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Theoretical Investigation of the Sound Field produced  
Downstream of a Choked Two-Dimensional Channel  
due to Unsteady Upstream Entropy Fluctuations

By

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TWO SHILLINGS NET



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29th October, 1958

SUMMARY

The two-dimensional flow considered is that of an ideal, compressible fluid bounded by two rigid, non-conducting, parallel walls of infinite extent. A shock wave normal to the rigid boundaries divides the flow into subsonic and supersonic régimes. In the supersonic region a normal static temperature variation, which is sectionally invariant but fluctuates periodically with time, when convected through the shock wave results in a perturbation of the flow in the subsonic region. By assuming that the magnitude of the fluctuations of the static temperature are small an expression for the acoustic pressure fluctuation in the downstream part of the gas is deduced.

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Introduction

Experimental evidence obtained in recent years has firmly established the fact that a large increase in jet noise arises when the jet is running beyond or at least close to its choking condition. Powell<sup>1</sup>, Burgers<sup>2</sup>, Ribner<sup>3</sup>, Lighthill<sup>4</sup> and many others have endeavoured to explain analytically how this large increase in jet noise occurs. The outcome of their investigations suggests in general that any one of the upstream fluctuations of entropy, pressure and vorticity, on being convected through a shock-wave system, give rise to all three modes of disturbances in the subsonic downstream region.

The analysis contained herein is an advance on a paper by Pcowell<sup>1</sup> in that it considers two-dimensional motion, but it concerns itself with the acoustic disturbances produced downstream due to the convection of entropy fluctuations only through a normal shock wave.

The analytic model chosen for investigation consists of a two-dimensional bounded channel within which supersonic and subsonic flows are separated by a normal shock wave. The boundaries are two parallel rigid non-conducting walls of infinite extent. An upstream static temperature fluctuates in a periodic manner about a mean value. This gives rise to disturbances of the normal shock wave profile which in turn produces a sound field within the subsonic downstream region. The mean value of the upstream static temperature is dependent only on the co-ordinate normal to the channel centre line and is symmetric about this centre line. As the reduced equations governing the flow are linear the perturbation of the downstream flow due to the mean value of the upstream static temperature is separable from the unsteady perturbation due to the fluctuations of the upstream static temperature about the mean value.

Notation/

Notation

$A_s$	speed of sound
$C_p$	specific heat at constant pressure
$D$	width of channel
$M_s$	Mach number
$P_s$	static pressure
$\bar{Q}_s$	velocity vector
$R_s$	density
$\overleftrightarrow{R}$	symbol denoting rarefaction wave; arrow denotes direction
$T_s$	absolute temperature
$U_s$	velocity parallel to channel centre line
$a$	perturbation of the speed of sound
$p$	perturbation of static pressure
$\bar{q}$	perturbation velocity vector
$s$	= $\begin{cases} 1 - \text{suffix denoting supersonic régime} \\ 2 - \text{suffix denoting subsonic régime} \end{cases}$
$t$	time variable
$u$	perturbation of velocity parallel to channel centre line
$v$	perturbation of velocity normal to channel centre line
$x, y$	orthogonal cartesian co-ordinates
$\gamma$	ratio of specific heats
$\rho$	perturbation of density

Theory

The physical state contemplated in the theoretical investigation is shown in Fig.1. The origin of the co-ordinate system used in this analysis is located on the centre line of the channel at the undisturbed shock front.

The unsteady equations expressing conservation of mass, momentum and energy are, for an ideal gas

$$\frac{DR}{Dt} + R \operatorname{div} \bar{Q} = 0 \quad \dots(1)$$

$$\frac{D\bar{Q}}{Dt} + \frac{1}{R} \operatorname{grad} P = 0 \quad \dots(2)$$

$$\frac{DP}{Dt} - A^2 \frac{DR}{Dt} = 0 \quad \dots(3)$$

where/

where 
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\bar{Q} \cdot \text{grad}).$$

