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The Estimation of Transient Temperature Distributions and Thermal Stresses in Turbine and Compressor Discs

By

M. Cox

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SUMMARY

A step-by-step graphical construction developed by E. Schmidt is used to calculate the transient temperature distributions in a compressor disc under start-up conditions. A method is described by which allowance may be made for the transfer of heat to the disc from the air in the spaces between adjacent discs.

It is concluded, for the particular case evaluated, that the temperature difference producing thermal stress at the disc bore is about one third of the difference between the initial and final steady-state disc temperatures. Some results obtained by an electrical analogue are presented which indicate the effects of a change in heat transfer coefficient and in the manner in which the disc receives heat over the disc face.

The relationship between the temperature differences and the calculated thermal stresses in a disc are discussed, and formulae are derived which give approximately the thermal stresses at the bore and rim of discs as a function of the disc dimensions and temperature distribution.

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1.0 Introduction

Thermal stresses arise in the rotor of the turbine of a gas turbine engine when there are radial variations in temperature due to the action of cooling air which is applied to the surfaces of the disc in order to prevent it from reaching an excessive temperature. Under steady running conditions there is usually a higher temperature at the rim than at the bore of the disc, and the associated thermal strains are compressive at the rim and tensile at the bore. In the turbine, the effect is of interest because it tends to offset the high level of tensile stress at the rim caused by rotation.

Axial multi-stage compressors may have rotors formed from a number of discs; often the centrifugal stress at the bore is high because the diameter of the bore may be a substantial fraction of the disc outside diameter, so that any additional stress resulting from a radial temperature gradient may increase the level of tensile stress at the bore by a critical amount. Under steady running conditions the disc will usually be at a uniform temperature but, on starting the engine, transient radial temperature gradients may occur because the disc is heated mainly at the rim. This effect may be encountered to a marked degree, particularly in the high pressure compressor stages of double-compound engines, both by reason of the high compression temperatures and because the compressors may have to accommodate shafting which leads to a large bore being required through the discs.

It is with the calculation of transient temperatures in such a case that this paper is mainly concerned.

2.0 The "Schmidt" graphical construction

The method used is that developed by E. Schmidt and others[1,2]. A sector of the disc is considered to be a bar of varying cross section in which the heat flow along its length represents radial heat flow in the disc. Figure 1 shows such a sector of a disc in cross section; Figure 2 is a plot in which the vertical scale represents the cross-sectional area of a sector of the disc, that is the thickness of the disc in Figure 1 has been multiplied at each radial station by r/r_0. This equivalent bar is now subdivided into elements of length Δx. If the material of the bar has a constant thermal diffusivity, then all the elements have an equal length. If the thermal diffusivity varies, then the subdivision must be made such that

$$\Delta x = \Delta x_1 \sqrt{\frac{a}{a_1}}$$

In the present instance, it is assumed that the disc material has uniform thermal properties so that Δx is constant except at the point in the disc where there is a circumferential flange which may represent an integral spacing ring or an extension to carry a sealing member. This flange does not greatly alter the cross-sectional area for radial heat flow, but it does constitute an additional mass of metal which acts as a heat sink at this point. Its presence may be regarded as being equivalent to an increase in the specific heat of the disc material at this point.

Figure 3 shows the region of the flange drawn to a larger scale. One assumes that the apparent thickness is y' in this region, which is divided into n elements of length $\Delta x'$. The additional amount of material M represented by the shaded area is distributed equally among the elements in which the specific heat is taken to be

$$\frac{C'}{C_1} = 1 + \frac{M}{ny'\Delta x'}$$

Since

$$\frac{a'}{a_1} = \frac{C_1}{C'}$$

(assuming constant density and thermal conductivity)

$$\frac{\Delta x'}{\Delta x_1} = \left[\frac{1}{1 + \frac{M}{ny'\Delta x'}}\right]^{\frac{1}{2}}$$

or

$$\Delta x' = \left[\left(\frac{M}{ny'}\right)^2 + \Delta x_1^{\,2}\right]^{\frac{1}{2}} - \frac{M}{ny'}$$

(ignoring the negative root).

Since n must be an integer, we proceed to try several values of n and y' to find a solution which gives a reasonable compromise between the values of $n\Delta x'$ and y' drawn on Figure 3.

The whole length of the bar on Figure 2 may now be subdivided into elements of length Δx_1 or $\Delta x'$. Ideally the length should be made up of an integral number of elements; however, in the present instance the addition or loss of the part of an element at the inner radius needed to make an integral number of elements has an insignificant effect on the process, provided that the elements are small.

Now the mean heights of each element, f_1, f_2, etc. are measured; these values are used to determine the intervals $\Delta\xi$ on the "Schmidt" diagram, which are the intervals Δx corrected in inverse proportion to f. That is

$$\Delta\xi = \Delta x \frac{f_1}{f}$$

as indicated in Figure 4.

