C.P. No. 394
(19,485)
A.R.C. Technical Report

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MINISTRY OF SUPPLY

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CURRENT PAPERS

A Technique for Improving the Predictions of Linearised Theory on the Drag of Straight-Edged Wings

by

D.G.Randall, B.Sc.

LONDON: HER MAJESTY'S STATIONERY OFFICE

1958

SIX SHILLINGS NET

C.P. No. 394

U.D.C. No. 533.691.11:533.6.013.12:533.6.011.35/5

Technical Note No. Aero 2474

January, 1957

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ERRATA

The Mach number, M, in equation (28a) on page 11, is the free stream Mach number and not the local Mach number. Equation (28b) on page 12 is, therefore, unnecessary.

As a result of this aberration, Figures 3,4,5,6 and 7 require correction. The curve labelled "Modified Theory" in Fig. 3 should be redrawn to pass through the points (0,0.638), (0.1, 0.371), (0.2, 0.303), (0.3, 0.272), (0.4, 0.255), (0.5, 0.244), (0.6, 0.237), (0.7, 0.233), (0.8, 0.231), (0.9, 0.229), (1.0, 0.229). In Fig 4, the dashes at the left-hand side of the figure, which are labelled "Modified Result", should be changed as follows: for B = 1, from 0.554 to 0.638; for B = 3, from 0.364 to 0.383. The points of tangency (the crosses) are now approximately given by $\zeta = 2.5$ and not $\zeta = 3.0$, so that the factor 3/2 in equations (49) to (52) inclusive would be better replaced by 5/4. This will, however, have a negligible effect on the calculations.

Finally, the crosses in Figures 5,6 and 7 are incorrect. The following changes should be made, (reading from left to right in each case). In Fig 5: 1.92 to 2.15; 1.75 to 1.80. In Fig 6: 1.84 to 2.03; 1.70 to 1.72. In Fig 7. 0.98 to 1.06; 1.72 to 1.87; 1.68 to 1.76; 1.62 to 1.67; 1.47 to 1.50.

The conclusions in section 6 remain unaltered, since the above errors are all small in comparison with the deviations between linearised theory and experiment.

Addenda

Although it has been shown that (24) satisfies (22) together with (21), the uniqueness of this solution has not been proved A "solution" of (22) in powers of δ can be obtained by successive substitution, the first substitution being $f = \delta$. It can be shown that the series so obtained corresponds to the expansion of (24). It is, therefore, probable that (24) is the unique solution of (22).

After equation (44b) the words "where terms of higher order than ε have been omitted' should be inserted.

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SUMMARY

The curve of drag against Mach number for straight-edged wings, calculated by using the linearised theory of supersonic flow, displays discontinuities in slope at the various Mach numbers for which the edges are sonic. These features, which are not observed in practice, are due to the fact that linearised theory predicts an infinite pressure along a subsonic or sonic edge. It is shown that if the linearised equation of supersonic flow is used to determine the flow over straight-edged wings, but the linearised boundary condition is replaced by the full (non-linear) boundary condition, these infinities disappear and are replaced by plausible values. On this basis a simple method is derived for improving the linearised predictions of the drag of straight-edged wings which exhibits satisfactory agreement with experimental results.

While the technique is not directly applicable to ridge lines, an artifice renders them amenable to similar treatment.

LIST OF CONTENTS

LIST OF APPENDICES

LIST OF ILLUSTRATIONS

---

1 Introduction

Linear theory has been used with success to determine the flow over thin wings at moderate supersonic Mach numbers; a full description of the methods which have been used is given in Ref.1. In general, the results obtained for aerodynamic forces and moments are physically plausible and agree with experiment as well as can be expected, but there are cases where the linear theory produces seemingly incorrect results. These occur in the study of the flow over a wing the planform of which has straight edges* and/or ridge lines. When the free stream Mach number is such that the Mach lines are parallel to one of the straight edges or ridge lines of the wing a discontinuity in slope (vulgarly, a "kink") appears in the graph of, for example, the wave drag of the wing against Mach number. Further, the drag curve rises very steeply to the value at the discontinuity as the appropriate Mach number is approached from below. Neither the discontinuities nor the steep rises are observed in practice.

The explanation of this phenomenon lies in the nature of the pressure distribution predicted by linear theory. The flow over a delta wing with wedge section placed in a supersonic free stream will serve for an example. If the leading edge of the wing is supersonic, then over the region between the Mach line from the apex and the leading edge the pressure is constant. This constant value becomes arbitrarily large as the Mach number decreases until, in the limit, when the leading edge becomes sonic, the constant becomes infinite. At this stage the region of constant pressure has become vanishingly small and the pressure close to the leading edge tends to infinity as the square root of the reciprocal of the distance from the leading edge. As the free stream Mach number drops further the pressure still tends to infinity at the leading edge but the infinity becomes less severe. This is the case of a subsonic leading edge and the pressure tends to infinity as the logarithm of the distance from the leading edge. It is this failure of the linear theory to predict the pressures in the region of straight edges and ridge lines of a wing correctly, (at any rate for a range of Mach numbers), which leads to the spurious discontinuities mentioned above. (Examples of pressure distributions on linear theory are drawn in Fig.1).

The reason why linear theory gives such completely wrong answers for the pressures near straight edges is usually stated to be that the deviation of flow quantities (velocities etc.) from their free stream values is, in actual fact, large, and so the conditions for validity of the linear theory are not fulfilled. It will be suggested further on that the answer is not, perhaps, quite so simple as this; here, it is sufficient to note that the pressures near the leading edge may in reality be much larger than those over the rest of the wing.

Any attempt to eliminate the discontinuities from the linear theory predictions, (and such an attempt is clearly desirable), must almost certainly involve a limiting of the values of the linear theory pressures near straight edges. This note presents a method for doing this which involves satisfying the full boundary condition on the surface of the wing as distinct from the linearised boundary condition; the governing differential equation is still the linear equation of supersonic flow.

2 An Incompressible Analogy

The appearance of the infinities in the linearised pressure distribution over a wing with straight edges can be more easily understood by considering a problem in two-dimensional incompressible flow. This

* If the edges are cuspidal, no difficulties arise from the use of linearised theory. The technique described in this note is only necessary for wings with edges which are not cuspidal.

problem is the determination of the flow over the body drawn in Fig.2(a). The line of symmetry of the body (parallel to the free stream velocity) is taken as the X-axis, X being measured downstream from the nose of the body. The Y-axis is taken normal to the X-axis. The equation of the body is

$$0 < X < X_o, \qquad Y = \delta X, \tag{1a}$$

$$X_o < X, \qquad Y = \delta X_o, \tag{1b}$$

where only the upper half of the body has been described, since discussion of the flow can be confined to the region $Y > 0$, due to the symmetry of the problem. The free stream velocity is U and a perturbation velocity potential, ϕ, is introduced so that $U + u$ and v, the total velocities in the X and Y directions respectively, are given by

$$U + u = U\,(1 + \phi_X), \tag{2a}$$

$$v = U\,\phi_Y. \tag{2b}$$

ϕ can be determined, without any approximation, by means of a simple Schwarz-Christoffel transformation. If $\delta << 1$, ϕ_X and ϕ_Y will, in general, be small compared with one. At $X = 0$, $Y = 0$, however, $\phi_X = -1$ since $X = 0$ is a stagnation point.

The flow can be approximately determined by a different method. The boundary condition over the body is that the body boundary be a stream-line, i.e. that

$$0 < X < X_o, \qquad \left(\frac{\phi_Y}{1 + \phi_X}\right)_{body} = \delta, \tag{3a}$$

$$X_o < X \qquad \left(\frac{\phi_Y}{1 + \phi_X}\right)_{body} = 0. \tag{3b}$$

Since ϕ_X is, in general, small compared with one, this may be written as a linearised boundary condition,

$$0 < X < X_o \qquad (\phi_Y)_{body} = \delta, \tag{4a}$$

$$X_o < X \qquad (\phi_Y)_{body} = 0. \tag{4b}$$

The problem can now be solved by distributing sources along the X-axis from $X = 0$ to $X = \infty$ of strength $f(X)\,dX$ at the point $(X,0)$, $f(X)$ being an undetermined function of X. The potential becomes

$$\phi = \frac{1}{2\pi}\int_0^\infty f(X_1)\,\log\left[(X - X_1)^2 + Y^2\right]\,dX_1, \tag{5}$$

if a further approximation is made, namely that ϕ_X and ϕ_Y on the body can be evaluated on the line $Y = 0$. $f(X)$ can then be taken as equal to δ if $0 < X < X_o$, and equal to zero otherwise. It can then be shown that

$$(\phi_X)_{body} = \frac{\delta}{\pi} \log\left|\frac{X}{X_o - X}\right| \tag{6a}$$

$$0 < X < X_o, \qquad (\phi_Y)_{body} = \delta \tag{6b}$$

$$X_o < X \qquad (\phi_Y)_{body} = 0 \tag{6c}$$

ϕ_X on the body now tends to infinity as $X \to 0$ (and also as $X \to X_o$); over most of the body the formula for ϕ_X is approximately correct but near $X = 0$ and near $X = X_o$ the formula is quite invalid. The failure near these points must be caused by the introduction of one (or both) of the two approximations made above, the linearisation of the boundary condition and the evaluation of ϕ_X and ϕ_Y on $Y = 0$ instead of on the body. In fact, it is the first of these approximations which causes the spurious infinities in ϕ_X on the body; for instance, since $\phi_X = -1$ at $X = 0$, it is not permissible to neglect ϕ_X in comparison with unity near $X = 0$. If the second of the approximations is made but not the first, then

$$(\phi_X)_{body} = \frac{1}{\pi}\int_0^\infty \frac{f(X_1)}{(X - X_1)}\, dX_1, \tag{7}$$

and a known result for a distribution of two-dimensional sources gives

$$(\phi_Y)_{body} = f(X) \tag{8}$$

The boundary condition becomes

$$0 < X < X_o, \qquad \delta\left[1 + \frac{1}{\pi}\int_0^\infty \frac{f(X_1)\, dX_1}{X - X_1}\right] = f(X) \tag{9a}$$

$$X_o < X_1, \qquad 0 = f(X) \tag{9b}$$

Hence the following integral equation for $f(X)$ is obtained:

$$0 < X < X_o, \qquad \delta\left[1 + \frac{1}{\pi}\int_0^{X_o} \frac{f(X_1)\, dX_1}{(X - X_1)}\right] = f(X) \tag{10}$$

The solution of the equation is approximately $f(X) = \delta$, except for small regions near $X = 0$ and $X = X_o$. If the integral equation were solved for $f(X)$ and then ϕ_X on the body evaluated, using equation (7), it would be found that ϕ_X was finite for all X.

3 Supersonic Flow over a Sweptback Wing with Sonic Leading Edge

In this section supersonic flow over a sweptback wing will be considered. The apex of the wing will be taken as the origin of coordinates with the X-axis parallel to the free stream velocity. The wing will be assumed to lie approximately in the X-Y plane, i.e. in the plane $Z = 0$, X,Y,Z being rectangular coordinates. The equation of the (sonic) leading edge is $X = \pm BY$, where $B = (M_\infty^2 - 1)^{\frac{1}{2}}$ and M_∞ is the free stream Mach number. The equation of the upper surface of the wing in the region to be considered is taken as

$$Z = \delta(X - BY), \qquad Y > 0 \tag{11a}$$

$$Z = \delta(X + BY), \qquad Y < 0. \tag{11b}$$

where δ is a constant small compared with unity. These equations represent the upper surface of a delta wing with a wedge section. If the magnitude of the free stream velocity is U, then a perturbation velocity potential can be introduced so that U + u, v and w, the velocities in the X, Y and Z directions respectively, are given by

$$U + u = V(1 + \phi_X) \tag{12a}$$

$$v = U\,\phi_Y, \tag{12b}$$

$$w = U\,\phi_Z. \tag{12c}$$

The governing partial differential equation is known to be

$$B^2\,\phi_{XX} = \phi_{YY} + \phi_{ZZ} \tag{13}$$

The boundary condition to be satisfied on that part of the wing for which $Y > 0$ is that there should be no flow velocity normal to the surface, i.e. that

$$Z = 0, \qquad \delta(1 + \phi_X) - B\delta\,\phi_Y - \phi_Z = 0, \tag{14}$$

the boundary condition being applied on the plane $Z = 0$ rather than on the wing itself. A solution of equation (13) is required which satisfies equation (14).

The integrals which arise in the solution of equation (13) are handled more easily if the following transformation of coordinates is made.

$$x = \frac{X - BY}{\sqrt{2B}}, \quad y = \frac{X + BY}{\sqrt{2B}}, \quad z = Z \qquad (15a), (15b), (15c)$$

The governing partial differential equation becomes

$$2\,\phi_{xy} = \phi_{zz} \tag{16}$$

and the surface boundary condition becomes

$$\delta\left\{1 + \frac{B^2 + 1}{\sqrt{2B}}\,\phi_x - \frac{B^2 - 1}{\sqrt{2B}}\,\phi_y\right\} = \phi_z \tag{17}$$

An elemental source solution of equation (13) is

$$d\phi = -\frac{dS}{\pi\,[(X-X_1)^2 - B^2(Y-Y_1)^2 - B^2(Z-Z_1)^2]^{\frac{1}{2}}}$$

where (X,Y,Z) is the position of the infinitesimal element of surface, dS, and(X_1, Y_1, Z_1) are running coordinates. If the problem is solved by distributing sources over the surface of the wing, then the solution of equation (13) is

$$\phi = -\frac{1}{\pi}\iint_S \frac{f(X_1,Y_1)\;dX_1,dY_1}{[(X-X_1)^2 - B^2(Y-Y_1)^2 - B^2\;\;Z^2]^{\frac{1}{2}}} \tag{18}$$

S being that part of the wing surface lying within the Mach fore-cone from the point (X,Y,Z). The wing surface has been assumed to lie in the plane Z = 0 and f(X,Y) is an unknown function to be found by substituting in equation (14). After the transformation of equations (15) has been applied to equation (18), the potential becomes

$$\phi = -\frac{1}{\pi}\iint_S \frac{f(x_1,y_1)\;dx_1\;dy_1}{[2(x-x_1)(y-y_1) - z^2]^{\frac{1}{2}}} \tag{19}$$

The new arbitrary function is now to be determined by substituting in equation (17).