Following a linear treatment for two-dimensional flow of an ideal compressible fluid<sup>4</sup>, the flow variables downstream of the shock may be written as

$$P_2 + p(xyt), R_2 + \rho(xyt), A_2 + a(xyt), \bar{Q}_2 + \bar{q}(xyt)$$

where the components of the velocity vector are  $U_2 + u(xyt), v(xyt)$ . Equations (1), (2) and (3) then simplify to the form

$$\frac{d\rho}{dt} + R_2 \text{div } \bar{q} = 0 \quad \dots(4)$$

$$\frac{d\bar{q}}{dt} + \frac{1}{R_2} \text{grad } p = 0 \quad \dots(5)$$

$$\frac{dp}{dt} - A_2^2 \frac{d\rho}{dt} = 0 \quad \dots(6)$$

where 
$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x}.$$

Elimination of  $\rho$  and  $\bar{q}$  yields the pressure perturbation equation

$$\frac{d^2 p}{dt^2} - M_2^2 \text{div} \cdot \text{grad } p = 0 \quad \dots(7)$$

where the non-dimensional co-ordinates

$$x^* = x/D, \quad y^* = y/D, \quad \tau = U_2 t/D$$

are used in the expanded form. In the remainder of this work the suffix 2 on the Mach number,  $M_2$ , and velocity,  $U_2$ , will be omitted.

Steady Perturbation

The total pressure perturbation may be regarded as the sum of two parts: (i) the steady perturbation of the downstream flow due to the mean of the upstream static temperature and (ii) the unsteady perturbation due to the time fluctuations of the upstream static temperature about the mean value.

In the steady case, equation (7) reduces to

$$(1 - M^2) \frac{\partial^2 p}{\partial x^{*2}} + \frac{\partial^2 p}{\partial y^{*2}} = 0. \quad \dots(8)$$

In order to solve this equation it is necessary to obtain an expression for  $p$  at the shock front in terms of the disturbing agent which, in this case, is the upstream static temperature variation,  $T(y)$ . This temperature variation may be represented by

$$T(y) = T_0 [1 + T'(y)], \quad T'(y) \ll 1.$$

The Mach number of the flow along a general streamline in the subsonic region of the channel is then given by

$$M^2 = M_1^2 [1 - T'(y)] \quad \dots(9)$$

neglecting/

neglecting  $O\{T'(y)^2\}$ . The pressure ratio across an oblique shock wave, correct to the first order of the inclination of the shock to the y-axis is given by

$$\frac{P_2 + p(xy)}{P_1} = \frac{2\gamma}{\gamma+1} M^2 - \frac{\gamma-1}{\gamma+1}$$

where ... (10)

$$\frac{P_2}{P_1} = \frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1}$$

Hence, from (9) and (10) the pressure perturbation in the subsonic region at the shock front is given by

$$p(xy) = - \frac{2\gamma M_1^2 P_1}{\gamma+1} T'(y)$$

In what follows this boundary condition will be taken on the y-axis, i.e., the above boundary condition will be interpreted as

$$p(0, y) = - \frac{2\gamma M_1^2 P_1}{\gamma+1} T'(y)$$

In particular, assuming a static temperature distribution within the channel  $|y^*| \leq 1$  which is symmetric about  $y^* = 0$  the pressure perturbation at the undisturbed shock front may be written in the form

$$p(0, y) = - \frac{2\gamma M_1^2 P_1}{\gamma+1} \sum_{n=0}^{\infty} T'_n \cos 2n\pi y^*$$

This condition together with that at infinity, i.e.,  $p(xy) \rightarrow 0$  as  $x^* \rightarrow \infty$  are sufficient to uniquely determine the solution of (8) by the method of separation of variables. Thus, the steady pressure perturbation of the downstream flow due to the mean of the upstream static temperature is given by

$$p(xy) = - \frac{2\gamma}{\gamma+1} P_1 M_1^2 \sum_{n=0}^{\infty} T'_n \exp \left\{ - \frac{2n\pi}{D} \cdot \frac{x}{(1-M^2)^{\frac{1}{2}}} \right\} \cdot \cos \frac{2n\pi}{D} y \quad \dots (11)$$

The isobaric contours of the subsonic region for the particular case when the upstream static temperature profile is given by a simple cosine wave are shown in Fig. 2.