In the construction, the mid points of these elements are needed, and these are placed at intervals $\frac{\Delta\xi_1}{2}$, $\frac{\Delta\xi_1 + \Delta\xi_2}{2}$ etc. as shown to one side of the ordinate representing the surface through which heat is entering. The first and last ordinates are repeated on the other side of the surface ordinates at the rim and bore positions. The vertical direction represents temperature and any suitable scale is chosen. Thus, the horizontal base line represents the uniform temperature of the disc at time zero.

To carry out the construction a point A is chosen which represents a sudden increase in the gas temperature at time zero. The distance of A from the rim surface ordinate is chosen so that

$$\frac{s}{\Delta\xi_1} = \frac{k/h}{\Delta x_1}$$

The straight line ABO is drawn (see Figure 5). The temperature at X_1 after a time Δt is given by D, where the line joining B and C intersects X_1 and the virtual temperature at X_{-1} by E on the line joining AD. After a further time interval Δt, the temperature at X_1 is given by F on the line joining E and C, and the temperature at X_2 by H on the line joining D and G. Thus, the temperature at an ordinate X_n at any time $(t + \Delta t)$ is found by the intersection of that ordinate with the line joining the temperatures at ordinates X_{n-1} and X_{n+1} for time t.

After a number of time intervals equal to the number of ordinates less one, the construction will have penetrated to the innermost element, X_{21}, as in Figure 6. At the next interval, the temperature M at X_{21} is found by joining K with point L on the other side of the bore surface. This temperature is then transferred to N, the horizontal line MN representing the required condition of zero temperature gradient at the bore. After a further time interval, the temperature at X_{21} is found by the line adjoining P with N.

The construction is continued for any desired number of time intervals. The interval Δt is given by

$$\Delta t = \frac{1}{2a_1} (\Delta x_1)^2$$

We have assumed that the gas temperature and heat transfer coefficient remain constant with time, but any required variation in these quantities may be represented by shifting point A to a position appropriate to the value of heat transfer coefficient and gas temperature obtaining for the particular time interval under consideration.

3.0 Allowance for heat transfer at the disc face

In a turbine rotor, cooling air is often caused to flow up the faces of the disc so that the disc will be maintained at a temperature below that of the gas at the rim during steady running conditions. In the case of the compressor, the air in the spaces between the discs may be at a different temperature from the disc, particularly during the transient heating and cooling phases.

If air at a temperature θ_c is in contact with the disc face, then an element of radial width Δx at radius r will receive heat at a rate

$$2\,\pi r\,\Delta x\,h_c\,(\theta_c - \theta)\quad \text{C.h.u./sec}$$

For a temperature rise $\Delta\theta$ of this element, the amount of heat to be added is

$$2\,\pi r\,\Delta x\,y\rho C\,\Delta\theta\ \text{C.h.u.}$$

If $\Delta\theta$ is the temperature rise in time Δt, then

$$\Delta\theta = \frac{2\,\pi r\,\Delta x\,h_c\,(\theta_c - \theta)}{2\,\pi r\,\Delta x\,y\rho C}\,\Delta t$$

$$= \frac{h_c\,(\theta_c - \theta)}{y\rho C}\cdot\Delta t$$

In drawing the diagram, allowance can be made for heat added in this way by adding on to the temperature of each element at the end of each time interval a temperature change $\Delta\theta$. Usually $\Delta\theta$ will be small compared with $(\theta_c - \theta)$, but if this is not so, a better approximation may be made by taking

$$\Delta\theta = \frac{h_c\,\Delta t}{y\rho C}\left(\theta_c - \left(\theta + \frac{\Delta\theta}{2}\right)\right)$$

$$\Delta\theta = \frac{z}{1 + \frac{z}{2}}\,(\theta_c - \theta)$$

where $$z = \frac{h_c\,\Delta t}{y\rho C}$$

If $\Delta\theta$ is small, then the amount of work involved in the construction may be reduced if an allowance $n\Delta\theta$ is made at time intervals $\frac{n}{2}\ \frac{3n}{2}\ \frac{5n}{2}$ etc.

4.0 Temperature of the air at the disc face

It will usually be the case that the temperature of the air in contact with the disc face is known only at the point at which it enters the rotor system; thereafter, heat will be transferred between the disc and the air, and we require to know the temperature of this at all points on the disc. Two cases will be considered; firstly, the particular one under review, and secondly, the turbine rotor where cooling air enters at the bore and flows outwards over the disc face.

4.1 The compressor disc with air entering at the disc rim

At any annular element of the disc face, the rate of heat transfer from air to the disc is

$$2\,\pi r\,\Delta x\,h_c\,(\theta_c - \theta)\quad \text{C.h.u./sec}$$

In the present example, illustrated diagrammatically in Figure 7, the space between discs is divided into two chambers by an abutment flange. Air leaks into the outer chamber, then into the inner chamber through a small aperture and thence away through the bore of the disc. It is assumed that the air in each chamber is completely mixed to a uniform temperature.

