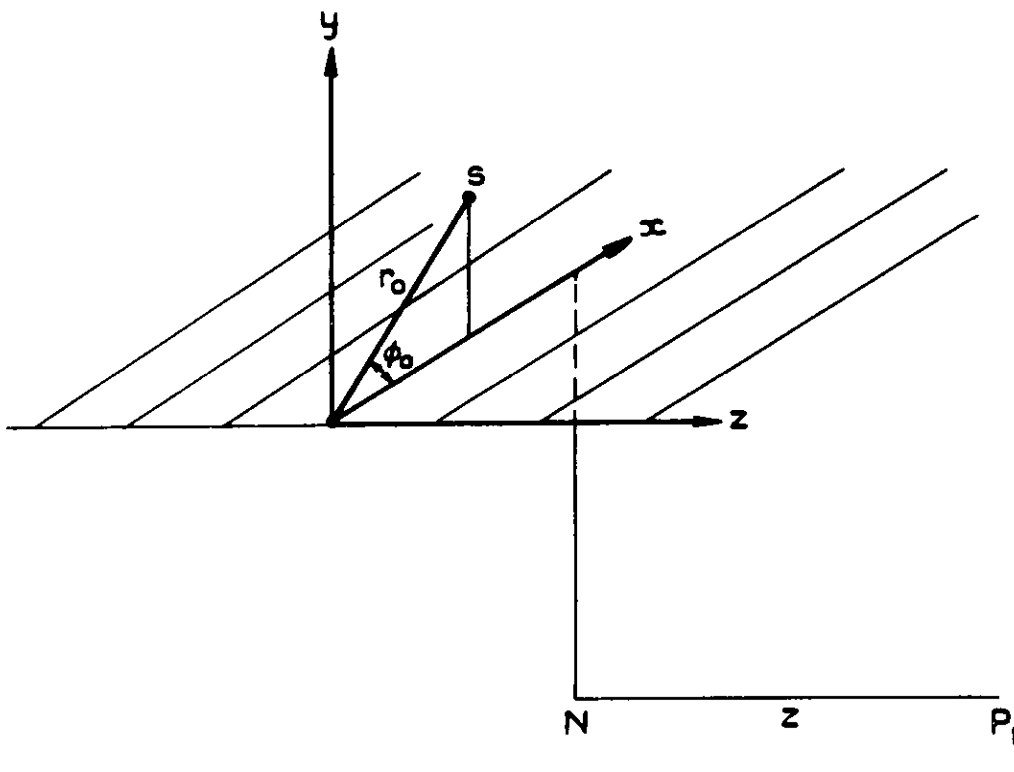



Fig. I




N is on the xy plane

Fig. 2

# PART B

# MOVING SOUND SOURCE

## 1 BASIC ANALYSIS

We start with a source at rest in a fluid at rest in the presence of a rigid body. We then apply the Lorentz transformation in the case where the source and body are moving with the recipient at rest. Other situations may then be dealt with by straightforward methods.

(a) Acoustic source emitting in the presence of a rigid body fixed in a quiescent fluid.

In terms of the velocity potential, the solution is governed by the equation (the wave equation with Dirac delta functions on the right)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = f(t)\,\delta(x - x_s)\,\delta(y - y_s)\,\delta(z - z_s) ,$$

with the source at $(x_s, y_s, z_s)$, together with the boundary condition of zero normal derivative on the body. (In the special case of a periodic source $f(t)$ is simply $e^{i\omega t}$, but it is just as convenient to leave it arbitrary.)