The equation of the wing differs according as to whether Y < 0 or Y > 0, i.e. as to whether y < x or y > x, and so will the source distribution f(x,y). A consideration of the symmetry of the problem shows that f(x,y) = f(y,x). The integral for ϕ on the wing can now be written as

$$\phi = -\frac{1}{\sqrt{2}\pi}\int_0^x\int_0^{x_1}\frac{f(y_1,x_1)\;dy_1\;dx_1}{(x-x_1)^{\frac{1}{2}}\;(y-y_1)^{\frac{1}{2}}} - \frac{1}{\sqrt{2}\pi}\int_0^x\int_{x_1}^{y}\frac{f(x_1,y_1)\;dy_1\;dx_1}{(x-x_1)^{\frac{1}{2}}\;\;(y-y_1)^{\frac{1}{2}}}$$

considering only that part of the wing for which y > x.

To obtain ϕ_x and ϕ_y on the wing it is permissible to differentiate the above formulae directly, i.e. to put z = 0 in equation (19) and differentiate afterwards. After a partial integration there follows:

$$\phi_x = -\frac{1}{\sqrt{2}\pi\;x^{\frac{1}{2}}}\int_0^y\frac{f(0,y_1)\;dy_1}{(y-y_1)^{\frac{1}{2}}} - \frac{1}{\sqrt{2}\pi}\int_0^x\int_0^{x_1}\frac{f_{x_1}(y_1,x_1)\;dy_1\;dx_1}{(x-x_1)^{\frac{1}{2}}\;(y-y_1)^{\frac{1}{2}}}$$

$$-\frac{1}{\sqrt{2}\pi}\int_0^x\int_{x_1}^{y}\frac{f_{x_1}(x_1,y_1)\;dy_1\;dx_1}{(x-x_1)^{\frac{1}{2}}\;(y-y_1)^{\frac{1}{2}}} \tag{20a}$$

and

$$\phi_y = -\frac{1}{\sqrt{2\pi}\, y^{\frac{1}{2}}} \int_0^x \frac{f(0,x_1)\, dx_1}{(x-x_1)^{\frac{1}{2}}} - \frac{1}{\sqrt{2\pi}} \int_0^x \int_0^{x_1} \frac{f_{y_1}(y_1,x_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}}$$
$$- \frac{1}{\sqrt{2\pi}} \int_0^x \int_{x_1}^y \frac{f_{y_1}(x_1,y_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} \qquad (20b)$$

A known result for a distribution of sources gives, on the wing,

$$\phi_z = f(x,y) \qquad (20c)$$

An integral equation for $f(x,y)$ is obtained when these values of ϕ_x, ϕ_y and ϕ_x are substituted in equation (17).

From the formula for ϕ_x it follows that ϕ_x (and hence the pressure) will tend to infinity as x tends to zero unless

$$\int_0^y \frac{f(0,y_1)\, dy_1}{(y-y_1)^{\frac{1}{2}}} = 0 \qquad (21)$$

The solution of this Abel equation is simply $f(0,y) = 0$; therefore, the infinity in the pressure distribution predicted by linear theory can only be removed if $f(x,y)$, the solution of the integral equation derived from equation (17), is such that $f(0,y) = 0$. Assuming that this is so, the integral equation becomes

$$\delta \left\{ 1 - \frac{B^2+1}{2\pi B} \int_0^x \int_0^{x_1} \frac{f_{x_1}(y_1,x_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} - \frac{B^2+1}{2\pi B} \int_0^x \int_{x_1}^y \frac{f_{x_1}(x_1,y_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} \right.$$
$$\left. + \frac{B^2-1}{2\pi B} \int_0^x \int_0^{x_1} \frac{f_{y_1}(y_1,x_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} + \frac{B^2-1}{2\pi B} \int_0^x \int_{x_1}^y \frac{f_{y_1}(x_1,y_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} \right\} = f(x,y) \qquad (22)$$

and a solution of this is required such that $f(0,y) = 0$.

$f(x,y)$ and the various flow quantities will be constant along straight lines drawn through the apex of the wing, i.e. they will be functions of x/y only. If it is assumed that $f(x,y)$ vanishes at $x = 0$ as $k(x/y)^{\frac{1}{2}}$, with a constant to be determined, then, near $x = 0$,

$$f_x(y,x) \sim -\frac{k}{2}\,\frac{y^{\frac{1}{2}}}{x^{3/2}}, \qquad f_x(x,y) \sim \frac{k}{2}\,\frac{1}{(xy)^{\frac{1}{2}}},$$

$$f_y(y,x) \sim \frac{k}{2}\,\frac{1}{(xy)^{\frac{1}{2}}}, \qquad f_y(x,y) \sim -\frac{k}{2}\,\frac{x^{\frac{1}{2}}}{y^{3/2}}.$$

The various integrals in equation (22) can be evaluated without any difficulty and the following results are obtained.

$$\int_0^x\int_0^{x_1} \frac{f_{x_1}(y_1,x_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}} \sim -\frac{2k}{3}\left(\frac{x}{y}\right)^{\frac{1}{2}}$$

$$\int_0^x\int_{x_1}^{y} \frac{f_{x_1}(x_1,y_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}} \sim \frac{\pi^2 k}{2} - 2k\left(\frac{x}{y}\right)^{\frac{1}{2}}$$

$$\int_0^x\int_0^{x_1} \frac{f_{y_1}(y_1,x_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}} \sim 2k\left(\frac{x}{y}\right)^{\frac{1}{2}}$$

$$\int_0^x\int_{x_1}^{y} \frac{f_{y_1}(x_1,y_1)\, dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}} \sim -2k\left(\frac{x}{y}\right)^{\frac{1}{2}}$$

If these are substituted in equation (22) and x/y allowed to tend to zero the integral equation reduces to

$$\delta\left\{1 - \frac{(B^2+1)}{4B}\, k\right\} = 0$$

$$\text{i.e.} \qquad k = \frac{4B}{\pi(B^2+1)} \tag{23}$$

Thus, f(x,y) vanishes as $\frac{4B}{\pi(B^2+1)}\left(\frac{x}{y}\right)^{\frac{1}{2}}$. It is to be expected that, over most of the wing surface, f(x,y) will be approximately equal to the value on linearised theory with the usual linearised boundary condition applied, and that only in a small region near the leading edge will it depart significantly from the usual linearised value; this value is f(x,y) = δ. A function satisfying these requirements is

$$f(x,y) = \frac{2\delta}{\pi}\ \sin^{-1}\frac{(x/y)^{\frac{1}{2}}}{(\nu^2 + x/y)^{\frac{1}{2}}}\ , \tag{24}$$

where

$$\nu = \frac{B^2+1}{2B}\,\delta\,. \tag{25}$$

When ν^2 can be neglected in comparison with x/y, (and this is so over most of the wing surface), f(x,y) = δ. There is, however, a small region near the leading edge where x/y is of the same order as ν^2 or of smaller order and then f(x,y) differs significantly from δ. When x/y is sufficiently small

$$\sin^{-1}\frac{(x/y)^{\frac{1}{2}}}{(\nu^2+x/y)^{\frac{1}{2}}} \sim \frac{1}{\nu}\left(\frac{x}{y}\right)^{\frac{1}{2}} = \frac{2B}{\delta(B^2+1)}\left(\frac{x}{y}\right)^{\frac{1}{2}}$$

so that f(x,y) vanishes as $\dfrac{4B}{\pi(B^2+1)}\left(\dfrac{x}{y}\right)^{\frac{1}{2}}$.

It is shown in Appendix I that the function defined in equation (24) satisfies the integral equation (22) to an accuracy of order δ everywhere on the wing including the region close to the leading edge where ordinary linearised theory breaks down. In appendix I the following formulae for the partial derivatives of ϕ on the wing are derived:

$$\phi_x = -\frac{2\sqrt{2B}}{\pi(B^2+1)}\sin^{-1}\frac{\nu y^{\frac{1}{2}}}{(x+\nu^2 y)^{\frac{1}{2}}},$$

$$\phi_y = -\frac{\sqrt{2}\delta}{\pi}\left(\frac{x}{y}\right)^{\frac{1}{2}},$$

$$\phi_z = \frac{2\delta}{\pi}\sin^{-1}\frac{x^{\frac{1}{2}}}{(\nu^2 y+x)^{\frac{1}{2}}}$$

Reverting to the original coordinates X,Y and Z by using equations (15), it is found that

$$\phi_X = \frac{1}{\sqrt{2B}}\phi_x + \frac{1}{\sqrt{2B}}\phi_y = -\frac{1}{\pi B}\left[\frac{2B}{(B^2+1)}\sin^{-1}\frac{\nu y^{\frac{1}{2}}}{(x+\nu^2 y)^{\frac{1}{2}}} + \delta\left(\frac{x}{y}\right)^{\frac{1}{2}}\right]$$

so that the X-velocity is given by

$$U+u = U\left\{1 - \frac{2}{\pi(B^2+1)}\sin^{-1}\frac{\nu y^{\frac{1}{2}}}{(x+\nu^2 y)^{\frac{1}{2}}} - \frac{\delta}{\pi B}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right\}. \qquad (26a)$$

The Y-velocity is given by

$$v = U\phi_Y = U\left[-\frac{1}{\sqrt{2}}\phi_x + \frac{1}{\sqrt{2}}\phi_y\right] = U\left[\frac{2B}{\pi(B^2+1)}\sin^{-1}\frac{\nu y^{\frac{1}{2}}}{(x+\nu^2 y)^{\frac{1}{2}}} - \frac{\delta}{\pi}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right]. \qquad (26b)$$

The Z-velocity is given by

$$w = U\phi_Z = U\phi_z = \frac{2\delta U}{\pi}\sin^{-1}\frac{x^{\frac{1}{2}}}{(\nu^2 y+x)^{\frac{1}{2}}} = \delta U\left[1 - \frac{2}{\pi}\sin^{-1}\frac{\nu y^{\frac{1}{2}}}{(x+\nu^2 y)^{\frac{1}{2}}}\right] \qquad (26c)$$

Over most of the wing surface, i.e. for $x/y >> \delta^2$, these results reduce to the usual linear theory results, but close to the leading edge they differ markedly from the latter results. In fact, at the leading edge $(x/y = 0)$,

$$U + u = \frac{B^2}{(B^2 + 1)} U, \qquad v = \frac{B}{(B^2 + 1)} U, \qquad w = 0$$

Now the component of velocity in the X, Y plane normal to the leading edge is

$$- v \cos\mu + (U + u) \sin\mu$$

here, $\mu = \cot^{-1} B$ and is the Mach angle or, since the leading edge is sonic, the semi-apex angle of the wing. But along the leading edge

$$- v \cos\mu + (U + u) \sin\mu = - \frac{B}{(1 + B^2)^{\frac{1}{2}}} v + \frac{1}{(1 + B^2)^{\frac{1}{2}}} (U + u) = 0.$$

The theory predicts that, at a point of the leading edge, the flow in the plane through this point normal to the leading edge is brought to rest. The velocity parallel to the leading edge is

$$(U + u) \cos\mu + v \sin\mu = U \left\{ \frac{B^2}{(B^2+1)} \cdot \frac{B}{(B^2+1)^{\frac{1}{2}}} + \frac{B}{(B^2+1)} \cdot \frac{1}{(B^2+1)^{\frac{1}{2}}} \right\} = U \cos\mu$$

which is simply the component of the free stream velocity along the leading edge. (The ordinary linearised theory gives this result but predicts an infinite velocity normal to the leading edge).

From the above formulae for the velocity components it is possible to evaluate the pressure distribution over the wing. The familiar linearised formula for the pressure coefficient,

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho U^2} = - 2 \phi_x , \tag{27}$$

where p, p_∞ and ρ are local pressure, the free stream pressure and free stream density respectively, cannot be used here because, in the vicinity of the leading edge, ϕ_x and ϕ_y are not necessarily small compared with one. If q is the speed of the flow, i.e. if

$$q^2 = (U + u)^2 + v^2 + w^2,$$

then

$$C_p = \frac{2}{\gamma M^2} \left\{ \left[1 + \frac{\gamma - 1}{2} M^2 \left(1 - \frac{q^2}{U^2} \right) \right]^{\gamma/\gamma-1} - 1 \right\} \tag{28a}$$

where M, the local Mach number, is given by

$$1 + \frac{2}{(\gamma-1)M^2} = \frac{U^2}{q^2}\left[1 + \frac{2}{(\gamma-1)\,M_\infty^2}\right], \tag{28b}$$

and γ is the ratio of the specific heats. The pressure as a function of x/y (which varies from 0 at the leading edge to 1 at the centre line) is plotted in Fig.3 for $\delta = 0.2$, $M = \sqrt{2}$ and $\gamma = 1.4$; the ordinary linearised theory curve is also drawn.

4 Further Examples of Flow over Straight-edged Wings

In this section some more examples of supersonic flow over wings the planforms of which have straight edges will be briefly discussed.