If w is the leakage rate into the disc space, then

$$(\theta_g - \theta_c)\,\frac{wC_p}{2} = \sum 2\,\pi r\,\Delta x\,h_c\,(\theta_c - \theta)$$

where the summation is made for the elements in the outer chamber.

$$\theta_c = \frac{\theta_g + \sum \dfrac{4\,\pi r\,\Delta x\,h_c\,\theta}{wC_p}}{1 + \sum \dfrac{4\,\pi r\,\Delta x\,h_c}{wC_p}}$$

Similarly for the inner chamber.

It will be seen that to calculate θ_c and θ_c' at any time interval we need to know θ, which depends upon the correction to be made for heat added to the disc, itself dependent upon θ_c. Usually the quantities are such that the effect of the change in wall temperature, due to the allowance for heat addition, is only a small one and so values of θ for the previous time interval may be used, but in some instances it may be desirable to make a second approximation to θ_c to check that this is so.

4.2 The turbine rotor with cooling air entering at the bore

In this case the cooling air increases gradually in temperature as it flows over the disc face (see Figure 8). Over an annular width Δx, the temperature rise is

$$\Delta\theta_c = \frac{4\,\pi r\,\Delta x\,h_c}{wC_p}\left(\theta - \left(\theta_c + \frac{\Delta\theta_c}{2}\right)\right)$$

$$\Delta\theta_c = \frac{Y}{1 + \frac{Y}{2}}\,(\theta - \theta_c)$$

where $$Y = \frac{4\,\pi r\,\Delta x\,h_c}{wC_p}$$

Starting at the bore where the air inlet temperature is known, we can estimate the temperature rise across the first element and hence the temperature at entry to the next. In this way the variation of cooling air temperature across the disc can be determined from bore to rim.

5.0 Heat transfer coefficients at the disc face

In the compressor disc, the heat transfer at the disc face has been considered to result from natural convection currents set up in the spaces between the disc. The air in the outer chamber is initially hotter than the disc; as the air cools at the disc face, the increase in density will cause an outward flow across the disc face and circulating currents similar to those indicated in Figure 7 will result. Because of the rotation of the disc, and the fact that the air enters at the rim with the angular velocity of the disc, gravitational forces are very high, so that the heat transfer coefficients are considerably above the values normally associated with natural convection.

The heat transfer coefficients were calculated from the formula[8]

$$\frac{h_{mean} \cdot H}{k} = 0.48 \left(\frac{gH^3 \, (\theta_c - \theta_w)}{\upsilon_w^2 \, T_c} \right)^{0.25}$$

For the turbine rotor, the heat transfer coefficients are calculated on the basis of forced convection from the cooling air, and formulae appropriate to the manner in which the air is ducted up the disc face must be chosen. Where the disc rotates in a stationary casing and the flow rates are small the data of Rotem[3] may be used.

$$\frac{hr}{k} = 0.017 \left(\frac{\omega r^2}{\upsilon} \right)^{0.8} \quad \text{(local value)}$$

In other instances a formula based on pipe flow data may be more appropriate.

6.0 Estimation of temperature distribution for maximum stress

The temperature distributions determined in the way described are needed for calculating thermal stresses in discs. Since we are concerned with a transient phenomenon, the temperature distribution is changing continuously with time and likewise the thermal stresses are changing, rising to a maximum and then approaching zero if the disc approaches a uniform temperature, as in the case of an uncooled disc. It would be an advantage to know in the transient temperature calculations when the construction had been carried sufficiently far to include the temperature distribution which gave the maximum thermal stress.

Stress calculations have been made[4] for the present disc using several different temperature distributions. These distributions are shown on Figure 9, in which the vertical scale represents the temperature and the horizontal scale the mass of the disc contained between the rim and the radius at which the temperature is given. Hence the area under a curve represents the total quantity of heat in the disc, and the mean height the mass-weighted mean temperature of the disc. On Figure 10 are shown the thermal stresses at the disc bore calculated for the temperature distributions of Figure 9, plotted against the difference between disc mean temperature and bore temperature. The inference from this is that the value of the difference between the bore temperature and the mean temperature of the whole disc on a mass basis is a good indication of the level of thermal stress at the bore, irrespective of the particular manner in which the temperature is assumed to vary with radius.

In Figure 11 is shown a series of temperature distributions evaluated at different stages of a heating cycle. Figure 12 shows the resultant change of bore-to-mean temperature difference against time. It is seen that for the particular geometry and rate of rim heat transfer selected, the maximum temperature difference is reached in about 5 minutes from the start of the cycle.