Call this solution

$$\phi_F = \phi_F\,(x, y, z, t, x_s, y_s, z_s) . \tag{1}$$

(b) Now let the source and the body move with velocity $-V$ parallel to the x axis. Coordinates axes are still fixed. Let the source be at $(x_s - V\,t, y_s, z_s)$ at time $t$. Then the potential satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = f(t)\,\delta(x - x_s + V\,t)\,\delta(y - y_s)\,\delta(z - z_s) .$$

(c) Apply to this equation a modified Lorentz transformation

$$x' = \gamma^2\,(x + V\,t), \qquad y' = \gamma\,y, \qquad z' = \gamma\,z, \qquad t' = \gamma^2\left(t + \frac{V\,x}{c^2}\right)$$

$$\gamma = (1 - M^2)^{-\frac{1}{2}}, \qquad M = \frac{V}{c} ;$$

with inverse

$$x = x' - V\,t', \qquad y = \frac{y'}{\gamma}, \qquad z = \frac{z'}{\gamma}, \qquad t = t' - \frac{V\,x'}{c^2}\,.$$

Then we have

$$\gamma^2\left(\nabla'^2 - \frac{\partial^2}{\partial t'^2}\right)\phi = f\left(t' - \frac{V\,x'}{c^2}\right)\delta\left(\frac{x'}{\gamma^2} - x_s\right)\delta\left(\frac{y'}{\gamma} - y_s\right)\delta\left(\frac{z'}{\gamma} - z_s\right)$$

$$= \gamma^4 f\left(t' - \frac{V\,\gamma^2\,x_s}{c^2}\right)\delta(x' - \gamma^2\,x_s)\,\delta(y' - \gamma\,y_s)\,\delta(z' - \gamma\,z_s)$$

noting that

$$\delta\left(\frac{x}{\gamma}\right) = \gamma\,\delta(x), \qquad f\,(a - b\,x)\,\delta(x - c) = f\,(a - b\,c)\,\delta(x - c)\,.$$

Take the body to be an infinite plate with origin of coordinates in the leading edge, the plate being $y = 0$, $x \geqslant 0$ at $t = 0$. Then the boundary condition is $\partial\phi/\partial y = 0$ for $x + V\,t \geqslant 0$, $y = 0$. In the Lorentz space it is $\partial\phi/\partial y' = 0$ for $x' \geqslant 0$, $y' = 0$ and so is the same as in (a). Call the solution $\phi_M$. Then

$$\phi_M = \gamma^2\,\phi_F\left(x', y', z', t' - \frac{\gamma^2\,V\,x_s}{c^2}, \gamma^2\,x_s, \gamma\,y_s, \gamma\,z_s\right)$$

or

$$\phi_M = \gamma^2\,\phi_F\left(\gamma^2\,(x + V\,t), \gamma\,y, \gamma\,z, \gamma^2\left(t + \frac{V\,x}{c^2} - \frac{V\,x_s}{c^2}\right), \gamma^2\,x_s, \gamma\,y_s, \gamma\,z_s\right)$$

$$= \gamma^2\,\phi_F\left(X + V\,T, Y, Z, T + \frac{V\,X}{c^2} - \frac{V\,x_s}{c^2}, X_s, Y_s, Z_s\right) \qquad (2)$$

on writing

$$X = \gamma^2\,x, \qquad Y = \gamma\,y, \qquad Z = \gamma\,z, \qquad T = \gamma^2\,t\,. \qquad (3)$$

*Note that this is the potential for a moving body, fixed axes and quiescent fluid.*

One must now decide what the observer is doing. If he is at rest at $(x_o, y_o, z_o)$ then the velocity potential is as given above with $x = x_o$, $y = y_o$, $z = z_o$ or $X = X_o$, $Y = Y_o$, $Z = Z_o$. If he is moving with velocity components $(u, v, w)$ then the velocity potential is as given above with $X = X_o + u\,T$, $Y = Y_o + \frac{v\,T}{\gamma}$, $Z = Z_o + \frac{w\,T}{\gamma}$.

The above analysis is essentially due to Professor W. Chester of Bristol University.