4.1 Flow over a wing with supersonic leading edges

The angle of sweepback of the wing is denoted by Λ_o and, with notation as before, the equation of the upper surface of the wing is

$$Y > 0, \quad Z = \delta(X - Y \tan \Lambda_o) \tag{29a}$$

$$Y < 0, \quad Z = \delta(X + Y \tan \Lambda_o) \tag{29b}$$

The surface boundary condition can be written as

$$Z = 0, \quad \delta(1 + \phi_X) - \delta\, \phi_Y \tan \Lambda_o - \phi_Z = 0, \tag{30}$$

considering only that part of the wing for which $Y > 0$. If the transformation of equations (15) is made, the boundary condition becomes

$$\delta\left\{1 + \frac{B \tan \Lambda_o + 1}{\sqrt{2}\,B}\,\phi_x - \frac{B \tan \Lambda_o - 1}{\sqrt{2}\,B}\,\phi_y\right\} = \phi_z \tag{31}$$

The solution of the problem will first be found for the region between the Mach cone from the apex of the wing and the leading edge. ϕ is given, as above, by

$$\phi = -\frac{1}{\pi}\iint_S \frac{f(x_1,y_1)\,dx_1\,dy_1}{[2(x-x_1)(y-y_1) - z^2]^{\frac{1}{2}}},$$

$f(x,y)$ being the unknown source distribution, and S being that part of the wing surface lying within the Mach fore-cone from (X,Y,Z). The equation of the leading edge ($X = Y \tan \Lambda_o$), is in the new coordinates

$$(B + \tan \Lambda_o)\,x + (B - \tan \Lambda_o)\,y = 0 \tag{32}$$

ϕ becomes from Fig.(2b)

$$\phi = -\frac{1}{\pi}\int_{-x\cot\beta}^{y}\int_{-y_1\tan\beta}^{x} \frac{f(x_1,y_1)\,dx_1\,dy_1}{[2(x-x_1)(y-y_1) - z^2]^{\frac{1}{2}}} \tag{33}$$

where

$$\tan\beta = \frac{B - \tan\Lambda_o}{B + \tan\Lambda_o} \tag{34}$$

Since the flow is locally two-dimensional in the region under consideration flow quantities along lines parallel to the leading edge are constant. They are also constant along lines through the apex, and so are constant everywhere in the region. This suggests putting $f(x,y) = K$ where K is a constant to be determined from the surface boundary condition. ϕ on the wing surface, (taken to be $z = 0$), is given by

$$\phi = -\frac{1}{\pi}\int_{-x\cot\beta}^{y}\int_{-y_1\tan\beta}^{x}\frac{K\,dx_1\,dy_1}{\sqrt{2}\,(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}}$$

$$= -\frac{K}{\sqrt{2}\pi}\int_{-x\cot\beta}^{y}\frac{dy_1}{(y-y_1)^{\frac{1}{2}}}\left[-2(x-x_1)^{\frac{1}{2}}\right]_{-y_1\tan\beta}^{x}$$

$$= -\frac{\sqrt{2}K}{\pi}\int_{-x\cot\beta}^{y}\frac{(x+y_1\tan\beta)^{\frac{1}{2}}\,dy_1}{(y-y_1)^{\frac{1}{2}}}$$

$$= -\frac{2\sqrt{2}K}{\pi}\int_{0}^{(y+x\cot\beta)^{\frac{1}{2}}}(x+y\tan\beta-u^2\tan\beta)\,du$$

$$= -\frac{2\sqrt{2}K}{\pi}(y+x\cot\beta)\,(\tan\beta)^{\frac{1}{2}}\int_{0}^{\pi/2}\cos^2\theta\,d\theta$$

$$= -\frac{K}{\sqrt{2}}(y+x\cot\beta)\,(\tan\beta)^{\frac{1}{2}}$$

Thus,

$$\phi_x = -\frac{K}{(2\tan\beta)^{\frac{1}{2}}} = -\frac{K}{\sqrt{2}}\left(\frac{B+\tan\Lambda_o}{B-\tan\Lambda_o}\right)^{\frac{1}{2}} \tag{35a}$$

$$\phi_y = -\frac{K}{\sqrt{2}}(\tan\beta)^{\frac{1}{2}} = -\frac{K}{\sqrt{2}}\left(\frac{B-\tan\Lambda_o}{B+\tan\Lambda_o}\right)^{\frac{1}{2}} \tag{35b}$$

Substituting in the boundary condition, equation (31),

$$\delta\left\{1-\frac{K}{2B}(B\tan\Lambda_o+1)\left(\frac{B+\tan\Lambda_o}{B-\tan\Lambda_o}\right)^{\frac{1}{2}}+\frac{K}{2B}(B\tan\Lambda_o-1)\left(\frac{B-\tan\Lambda_o}{B+\tan\Lambda_o}\right)^{\frac{1}{2}}\right\} = K,$$

since on the wing, as before,

$$\phi_z = f(x,y).$$

K is given by

$$K = \frac{2B(B^2 - \tan^2 \Lambda_o)^{\frac{1}{2}} \delta}{2B(B^2 - \tan^2 \Lambda_o)^{\frac{1}{2}} + 2B\,\delta\,\sec^2 \Lambda_o}$$

$$= \frac{\cot \Lambda_o\,(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}}\,\delta}{\cot \Lambda_o\,(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}} + \delta\,\mathrm{cosec}^2 \Lambda_o} \qquad (36)$$

If δ is small in comparison with $(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}}$ this is very nearly the same result as is obtained by ordinary linearised theory, (i.e. $K = \delta$); but when $(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}}$ is at most $O(\delta^2)$, i.e. when the leading edge is almost sonic, then this result for K differs significantly from that of the ordinary linearised theory. When the leading edge is sonic the formula reduces to $K = 0$, in agreement with the result of section 3. Using equations (35) the formulae for the velocity components are

$$U + u = U(1 + \phi_X) = U(1 + \frac{1}{\sqrt{2B}}\,\phi_x + \frac{1}{\sqrt{2B}}\,\phi_y)$$

$$= U\left\{1 - \frac{\delta \cot^2 \Lambda_o}{\left[\cot \Lambda_o\,(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}} + \delta\,\mathrm{cosec}^2 \Lambda_o\right.}\right\}, \qquad (37a)$$

$$v = U\,\phi_Y = U\left(-\frac{1}{\sqrt{2}}\,\phi_x + \frac{1}{\sqrt{2}}\,\phi_y\right) = \frac{\delta \cot \Lambda_o}{\left[\cot \Lambda_o\,(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}} + \delta\,\mathrm{cosec}^2 \Lambda_o\right]}, \qquad (37b)$$

$$w = U\,\phi_Z = U\,\phi_z = \frac{\delta \cot \Lambda_o\,(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}}}{\left[\cot \Lambda_o\,(B^2 \cot^2 \Lambda_o - 1)^{\frac{1}{2}} + \delta\,\mathrm{cosec}^2 \Lambda_o\right]} \qquad (37c)$$

The pressure coefficient can now be worked out from these velocity components in the manner described at the end of section 3. In general it will agree with the ordinary linearised theory (to within the accuracy of that theory), but, if the leading edge is imagined to become closer and closer to a sonic leading edge, the ordinary linearised pressure coefficient becomes arbitrarily large along this edge. The pressure coefficient on the present theory tends to the finite value given at the end of section 3.

As Λ_o tends to zero (in which case the problem becomes two-dimensional) K tends to

$$\frac{B\delta}{B + \delta} = \frac{\delta}{1 + \delta/B},$$

which compares with the ordinary linearised theory result of δ. As might be expected the two results differ by a term of order δ^2 only, so that the pressure coefficients differ in the same way. Although this is theoretically of no consequence it is obviously desirable to have the modified theory agreeing with the linear theory exactly in this limiting case. One way of achieving this is as follows. Writing $(C_p)_m$ for the modified pressure coefficient and $(C_p)_\ell$ for the linearised pressure coefficient, the quantity

$$(C_p)_m + \left[(C_p)_\ell - (C_p)_m\right]\frac{(B^2 \cot^2 \Lambda_o - 1)}{B^2 \cot^2 \Lambda_o} \qquad (38)$$

has the following properties. If the leading edge is sonic it is equal to $(C_p)_m$, since $(C_p)_\ell$ behaves as $(B^2 \cot^2 \Lambda_o - 1)^{-\frac{1}{2}}$ when the leading edge is almost sonic. On the other hand, when $\Lambda_o = 0$ and the problem is two-dimensional, the above quantity reduces exactly to $(C_p)_\ell$. In fact, this quantity never differs significantly from $(C_p)_m$ but it has the advantage that it is exactly equal to the linearised value in the limiting case of a two-dimensional problem.

Over the region of the wing between the Mach cone from the apex and the centre line $f(x,y)$ is no longer a constant everywhere, although it is still constant along lines through the apex. It will start with the value given by equation (36) on the trace of the Mach cone ($X = BY$) and, in a region close to this line, it will fall rapidly to approximately the value given by ordinary linearised theory, which is simply $f(x,y) = \delta$. The exact variation of $f(x,y)$ can be found only by solving a rather complicated integral equation derived from the boundary condition of equation (31). For the moment it is sufficient to remark that the pressure coefficient has a value on the line X=BY which can be obtained from equations (37) and then also falls rapidly, in a region close to this line, to approximately the value of ordinary linearised theory.

4.2 Flow over a wing with subsonic leading edges

The equation of the upper surface of the wing and the notation used are the same as in section (4.1); the surface boundary condition is again

$$\delta\left\{1 + \frac{B \tan \Lambda_o + 1}{\sqrt{2}B}\,\phi_x - \frac{B \tan \Lambda_o - 1}{\sqrt{2}B}\,\phi_y\right\} = \phi_z$$

There is now only one region to consider since the Mach cone lies outside the leading edge. Ordinary linearised theory gives for the source distribution over the wing,

$$f(x,y) = \delta$$

This leads to logarithmic infinities in ϕ_x and ϕ_y at the leading edge; these arise because the linearised boundary has been used instead of the full boundary condition and these two conditions differ considerably from one another in the vicinity of the leading edge. If ϕ_x is written out in terms of $f(x,y)$, as in section 3, it can be shown that ϕ_x will be infinite at the leading edge unless $f(x,y) = 0$ there. Thus, the integral equation for $f(x,y)$ obtained from equation (31) must possess a solution which vanishes along the leading edge if the pressure coefficient is to remain finite there. It will be assumed that such a solution exists.

At the leading edge, then,

$$f(x,y) = \phi_z = 0,$$

and from equation (31)

$$1 + \frac{B \tan \Lambda_o + 1}{\sqrt{2}B} \phi_x - \frac{B \tan \Lambda_o - 1}{\sqrt{2}B} \phi_y = 0$$

Now the velocity normal to the leading edge is

$$(U + u) \cos \Lambda_o - v \sin \Lambda_o = U \cos \Lambda_o \left[1 + \phi_X - \phi_Y \tan \Lambda_o\right]$$

$$= U \cos \Lambda_o \left[1 + \left(\frac{1}{\sqrt{2}B} \phi_x + \frac{1}{\sqrt{2}B} \phi_y\right) + \left(\frac{1}{\sqrt{2}} \phi_x - \frac{1}{\sqrt{2}} \phi_y\right) \tan \Lambda_o\right]$$

$$= U \cos \Lambda_o \left[1 + \frac{B \tan \Lambda_o + 1}{\sqrt{2}B} \phi_x - \frac{B \tan \Lambda_o - 1}{\sqrt{2}B} \phi_y\right]$$

$$= 0$$

Therefore, as in the case of a sonic leading edge, the flow at a point of the leading edge in a plane normal to the edge is brought to rest. The velocity along the edge is

$$(U + u) \sin \Lambda_o + v \cos \Lambda_o = U \sin \Lambda_o \left(1 + \phi_X + \phi_Y \cot \Lambda_o\right).$$

As in section 3 ordinary linear theory does not break down in the determination of the component of velocity along the leading edge. In this case it gives

$$\phi = -\frac{\delta}{\pi(1-B^2\cot^2\Lambda_o)^{\frac{1}{2}}}\left[(X\cot\Lambda_o+Y)\cosh^{-1}\frac{(X+B^2Y\cot\Lambda_o)}{B(X\cot\Lambda_o+Y)} + (X\cot\Lambda_o-Y)\cosh^{-1}\frac{(X-B^2Y\cot\Lambda_o)}{B(X\cot\Lambda_o-Y)}\right],$$

$$\phi_X = -\frac{\delta\cot\Lambda_o}{\pi(1-B^2\cot^2\Lambda_o)^{\frac{1}{2}}}\left[\cosh^{-1}\frac{(X+B^2Y\cot\Lambda_o)}{B(X\cot\Lambda_o+Y)} + \cosh^{-1}\frac{(X-B^2Y\cot\Lambda_o)}{B(X\cot\Lambda_o-Y)}\right],$$

$$\phi_Y = -\frac{\delta}{\pi(1-B^2\cot^2\Lambda_o)^{\frac{1}{2}}}\left[\cosh^{-1}\frac{(X+B^2Y\cot\Lambda_o)}{B(X\cot\Lambda_o+Y)} - \cosh^{-1}\frac{(X-B^2Y\cot\Lambda_o)}{B(X\cot\Lambda_o-Y)}\right],$$

and so

$$U\sin\Lambda_o(1+\phi_X+\phi_Y\cot\Lambda_o) = U\sin\Lambda_o\left[1 - \frac{2\delta\cot\Lambda_o}{\pi(1-B^2\cot^2\Lambda_o)^{\frac{1}{2}}}\cosh^{-1}\frac{(X+B^2Y\cot\Lambda_o)}{B(X\cot\Lambda_o+Y)}\right].$$

Along the leading edge, then, the velocity is

$$U\sin\Lambda_o\left[1 - \frac{2\delta\cot\Lambda_o}{\pi(1-B^2\cot^2\Lambda_o)^{\frac{1}{2}}}\cosh^{-1}\frac{1+B^2\cot^2\Lambda_o}{2B\cot\Lambda_o}\right] \qquad (39)$$

The pressure coefficient can now be obtained as at the end of section 3. To determine the pressure coefficient over the rest of the wing, $f(x,y)$ would have to be found by satisfying the boundary condition, equation (31). This condition yields a complicated integral equation for $f(x,y)$; if this were solved it would be found that, except for a small region close to the leading edge, $f(x,y)$ would be almost equal to δ, its value on ordinary linearised theory. Close to the leading edge $f(x,y)$ would rise rapidly from the value there of zero to approximately the value δ. The pressure coefficient has a value at the leading edge which can be obtained from the theory of this section and then falls rapidly from this value to approximately the ordinary linearised value in a region close to the leading edge. Thereafter, up to the centre line, it remains approximately the same as the ordinary linearised value.

4.3 Supersonic flow in the presence of a ridge line

Further difficulties arise in the application of linear theory to the supersonic flow over thin wings when straight ridge lines are present, (as in the case of a delta wing of double wedge section). If the ridge line is sonic or subsonic the pressure coefficient along the ridge line is infinite on linearised theory; if the ridge line is supersonic but nearly sonic the pressure coefficient can become arbitrarily large. The curve of the drag of the wing plotted against free stream Mach number displays a discontinuity in slope at the Mach number at which the ridge line becomes sonic; this discontinuity is not obtained in practice. The presence of this discontinuity can be traced to the use of a linearised boundary condition which neglects a term not negligible near the ridge line.