Unsteady Perturbation

In this case the downstream perturbation, which is due to the oscillation of the temperature variation about the mean, satisfies equation (7). As in the steady case, the condition at the shock front is applied along the line  $x = 0$  and on the assumption that the inclination of the shock to the y-axis is small. This is justified provided that the variation of upstream static temperature is small<sup>5</sup>. It remains to investigate the interaction problem of a shock wave with a temperature gradient in the x-direction constituted by a number of plane entropy waves parallel to the normal shock wave, the shock wave being taken to move into a fluid which is at rest. Fig. 5 illustrates this physical situation in the x,t-plane where  $\delta x$  is taken to be sufficiently small such that the

static/

static temperature profile in the x-direction can be represented approximately by a continuous function. In developing this argument it is also necessary to assume that reflection, refraction and small alterations in strength of the resulting pressure wave system, on being convected through the subsequent entropy wave, is sufficiently small to be neglected<sup>6</sup>.

A note by Powell<sup>1</sup> and a subsequent report by Appleton<sup>5</sup> develop an expression for the acoustic disturbance produced by the passage of a single plane entropy wave through a normal shock wave. It was shown that within the limitations of the linear theory, the magnitude of the resulting acoustic disturbance was directly proportional to the magnitude of the entropy disturbance and was given by

$$\delta p_2 = R_2 \cdot \delta T_1 \cdot C_p \cdot \sigma$$

where  $\delta T_1$  is the static temperature change which defines the entropy wave in the fluid ahead of the shock,  $\delta p_2$  is the magnitude of the resulting pressure perturbation in the region behind the shock wave, and  $\sigma$  is a transmission coefficient dependent only on the shock wave Mach number (see Ref. 4, Fig. 7 for variation of  $\sigma$  with M).

The change in the static pressure behind the shock wave immediately after the general entropy wave designated by  $r$  (see Fig. 3) has been convected through it, is given by the expression

$$P_A - P_C = \delta p_2 = R_2 C_p \sigma \sum_{r=1}^r \delta T_r.$$

This expression is true to within the degree of accuracy implied in the previous assumptions, i.e., terms such as  $\delta \sigma \cdot \delta T_r$  are of second order importance and can therefore be neglected. If a sufficiently large number of entropy waves are considered so that the static temperature profile in the x-direction can be represented by a continuous function, which will be assumed to be sinusoidal, then  $\delta p_2$  may be written in the form

$$\delta p_2 = R_2 C_p \sigma \Delta T \cdot \exp (i2\pi x/c).$$

A transformation of the form  $x = U_1 t$  fixes the shock wave relative to a stationary observer. The pressure perturbation just downstream of the shock wave may then be written

$$p_2 = R_2 C_p \sigma \Delta T \exp (i\omega t) \quad \dots (12)$$

where

$$\omega = 2\pi U_1 / c.$$

The above argument has been developed for a one-dimensional flow system confined to an element of the shock front. However, provided that natural convection of the fluid in the supersonic region upstream of the shock wave is neglected and also that the disturbed shock wave remains normal to a first order approximation equation (12) may be generalised to give the pressure perturbation immediately downstream of the shock wave. Then  $\Delta T$  is given as a function of the normal co-ordinate  $y$ . Thus, with an upstream static temperature symmetrical about the centre line  $y = 0$ , the downstream pressure perturbation at the undisturbed shock front may be written as

$$p(yt)_{x=0} = K \exp (i\omega t) \sum_{n=0}^{\infty} T_n' \cos 2n\pi \frac{y}{D} \quad \dots (13)$$

where

$$K = R_2 C_p \sigma \Delta T.$$

A general solution of equation (7) for simple harmonic oscillations can be obtained following Temple<sup>7</sup> to give

$$p = \exp i(\lambda\tau + kx^*) f(x^*y^*)$$

where  $\lambda = \omega D/U$ ,  $k = \lambda M^2/(1 - M^2)$ , and the reduced pressure function  $f(\bar{x} \bar{y})$  satisfies the equation

$$\nabla^2 f = -n'^2 f$$

in the co-ordinate system  $\bar{x} = x^*/(1 - M^2)^{1/2}$ ,  $\bar{y} = y^*$  and where  $n'^2 = \lambda^2 M^2/(1 - M^2)$ . This equation may be solved by separation of the variables. The particular solution of (7) which satisfies the boundary condition (13) and is finite for all values of  $x$  is then obtained in the form

$$p(xyt) = K \exp i \omega \left( t + \frac{M^2}{1-M^2} \frac{x}{U} \right) \sum_{n=0}^{\infty} T'_n \exp(-mx/D) \cos(2n\pi y/D) \quad \dots(14)$$

where  $K = R_a C_p \sigma \Delta T$  and

$$m^2 = 4n^2 \pi^2 - \frac{\lambda^2 M^2}{1-M^2} \quad \dots(15)$$

Conclusion

Equation (14) gives the magnitude of the pressure perturbation at any point in the subsonic downstream region due to periodic upstream static temperature fluctuations. The steady pressure perturbation of the downstream region due to the mean value of the upstream static temperature distribution is given by equation (11). As these are solutions of a linearised field the total downstream pressure perturbation is given by the sum of (11) and (14).