## 2 A FREE SOURCE

Let us consider as an illustration a source with no body present, which at time $t = 0$ is at $(x_s, y_s, z_s)$ and is moving with speed $V$ in the direction of the negative real axis, the fluid being at rest. We consider two cases: (a) P at rest at (x, y, z) and (b) P moving with the source, being at (x − V t, y, z) so that, at time $t = 0$, P is at the same point in the two cases.

Then we have $f(t) = A\, e^{i\omega t}$ and

$$\phi_F = \frac{A}{s} e^{i\ell}$$

with

$$s^2 = (x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2, \qquad \ell = \omega\left(t - \frac{s}{c}\right).$$

Hence

$$\phi_M = \frac{A\,\gamma^2}{S} e^{iL}$$

with

$$S^2 = (X + V\,T - X_s)^2 + (Y - Y_s)^2 + (Z - Z_s)^2$$

$$L = \omega\left(T + \frac{V\,X}{c^2} - \frac{V\,X_s}{c^2} - \frac{S}{c}\right), \qquad (4)$$

and this gives the value of $\phi$ at P when P is fixed.

When P is moving with the source replace X by X − V T. Naturally at time $t = 0$ the two values of $\phi$ are the same, but this will not apply to their derivatives with respect to $t$, so that the values of pressure and frequency are different in the two cases.

We shall assume that P is sufficiently distant from the source to ignore terms of order $1/S^2$, and will calculate the pressures at time $t = 0$ for the two cases.

In case (a) we find

$$p_a = -\rho_o \frac{\partial \phi}{\partial t} = \frac{-i\,\omega\,\rho_o\,A\,\gamma^4}{S'} \left[1 - \frac{V\,(X - X_s)}{c\,S'}\right] e^{iL'}$$

$$S'^2 = (X - X_s)^2 + (Y - Y_s)^2 + (Z - Z_s)^2, \qquad L' = \omega\left(\frac{V\,X}{c^2} - \frac{V\,X_s}{c^2} - \frac{S'}{c}\right).$$

and in case (b) we have

$$p_b = -\frac{i\,\omega\,\rho_o\,A\,\gamma^4}{S'}\,(1 - M^2)\,e^{iL'}\ .$$

Hence the ratio of the pressure amplitudes is given by

$$\frac{p_a}{p_b} = \frac{1 - \dfrac{V\,(X - X_s)}{c\,S'}}{1 - M^2}\ .$$

In the particular case where the source is moving along the line $P\,S$ we have $y_s = y = z_s = z = 0$ and $S' = X - X_s$ and so

$$\frac{p_a}{p_b} = \frac{1}{1 + M}\ .$$

It is of interest to find the frequencies at time $t = 0$ in the two cases. In case (a) $L$ is given by equation (4) and the frequency, defined to be $\partial L/\partial t$ is equal to

$$\gamma^2\,\omega\left[1 - \frac{V\,(X - X_s)}{c\,S'}\right].$$

In case (b) $L$ is given by equation (4) with $X - V\,T$ instead of $X$ and we then find that the frequency is simply $\omega$, as we might expect, since $P$ and $S$ have no relative motion.

The change in frequency in case (a) is of course the Doppler effect. When $S$ is moving in the line $P\,S$ we find that the frequency is $\omega/(1 + M)$ thus verifying the well-known case for a fixed observer and a source moving directly away from him.

## 3 SOURCE AND SEMI-INFINITE PLANE

We shall only consider cases in which the plane moves parallel to itself with constant speed V and will first suppose it moves in a direction perpendicular to its edge, leaving the case of a swept plane to the next section. We must work in terms of the velocity potential $\phi$ since in the moving system p and $\phi$ do not satisfy the same differential equation. They did do this in section A of these notes and so the solution given there applied equally for $\phi$ or p. Now we shall use this solution as applying to $\phi$. If we take the source to be at $(x_s, y_s, 0)$ and if the source and body are at rest then the solution for $\phi$ can be written down in the form (12) or (13) of section A. Calling that solution $\phi_F$ and the solution we now require $\phi_M$ then we have shown that

$$\phi_M = \gamma^2 \phi_F\left(X + V T, Y, Z, T + \frac{V X}{c^2} - \frac{V X_s}{c^2}, X_s, Y_s, Z_s\right) .$$