7.0 Evaluation by electrical analogue

An electrical analogue circuit built to study temperature changes in turbine blades[5] was set up to represent the one-dimensional problem of heat flow in a bar of varying cross section, the bar being the equivalent of the segment of the disc, (see Figure 2). Three cases were studied; heat flow in at the disc rim only, rim heating and heat transfer from air leaking through the space between discs, and a third case in which the air between the discs was assumed to be sealed in but still to exchange heat with the disc face as in the second case. The measure of the agreement between the analogue and the step-by-step construction method is shown on Figure 13.

The heating conditions at the disc rim could be varied easily on the analogue, and Figure 14 shows the effect of changing the heat transfer coefficient at the rim for the case of rim heating only. In Figure 15 the three different modes of heating are compared. From Figure 14 it is clear that at the highest values of heat transfer coefficient considered here (k/h = 0.15 and 0.3 in.), the precise value of h used has very little effect upon the value of maximum temperature difference at the bore. The effect of heat transfer at the disc face is seen from Curve 3 of Figure 15 to be one of reducing the maximum temperature difference. This might be expected, since this mode of heat transfer is equivalent to increasing the conduction of heat from rim to bore. When air at the compressor stage temperature leaks into the space between the disc the effect is to raise the maximum temperature difference (Curve 2) because more heat is added to the outer parts of the disc than at the bore. The maximum temperature difference producing strain at the bore is about the same in this instance as for Curve 1, heating assumed to be only at the rim, but the time to reach the maximum is only about one third of that in the latter case. If the leakage rate were increased, the maximum temperature difference would be increased up to a limiting value of about 0.38, as indicated by Curve 4 on Figure 15.

One concludes from these considerations that the maximum temperature difference producing strain at the bore of the disc is likely to be 0.3 to 0.34 of the change in temperature of the air in the compressor stage during the starting cycle, and that errors in the assumptions made concerning the heat transfer conditions at the disc surfaces are not likely to affect this conclusion seriously. The most favourable condition is one in which there is zero leakage of air from the compressor annulus and where the greatest opportunity is given for allowing natural circulation within the space between adjacent compressor discs.

8.0 Discussion

Although the results presented have referred only to one particular design of disc, one may enquire whether they might not be applied more generally to the problem of thermal stresses in discs. In particular, two questions arise. Firstly, it is clear that if h were reduced to a small value, the thermal strains would be reduced; how far then must h be reduced to give a significant reduction in thermal strain? Secondly, is the difference between the bore temperature and the mean temperature of the disc related quantitatively to the thermal stresses at the bore in any simple, approximate manner, irrespective of the detailed shape of the of the disc?

8.1 Limiting value of h

Since most of the temperature drop from rim to bore occurs in the thin part of the disc, we may liken this to temperature difference along a bar, heated at one end by a gas, the temperature of which is suddenly changed. An examination of data[2] for transient temperature changes of this type shows that the maximum transient difference occurs when the bore is still scarcely changed in temperature, and from the same source the data required to draw Figure 16 may be obtained. The curve shows the relation between maximum temperature difference across the ends of a bar and the factor $h\ell/k$, where h is the heat transfer coefficient at the exposed end of the bar, ℓ its length and k the material conductivity. By taking ℓ to be the radial length of the thin part of the disc, and by increasing the rim heat transfer coefficient by a factor to allow for the fact that the rim is wider than the thin part of the disc, the maximum values of the curves of Figure 14 may also be plotted on Figure 16. The indications are then that the parameter $h\ell/k$ must be reduced to values less than about 15 in order to secure worthwhile reductions in thermal stress and that in the range $10 > h\ell/k > 0.5$ the thermal stress is significantly dependent upon the value of heat transfer coefficient.

8.2 Correlation between thermal stress and temperature difference

Figure 17 is a replot of Figure 10 in which effective strain (= stress ÷ E) is plotted against one-dimensional thermal strain based on the difference between bore temperature and mean disc temperature. Other results[4] for a turbine disc with a small ratio of bore radius/disc outer radius are given and these likewise show a practically linear relationship between stress and temperature difference; however the proportionality between stress and thermal strain is different in the two cases. One might anticipate that, when the ratio of bore radius/disc radius approaches unity, the thermal strain would give directly the resultant stress, so that a correlation of the sort shown on Figure 18 might be proposed. However, the inclusion of other points taken from calculations[6] made for a number of different turbine discs, each with different temperature distributions, indicates that this does not give an exact basis for correlation.

The theory of thermal stresses[7] in circular discs of uniform thickness gives that the elastic thermal stresses at the bore and rim of a disc are to be calculated from the thermal strains due to the differences in temperature from the mean temperature of the disc, evaluated on a mass-weighted basis, for any bore/rim radius ratio. Any apparent correlation with disc bore/rim radius ratio must, therefore, be associated with other factors, such as the radial variation in thickness of the disc.

