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An Examination of an Iterative Procedure for Determining the Characteristic Exponents of Linear Differential Equations with Perodic Coefficients

by

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AN EXAMINATION OF AN ITERATIVE PROCEDURE FOR DETERMINING THE CHARACTERISTIC EXPONENTS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

by

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SUMMARY

An iterative procedure for the determination of the characteristic exponents of linear differential equations with periodic coefficients is developed and applied to a number of examples. The conditions under which convergence is obtained are considered and it is shown that these do not necessarily exclude unstable motions of the system.

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* Replaces RAE Technical Report 71115 - ARC 33106

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CONTENTS

1 INTRODUCTION

In the dynamics of rotorcraft, and no doubt in many other problems, one is often concerned with the timewise stability of solutions of differential equations with periodic coefficients (see e.g. Ref.1). Consider the linear system

$$\frac{dX(t)}{dt} = A(t)X(t) \tag{1}$$

where $A(t)$ denotes an $n \times n$ matrix whose elements are periodic functions of period τ, i.e.

$$A(t + \tau) = A(t) \quad . \tag{2}$$

Now it is established (Ref.2, pp.336-8) that the equation (1) can be transformed into a similar equation with constant coefficients (i.e. it is reducible in the sense of Lyapunov)

$$\frac{dY(t)}{dt} = BY(t) \tag{3}$$

where

$$X = L(t)Y(t) \tag{4}$$

and B is an $n \times n$ matrix whose elements are independent of t. The matrix $L(t)$ is called a Lyapunov matrix. It satisfies the equation

$$\frac{dL}{dt} = AL - LB \tag{5}$$

and is periodic of the same period as A. Thus (1) has the solution

$$X = L(t)e^{Bt} \quad . \tag{6}$$

This is a statement of the well known Floquet theorem (see e.g. Ref.3, p.55).

It follows that the stability of (1) can be determined by studying the eigenvalues of the constant matrix B. A simple and useful way of determining these (cf. Ref.1) is by integration over a period. If we write

$$E = X(\tau)X^{-1}(0) \tag{7}$$

then

$$E = L(0)e^{B\tau}L^{-1}(0) \tag{8}$$

and so the two matrices E and $e^{B\tau}$ are similar (i.e. one is obtained from the other by a similarity transformation). Thus the eigenvalues of E are equal to the eigenvalues of $e^{B\tau}$; and hence the eigenvalues of B are equal to τ^{-1} times the logarithm of the eigenvalues of E, and are determined modulo $2\pi i/\tau$. Integrations are a period, with the initial condition $X = X(0)$, immediately given E and so enables one to determine the eigenvalues of B which are called the characteristic exponents of the system. If none of the characteristic exponents have positive real parts then the system will be stable*.

This integration method works well provided none of the eigenvalues of E are small; but in the latter case it is extremely difficult to determine the corresponding characteristic exponents at all accurately. In particular if the system is reciprocal - that is the eigenvalues of E are either unity or in reciprocal pairs - this difficulty does not in general occur; but non-reciprocal systems have been encountered by the authors which possessed an almost singular E matrix. In such circumstances it would be advantageous if B could be found directly and in this connection an iteration procedure based on Coakley's[4] work was investigated.

2 THE ITERATIVE PROCEDURE

In Ref.4 Coakley devised an iterative procedure for transforming an almost periodic system with rapidly varying coefficients into one with more slowly varying coefficients. For a periodic system this method can, in principle, be used to make the transformation to a constant coefficient system and hence to obtain the matrix B directly. For this case the procedure is as follows.

The Lyapunov matrix of the transformation is written as

$$L(t) = I + V(t) \tag{9}$$

where $V(t)$ is periodic of period τ, and I is the unit matrix. Then from (5)

$$\frac{dV}{dt} = U - B \tag{10}$$

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*Assuming the characteristic exponents are all distinct. If not there is the possibility of secular instability specified by the appearance in the general integral of terms t^r.

where $$U = A + AV - VB \tag{11}$$

V can obviously be written in the form

$$V = \sum_{n=1} \frac{1}{in\omega} \{A_n e^{in\omega t} - B_n e^{-in\omega t}\} \tag{12}$$

(there is no need to include a constant term since this is catered for by the I term in equation (9)). So to satisfy (10) U is given by

$$U = B + \sum_{n=1} \{A_n e^{in\omega t} + B_n e^{-in\omega t}\} \quad . \tag{13}$$