All we have to do is to stretch all lengths by a factor $\gamma^2$ in the x direction and by a factor $\gamma$ in directions perpendicular to this, followed by putting $X + V T$ instead of $X$ (where T is t stretched by a factor $\gamma^2$) and $T + V (X - X_s)/c^2$ instead of the original t. Also multiply $\phi_F$ by $\gamma^2$.

We will now calculate $\phi_M$ from equation (13) of section A. Equation (12) without the terms of the form $3\beta/8k R$ is included in this equation.

The value of $\phi_F$, writing $k = \omega/c$, $R = s$, is

$$\phi_F = \frac{A}{s} e^{i\omega\left(t-\frac{s}{c}\right)} \left\{\tfrac{1}{2} - \frac{1 + i}{2} [C(\beta) - i S(\beta)]\right\} \qquad (5)$$

$$+ \text{ a similar term with } \bar{s} \text{ and } \bar{\beta} .$$

For simplicity in presentation, we shall omit the term with bars over the letters in the work that follows. We have replaced R and $\bar{R}$ by s and $\bar{s}$, because we wish to reserve capital letters for values in the stretched coordinate system.

Hence in the new situation

$$\phi_M = \frac{A \gamma^2}{S} e^{i\omega\left(T+\frac{VX}{c^2}-\frac{VX_s}{c^2}-\frac{S}{c}\right)} \left\{\tfrac{1}{2} - \frac{1 + i}{2} [C(B) - i S(B)]\right\}$$

where S and B are calculated in the stretched coordinate system in exactly the same way as s and β were calculated in the old system, with the proviso that X must be replaced by X + V T wherever it occurs. Thus

$$B^2 = \frac{2\omega}{\pi\ c}\{[Z^2 + (R + R_s)^2]^{\frac{1}{2}} - S\}\ ,$$

$$S^2 = (X + V\,T - X_s)^2 + (Y - Y_s)^2 + Z^2\ ,$$

$$R^2 = (X + V\,T)^2 + Y^2, \quad R_s^2 = X_s^2 + Y_s^2\ .$$

We are interested in the pressure at large distances and we use the result $p = -\rho_o\ \partial\phi/\partial t$ and will put $t = 0$ after differentiation. We suppose that a prime attached to any symbol represents its value at time $t = 0$. We then have

$$\frac{\partial\phi'}{\partial t} = \frac{A\,\gamma^2}{S'}\,e^{iL'}\left[\left\{\frac{1}{2} - \frac{1+i}{2}\,[C(B')\ i\ S(B')]\right\}\left(i\,\gamma^2\,\omega - \frac{i\,\gamma^2\,\omega}{c}\,\frac{\partial S'}{\partial T}\right)\right.$$
$$\left. - \gamma^2\,\frac{1+i}{2}\,e^{-i\frac{\pi}{2}B'^2}\,\frac{\partial B'}{\partial T}\right] + O\left(\frac{1}{S'^2}\right)$$

where

$$L' = \omega\left(\frac{V\,X}{c^2} - \frac{V\,X_s}{c^2} - \frac{S'}{c}\right), \qquad S'^2 = (X - X_s)^2 + (Y - Y_s)^2 + Z^2\ ,$$

$\partial B'/\partial T$ and $\partial S'/\partial T$ are the values of $\partial B/\partial T$ and $\partial S/\partial T$ when $t$ is put equal to zero after differentiation.

Note that

$$C(B) - i\,S(B) = \int_0^B e^{-\frac{i\pi\alpha^2}{2}}\,d\alpha$$

and so

$$\frac{\partial}{\partial t}(C - i\,S) = \gamma^2\,e^{-\frac{i\pi B^2}{2}}\,\frac{\partial B}{\partial T}\ .$$

We have used this in deriving the above equation for $\partial\phi_M/\partial t$.