From equation (14) it is seen that the acoustic disturbances of the downstream region are attenuated only if  $m^2 > 0$ . There is thus a critical frequency given by  $\omega_0/2\pi$  satisfying  $m^2 = 0$ , i.e., from (15)

$$\frac{\lambda_0 U}{2\pi D} = \frac{\omega_0}{2\pi} = \frac{nU}{D} \frac{(1 - M^2)^{1/2}}{M}$$

Below this frequency acoustic waves can be propagated; above this frequency the wave undergoes attenuation.

References/



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FIG. 1.

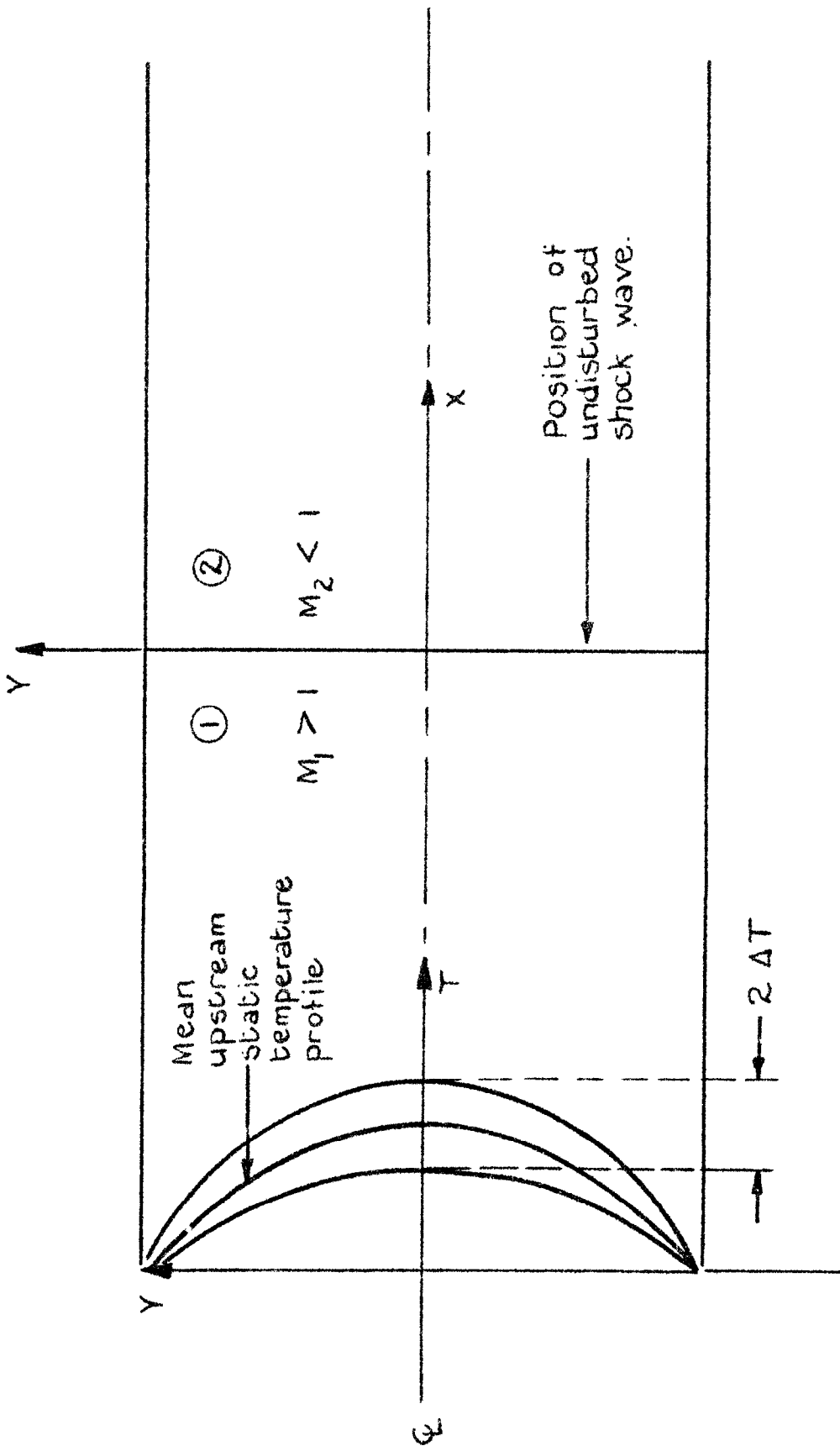
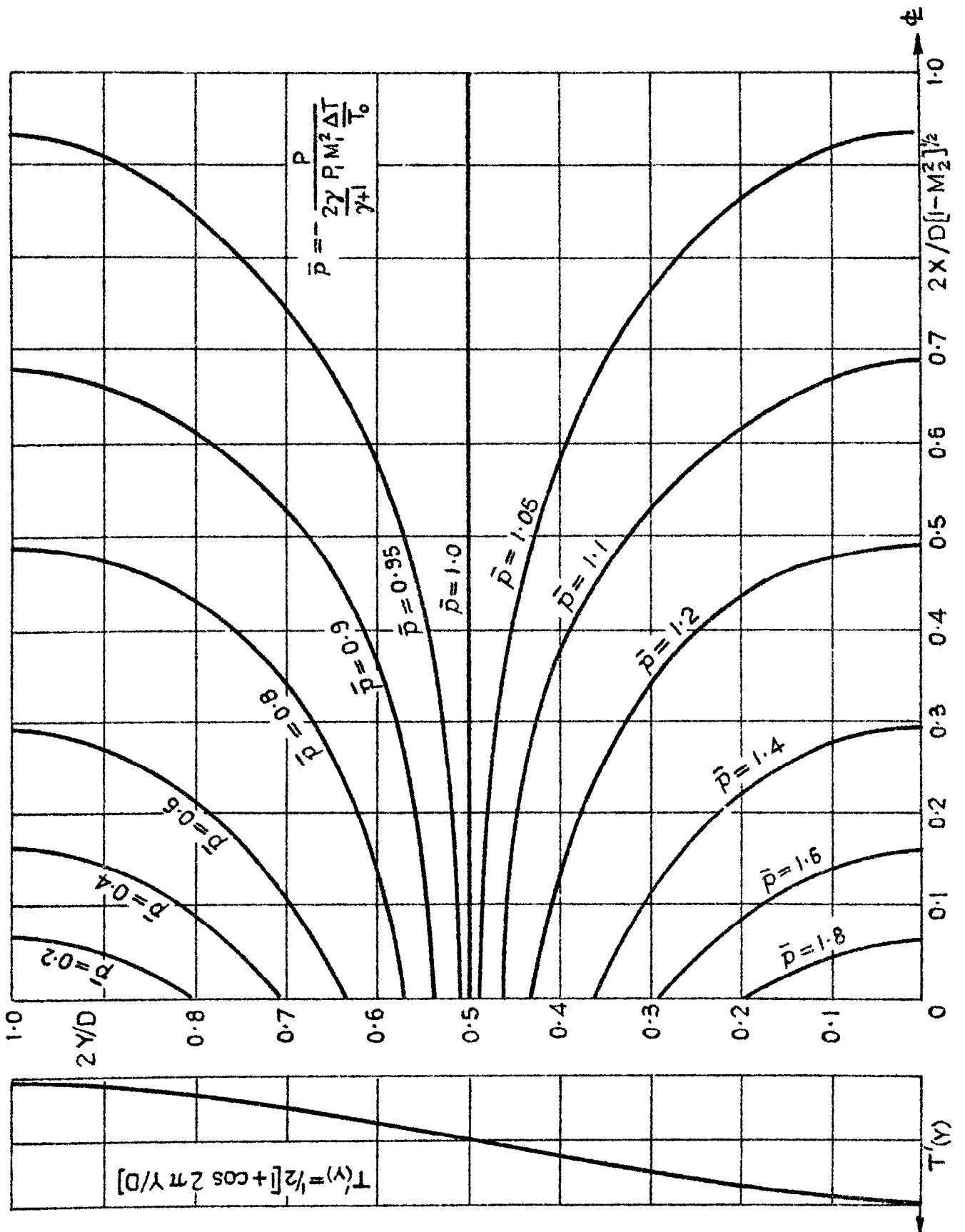


FIG. 2.