In order to examine the effect of bore/rim radius ratio on the thermal stress in a disc of varying thickness, the calculations laid out in detail in Appendix I have been made. It is shown that, for a disc in which the thickness varies inversely as the radius, the range of stress between bore and rim produced by a temperature distribution similar to that occurring in practical cases decreases as the bore becomes a smaller fraction of the rim diameter. A comparison with stresses calculated for a number of real discs shows quite close agreement between the stress range in real cases and in the assumed model. The values are recorded in Tables I and II.

9.0 Approximate determination of thermal stress

It may be concluded from the preceding section that an approximate estimate of the maximum transient thermal stress at the bore of a disc, when subjected to sudden heating at the rim, may be determined rapidly as follows.

9.1 Uncooled disc, $\frac{h\ell}{k} > 20$

$$\text{Stress} \simeq Z \times E \times \alpha \times \frac{\Delta T_{gas}}{3}$$

where

ΔT_{gas} = step change in gas temperature at the rim

Z = a constant dependent upon the ratio of bore radius/rim radius (see Figure 18)

If $h\ell/k$ is substantially less than 20, the above formula will overestimate the stress. In such an instance the maximum value of $(T_{mean} - T_{bore})$, replacing the term $\Delta T_{gas}/3$ in the above equation, may be determined more accurately by the graphical construction described in Section 2.0.

9.2 Cooled discs

Employing the graphical constructions described in Sections 2.0 to 5.0 the temperature distribution at a time when $(T_{mean} - T_{bore})$ reaches a maximum value may be determined. This radial temperature distribution and also the radial form of the disc may then be approximated by equations of the form

$$T = Kr^{\gamma-1}$$

$$y = c/r^n$$

in which the appropriate values of K, γ, c and n are determined by trial as described in Appendix I.

The thermal stresses at the bore and rim may then be determined directly from Equation (10) of Appendix I.

10.0 Conclusions

The transient changes in temperature distribution in a compressor disc resulting from sudden changes in temperature of the air in the compressor stage have been calculated using a well-known step-by-step graphical construction. A method of allowing for heat transfer at the disc face is explained.

In the case of the compressor disc studied, the maximum value of temperature difference producing strain at the bore (mean disc temperature - bore temperature) is about one third of the step change in temperature of the air heating the disc at the rim. Convection between the air in the space between adjacent discs and the disc face reduces this maximum difference, but if air is allowed to leak through this space from the compressor annulus the temperature difference increases. These variations do not exceed ±12 per cent approximately.

Results from an electrical analogue are used to extend the range of conditions covered. There was good agreement between the analogue measurements and the temperature distributions calculated by the graphical method.

A parameter is given from which it may be decided approximately at what level of heat transfer coefficient at the rim the thermal stresses cease to be significantly affected by changes in heat transfer coefficient.

By adducing values of thermal stress calculated elsewhere, it is shown that the value of the difference between the surface temperatures and the mass-weighted mean temperature of a disc is related approximately linearly to the stresses produced at the bore and rim surfaces.

A theoretical calculation for a disc of varying thickness shows that the stresses produced become less than the one-dimensional thermal stresses as the bore radius becomes a small fraction of the rim radius. A comparison between stresses calculated for real discs and those of the theoretical model indicates that the latter are good approximate indication of the thermal stress in real turbine discs.

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List of symbols

a	thermal diffusivity $= \frac{k}{\rho C}$	$\frac{ft^2}{sec}$
C	specific heat of disc material	$\frac{C.h.u.}{lb\ ^oC}$
C_p	specific heat of cooling air	$\frac{C.h.u.}{lb\ ^oC}$
f	cross-sectional area	ft^2
g	acceleration	$\frac{ft^2}{sec}$
h	heat transfer coefficient	$\frac{C.h.u.}{ft^2\ sec\ ^oC}$
H	height of surface for natural convection	ft
k	thermal conductivity	$\frac{C.h.u.}{ft\ sec\ ^oC}$
ℓ	length	ft
r	radius of point on disc face	ft
s	$= \frac{k}{h} \times \frac{\Delta\xi_1}{\Delta x_1}$	ft
Δt	time interval	sec
w	flow rate of air over disc (both faces)	$\frac{lb}{sec}$
Δx	radial interval of element of disc sector	ft
y	thickness of disc	ft
$\Delta\xi$	interval between ordinates on diagram	ft
υ	kinematic viscosity	$\frac{ft^2}{sec}$
ρ	specific weight of disc material	$\frac{lb}{ft^3}$
θ, T	temperature	^oC
ω	rotational speed	radian/sec
α	coefficient of thermal expansion	$\frac{1}{^oC}$

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TABLE I

Comparison between thermal stresses calculated from Equation (10) and by step-by-step method

Turbine disc with bore/tip radius ratio = 0.1126

Temperature distributions $T = Kr^{\gamma-1}$

Thickness assumed to be as $y = \frac{c}{r^{0.62}}$

$\gamma - 1$		2	4	6	8
Pb	Equation (10)	-20,200	-26,070	-29,450	-31,250
Pa	Equation (10)	10,520	6,260	4,520	3,530
Pb	Step-by-step method	-19,530	-26,320	-29,031	-30,848
Pa	Step-by-step method	9,330	5,751	4,158	3,060
% error, at rim		2.3	0.8	1.3	1.2
% error, at bore		4.1	1.6	1.1	1.4

Pb stress at rim, lb/sq.in.