Noting that

$$\frac{\partial S'}{\partial T} = \frac{V\,(X - X_s)}{S'}$$

$$\frac{\partial B'}{\partial T} = \frac{\omega}{\pi\, c\, B'}\left\{\frac{V\,X}{\sqrt{Z^2 + (R' + R_s)^2}}\,\frac{R' + R_s}{R'} - \frac{V\,(X - X_s)}{S'}\right\}$$

we have finally

$$\frac{\partial\phi'}{\partial t} = \frac{A\,\gamma^4\,\omega}{S'}\, e^{i\omega\left[\frac{VX}{c^2} - \frac{VX_s}{c^2} - \frac{S'}{c}\right]}\Bigg[ i\left(1 - \frac{(X - X_s)\,V}{c\,S'}\right)\left(\tfrac{1}{2} - \frac{1 + i}{2}\,[C(B') - i\,S(B')]\right)$$
$$- \frac{1 + i}{2}\, e^{-\frac{i\pi B'^2}{2}}\,\frac{V}{\pi\, c\, B'}\left\{\frac{X}{\sqrt{Z^2 + (R' + R_s)^2}}\,\frac{R' + R_s}{R'} - \frac{X - X_s}{S'}\right\}\Bigg]$$

.....(6)

If $x$ and $x_s$ are small compared with $y$ then $V\,X/c\,S'$ is small and

$$\frac{\partial\phi'}{\partial t} = \frac{i\,\omega\,A\,\gamma^4}{S'}\, e^{i\,\omega\left[\frac{VX}{c^2} - \frac{VX_s}{c^2} - \frac{S'}{c}\right]}\left[\tfrac{1}{2} - \frac{1 + i}{2}\,[C(B') - i\,S(B')] + O\left(\frac{V\,X}{c\,S'}\right)\right]. \quad (7)$$

*With no motion of source or body we would find that*

$$\frac{\partial\phi'}{\partial t} = \frac{i\,\omega\,A}{s}\, e^{-\frac{i\omega s}{c}}\left[\tfrac{1}{2} - \frac{1 + i}{2}\,[C(\beta) - i\,S(\beta)] + O\left(\frac{1}{\omega\, s}\right)\right],$$

as in section A of these notes.

If the point P is moving at the same speed as the source and the body then B and S are independent of t and we obtain simply

$$\frac{\partial\phi'}{\partial t} = \frac{i\,\omega\,A\,\gamma^2}{S'}\, e^{i\left[\frac{VX}{c^2} - \frac{VX_s}{c^2} - \frac{S'}{c}\right]}\left[\tfrac{1}{2} - \frac{1 + i}{2}\,[C(B') - i\,S(B')] + O\left(\frac{1}{\omega' S'}\right)\right]. \quad (8)$$

There are of course second terms to be added to each of equations (6), (7) and (8), corresponding to the image source. Under the approximation used we can show that the ratio of the pressure amplitudes for what we called cases (a) and (b) is

$$\frac{p_a}{p_b} = \gamma^2 = \frac{1}{1 - M^2}$$

as in the case of a single free source, with small $V X/c S'$.

To summarise then we have found that if $M x/s$ and $M x_s/s$ are small then for a moving source and body the formula for the pressure has a form closely akin to that which it would have if there were no movement, provided that distances are suitably redefined and a factor $\gamma^4$ multiplies the expression when the recipient is at rest; the factor is $\gamma^2$ if the recipient is moving with the rest of the system. There is also a phase change.