It is not possible directly to improve the results of linearised theory by using the technique of this note. The reason for this can best be demonstrated by an example, that of the flow over the rear part of a delta wing of double wedge section with a supersonic ridge line. The trailing edge will be taken to be a line in the X, Y plane normal to the direction of the free stream. The flow upstream of the ridge line will be ignored in this example; the results obtained are unaffected if its effect is included. The equation of the rear part of the upper surface of the wing is

$$Z = -\delta X$$

The origin of coordinates has been moved to the point where the ridge line meets the centre line, and a value of $-\delta$ has been taken for the slope of the rear part of the wing. The boundary condition becomes

$$\delta\,(1 + \phi_X) + \phi_Z = 0.$$

The transformation of equations (15) turns the above equation into

$$\delta\left(1 + \frac{1}{\sqrt{2B}}\,\phi_x + \frac{1}{\sqrt{2B}}\,\phi_y\right) + \phi_z = 0. \tag{40}$$

If the problem of the flow over the wing is solved by distributing sources over the wing, then the source distribution function in the region between the trace of the Mach cone from the origin and the ridge line is a constant, K say, as in section 4.1. Using equations (35), equation (40) becomes

$$\delta\left\{1 - \frac{K}{2B}\left(\frac{B + \tan\Lambda_o}{B - \tan\Lambda_o}\right)^{\frac{1}{2}} - \frac{K}{2B}\left(\frac{B - \tan\Lambda_o}{B + \tan\Lambda_o}\right)^{\frac{1}{2}}\right\} + K = 0,$$

$$\therefore \quad K = \frac{-2B\,(B^2 - \tan^2\Lambda_o)^{\frac{1}{2}}\,\delta}{2B(B^2 - \tan^2\Lambda_o)^{\frac{1}{2}} - (B + \tan\Lambda_o)\,\delta - (B - \tan\Lambda_o)\delta}$$

$$= \frac{-(B^2 - \tan^2\Lambda_o)^{\frac{1}{2}}\,\delta}{(B^2 - \tan^2\Lambda_o)^{\frac{1}{2}} - \delta} \qquad (41)$$

Thus, when $(B^2 - \tan^2\Lambda_o) = \delta^2$, the source distribution over a finite region of the wing becomes infinite, together with the velocity components and the theory breaks down completely. If the ridge line is subsonic or sonic the preceding theory gives sensible results for all cases. Nevertheless, it is clear that the theory cannot be applied directly to flow over a wing with straight ridge lines.

The difficulty can be overcome by the following subterfuge. When ordinary linearised theory is applied to determine the flow over a thin delta wing of double wedge section the wing is regarded as one delta wing superimposed on another, the first delta wing having a positive slope, and the second having a negative slope. The modified theory already developed in this note can be used to obtain the flow over the first wing since this has a positive slope. Again, considering two delta wings of the same planform and with equal and opposite slopes, it is evident that the distributions of pressure coefficient on ordinary linearised theory will be identical except for sign. It will be assumed that this result holds true in practice, so that the pressure coefficient of a wing with negative slope will be everywhere taken as the negative of the pressure coefficient of the wing which is the image in the X, Y plane of the original wing.

The justification for this step is first that it is correct on ordinary linearised theory and it is therefore to be expected that in practice the pressure coefficients will have approximately the same absolute value. Secondly, the method developed in this note is no more than an artifice to eliminate the spurious discontinuities present in the curve of drag against Mach number. All that is required of such a method is that it shall remove the discontinuities and shall replace them by a plausible curve; the method of this note does fulfil both these requirements and this is its ultimate justification.

The technique of this note can be used, then, to obtain the drag of a delta wing of double-wedge section. As in ordinary linearised theory the wing is assumed to consist of two deltas superimposed and the technique is applied to each delta separately. This means that the surface boundary condition is not exactly satisfied since the interference of one delta on the other is ignored. It can be shown, however, that this interference can be neglected, without involving an error any larger than that normally tolerated in linearised theory.

4.4 Supersonic flow over a wing of biconvex section

So far all the wings considered have had regions of constant slope; the example of this section is a wing the slope of which varies. The equation of the upper surface of this wing is

$$0 < Y < s, \qquad Z = 2\varepsilon\ (X - BY)\ [1 - (X-BY)], \qquad \varepsilon << 1 \tag{42a}$$

$$- s < Y < 0, \qquad Z = 2\varepsilon\ (X + BY)\ [1 - (X + BY)] \tag{42b}$$

which represents a wing of constant chord, this constant being taken as the unit of length, span 2s, and thickness-chord ratio ε. The leading edge is sonic. The wing is symmetrical about the line Y = 0 and only that part of the wing defined by Y > 0 will be considered. The boundary condition becomes

$$2\varepsilon\ [1 - 2(X - BY)]\ [1 + \phi_X - B\ \phi_Y] - \phi_Z = 0$$

With the same transformation as before, i.e. that of equations (15), this becomes

$$2\varepsilon\ [1 - 2\sqrt{2B}\ x] \left[1 + \frac{B^2 + 1}{\sqrt{2B}}\ \phi_x - \frac{B^2 - 1}{\sqrt{2B}}\ \phi_y \right] - \phi_z = 0 \tag{43}$$

ϕ may be put equal to $\phi_1 + \phi_2$, where ϕ_1 and ϕ_2 both satisfy the linearised equation of supersonic flow while ϕ_1 satisfies the boundary condition

$$2\varepsilon \left[1 + \frac{B^2 + 1}{\sqrt{2B}}\ \phi_{1x} - \frac{B^2 - 1}{\sqrt{2B}}\ \phi_{1y}\right] - \phi_{1z} = 0 \tag{44a}$$

and ϕ_2 satisfies the boundary condition

$$-4\ \sqrt{2}\varepsilon x B \left[1 + \frac{B^2 + 1}{\sqrt{2B}}\ \phi_{2x} - \frac{B^2 - 1}{\sqrt{2B}}\ \phi_{2y}\right] - \phi_{2z} = 0 \tag{44b}$$

ϕ_1 satisfies a boundary condition which is the same as equation (31) except that δ has been replaced by 2ε and so ϕ_{1x}, ϕ_{1y} and ϕ_{1z} have effectively been found already. If ϕ_2 is determined by distributing sources over the wing surface, f(x,y) being the source distribution function, ordinary linearised theory gives

$$f(x,y) = - 4\sqrt{2}\ B\,\varepsilon x,$$

using equations (20c) and (44b). Now, as in equation (19),

$$\phi_2 = - \frac{1}{\sqrt{2}\pi} \int_0^x \int_0^{x_1} \frac{f(y_1,x_1)\ dy_1\ dx_1}{(x-x_1)^{\frac{1}{2}}\ \ (y - y_1)^{\frac{1}{2}}} - \frac{1}{\sqrt{2}\pi} \int_0^x \int_{x_1}^y \frac{f(x_1,y_1)\ dy_1\ dx_1}{(x-x_1)^{\frac{1}{2}}\ (y-y_1)^{\frac{1}{2}}},$$

and so

$$\phi_2 = \frac{4\varepsilon B}{\pi}\int_0^x\int_0^{x_1}\frac{y_1\,dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}} + \frac{4\varepsilon B}{\pi}\int_0^x\int_{x_1}^{y}\frac{x_1\,dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}}$$

$$= \frac{4\varepsilon B}{\pi}\int_0^x\frac{dx_1}{(x-x_1)^{\frac{1}{2}}}\int_{(y-x_1)^{\frac{1}{2}}}^{y^{\frac{1}{2}}}2(y-u^2)du + \frac{4\varepsilon B}{\pi}\int_0^x\frac{x_1\,dx_1}{(x-x_1)^{\frac{1}{2}}}\left[-2(y-y_1)^{\frac{1}{2}}\right]_{x_1}^{y}$$

$$= \frac{8\varepsilon B}{\pi}\int_0^x\frac{dx_1}{(x-x_1)^{\frac{1}{2}}}\left[y\,u - \frac{u^3}{3}\right]_{(y-x_1)^{\frac{1}{2}}}^{y^{\frac{1}{2}}} + \frac{8\varepsilon B}{\pi}\int_0^x\frac{x_1(y-x_1)^{\frac{1}{2}}}{(x-x_1)^{\frac{1}{2}}}dx_1$$

$$= \frac{16\varepsilon B}{3\pi}y^{3/2}\int_0^x\frac{dx_1}{(x-x_1)^{\frac{1}{2}}} - \frac{8\varepsilon B}{\pi}y\int_0^x\frac{(y-x_1)^{\frac{1}{2}}\,dx_1}{(x-x_1)^{\frac{1}{2}}} + \frac{8\varepsilon B}{3\pi}\int_0^x\frac{(y-x_1)^{3/2}}{(x-x_1)^{\frac{1}{2}}}dx_1$$

$$+ \frac{8\varepsilon B}{\pi}\int_0^x\frac{x_1(y-x_1)^{\frac{1}{2}}}{(x-x_1)^{\frac{1}{2}}}dx_1$$

$$= \frac{16\varepsilon B}{3\pi}y^{3/2}\left[-2(x-x_1)^{\frac{1}{2}}\right]_0^x - \frac{16\varepsilon B}{3\pi}\int_0^x\frac{(y-x_1)^{3/2}}{(x-x_1)^{\frac{1}{2}}}dx_1$$

$$= \frac{32\varepsilon B}{3\pi}y^{3/2}x^{\frac{1}{2}} - \frac{32\varepsilon B}{3\pi}\int_0^{x^{\frac{1}{2}}}(y-x+u^2)^{3/2}du$$

$$= \frac{32\varepsilon B}{3\pi}y^{3/2}x^{\frac{1}{2}} - \frac{32\varepsilon B}{3\pi}\int_0^{\sinh^{-1}\left(\frac{x}{y-x}\right)^{\frac{1}{2}}}(y-x)^2\cosh^4\theta\,d\theta$$

$$= \frac{32\varepsilon B}{3\pi}y^{3/2}x^{\frac{1}{2}} - \frac{4\varepsilon B}{3\pi}(y-x)^2\left[3\sinh^{-1}\left(\frac{x}{y-x}\right)^{\frac{1}{2}} + 3\frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{y-x} + \frac{2x^{\frac{1}{2}}y^{3/2}}{(y-x)^2}\right]$$

$$= \frac{32\varepsilon B}{3\pi}y^{3/2}x^{\frac{1}{2}} - \frac{4\varepsilon B}{3\pi}\left[3(y-x)^2\tanh^{-1}\left(\frac{x}{y}\right)^{\frac{1}{2}} + 5x^{\frac{1}{2}}y^{3/2} - 3x^{3/2}y^{\frac{1}{2}}\right]$$

$$= \frac{4\varepsilon B}{\pi}y^{3/2}x^{\frac{1}{2}} - \frac{4\varepsilon B}{\pi}(y-x)^2\tanh^{-1}\left(\frac{x}{y}\right)^{\frac{1}{2}} + \frac{4\varepsilon B}{\pi}x^{3/2}y^{\frac{1}{2}}$$

$$\therefore\quad \phi_2 = \frac{4\varepsilon B}{\pi}x^{\frac{1}{2}}y^{\frac{1}{2}}(x+y) - \frac{4\varepsilon B}{\pi}(y-x)^2\tanh^{-1}\left(\frac{x}{y}\right)^{\frac{1}{2}}.$$

Hence,

$$\phi_{2x} = \frac{2\varepsilon B}{\pi} \frac{y^{3/2}}{x^{\frac{1}{2}}} - \frac{2\varepsilon B}{\pi} \frac{y^{\frac{1}{2}}(y-x)}{y^{\frac{1}{2}}} + \frac{8\varepsilon B}{\pi} (y-x) \tanh^{-1} \left(\frac{x}{y}\right)^{\frac{1}{2}} + \frac{6\varepsilon B}{\pi} x^{\frac{1}{2}} y^{\frac{1}{2}}$$

$$= \frac{8\varepsilon B}{\pi} x^{\frac{1}{2}} y^{\frac{1}{2}} + \frac{8\varepsilon B}{\pi} (y-x) \tanh^{-1} \left(\frac{x}{y}\right)^{\frac{1}{2}} \tag{45a}$$

while

$$\phi_{2y} = \frac{6\varepsilon B}{\pi} y^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{2\varepsilon B}{\pi} \frac{x^{\frac{1}{2}}(y-x)}{y^{\frac{1}{2}}} - \frac{8\varepsilon B}{\pi} (y-x) \tanh^{-1} \left(\frac{x}{y}\right)^{\frac{1}{2}} + \frac{2\varepsilon B}{\pi} \frac{x^{3/2}}{y^{\frac{1}{2}}}$$

$$= \frac{8\varepsilon B}{\pi} x^{\frac{1}{2}} y^{\frac{1}{2}} - \frac{8\varepsilon B}{\pi} (y - x) \tanh^{-1} \left(\frac{x}{y}\right)^{\frac{1}{2}} \tag{45b}$$

ϕ_{2x} and ϕ_{2y} are $O(\varepsilon)$ everywhere, even at the leading edge $x = 0$ where, in fact, they vanish; this means that ordinary linearised theory predicts a plausible value for the pressure coefficient (or at least for that part of the pressure coefficient involving ϕ_2). The part of the pressure coefficient involving ϕ_1 can be dealt with by the technique already developed in this note and so the problem of flow over a wing of parabolic arc section requires no extension of the theory.

The above discussion ignores tip effects. If the wing is of large span it is probably sufficiently accurate to work out the flow in the region influenced by the tip on ordinary linearised theory; if it is felt necessary to improve the linearised results in this region also, this can be done by using the method of this note since the tip can be regarded as a subsonic edge.

5 Further Simplifications of the Method

In Fig.4 the pressure coefficient determined from ordinary linearised theory, over the wing of section 3 is shown; this wing was a symmetrical delta wing with sonic leading edge and a constant slope of δ. BC_p is plotted against ζ where

$$\zeta = \frac{4B^2}{(B^2+1)^2\,\delta^2}\,\frac{x}{y} = \frac{1}{\nu^2}\,\frac{x}{y}\,, \tag{46}$$

for two values of B (B = 1 and 3) and three values of δ (δ = 0, 0.1 and 0.2). The pressure coefficient is given by

$$C_p = \frac{2\delta}{\pi B} \left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + \left(\frac{y}{x}\right)^{\frac{1}{2}} \right] = \frac{(B^2+1)\,\delta^2}{\pi B^2} \zeta^{\frac{1}{2}} + \frac{4}{\pi(B^2+1)} \zeta^{-\frac{1}{2}} \tag{47}$$

on ordinary linearised theory and so $\delta = 0$ is a limiting case in which

$$C_p = \frac{4}{\pi(B^2 + 1)} \zeta^{-\frac{1}{2}} \qquad (48)$$

On the axis $\zeta = 0$ two points are marked; these are the values which the modified pressure coefficient takes at the leading edge, using the theory of section 3. This pressure coefficient at $\zeta = 0$ depends only on B and not on δ. The pressure coefficient on modified theory becomes approximately the same as the ordinary linearised coefficient at a very small value of $\frac{X-BY}{X+BY}$, this value being $O(\delta^2)$, so that ζ is $O(1)$. A considerable amount of work would be involved if the pressure coefficient derived from equations (26) was used to work out the drag of the wing. It is proposed instead to draw the tangent from the point on the axis $\zeta = 0$ which represents the modified pressure coefficient at the leading edge to the linearised curve and then continue with this curve until the centre line (Y = 0) is reached. The error incurred in doing this should be negligible. It is possible to simplify the calculation still further. Each of the linearised curves of Fig.4 (drawn for various values of B and δ) has one point marked on it with a cross; this is the point at which the tangent described above touches the curve. It will be seen that to sufficient accuracy all these points are given by $\zeta = 3$, i.e.

$$\frac{4B^2}{(B^2+1)^2\delta^2}\frac{x}{y} = \frac{4B^2}{(B^2+1)^2\delta^2}\frac{X-BY}{X+BY} = 3,$$

or

$$\frac{X}{Y} = B\,\frac{1 + \frac{3(B^2+1)^2\delta^2}{4B^2}}{1 - \frac{3(B^2+1)^2\delta^2}{4B^2}} \sim B\left[1 + \frac{3(B^2+1)^2}{2B^2}\delta^2\right] \qquad (49)$$

Thus, the pressure coefficient over the wing will be taken as falling linearly in $\frac{X-BY}{X+BY}$ from the modified value at the leading edge to the value given by ordinary linear theory when X/Y becomes equal to the quantity in equation (49). From then until the centre line (Y = 0) the ordinary linearised curve will be used.