We need therefore to determine values of the matrices B, A_n, B_n which will result in equation (11) being satisfied by the above expressions for U and V. Suppose $B^{(r)}$, $A_n^{(r)}$, $B_n^{(r)}$ are approximations to these matrices; and U_r, V_r the corresponding approximations to U, V. Then we can use (11) to get a further approximation U_{r+1}:-

$$U_{r+1} = A + AV_r - V_r B^{(r)} \quad . \tag{14}$$

Having got U_{r+1} we can immediately get the corresponding further approximation $B^{(r+1)}$, $A_n^{(r+1)}$, $B_n^{(r+1)}$, V_{r+1} from equations (13) and (12). This then is the iteration procedure and, provided it converges, it will determine the matrix B etc.

In the particular case when the matrix A is real, it follows from (8) and (7), since X is always real, that B will be real provided we make L(0), which is arbitrary, real. Also, from (6), L(t) will be real for any t. Consequently U and V will also be real and so each matrix B_n will be the complex conjugate of A_n, which we denote by $\bar{A}_n$.

If we take

$$U_0 = A \tag{15}$$

as the first approximation to U, where, say

$$A = B^{(0)} + \sum_{n=1}^{m} \{A_n^{(0)} e^{in\omega t} + \bar{A}_n^{(0)} e^{-in\omega t}\} \tag{16}$$

then the (r+1)th approximation will have the form

$$U_r = B^{(r)} + \sum_{n=1}^{(r+1)m} \{A_n^{(r)} e^{in\omega t} + \bar{A}_n^{(r)} e^{-in\omega t}\} \quad . \tag{17}$$

The associated form of V_r is

$$V_r = \sum_{n=1}^{(r+1)m} \frac{1}{in\omega} \{A_n^{(r)} e^{in\omega t} - \bar{A}_n^{(r)} e^{-in\omega t}\} \quad . \tag{18}$$

3 THE CONVERGENCE CRITERION

In Ref.4, Coakley states, without proof, an explicit condition which, he says, guarantees the existence of solutions to equation (11). This condition is

$$\|A\| < (3 - 2\sqrt{2})\omega \tag{19}$$

where the matrix norm $\|A\|$ is any norm which satisfies the conditions

$$\left.\begin{aligned} \|A\| &> 0 \quad \text{unless} \quad A = 0 \\ \|kA\| &= |k| \,.\, \|A\| \\ \|A + B\| &\leqslant \|A\| + \|B\| \\ \|AB\| &\leqslant \|A\| \,.\, \|B\| \quad . \end{aligned}\right\} \tag{20}$$

We know however that such a solution always exists. There always is a Lyapunov matrix which will give the desired transformation (cf. Ref.2, p.338). The above condition is rather a sufficient condition for the convergence of the iterative procedure described in the previous section. It is obtained as follows.

Taking the 'norm' of both sides of equation (14) gives

$$\|U_{r+1}\| \leqslant \|A\| + \|A\|\|V_r\| + \|V_r\|\|B^{(r)}\| \quad . \tag{21}$$

Now, from (17) and (18)

$$\|V_r\| \leqslant \frac{1}{\omega} \|U_r\| \tag{22}$$

$$\|B^{(r)}\| \leqslant \|U_r\| \tag{23}$$

and so

$$\|U_{r+1}\| \leqslant \|A\| + \|A\| \frac{\|U_r\|}{\omega} + \frac{\|U_r\|^2}{\omega} \quad . \tag{24}$$

Suppose

$$\|A\| < k\omega \tag{25}$$

then, since $U_0 = A$, using (24), we find that

$$\|U_1\| < (k + 2k^2)\omega \tag{26}$$

$$\|U_2\| < (k + 2k^2 + 6k^3 + 4k^4)\omega$$

etc.