## 4 SOURCE AND SEMI-INFINITE SWEPT PLANE

For this case there is considerably increased complication. Taking the coordinate system as before we suppose the source and plane have velocity components $(-u, 0, -w)$. The source will then be taken to be at $(x_s - u\,t, y_s, z_s - w\,t)$. We write

$$V^2 = u^2 + w^2, \quad \gamma^2 = \left(1 - \frac{V^2}{c^2}\right)^{-1}, \quad \delta^2 = \left(1 - \frac{u^2}{c^2}\right)^{-1}, \quad \varepsilon = \frac{u\,w}{c^2} .$$

Then the equation to be solved is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = f(t)\,\delta(x + u\,t - x_s)\,\delta(y - y_s)\,\delta(z + w\,t - z_s). \quad (9)$$

In effect we turn the coordinate system about the y axis through an amount equal to the angle of sweep, then apply the Lorentz transformation. Unfortunately this alters the angle of sweep in the new system so that a different coordinate transformation is required to get back to the standard non-moving situation.

The first coordinate rotation and Lorentz transformation may be combined into

$$\xi = \gamma^2 \left[ \frac{x\,u}{V} + \frac{z\,w}{V} + V\,t \right] = \frac{\gamma^2}{V} [u\,(x + u\,t) + w\,(z + w\,t)]$$

$$\eta = \gamma\,y$$

$$\zeta = \gamma \left[ \frac{-x\,w + z\,u}{V} \right] = \frac{\gamma}{V} [-w\,(x + u\,t) + u\,(z + w\,t)]$$

$$\tau = \gamma^2 \left[ t + \frac{x\,u + z\,w}{c^2} \right] = t + \frac{\gamma^2}{c^2} [u\,(x + u\,t) + w\,(z + w\,t)] .$$

This transforms

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \qquad \text{into} \qquad \gamma^2 \left[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \right]$$

where the second $\nabla^2$ is evaluated in $\xi$, $\eta$, $\zeta$ coordinates.

Now we make the transformation

$$x' = \frac{u\,\xi - \gamma\,w\,\zeta}{W} = \gamma\,\delta(x + u\,t)$$

$$y' = \eta$$

$$z' = \frac{\gamma\,w\,\xi + u\,\zeta}{W} = \gamma^2\,\delta\varepsilon\,(x + u\,t) + \frac{\gamma^2}{\delta}\,(z + w\,t)$$

$$t' = \tau ,$$

where

$$W^2 = u^2 + \gamma^2\,w^2 = \frac{\gamma^2\,V^2}{\delta^2} .$$

This makes

$$z + w\,t = \frac{\delta z'}{\gamma^2} - \frac{\delta x'\,\varepsilon}{\gamma} , \qquad x + u\,t = \frac{x'}{\gamma\,\delta} .$$

The change is a pure rotation and so the differential operator is now

$$\gamma^2 \left[ \nabla'^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \right] .$$

Equation (9) then becomes

$$\gamma^2 \left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\phi = f\left[t - \frac{\delta}{c^2\gamma}(u\,x' + \gamma\,w\,z')\right]\delta\left(\frac{x'}{\delta\gamma} - x_s\right)\delta\left(\frac{y'}{\gamma} - y_s\right)\delta\left(\frac{\delta z'}{\gamma^2} - \frac{\delta x'\,\varepsilon}{\gamma} - z_s\right)$$

$$= \gamma^4 f\left[t' - \frac{u\,\gamma^2 x_s}{c^2} - \frac{w\,\gamma^2 z_s}{c^2}\right]\delta(x' - \delta\gamma\,x_s)\,\delta(y' - \gamma\,y_s)\,\delta\left[z' - \frac{\gamma^2}{\delta}(\delta^2\varepsilon\,x_s + z_s)\right]$$

Here we have used the fact that

$$\delta\left(\frac{x}{a}\right) = a\,\delta(x)\,,$$

$$f\,(t' - a\,x' - b\,z')\,\delta(x' - c)\,\delta(z' - dx' - e) = f\,[t - a\,c - b\,(dc + e)]\,\delta(x' - c)\,\delta(z' - dc - e)\,.$$

The boundary condition in the physical plane is $\partial\phi/\partial y = 0$ for $y = 0$, $x + u\,t \geqslant 0$ which reduces to $\partial\phi/\partial y' = 0$ for $y' = 0$, $x' \geqslant 0$ and so is the same as in the original non-moving problem.