For the cases of subsonic and supersonic edges a similar procedure is suggested. Thus, for a supersonic edge the pressure coefficient will be taken as falling linearly in $\frac{X-BY}{X+BY}$ from the modified value at the Mach line (determined in section 4.1) to the value given by ordinary linear theory when

$$\frac{X}{Y} = B\left[1 + \frac{3(B^2+1)^2\,\delta^2}{2B^4\cot^2\Lambda_o}\right] \qquad (50)$$

and ordinary linearised theory will be used from then on. The factor $\frac{1}{B^2\cot^2\Lambda_o}$ has been introduced since it is to be expected that, keeping the free stream Mach number constant and varying the sweepback angle, the region in which ordinary linearised theory is invalid will become progressively smaller as the edge becomes more supersonic. For a subsonic edge the pressure coefficient will be taken as falling linearly in $\frac{X-BY}{X+BY}$ from the modified value at the edge (determined in section 4.2) to the value given by ordinary linearised theory when

$$\frac{X}{Y} = \tan \Lambda_o \left[1 + \frac{3(B^2 + 1)^2\delta^2}{2} \cot^2 \Lambda_o\right] \tag{51}$$

The factor $B^2 \cot^2 \Lambda_o$ has been introduced since it is to be expected that, keeping the free stream Mach number constant and varying the sweepback angle, the region in which ordinary linearised theory is invalid will become progressively smaller as the edge becomes more subsonic. In some cases the behaviour of the pressure coefficient in the region lying within the Mach cone from the apex but outside the edge is required; in this region $\tan \Lambda_o > X/Y > B$. The pressure coefficient will be taken as falling linearly in $\frac{X-BY}{X+BY}$ from the modified value at the edge (determined in section 4.2) to the value given by ordinary linearised theory when

$$\frac{X}{Y} = \tan \Lambda_o - (\tan \Lambda_o - B) \frac{3(B^2 + 1)^2\delta^2}{2} \cot^2 \Lambda_o \tag{52}$$

The factor $B^2 \cot^2 \Lambda_o$ has been introduced for the same reason as before. In both these cases ordinary linear theory is to be used for values of X/Y respectively greater and less than those given by equations (51) and (52).

The above formulae may seem arbitrary, and so they are, but they simply involve replacing the curve of the modified pressure by one which has the correct value at $\frac{X-BY}{X+BY} = 0$, is then incorrect (probably only slightly so) for an interval of order δ^2, and then from $\frac{X-BY}{X+BY} = O(\delta^2)$ to $\frac{X-BY}{X+BY} = 1$ has an error negligible in a linearised theory.

The calculation of the aerodynamic forces now involves integrating the pressure coefficient multiplied by the local slope over the wing. The integration over that part of the wing where the pressure coefficient is to be taken as having the ordinary linearised value leads to a double integral which is not very difficult to evaluate, but the integration over the rest of the wing does give a rather complicated result. Since this region is small it is possible to make a further simplification; the mean of the integrand at the two extremes of the region (the line given by the appropriate formula of equations (49) to (52) inclusive, and the edge itself) is multiplied by the area of the region. The validity of this approximation was checked for the examples of the following section, and was found to be satisfactory.

6 Results and Discussion

It is clearly desirable that the theory should be checked against experimental determinations of the drag of straight-edged wings; such determinations are unfortunately, very rare. Ref.2 gives the results of measurements of, among other things, the drags of certain delta wings. The wings are of double wedge section, having thickness-chord ratio of

8% with the maximum thickness occurring at 18% of the root chord measured from the apex. The measurements were at Mach numbers of 1.62, 1.92 and 2.40, and were made for a range of sweepback angles.

Both theoretical and experimental results are plotted in Figs.5 to 7 inclusive. The ordinary linearised curve is shown as a full line and the two sharp discontinuities in slope can be seen clearly. Experimental points are shown by circles and a chain-dotted curve has been drawn through these points. Finally, some points calculated using the modified theory of section 5 are shown by crosses and a dashed line has been drawn through these points. Although there is still a sharp rise at the Mach number at which the ridge line becomes sonic, the modified curve lies much closer to the experimental results than does the ordinary linearised curve.

A few remarks about these figures must be made here. The customary form of plotting has been employed, i.e. $C_D/A\tau^2$ against AB, where A is the aspect ratio of the wing τ the thickness-chord ratio and C_D, the drag coefficient, is given by

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 S}$$

S being the area of the wing and D the drag. The theoretical drag is the wave drag, of course, whereas the measurements of Ref.2 included skin friction drag. For each Mach number an estimated value of the skin friction drag coefficient has been subtracted from the experimental result; this value was 0.009 for M = 1.62, 0.0085 for M = 1.92 and 0.008 for M = 2.40. The final point to be mentioned concerning these figures is the construction of the modified curve. Five points were calculated for Fig.7 (M = 2.40) and this enabled the curve to be drawn fairly accurately, but for the remaining two figures only the points corresponding to a sonic ridge line and a sonic leading edge were calculated. These two points are obviously the two most important and, together with a knowledge of the ordinary linearised curve and an example in Fig.7 of a complete modified curve, there should not be any difficulty in drawing a curve to pass through these two points and to fair into the ordinary linearised curve for values of AB both small and large compared with unity.

Finally, a brief discussion of the theory of this note will be given. It is obvious that the theory can be immediately extended to a swept trailing edge; the wing (assuming it to be a fully tapered wing) is regarded as being composed of three superimposed deltas and the modified theory applied to each of the deltas. Separating the flow field into three distinct flow fields in this way means that the surface boundary condition is not exactly satisfied but, as in section 4.3, this results in a negligible error. It might also be supposed that the extension to the incidence case, i.e. to the removal of the discontinuity in slope in the curve of lift coefficient against Mach number, is very simple; this, however, is not the case. Suppose that the incidence of the wing is α and that the slope at a certain point is δ while the pressure coefficient at this point is $(C_p)_u$ on the upper surface and $(C_p)_\ell$ on the lower surface; (this pressure coefficient is the coefficient on an exact theory). Now the required quantity is C_{L_α} the lift-curve slope, and this requires the evaluation first of

$$\left[\frac{(C_p)_u - (C_p)_\ell}{\alpha}\right]_{\alpha\to 0} = \left[\frac{C_p(\delta+\alpha) - C_p(\delta-\alpha)}{\alpha}\right]_{\alpha\to 0}$$

or of $\frac{d}{d\delta}(C_p)$.

In evaluating the drag of straight-edged wings the exact pressure coefficient was replaced by a modified coefficient which behaved everywhere in a manner which was at any rate plausible and the results of this section suggest that this was quite sufficient. The evaluation of the lift-curve slope, however, involves the derivative of the exact pressure coefficient and much more care must be exercised in replacing this coefficient by a "suitably chosen" function. It is very doubtful whether any method of the type employed in this note (that is, any method involving the limiting of pressures near the leading edge) could justifiably be used to improve the values of lift-curve slope predicted by linearised theory when a straight edge is almost sonic. No attempt has been made to do so in this note.

The justification for the method as applied to the evaluation of the wave drag of wings lies almost entirely in the agreement of the results obtained by the theory with those obtained by experiment. The agreement has been shown to be satisfactory. It would have been interesting to have compared the pressures predicted at the leading edges of delta wings with those obtained in practice but it is, of course, very difficult to measure pressures at leading edges and so the comparison cannot be made. All that can be said is that the overall aerodynamic forces exhibit satisfactory agreement with experiment, while the method has the advantage that it is quick and simple to apply in any particular case. It is, of course, no longer possible by judicious choice of parameters to reduce the plotting of the drag of a large class of delta wings against Mach number to a single curve as can be done using ordinary linearised theory. On the other hand, it is difficult to conceive of an improved theory which would retain this property, and it is certainly more realistic to have $C_D/A\tau^2$ for a fixed Mach number varying slightly with τ.

LIST OF SYMBOLS

A	Aspect ratio
B	$(M_\infty^2 - 1)^{\frac{1}{2}}$
C_D	$D/\frac{1}{2}\rho U^2 S$
C_{L_α}	Lift-curve slope
C_p	$p-p_\infty/\frac{1}{2}\rho U^2$
$(C_p)_\ell, (C_p)_m$	Linearised and modified C_p respectively, (Section 4.1)
$(C_p)_\ell, (C_p)_u$	C_p on lower and upper surface respectively (Section 6)
c	Root chord
D	Wave drag
f	Source distribution function
K	A constant source distribution in section 4.1

k	Defined in Section 3
M	Local Mach number
M_∞	Free stream Mach number
p	Local pressure
p_∞	Free stream pressure
q	Local speed
S	Area of wing in Figs. 5, 6 and 7
s	Semi-span of wing of Section 4.4
U	Free stream velocity
u,v,w	Local perturbation velocities in X,Y and Z directions respectively
X,Y,Z	Cartesian coordinates
X_o	See Fig.2a
x,y,z	Defined by equations (15)
α	Incidence
β	$\tan\beta = \dfrac{B - \tan\Lambda_o}{B + \tan\Lambda_o}$
γ	Ratio of specific heats
δ	(Constant) slope of wings at Sections 3 and 4
ε	Thickness-chord ratio of wing of Section 4.4
ζ	$\zeta = \dfrac{4B^2}{(B^2+1)\delta^2}\dfrac{x}{y}$
Λ_o	Sweepback angle of edge
μ	$\cot\mu = B$
ν	$\nu = \dfrac{B^2+1}{2B}\delta$
ρ	Free stream density
τ	Thickness-chord ratio of wing in Figs.1,5,6 and 7
ϕ	Perturbation velocity potential
ϕ_1,ϕ_2	See equations (44)

REFERENCES

No.	Author	Title, etc.
1	W. R. Sears	General Theory of High Speed Aerodynamics Oxford, 1955, Section D, Chapter 3
2	E. S. Love	Investigations at Supersonic Speeds of 22 Triangular Wings Representing Two Airfoil Sections for each of 11 Apex Angles NACA Report 1238, 1955.

APPENDIX I

The Solution of an Integral Equation

In this Appendix it will be shown that

$$f(x,y) = \frac{2\delta}{\pi} \sin^{-1} \frac{(x/y)^{\frac{1}{2}}}{(\nu^2 + x/y)^{\frac{1}{2}}} \tag{I.1}$$

where

$$\nu = \frac{B^2+1}{2B}\,\delta \tag{I.2}$$

is a solution to first order everywhere of

$$\delta\left\{1 - \frac{B^2+1}{2\pi B}\int_0^x\int_0^{x_1} \frac{f_{x_1}(y_1,x_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} - \frac{B^2+1}{2\pi B}\int_0^x\int_{x_1}^y \frac{f_{x_1}(x_1,y_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}}\right.$$

$$\left.+ \frac{B^2-1}{2\pi B}\int_0^x\int_0^{x_1} \frac{f_{y_1}(y_1,x_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} + \frac{B^2-1}{2\pi B}\int_0^x\int_{x_1}^y \frac{f_{y_1}(x_1,y_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}}\right\} = f(x,y) \tag{I.3}$$

Now

$$f_x(y,x) = -\frac{\nu\,\delta y^{\frac{1}{2}}}{\pi x^{\frac{1}{2}}\,(\nu^2 x + y)} = -\frac{(B^2+1)\delta^2}{2\pi B}\,\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}\,(\nu^2 x + y)}, \tag{I.4a}$$

$$f_x(x,y) = \frac{\nu\,\delta y^{\frac{1}{2}}}{\pi\, x^{\frac{1}{2}}\,(\nu^2 y + x)} = \frac{(B^2+1)\delta^2}{2\pi B}\,\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}\,(\nu^2 y + x)}, \tag{I.4b}$$

$$f_y(y,x) = \frac{\nu\,\delta x^{\frac{1}{2}}}{\pi\, y^{\frac{1}{2}}\,(\nu^2 x + y)} = \frac{(B^2+1)\delta^2}{2\pi B}\,\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}\,(\nu^2 x + y)}, \tag{I.4c}$$

$$f_y(x,y) = -\frac{\nu\,\delta x^{\frac{1}{2}}}{\pi\, y^{\frac{1}{2}}\,(\nu^2 y + x)} = -\frac{(B^2+1)\delta^2}{2\pi B}\,\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}\,(\nu^2 y + x)}. \tag{I.4d}$$

The first integral in equation (I.3) is, using equation (I.4a),

$$\frac{(B^2+1)}{2\pi B}\int_0^x\int_0^{x_1} \frac{f_{x_1}(y_1,x_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} = -\frac{(B^2+1)^2\delta^2}{4\pi^2\,B^2}\int_0^x\int_0^{x_1} \frac{y_1^{\frac{1}{2}}\,dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}\,x_1^{\frac{1}{2}}\,(\nu^2 x_1+y_1)}$$