and for the rth approximation

$$\|U_r\| < \omega \left\{ \sum_{i=1}^{r+1} a_i k^i + \sum_{i=r+2}^{2^r} a_i^{(r)} k^i \right\} \tag{27}$$

where

$$a_1 = 1 \tag{28}$$

$$a_i = a_{i-1} + \sum_{j=1}^{i-1} a_j a_{i-j} \qquad i = 2 \rightarrow (r+1) \tag{29}$$

$$a_i^{(r)} = a_{i-1}^{(r-1)} + \sum_{j=1}^{i-1} a_j^{(r-1)} a_{i-j}^{(r-1)} \qquad i = (r+2) \rightarrow 2^{r-1} + 1$$

$$= \sum_{j=i-2^{r-1}}^{2^{r-1}} a_j^{(r-1)} a_{i-j}^{(r-1)} \qquad i = (2^{r-1} + 2) \rightarrow 2^r \tag{30}$$

and on the right-hand side of the last two equations

$$a_j^{(r-1)} = a_j \quad \text{for} \quad j = 1 \rightarrow r \quad . \tag{31}$$

Thus

$$\lim_{r\to\infty} \|U_r\| < \omega \sum_{i=1}^{\infty} a_i k^i \tag{32}$$

where the a_i are given by (28) and (29). Calculated values of the ratio of successive coefficients in the series on the right hand side of this inequality are:-

n	1	2	3	....	103	104
$\frac{a_n}{a_{n+1}}$	0.5	$0.\dot{3}$	$0.\dot{2}\dot{7}$		0.1740833	0.1740591

These appear to be converging to the number $(3 - 2\sqrt{2})$ quoted by Coakley[4]. Consequently we can say with virtual certainty that the series will converge for

$$k < (3 - 2\sqrt{2}) \tag{33}$$

which gives, from (25) and (32) the condition (19), i.e.

$$\|A\| < (3 - 2\sqrt{2})\omega$$

as sufficient to ensure convergence of the iterative procedure. This criterion is very restrictive but from the very nature of our calculations, i.e. 'taking norms', it can be seen that it will often be far too conservative, and, as results show later, convergence can be obtained even when the criterion is far from being satisfied.

4 APPLICATIONS

Some fairly simple examples were taken to see how and when the method worked, and in particular to get some idea about how conservative the convergence criterion (19) is. In each case they were systems for which the matrix A contained no overtones, i.e. $m = 1$ in equation (16). For this case it can easily be shown that, with the iterative method of section 2

$$B^{(r+1)} = B^{(0)} - \frac{i}{\omega}\{\bar{A}_1^{(0)}A_1^{(r)} - A_1^{(0)}\bar{A}_1^{(r)}\} \tag{34}$$

$$A_1^{(r+1)} = A_1^{(0)} - \frac{i}{\omega}\left\{B^{(0)}A_1^{(r)} + \frac{\bar{A}_1^{(0)}A_2^{(r)}}{2} - A_1^{(r)}B^{(r)}\right\} \tag{35}$$

$$A_n^{(r+1)} = -\frac{i}{\omega}\left\{B^{(0)}\frac{A_n^{(r)}}{n} + \frac{A_1^{(0)}A_{n-1}^{(r)}}{n-1} + \frac{\bar{A}_1^{(0)}A_{n+1}^{(r)}}{n+1} - \frac{A_n^{(r)}B_r}{n}\right\} \qquad (2 \leqslant n \leqslant r) \tag{36}$$

$$A_{r+1}^{(r+1)} = -\frac{i}{\omega}\left\{B^{(0)}\frac{A_{r+1}^{(r)}}{r+1} + \frac{A_1^{(0)}A_r^{(r)}}{r} - \frac{A_{r+1}^{(r)}B_r}{r+1}\right\} \tag{37}$$

$$A_{r+2}^{(r+1)} = -\frac{i}{\omega}\frac{A_1^{(0)}A_{r+1}^{(r)}}{r+1} \quad . \tag{38}$$

It is easily seen that all the $B^{(r)}$ are real, though the other matrices will in general be a complex. A computer program was written to perform iteratively the operations given by the above set of equations.

4.1 First example - a second order equation of dimension one

The equation considered was Mathieu's equation with the addition of a viscous damping term, i.e.

$$\ddot{f} + 2\varepsilon\dot{f} + \Omega^2(1 - 2\mu\cos\omega t)f = 0 \quad . \tag{39}$$

Making the substitution

$$\dot{f} = \Omega g \tag{40}$$

it becomes

$$\dot{g} = -\Omega(1 - 2\mu\cos\omega t)f - 2\varepsilon g \tag{41}$$

and so putting

$$x = \{f, g\} \tag{42}$$

we obtain the equation in the desired form

$$\dot{x} = A(t)x \tag{43}$$

where

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -2\varepsilon \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \Omega\mu & 0 \end{bmatrix} e^{i\omega t} + \begin{bmatrix} 0 & 0 \\ \Omega\mu & 0 \end{bmatrix} e^{-i\omega t}$$

$$\equiv B^{(0)} + A_1^{(0)}e^{i\omega t} + \bar{A}_1^{(0)}e^{-i\omega t} \quad . \tag{44}$$

It can easily be shown that this is a reciprocal system only when ε is zero.