If the solution for this is

$$\phi_F = \phi_F\,(x, y, z, t, x_s, y_s, z_s)$$

then the solution of our problem is

$$\phi_M = \gamma^2\,\phi_F\left\{x', y', z', t' - \frac{u\,\gamma^2 x_s}{c^2} - \frac{w\,\gamma^2 z_s}{c^2}, \delta\gamma\,x_s, \gamma\,y_s, \frac{\gamma^2}{\delta}(\delta^2\varepsilon\,x_s + z_s)\right\}.$$

Hence transforming back we have

$$\phi_M = \gamma^2\,\phi_F\left\{\gamma\,\delta(x + u\,t), \gamma\,y, \frac{\gamma^2}{\delta}(\delta^2\varepsilon\,x + z + w\,\delta^2\,t), \gamma^2\left(t + \frac{u\,x + w\,z}{c^2} - \frac{u\,x_s + w\,z_s}{c^2}\right), \gamma\,\delta x_s, \gamma\,y_s, \frac{\gamma^2}{\delta}(\delta^2\varepsilon\,x_s + z_s)\right\}$$

..... (10)

Note that when $w = 0$ then $u = V$, $\delta = \gamma$ and this reduces to our previous value (2). It does not seem worth while this time to make a transformation analogous to (3)

If the recipient is moving with the fluid we replace $x, z$ by $x - u\,t$, $z - w\,t$ respectively and obtain

$$\phi_M = \gamma^2 \phi_F \left\{\gamma\,\delta x,\ \gamma\,y,\ \frac{\gamma^2}{\delta}(\delta^2\varepsilon\,x + z),\ t + \gamma^2\left(\frac{u\,x + w\,z}{c^2} - \frac{u\,x_s + w\,z_s}{c^2}\right)\right.$$
$$\left.\gamma\,\delta x_s,\ \gamma\,y_s,\ \frac{\gamma^2}{\delta}(\delta^2\varepsilon\,x_s + z_s)\right\}. \qquad (11)$$

To the same approximations as equations (7) and (8) we find for a stationary observer and moving source and body, the source being at $(x_s, y_s, 0)$ at time $t = 0$

$$\frac{\delta\phi'}{\delta t} = \frac{i\,\omega\,A\,\gamma^4}{S'}\,e^{i\omega\gamma^2\left[\frac{ux+wz}{c^2} - \frac{ux_s+wz_s}{c^2}\right]}\left\{\tfrac{1}{2} - \frac{1+i}{2}\,[C(B') - i\,S(B')] + O\left(\frac{V\,X}{c\,S'}\right)\right\}$$

..... (12)

+ terms corresponding to the image of the source,

where

$$s'^2 = \gamma^2\,\delta^2\,(x - x_s)^2 + \gamma^2\,(y - y_s)^2 + \left(\frac{\gamma^2}{\delta}\right)^2 \{\delta^2\varepsilon\,(x - x_s) + z\}^2\ ,$$

$$B'^2 = \left\{\left(\frac{\gamma^2}{\delta}\right)^2 [\,\delta^2\varepsilon\,(x - x_s) + z]^2 + (r' + r_s)^2\right\}^{\frac{1}{2}} - s'$$

$$r_s^2 = \gamma^2\,(\delta^2 x_s^2 + y_s^2)$$

$$r'^2 = \gamma^2\,(\delta^2 x^2 + y^2)\ .$$

For a moving observer the result is the same but with the factor $\gamma^4$ replaced by $\gamma^2$ and $O\,(V\,x/c\,s')$ replaced by $O\,(1/\omega\,s')$.

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