None of the integrations in this Appendix offers any difficulty and intermediate steps are omitted. First, then,

$$\int_0^x \frac{y_1^{\frac{1}{2}}\,dy_1}{(y-y_1)^{\frac{1}{2}}\,(\nu^2 x_1 + y_1)} = 2\sin^{-1}\left(\frac{x_1}{y}\right)^{\frac{1}{2}} - \frac{2\nu x_1^{\frac{1}{2}}}{(\nu^2 x_1 + y)^{\frac{1}{2}}}\sin^{-1}\frac{(\nu^2 x_1 + y)^{\frac{1}{2}}}{(1+\nu^2)^{\frac{1}{2}}\,y^{\frac{1}{2}}}$$

Next,

$$\int_0^x\int_0^{x_1} \frac{y_1^{\frac{1}{2}}\,dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}\,x_1^{\frac{1}{2}}\,(\nu^2 x_1 + y_1)} = 4\int_0^{\pi/2}\sin^{-1}\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}}\sin\theta\right\}d\theta$$

$$- 4\int_0^{\sin^{-1}\frac{\nu x^{\frac{1}{2}}}{(\nu^2 x+y)^{\frac{1}{2}}}}\sin^{-1}\left\{\frac{(\nu^2 x+y)^{\frac{1}{2}}\cos\theta}{(1+\nu^2)^{\frac{1}{2}}\,y^{\frac{1}{2}}}\right\}d\theta$$

Hence

$$\frac{(B^2+1)}{2\pi B}\int_0^x\int_0^{x_1}\frac{f_{x_1}(y_1,x_1)\,dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} = -\frac{(B^2+1)\delta^2}{\pi^2 B^2}\left[\int_0^{\pi/2}\sin^{-1}\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}}\sin\theta\right\}d\theta\right.$$

$$\left. - \int_0^{\sin^{-1}\frac{\nu x^{\frac{1}{2}}}{(\nu^2 x+y)^{\frac{1}{2}}}}\sin^{-1}\left\{\frac{(\nu^2 x+y)^{\frac{1}{2}}\cos\theta}{(1+\nu^2)^{\frac{1}{2}}\,y^{\frac{1}{2}}}\right\}d\theta\right]$$

If x/y is at most $O(\delta^2)$ this result is at most $O(\delta^3)$ (I.5a)

If $x/y \gg \delta^2$, this result is $O(\delta^2)$ (I.5b)

The second integral in equation (I.3) is, using (I.4b),

$$\frac{(B^2+1)}{2\pi B}\int_0^x\int_{x_1}^y\frac{f_{x_1}(x_1,y_1)\,dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} = \frac{(B^2+1)^2\delta^2}{4\pi^2 B^2}\int_0^x\int_{x_1}^y\frac{y_1^{\frac{1}{2}}\,dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}\,x_1^{\frac{1}{2}}\,(\nu^2 y_1 + x_1)}$$

First,

$$\int_{x_1}^{y} \frac{y_1^{\frac{1}{2}}\, dy_1}{(y-y_1)^{\frac{1}{2}}\,(\nu^2 y_1+x_1)} = \frac{4B^2\pi}{(B^2+1)^2\delta^2}\left[1 - \frac{x_1^{\frac{1}{2}}}{(x_1+\nu^2 y)^{\frac{1}{2}}}\right]$$

$$+ \frac{8B^2}{(B^2+1)^2\delta^2}\,\frac{x_1^{\frac{1}{2}}}{(x_1+\nu^2 y)^{\frac{1}{2}}}\,\sin^{-1}\frac{(x_1+\nu^2 y)^{\frac{1}{2}}}{y^{\frac{1}{2}}(1+\nu^2)^{\frac{1}{2}}}$$

$$- \frac{8B^2}{(B^2+1)^2\delta^2}\,\sin^{-1}\left(\frac{x_1}{y}\right)^{\frac{1}{2}}$$

Therefore

$$\frac{B^2+1}{2\pi B}\int_{0}^{x}\int_{x_1}^{y} \frac{f_{x_1}(x_1,y_1)\,dy_1\;dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} = 1 - \frac{2}{\pi}\sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}}$$

$$+ \frac{4}{\pi^2}\int_0^{\sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}}} \sin^{-1}\left\{\frac{(x+\nu^2 y)^{\frac{1}{2}}\cos\theta}{(1+\nu^2)^{\frac{1}{2}}\; y^{\frac{1}{2}}}\right\} d\theta$$

$$- \frac{4}{\pi^2}\int_0^{\pi/2} \sin^{-1}\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}}\cos\theta\right\} d\theta$$

If x/y is at most $O(\delta^2)$, i.e. $x/y = C\nu^2$, where C is at most $O(1)$, then, apart from a factor of $4/\pi^2$, the last two integrals become

$$\int_0^{\sin^{-1}\left(\frac{C}{C+1}\right)^{\frac{1}{2}}} \sin^{-1}\left\{\frac{(C+1)^{\frac{1}{2}}\nu\;\cos\theta}{(1+\nu^2)^{\frac{1}{2}}}\right\} d\theta - \int_0^{\pi/2} \sin^{-1}\left(C^{\frac{1}{2}}\nu\;\cos\theta\right) d\theta$$

$$= (C+1)^{\frac{1}{2}}\nu\int_0^{\sin^{-1}\left(\frac{C}{C+1}\right)^{\frac{1}{2}}} \cos\theta\; d\theta + O(\nu^3) - C^{\frac{1}{2}}\nu\int_0^{\pi/2}\cos\theta\; d\theta + O(\nu^3) = O(\nu^3)$$

If $x/y \gg \delta^2$, it is easier to consider the value of the two integrals as follows. They may be written

$$\int_0^{\frac{\pi}{2}-\sin^{-1}\frac{\nu y^{\frac{1}{2}}}{(x+\nu^2 y)^{\frac{1}{2}}}} \sin^{-1}\left\{\left(\frac{1+\frac{y}{x}\nu^2}{1+\nu^2}\right)^{\frac{1}{2}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\cos\theta\right\}d\theta - \int_0^{\pi/2}\sin^{-1}\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}}\cos\theta\right\}d\theta$$

$$= \int_0^{\pi/2}\left[\sin^{-1}\left\{\left(\frac{1+\frac{y}{x}\nu^2}{1+\nu^2}\right)^{\frac{1}{2}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\cos\theta\right\} - \sin^{-1}\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}}\cos\theta\right\}\right]d\theta$$

$$- \int_0^{\sin^{-1}\frac{\nu}{(x/y+\nu^2)^{\frac{1}{2}}}} \sin^{-1}\left\{\frac{\left(1+\frac{y}{x}\nu^2\right)^{\frac{1}{2}}}{(1+\nu^2)^{\frac{1}{2}}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\sin\theta\right\}d\theta$$

$$= \int_0^{\pi/2}\sin^{-1}\left[\frac{\left(1+\frac{y}{x}\nu^2\right)^{\frac{1}{2}}}{(1+\nu^2)^{\frac{1}{2}}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\cos\theta\left(1-\frac{x}{y}\cos^2\theta\right)^{\frac{1}{2}} - \left(\frac{x}{y}\right)^{\frac{1}{2}}\cos\theta\left(1-\frac{1+\frac{y}{x}\nu^2}{1+\nu^2}\,\frac{x}{y}\cos^2\theta\right)^{\frac{1}{2}}\right]d\theta + O(\nu^2)$$

$$= \int_0^{\pi/2}\sin^{-1}\left[\frac{(x/y)^{\frac{1}{2}}\cos\theta}{(1+\nu^2)^{\frac{1}{2}}}\left\{\left(1+\frac{y}{x}\nu^2\right)^{\frac{1}{2}}\left(1-\frac{x}{y}\cos^2\theta\right)^{\frac{1}{2}} - \left(1+\nu^2-\frac{x}{y}\cos^2\theta-\nu^2\cos^2\theta\right)^{\frac{1}{2}}\right\}\right]d\theta + O(\nu^2)$$

$$= \int_0^{\pi/2}\sin^{-1}\left[\frac{(x/y)^{\frac{1}{2}}\cos\theta}{(1+\nu^2)^{\frac{1}{2}}}\,\frac{\left(\frac{y}{x}-1\right)\nu^2}{\left(1+\frac{y}{x}\nu^2\right)^{\frac{1}{2}}\left(1-\frac{x}{y}\cos^2\theta\right)^{\frac{1}{2}}+\left(1-\frac{x}{y}\cos^2\theta+\nu^2\sin^2\theta\right)^{\frac{1}{2}}}\right]d\theta + O(\nu^2)$$

$$= O(\nu^2).$$

To summarise, this second integral in equation (I.3) is

$$1-\frac{2}{\pi}\sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}} + O(\delta^3),\ \text{if}\ \ x/y\ \text{is at most}\ O(\delta^2), \tag{I.6a}$$

and

$$1-\frac{2}{\pi}\sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}} + O(\delta^2),\ \text{if}\ \ x/y \gg \delta^2. \tag{I.6b}$$

The third integral in equation (I.3) is, using equation (I.4c),

$$\frac{B^4-1}{2\pi B}\int_0^x\int_0^{x_1}\frac{f_{y_1}(y_1,x_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}} = \frac{(B^4-1)\delta^2}{4\pi^2 B^2}\int_0^x\int_0^{x_1}\frac{x_1^{\frac{1}{2}}dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}\,y_1^{\frac{1}{2}}(\nu^2 x_1+y_1)}$$

First,

$$\int_0^{x_1} \frac{dy_1}{(y-y_1)^{\frac{1}{2}}\, y_1^{\frac{1}{2}}\,(\nu^2 x_1+y_1)} = \frac{2}{\nu x_1^{\frac{1}{2}}(\nu^2 x_1+y)^{\frac{1}{2}}} \tan^{-1} \frac{(\nu^2 x_1+y)^{\frac{1}{2}}}{\nu (y-x_1)^{\frac{1}{2}}} .$$

Therefore,

$$\frac{(B^2-1)}{2\pi B} \int_0^x \int_0^{x_1} \frac{f_{y_1}(y_1,x_1)dy_1\; dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}}$$

$$= \frac{2(B^2-1)\delta}{\pi^2 B\nu} \int_0^{\sin^{-1}\frac{\nu x^{\frac{1}{2}}}{(\nu^2 x+y)^{\frac{1}{2}}}} \sin^{-1}\left\{\frac{(\nu^2 x+y)^{\frac{1}{2}} \cos\theta}{(1+\nu^2)^{\frac{1}{2}}\, y^{\frac{1}{2}}}\right\} d\theta$$

The integrand varies between

$$\sin^{-1} \frac{(\nu^2 x+y)^{\frac{1}{2}}}{(1+\nu^2) y^{\frac{1}{2}}} = \frac{\pi}{2} - \sin^{-1} \frac{\nu (y-x)^{\frac{1}{2}}}{(1+\nu^2)^{\frac{1}{2}}\, y^{\frac{1}{2}}}$$

and

$$\sin^{-1} \frac{1}{(1+\nu^2)^{\frac{1}{2}}} = \frac{\pi}{2} - \sin^{-1} \frac{\nu}{(1+\nu^2)^{\frac{1}{2}}}$$

so that the integral may be written

$$\frac{2(B^2-1)\delta}{\pi^2 B\nu} \int_0^{\sin^{-1}\frac{\nu x^{\frac{1}{2}}}{(\nu^2 x+y)^{\frac{1}{2}}}} \left[\frac{\pi}{2} - O(\delta)\right] d\theta = \frac{(B^2-1)\delta}{\pi B \nu} \sin^{-1} \frac{\nu x^{\frac{1}{2}}}{(\nu^2 x+y)^{\frac{1}{2}}} - O(\delta^2)$$

$$= \frac{(B^2-1)\delta}{\pi B} \left(\frac{x}{y}\right)^{\frac{1}{2}} - O(\delta^2) \qquad (I.7)$$

The third integral in equation (I.3) then, is at most $O(\delta^2)$ when x/y is at most $O(\delta^2)$, (I.7a), and $O(\delta)$ when $x/y >> \delta^2$, (I.7b).

The fourth integral in equation (I.3) is, using equation (I.4d),

$$\frac{(B^2-1)}{2\pi B} \int_0^x \int_{x_1}^y \frac{f_{y_1}(x_1,y_1)dy_1\; dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} = -\frac{(B^4-1)\delta^2}{4\pi^2 B^2} \int_0^x \int_{x_1}^y \frac{x_1^{\frac{1}{2}}\, dy_1, dx_1}{(x-x_1)^{\frac{1}{2}}(y-y_1)^{\frac{1}{2}}\, y_1^{\frac{1}{2}}\,(\nu^2 y_1+x_1)}$$

First,

$$\int_{x_1}^{y} \frac{f_{y_1}}{(y-y_1)^{\frac{1}{2}}\, y_1^{\frac{1}{2}}\, (\nu^2 y_1 + x_1)} = \frac{2}{x_1^{\frac{1}{2}}(x_1+\nu^2 y)^{\frac{1}{2}}} \tan^{-1} \frac{(y-x_1)^{\frac{1}{2}}}{(x_1+\nu^2 y)^{\frac{1}{2}}} .$$

Therefore,

$$\frac{(B^2-1)}{2\pi B} \int_0^x \int_{x_1}^y \frac{f_{y_1}(x_1,y_1)dy_1 dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} = -\frac{(B^4-1)\delta^2}{\pi^2 B^2} \int_0^{\sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}}} \cos^{-1}\left\{\frac{(x+\nu^2 y)^{\frac{1}{2}}\cos\theta}{(1+\nu^2)^{\frac{1}{2}}\, y^{\frac{1}{2}}}\right\} d\theta$$

and is at most $O(\delta^2)$. (I.8)

Thus, in equation (I.3), if x/y is at most $O(\delta^2)$, the first, third and fourth integrals are at most $O(\delta^2)$ while the second is

$$1 - \frac{2}{\pi} \sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}} + O(\delta^2);$$

the left hand side of (I.3) becomes then

$$\frac{2\delta}{\pi} \sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}} + O(\delta^2)$$

which is, to first order, $f(x,y)$. If $x/y >> \delta^2$, the first and fourth integrals are $O(\delta^2)$, while the second and third combine to give

$$1 - \frac{2}{\pi} \sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}} + O(\delta^2) = 1 - \frac{2}{\pi}\left[\frac{\pi}{2} - O(\delta)\right] = O(\delta) .$$

The left hand side of (I.3) becomes then, $\delta + O(\delta^2)$; the right hand side is also $\delta + O(\delta^2)$, if $x/y >> \delta^2$. Hence, the integral equation (I.3) is satisfied everywhere to first order by

$$f(x,y) = \frac{2\delta}{\pi} \sin^{-1} \frac{(x/y)^{\frac{1}{2}}}{(\nu^2 + x/y)^{\frac{1}{2}}}$$

From equation (20a)

$$\phi_x = -\frac{1}{\sqrt{2\pi}} \int_0^x \int_0^{x_1} \frac{f_{x_1}(y_1,x_1)dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} - \frac{1}{\sqrt{2\pi}} \int_0^x \int_{x_1}^y \frac{f_{x_1}(x_1,y_1)dy_1\, dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}}$$

$$= \frac{-\sqrt{2}B}{(B^2+1)}\left[1 - \frac{2}{\pi}\sin^{-1}\left(\frac{x}{x+\nu^2 y}\right)^{\frac{1}{2}}\right] = -\frac{2\sqrt{2}B}{\pi(B^2+1)} \sin^{-1} \frac{\nu y^{\frac{1}{2}}}{(x+\nu^2 y)^{\frac{1}{2}}} , \qquad (I.9)$$

with an error $O(\delta^2)$; here, (I.5) and (I.6) have been used. From equation (20b)

$$\phi_y = -\frac{1}{\sqrt{2\pi}}\int_0^x\int_0^{x_1}\frac{f_{y_1}(y_1,x_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}} - \frac{1}{\sqrt{2\pi}}\int_0^x\int_{x_1}^{y}\frac{f_{y_1}(x_1,y_1)dy_1\,dx_1}{(x-x_1)^{\frac{1}{2}}\,(y-y_1)^{\frac{1}{2}}}$$

$$= -\frac{\sqrt{2}\delta}{\pi}\left(\frac{x}{y}\right)^{\frac{1}{2}}, \qquad (I.10)$$

with an error $O(\delta^2)$; here, (I.7) and (I.8) have been used. From equations (20c) and (I.1)

$$\phi_z = \frac{2\delta}{\pi}\sin^{-1}\frac{x^{\frac{1}{2}}}{(\nu^2 y + x)^{\frac{1}{2}}}\ . \qquad (I.11)$$

Equations (I.9), (I.10) and (I.11) are valid on the wing only.