4.1.1 Other methods of solution

Equation (39) can be put into the standard form of Mathieu's equation by the transformation

$$f = e^{-\varepsilon t} z \qquad (45)$$

which gives

$$\ddot{z} + \Omega^2 \left(1 - \frac{\varepsilon^2}{\Omega^2} - 2\mu \cos \omega t\right) z = 0 \quad . \qquad (46)$$

Two methods of finding the characteristic exponents of equation (46) were considered. The first, which we call method a, is given by Bolotin in Ref.5 (section 56, p.214). By expanding z as a Fourier series, with unknown coefficients, multiplied by an exponential term $e^{\nu t}$, he obtains an infinite set of equations for the coefficients. For the existence of a non-trivial solution the corresponding infinite determinant must be zero. An approximation to this condition is equation to zero of the 3×3 determinant of the elements in the top left hand corner of the larger determinant. In our notation this is

$$\begin{vmatrix} (\nu^2 - \omega^2) + \Omega^2 - \varepsilon^2 & -\mu\Omega^2 & 2\nu\omega \\ -2\mu\Omega^2 & \nu^2 + \Omega^2 - \varepsilon^2 & 0 \\ -2\nu\omega & 0 & \nu^2 - \omega^2 + \Omega^2 - \varepsilon^2 \end{vmatrix} = 0 \; . \qquad (47)$$

This approximation gives six values of the characteristic exponents instead of the expected two. Two of them should be good approximations to the expected values and so the method is useful for our comparison requirements.

The second method, method b, is to obtain the solution as a power series in μ (cf. e.g. Ref.6). This is[6]

$$\nu^2 = -\left\{\Omega^2 - \varepsilon^2 - \frac{2\Omega^4\mu^2}{4(\Omega^2 - \varepsilon^2) - \omega^2} + O\left(\frac{\Omega^8\mu^4}{\omega^6}\right)\right\} \quad . \qquad (48)$$

The characteristic exponents of equation (39) are of course given by

$$\lambda^2 = (\nu - \varepsilon)^2 \quad . \qquad (49)$$

4.1.2 Comparisons of solutions

Two examples are given below where the iterative method converged to produce a matrix B whose eigenvalues were very nearly equal to the characteristic exponents as calculated by the two methods of the last section.

(i) $\Omega = 1,\ \mu = 0.2,\ \omega = 10,\ \varepsilon = 0$

After six iterations B had converged to

$$B = \begin{bmatrix} 0 & 1 \\ -1.000833 & 0 \end{bmatrix}$$

which has eigenvalues given by

$$\lambda^2 = -1.000833 \ .$$

Method a gave $\lambda^2 = -1.000833,\ -80.995505,\ -121.003659.$
Method b gave $\lambda^2 = -1.000833.$

(ii) $\Omega = 1,\ \mu = 0.25,\ \omega = 10,\ \varepsilon = 0.25$

After six iterations B had converged to

$$B = \begin{bmatrix} 0 & 1 \\ -1.001299 & -0.5 \end{bmatrix}$$

which has eigenvalues given by

$$\lambda^2 = -0.876299 \pm \frac{i}{2}\sqrt{0.938799} \ .$$

Method a gave $\lambda^2 = -0.876301 - \frac{i}{2}\sqrt{0.938801}$

$$= -81.5022427 - \frac{i}{2}\sqrt{81.5647427}$$

$$= -120.2464561 - \frac{i}{2}\sqrt{120.3089561}.$$

Method b gave $\lambda^2 = -0.876301 - \frac{i}{2}\sqrt{0.938787}.$

Thus it can be safely assumed that for both these examples the iterative method is converging to the correct solution.