Pa stress at bore, lb/sq.in.

Error is calculated as a percentage of the total stress range obtained from step-by-step method.

TABLE II

Comparison between thermal stresses calculated from Equation (10) and by step-by-step method for six different turbine discs

Disc No.		1	2	3	4	5	6
Radius ratio D = a/b		0.143	0.100	0.244	0.254	0.151	0.115
Mean thickness/bore thickness		0.280	0.350	0.567	0.429	0.260	0.255
n		0.875	0.77	0.60	0.935	1.0	0.86
$\gamma - 1$		1.0	1.2	3.5	0.65	4.0	1.0
P_b	Equation (10)	4.66	7.00	10.1	10.3	19.7	5.60
P_a	Equation (10)	4.10	5.48	4.24	10.7	3.73	4.63
P_b	Step-by-step method	4.64	6.63	11.85	10.3	20.52	5.60
P_a	Step-by-step method	4.24	5.95	3.65	11.4	6.01	3.36
% error, at rim		+0.2	+3.0	-4.8	0	-3.1	0
% error, at bore		-1.6	-3.7	+4.7	-3.2	+8.6	+14.2

P_b compressive stress at rim tons/sq.in.

P_a tensile stress at bore tons/sq.in.

Error is calculated as a percentage of the total stress range obtained from step-by-step method.

APPENDIX I

The thermal stresses in a disc of varying radial thickness

Notation

a	radius at disc bore
b	radius at disc rim
c	a constant in $y = c/r^n$
C_1, C_2	constants of integration
D	radius ratio a/b
E	Young's modulus, stress ÷ strain
K	a constant in $T = Kr^{\gamma-1}$
m	1/m = Poisson's ratio
n	a constant in $y = c/r^n$
p_r	radial stress
p_t	tangential stress
r	radius
R	radius ratio r/b
T	temperature
u	radial displacement of a point originally at radius r
y	thickness of disc
α	coefficient of thermal expansion
β_1 β_2	indices
γ	a constant in $T = Kr^{\gamma-1}$

APPENDIX I (cont'd)

Let us consider the stresses in a stationary disc in which it is assumed that axial stresses are zero and the thickness y and temperature T vary with radius.

The displacement of an element in the disc originally at radius r is governed[9] by three equations.

$$E \frac{\partial u}{\partial r} = p_r - \frac{p_t}{m} + E \alpha T \qquad \ldots \quad \ldots \quad (1)$$

$$E \frac{u}{r} = p_t - \frac{p_r}{m} + E \alpha T \qquad \ldots \quad \ldots \quad (2)$$

$$p_r - p_t + r \frac{dp_r}{dr} + \frac{rp_r}{y} \times \frac{dy}{dr} = 0 \qquad \ldots \quad \ldots \quad (3)$$

The radial and tangential stresses p_r and p_t may be found from Equations (1) and (2) in terms of u and r

$$p_r = \frac{Em}{m^2 - 1} \left[\left(\frac{u}{r} + m \frac{du}{dr} \right) - \alpha T(m + 1) \right] \qquad \ldots \quad \ldots \quad (4)$$

$$p_t = \frac{Em}{m^2 - 1} \left[\left(m \frac{u}{r} + \frac{du}{dr} \right) - \alpha T(m + 1) \right] \qquad \ldots \quad \ldots \quad (5)$$

If the thickness is given by $y = c/r^n$, then Equation (3) becomes, by substitution from Equations (4) and (5)

$$r^2 \frac{d^2u}{dr^2} + (1 - n) r \frac{du}{dr} - \frac{m + n}{m} u$$

$$= \alpha T \cdot \frac{m + 1}{m} \left(\frac{r}{T} \cdot \frac{dT}{dr} - n \right) \cdot r \qquad \ldots \quad \ldots \quad (6)$$

We assume also that the temperature distribution follows a simple power law

$$T = Kr^{\gamma - 1}, \; \frac{r}{T} \cdot \frac{dT}{dr} = \gamma - 1$$

So that Equation (6) becomes

$$r^2 \frac{du}{dr} + (1 - n)\, r \frac{du}{dr} - \frac{m + n}{m} u = \alpha K \cdot \frac{m + 1}{m} (\gamma - 1 - n)\, r^{\gamma} \qquad \ldots \quad \ldots \quad (7)$$