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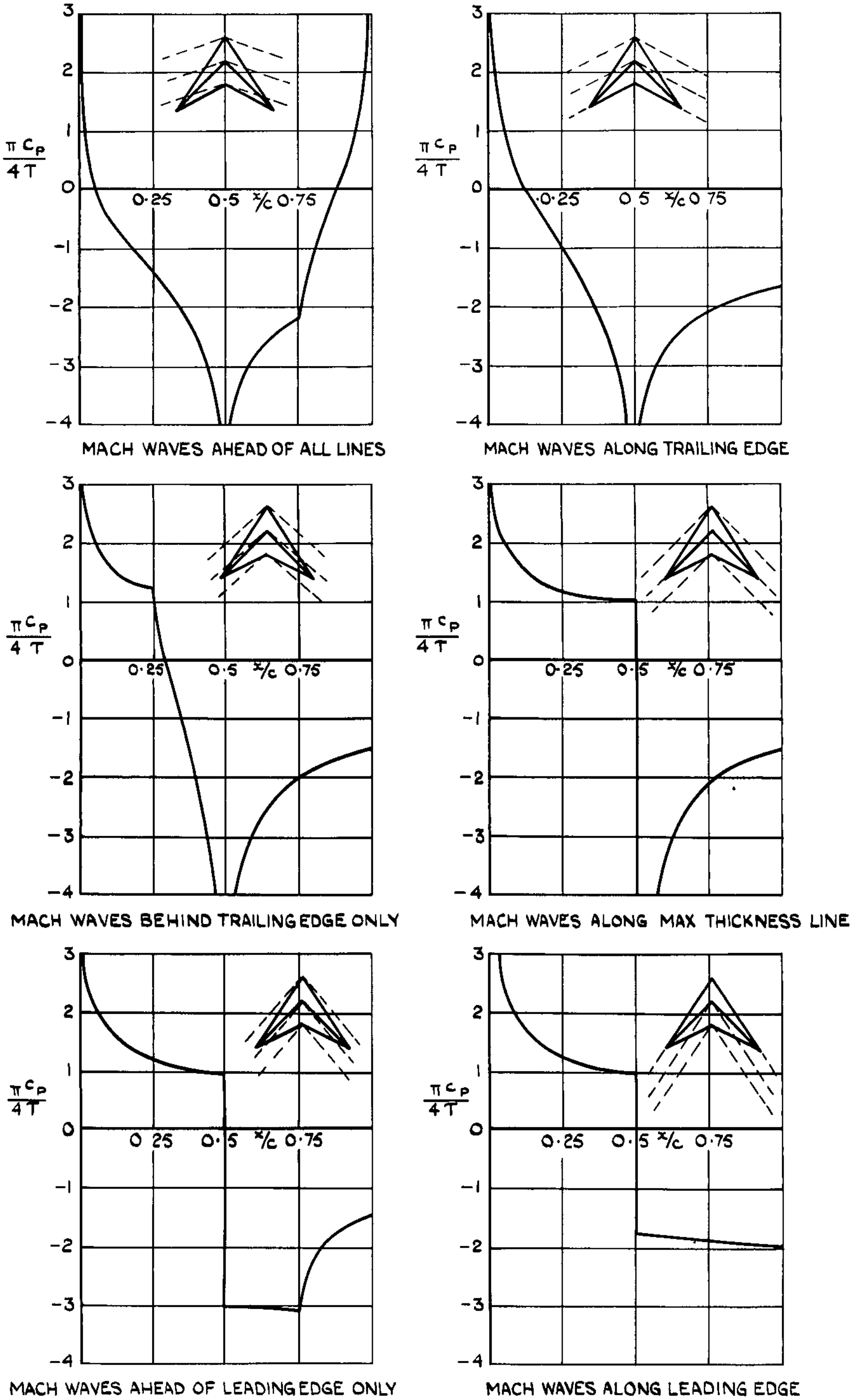

FIG.1. PRESSURE DISTRIBUTION AT MID SEMI-SPAN ON A WING OF DOUBLE-WEDGE SECTION.

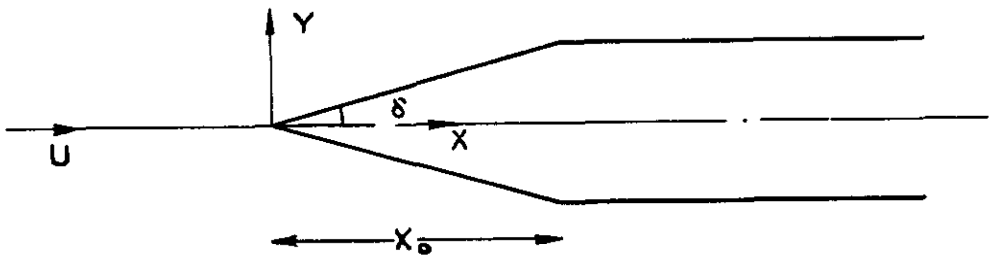

FIG.2 a THE BODY DESCRIBED IN SECTION 2

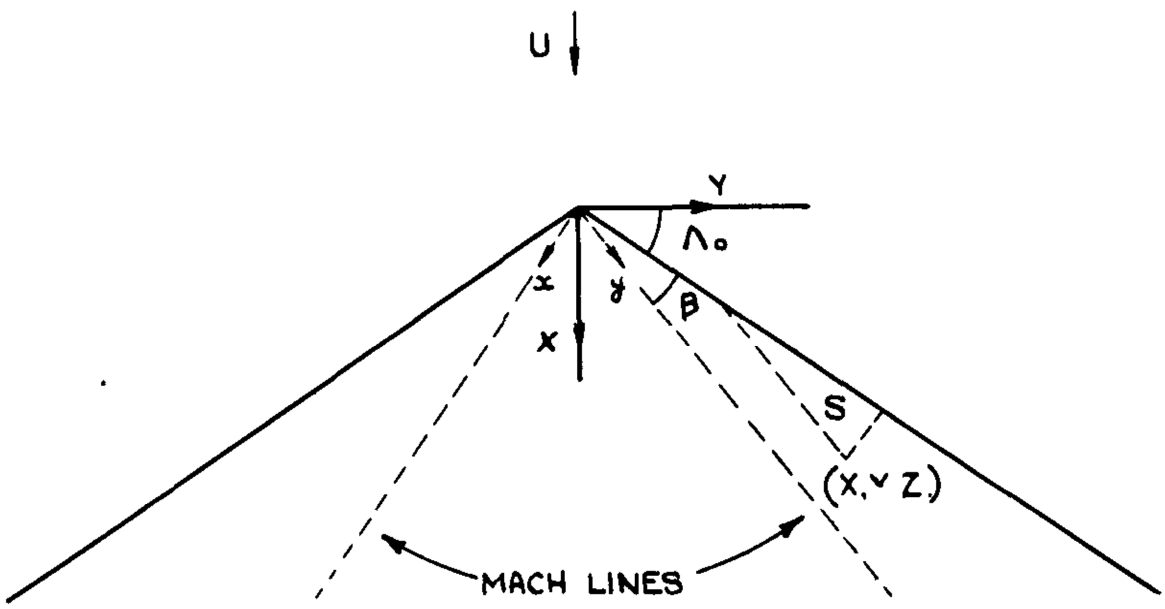

FIG. 2 b. THE WING OF SECTION 4.1

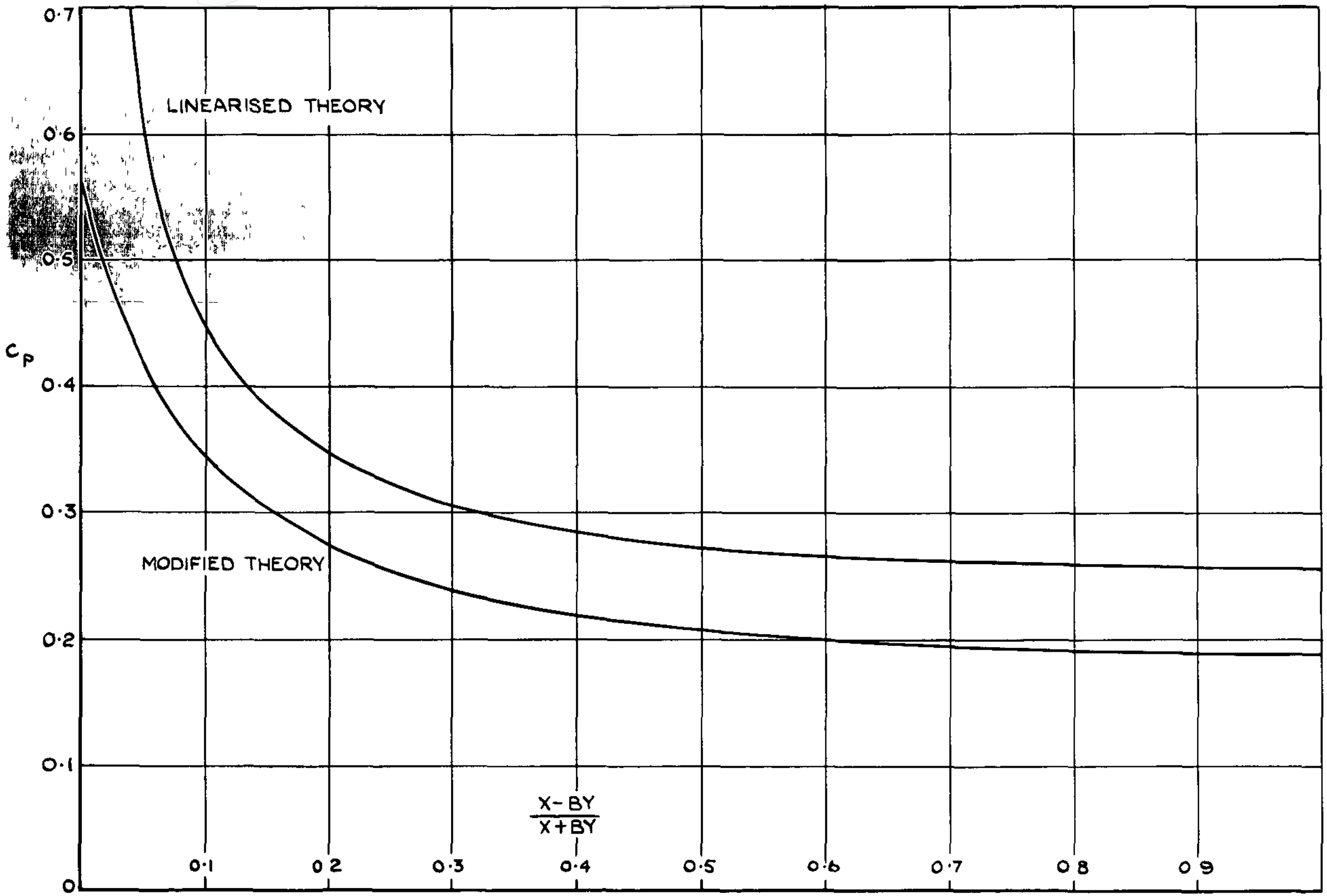

FIG.3. THE PRESSURE COEFFICIENT OVER A DELTA WING WITH SONIC LEADING EDGE.