4.1.3 The criterion for convergence

With the other parameters having the values used in the two examples of section 4.1.2 the frequency ω was varied to find the minimum value at which convergence could be obtained. Thus

(i) For $\Omega = 1$, $\mu = 0.2$, $\varepsilon = 0$ (which makes* $\|A\| = 1.4$) the iterative process converged for $\omega \geqslant 3$ approx., i.e. for $\|A\| \leqslant 0.47\omega$.

(ii) For $\Omega = 1$, $\mu = 0.25$, $\varepsilon = 0.25$ (which makes* $\|A\| = 2$) the iterative process converged for $\omega \geqslant 3$ approx., i.e. for $\|A\| \leqslant 0.67\omega$.

The criterion (19) is therefore extremely conservative for these two cases but there is no indication that we can replace the figure in the criterion ($3 - 2\sqrt{2} \approx 0.172$) by a larger figure. If a different specific norm**

$$\|A\| = \sqrt{\sum_{i,j} \max\,(A_{ij}^2)(t)}$$

is used the two limits of convergence become

(i) $\|A\| \leqslant 0.57\omega$

(ii) $\|A\| \leqslant 0.62\omega$

and the difference in the two numbers is much less. For a third set of parameters at one particular ω (i.e. $\Omega = 1$, $\mu = 0.005$, $\varepsilon = 0$, $\omega = 2$) the process did not converge and this was a case for which, with the second definition of norm,

$$\|A\| \approx 0.63\omega \quad .$$

All the above examples of convergence are to characteristic exponents with negative or zero real parts, which represent a stable motion of the system (cf. section 1). The instability regions of equation (39) have been fully determined (see e.g. Ref.7, p.98). An attempt was made to see if our iterative method could determine a solution in an unstable region. It was found however that the parameters could not be arranged so that the procedure would converge to an unstable root.

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*The specific norm used here is

$$\|A\| = \max_i \sum_j \max\,|A_{ij}(t)|$$

i.e. the maximum values of the modulus of each element of a row are summed, and the row is found for which this sum is a maximum. This maximum sum is then the value of the norm.

**This is the Euclidean norm which is the square root of the sum of the maximum values of the squares of each element of the matrix.

4.2 Second example - a first order equation of dimension four

As a second example a particular system was taken for which Cesari (Ref.3, p.74) has given the solution explicitly. The system is

$$\dot{x} = A(t)x \tag{50}$$

where

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ \sigma_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \sigma_2^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & -\varepsilon\sigma_1/\sigma_2 \\ 0 & \varepsilon/\sigma_1 & 0 & 0 \\ \varepsilon\sigma_2 & 0 & 0 & 0 \end{bmatrix} \cos \omega t$$

$$+ \begin{bmatrix} 0 & 0 & 0 & \varepsilon/\sigma_2 \\ 0 & 0 & \varepsilon\sigma_1 & 0 \\ -\varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon\sigma_2/\sigma_1 & 0 & 0 \end{bmatrix} \sin \omega t \quad . \tag{51}$$

The attraction of this system is that it is unstable at all frequencies ω. When σ_1 and σ_2 are equal it is a reciprocal system* (cf. section 4.21).

4.2.1 The explicit solution

In Ref.3 it is shown that

$$x = e^{\alpha t} \begin{bmatrix} \cos \gamma_1 t \\ -\sigma_1 \sin \gamma_1 t \\ \varepsilon\{(\omega + \gamma_1 - \sigma_2) \cos (\omega + \gamma_1)t - \alpha \sin (\omega + \gamma_1)t\}/\Delta \\ \varepsilon\sigma_2\{(\omega + \gamma_1 - \sigma_2) \sin (\omega + \gamma_1)t + \alpha \cos (\omega + \gamma_1)t\}/\Delta \end{bmatrix} \tag{52}$$

is a solution of (50) where

---

*With $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ then $A'P = -PA$ if $\sigma_1 = \sigma_2$ which is a condition[1] which ensures that the system is reciprocal.