A solution of this equation is

$$u = C_1 r^{\beta_1} + C_2 r^{\beta_2} + \frac{(m + 1)(\gamma - 1 - n)}{m(\gamma^2 - 1) - n(\gamma m + 1)} \cdot \alpha K r^{\gamma} \qquad \ldots \quad \ldots \quad (8)$$

where β_1 and β_2 are the roots of the equation

$$\beta^2 - n\beta - \frac{n + m}{m} = 0$$

$$\beta_1 = \frac{n + \left(n^2 + 4\frac{n + m}{m}\right)^{\frac{1}{2}}}{2}$$

$$\beta_2 = \frac{n - \left(n^2 + 4\frac{n + m}{m}\right)^{\frac{1}{2}}}{2}$$

The constants C_1 and C_2 may be determined from the two boundary conditions which are that p_r, the radial stress, is zero at the rim and bore surfaces of the disc.

That is, when $r = b$ and $r = a$, $p_r = o$.

By substituting for u from Equation (8) in Equation (4), we have

$$p_r = \frac{Em}{m^2 - 1}\left[C_1 (1 + m\beta_1) r^{\beta_1 - 1} + C_2 (1 + m\beta_2) r^{\beta_2 - 1} - \frac{(m^2 - 1)(\gamma - 1)}{m(\gamma^2 - 1) - n(m\gamma + 1)} \cdot \alpha K r^{\gamma - 1}\right] \qquad \ldots \quad \ldots \quad (9)$$

With the given boundary conditions we evaluate C_1 and C_2 to be

$$C_1 = \frac{1}{1 + m\beta_1} \cdot \frac{(m^2 - 1)(\gamma - 1)}{m(\gamma^2 - 1) - n(m\gamma + 1)} \cdot \frac{b^{\gamma - \beta_2} - a^{\gamma - \beta_2}}{b^{\beta_1 - \beta_2} - a^{\beta_1 - \beta_2}} \cdot \alpha K$$

$$C_2 = \frac{1}{1 + m\,\beta_2} \cdot \frac{(m^2 - 1)\ (\gamma - 1)}{m\ (\gamma^2 - 1) - n\ (m\,\gamma + 1)} \cdot \frac{b^{\gamma-\beta_1} - a^{\gamma-\beta_1}}{b^{\beta_2-\beta_1} - a^{\beta_2-\beta_1}} \cdot \alpha\, K$$

By making the necessary substitutions in Equation (5) we find p_t

$$p_t = \frac{m\,E\,\alpha\,K\,(\gamma - 1)}{m\ (\gamma^2 - 1) - n\ (m\,\gamma + 1)} \left[- \beta_2 \cdot \frac{b^{\gamma-\beta_2} - a^{\gamma-\beta_2}}{b^{\beta_1-\beta_2} - a^{\beta_1-\beta_2}}\, r^{\beta_1-1} \right.$$

$$\left. - \beta_1\ \frac{b^{\gamma-\beta_1} - a^{\gamma-\beta_1}}{b^{\beta_2-\beta_1} - a^{\beta_2-\beta_1}} \cdot r^{\beta_2-1} - (\gamma - n)\ r^{\gamma-1} \right]$$

For convenience we write $D = a/b$ and $R = r/b$

$$p_t = \frac{m\,E\,\alpha\,K\,(\gamma - 1) \cdot b^{\gamma-1}}{m\ (\gamma^2 - 1) - n\ (m\,\gamma + 1)} \left[- \beta_2\ \frac{1 - D^{\gamma-\beta_2}}{1 - D^{\beta_1-\beta_2}} \cdot R^{\beta_1-1} \right.$$

$$\left. - \beta_1 \cdot \frac{1 - D^{\gamma-\beta_1}}{1 - D^{\beta_2-\beta_1}} \cdot R^{\beta_2-1} - (\gamma - n)\ R^{\gamma-1} \right] \qquad \ldots\ \ldots\ (10)$$

If we consider a disc of uniform thickness, i.e. $n = o$, then $\beta_1 = 1$ and $\beta_2 = -1$, and it will be seen that m disappears from Equation (10), and it may be shown that, at the rim and bore surfaces, the stress is given by

$$E\ \alpha\ (T_m - T_{surface})$$

where T_m is the weighted mean temperature of the disc

$$T_m = \frac{\int_a^b T\ y\ r\ dr}{\int_a^b y\ r\ dr}$$

$$= K \cdot b^{\gamma-1} \cdot \frac{2 - n}{\gamma - n + 1}\ \frac{1 - D^{\gamma-n+1}}{1 - D^{2-n}}$$

If we choose appropriate values for n and γ, Equation (10) may be used to determine the variation of bore and rim tangential stresses as the ratio of bore radius/rim radius is altered. For the purposes of illustration, the thickness of the disc has been taken to be inversely proportional to radius, and the radial variation in temperature has been taken to be proportional to r^2 and r^8, values which cover most practical cases. m has been taken to be 10/3.