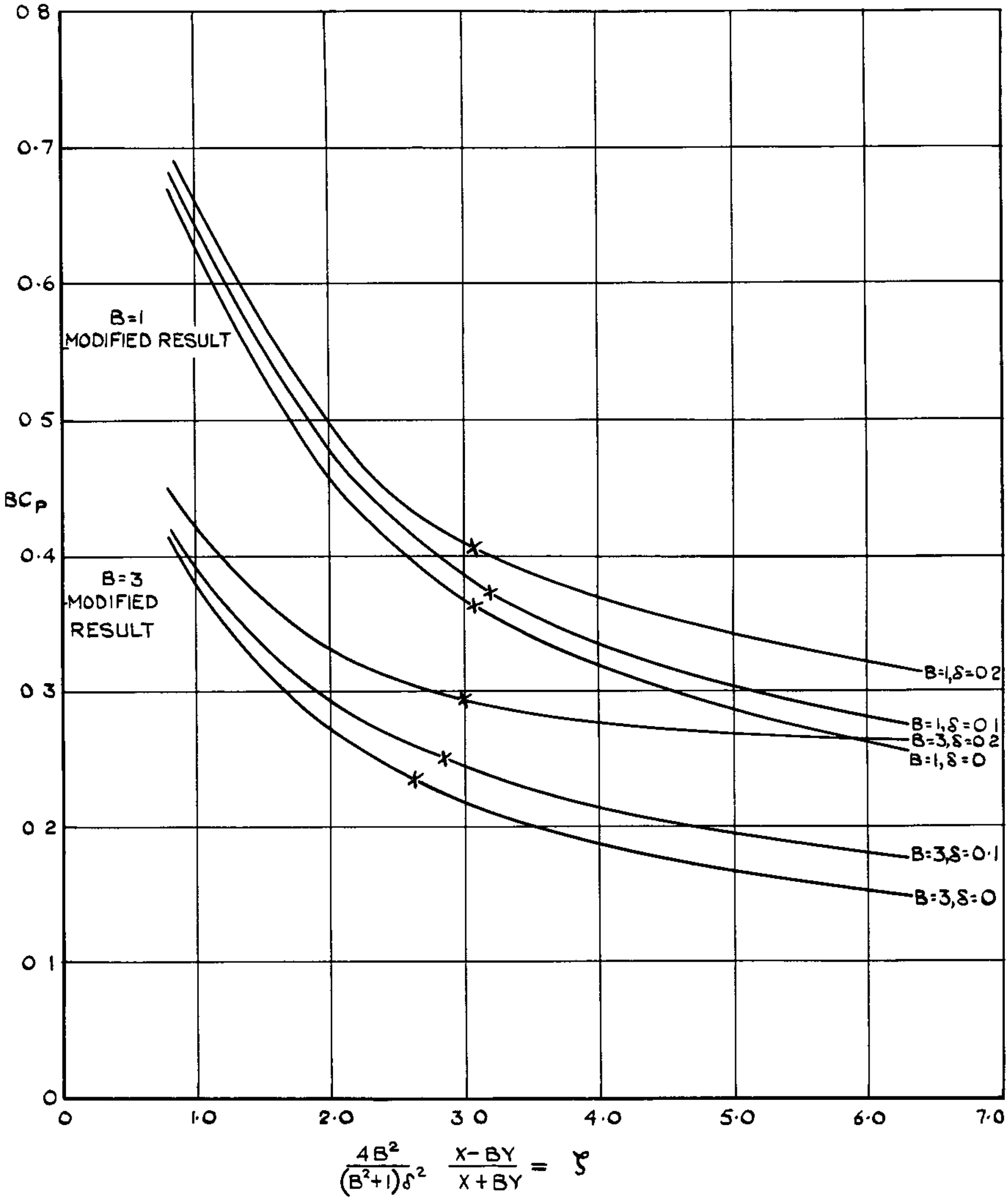

FIG 4 THE LINEARISED PRESSURE COEFFICIENT OVER A DELTA WING WITH SONIC LEADING EDGE

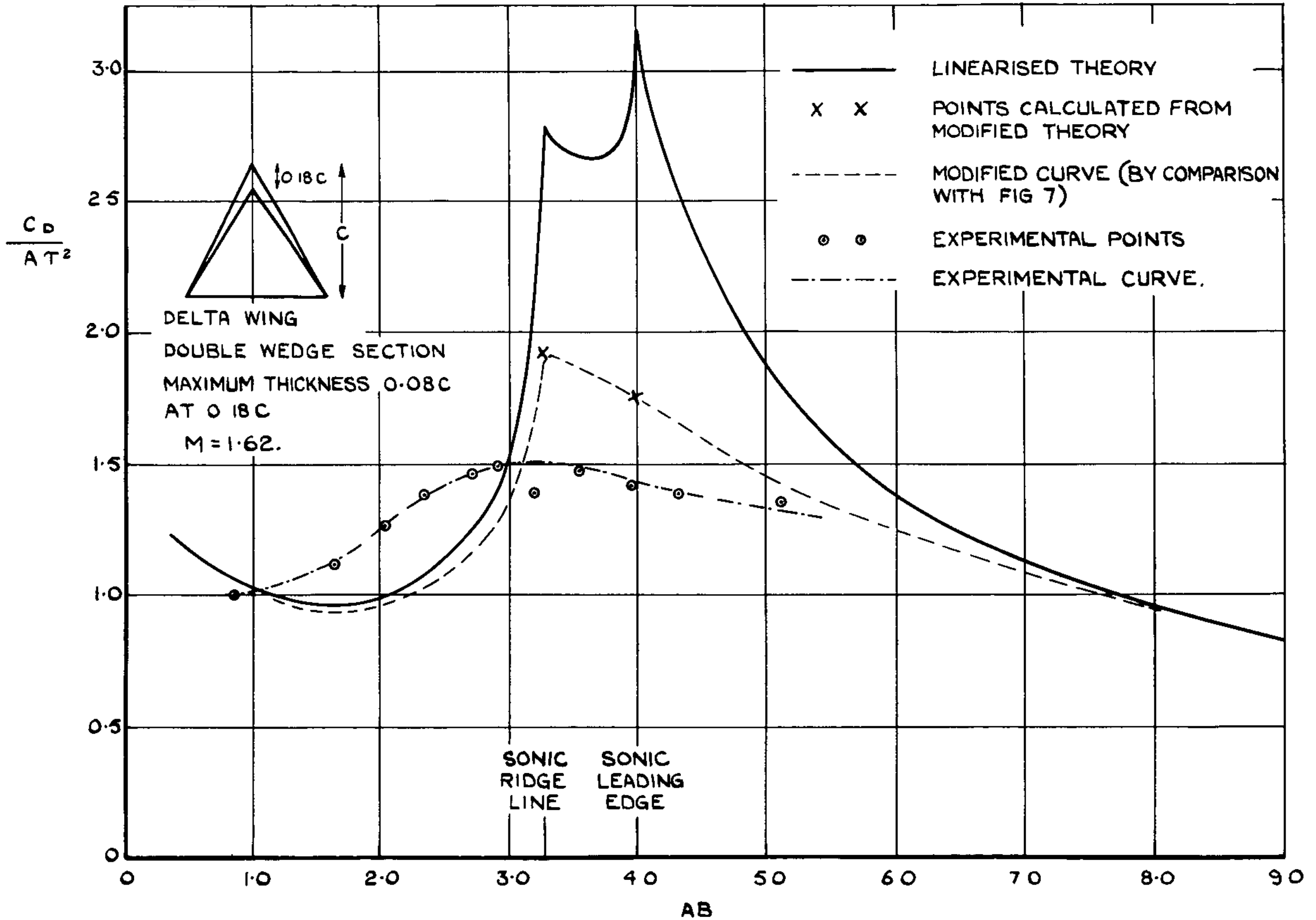

FIG.5. THE DRAG OF A DELTA WING OF DOUBLE-WEDGE SECTION M = 1·62

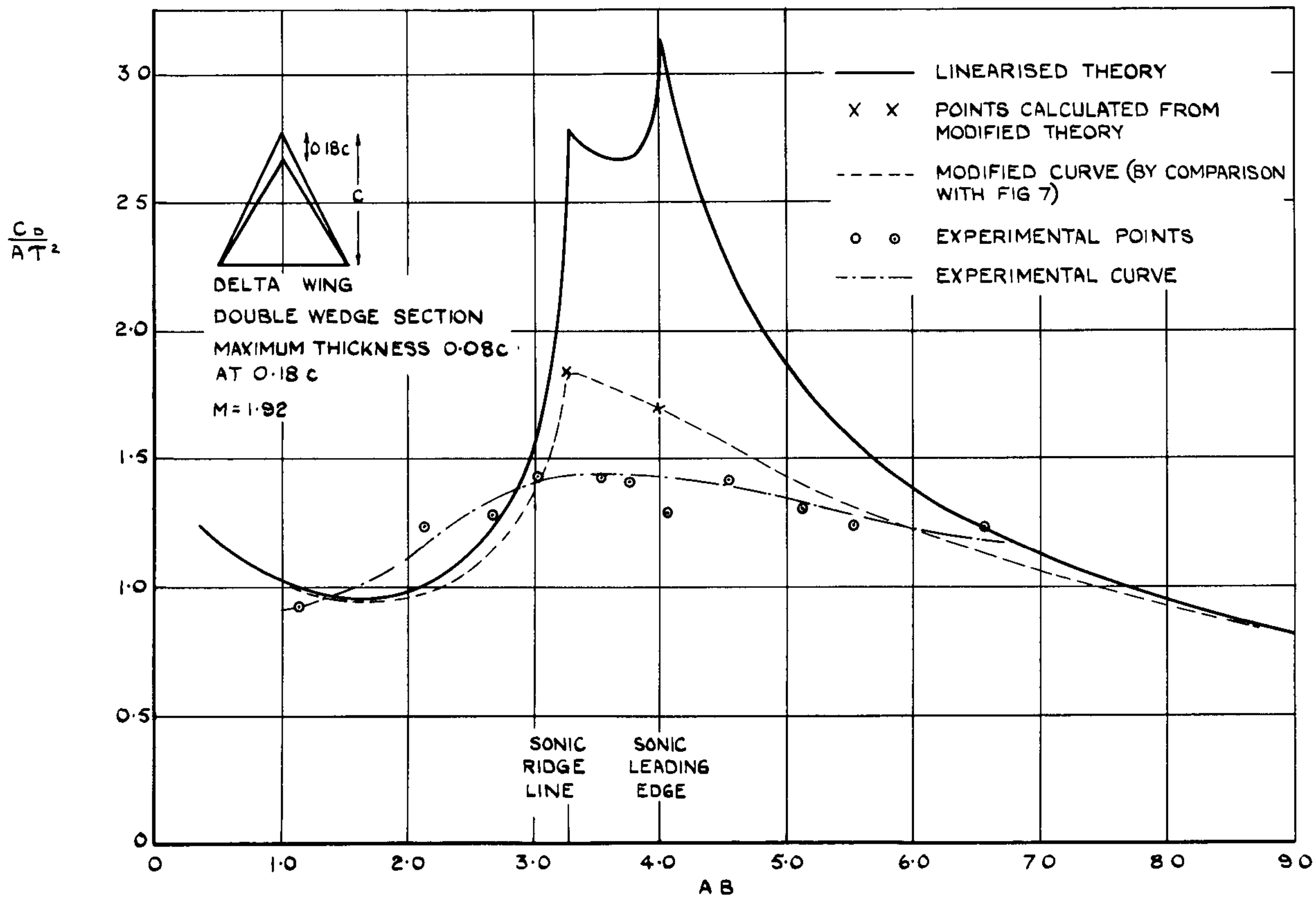

FIG.6. THE DRAG OF A DELTA WING OF DOUBLE-WEDGE SECTION M = 1·92

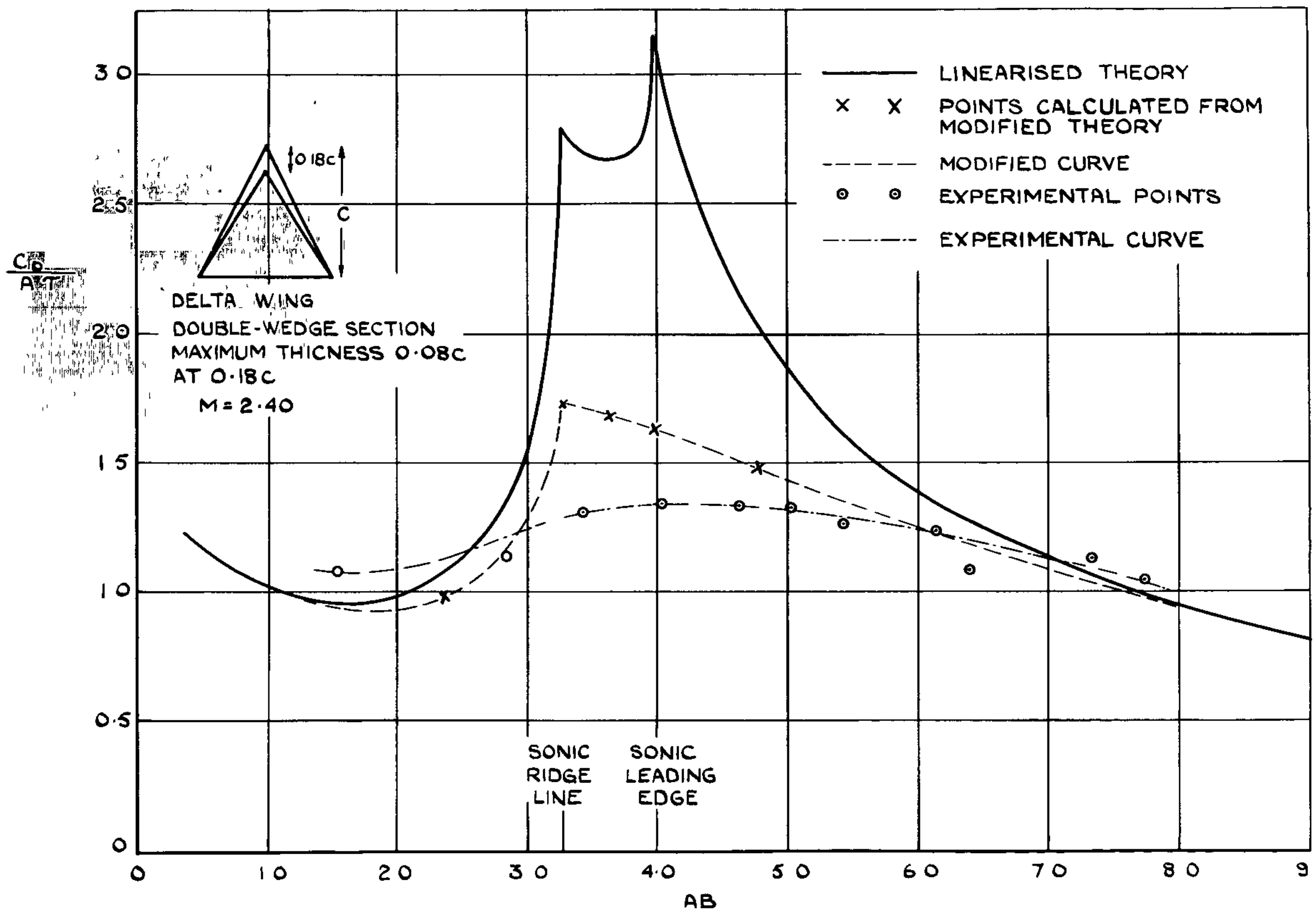

FIG. 7. THE DRAG OF A DELTA WING OF DOUBLE-WEDGE SECTION M = 2·40

C.P. No. 394
(19,485)
A.R.C. Technical Report

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