$$\gamma_1 = \beta - \sigma_1 \tag{53}$$

$$\Delta = (\omega + \beta - \sigma_1 - \sigma_2)^2 + \alpha^2 \tag{54}$$

and

$$\left.\begin{aligned} z &= \alpha - i\beta \\ \text{and} \quad z &= -\alpha + i(\omega + \beta - \sigma_1 - \sigma_2) \end{aligned}\right\} \tag{55}$$

are the roots of

$$z = \frac{\varepsilon^2}{\omega - \sigma_1 - \sigma_2 + iz} \quad . \tag{56}$$

Another solution is obtained by replacing cos by sin and sin by - cos in (52). Two other solutions are obtained by replacing α by $-\alpha$, σ_1 by σ_2 γ_1 by $\gamma_2 (= \beta - \sigma_2)$, and interchanging the last two rows of (52). Thus there always are two solutions with characteristic exponents* with positive real parts (except for $\varepsilon = 0$ when α is zero). The fact that (52) is a solution of (50) can be simply verified if one notes that from (55) and (56)

$$\alpha(2\beta + \omega - \sigma_1 - \sigma_2) = \varepsilon^2 \tag{57}$$

$$\alpha^2 = \beta(\beta + \omega - \sigma_1 - \sigma_2) \tag{58}$$

which two equations can be put in the form

$$\beta = \frac{\varepsilon^2 \alpha}{\Delta} \tag{59}$$

$$\alpha = \frac{\varepsilon^2(\beta + \omega - \sigma_1 - \sigma_2)}{\Delta} \quad . \tag{60}$$

To determine α one has to solve the quartic obtained by eliminating β from (57) and (58), viz.

$$4\alpha^4 + \alpha^2(\omega - \sigma_1 - \sigma_2)^2 - \varepsilon^4 = 0 \quad . \tag{61}$$

This gives, since α is real

*The characteristic exponents are $\alpha \pm i\gamma_1$, $-\alpha \pm i\gamma_2$.

$$\alpha = \pm \frac{\sqrt{\sqrt{16\varepsilon^4 + (\omega - \sigma_1 - \sigma_2)^4} - (\omega - \sigma_1 - \sigma_2)^2}}{2\sqrt{2}} \quad . \tag{62}$$

For $\varepsilon^2/(\omega - \sigma_1 - \sigma_2)^2$ small this gives

$$\alpha \approx \{\varepsilon^2/(\omega - \sigma_1 - \sigma_2)\} - 2\varepsilon^6/(\omega - \sigma_1 - \sigma_2)^5 \tag{63}$$

and the corresponding value of β is

$$\beta \approx \{\varepsilon^4/(\omega - \sigma_1 - \sigma_2)^3\} - \varepsilon^8/(\omega - \sigma_1 - \sigma_2)^7 \quad . \tag{64}$$

4.2.2 Particular solutions

Calculations were made for the system (50) with the parameters having the values

$$\left.\begin{aligned} \sigma_1 &= \sigma_2 = 1 \\ \varepsilon &= 0.2 \\ \omega &= 10 \quad . \end{aligned}\right\} \tag{65}$$

The iterative method converged after ten iterations to give a matrix B

$$B = \begin{bmatrix} 0.004999996 & -0.9999969 & 0 & 0 \\ 0.9999969 & 0.004999996 & 0 & 0 \\ 0 & 0 & -0.004999996 & -0.9999969 \\ 0 & 0 & 0.9999969 & -0.004999996 \end{bmatrix}$$

which has eigenvalues

$$\lambda = 0.004999996 \pm 0.9999969i \quad , \quad -0.004999996 \pm 0.9999969i \quad .$$

These compare perfectly with the explicit solution of the last section which gives exactly the same characteristic exponents as far as the quoted significant figures. Comparison was also made with values obtained by the integration method briefly described in the introduction (see also e.g. Ref.1). The agreement again was good the latter method giving characteristic exponents

$$\lambda = 0.004999986 \pm 0.9999969i \quad , \quad -0.004999986 \pm 0.999969i \quad .$$

Thus here we have a case where the iterative procedure converges to an unstable solution. It is a case which certainly should converge according to the convergence criterion of section 3, equation (19) for with the specific norm

$$\|A\| = \max_i \sum_j \max |A_{ij}(t)|$$

then

$$\|A\| = 1.4$$

which is less than $(3 - 2\sqrt{2})\omega$ for $\omega = 10$.

5 CONCLUSIONS

It has been shown that iterative procedure for the determination of the characteristic exponents of linear differential equations with periodic coefficients will obtain these with good accuracy provided convergence is obtained. However there is only a limited region in which the procedure will converge and this severely effects the value of the method. This region does not necessarily exclude all the regions where there is an unstable solution as was shown by the second example considered (section 4.2).

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