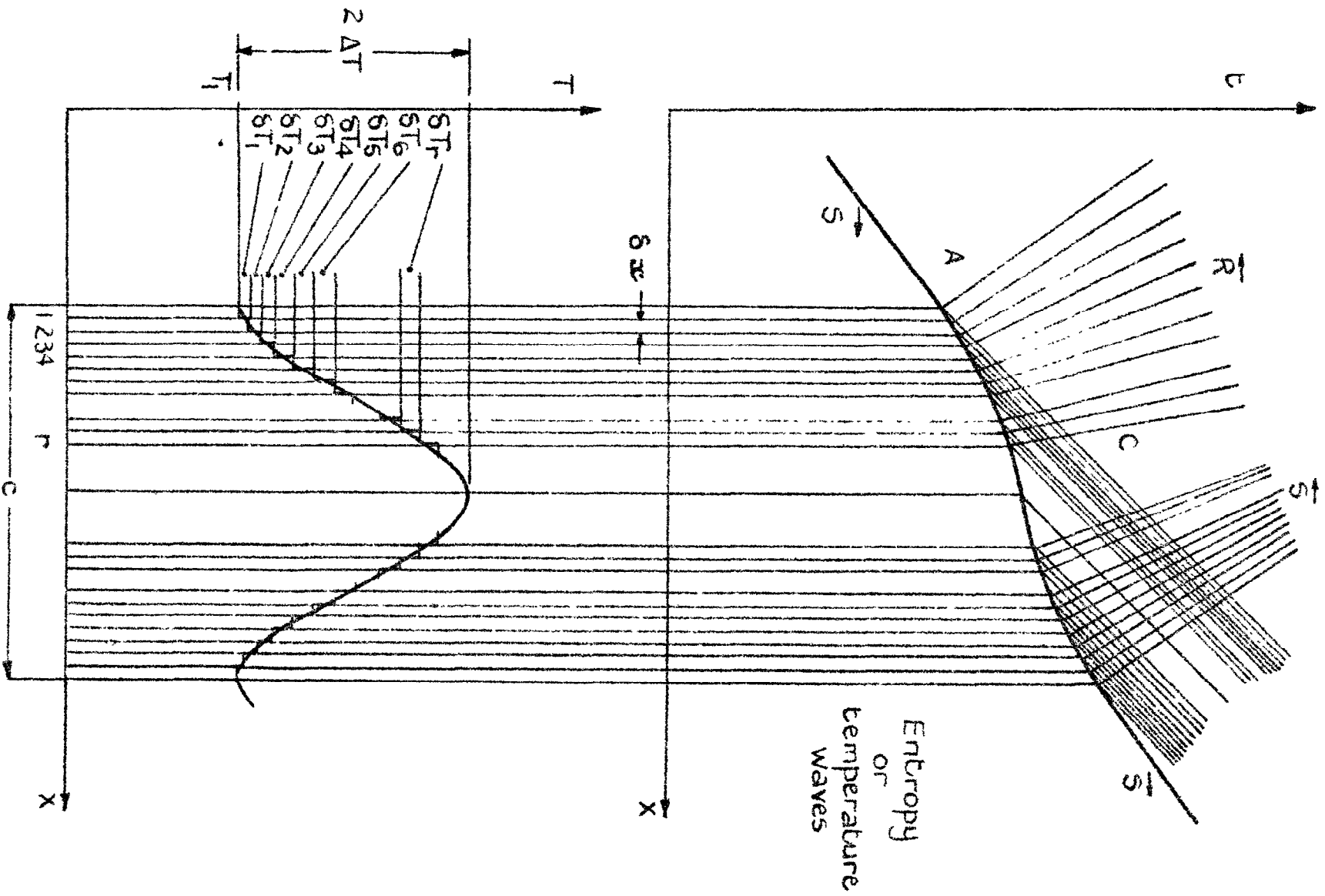


FIG. 3.





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