In Figure 19 the stress range $(p_{bore} - p_{rim})$, referred to the one-dimensional thermal stress $\left[E\,\alpha\,(T_{rim} - T_{bore})\right]$ is shown as a function of the radius ratio. The values of the same quantities calculated[4,6] by the step-by-step method[10] for eight different rotor discs are shown on the same graph.

The shapes of real discs do not usually follow a simple power law, and in order to use Equation (10) we must choose a value of n which approximates the real variation in thickness by the simple power law. The method used is to find a value of n which gives a disc of the same bore/rim radius ratio and with the same ratio of mean thickness/bore thickness. For the assumed equivalent disc this ratio is

$$\frac{y_{mean}}{y_a} = \frac{2}{2-n}\,\frac{D^n - D^2}{1 - D^2}$$

In the case of a disc[4] for which stress calculations had been made with radial temperatures following a series of power laws, the value of n was found to be 0.62. In Table I the values of stress calculated from Equation (10) are compared with the stresses calculated from the step-by-step method; the differences are not more than a few per cent of the overall thermal stress range in the disc.

In Table II a similar comparison is made for six different discs. Here the index $(\gamma - 1)$ of the radial variation in temperature has been chosen so that the assumed temperatures agree with the real ones at the rim and bore stations (equal overall temperature difference) and the assumed curve has a minimum mean deviation from the real curve. In most instances, the differences between calculated and "real" stresses amount to only a few per cent.

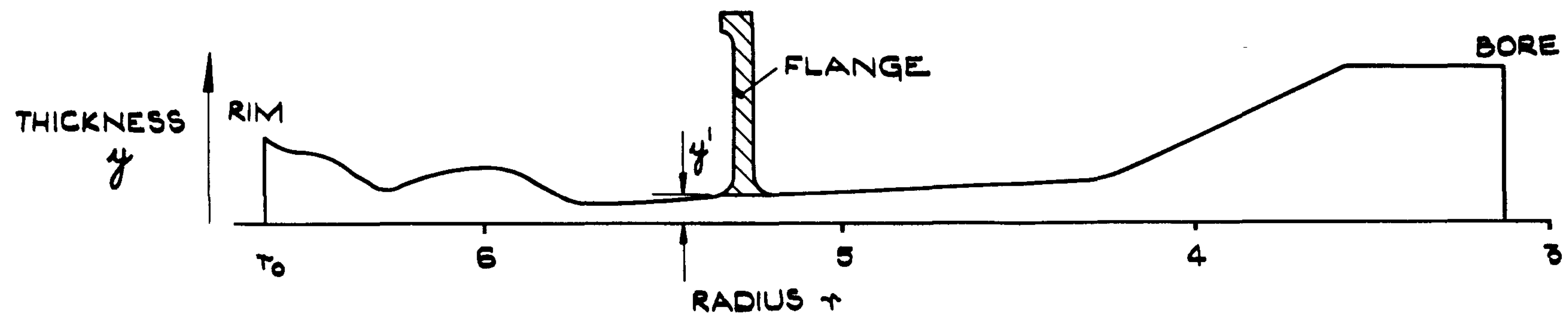


FIG.1 CROSS SECTION ALONG RADIUS OF DISC

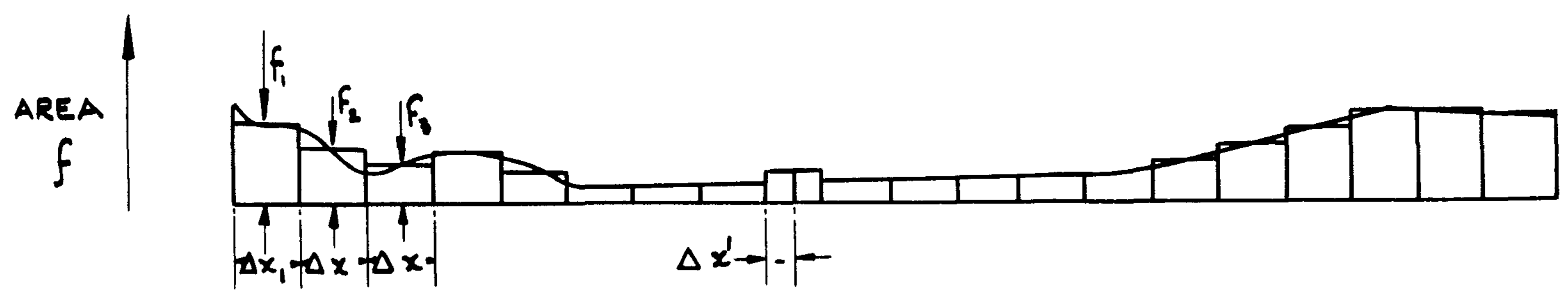


FIG.2 TRANSFORMATION AND SUBDIVISION OF SECTOR OF DISC

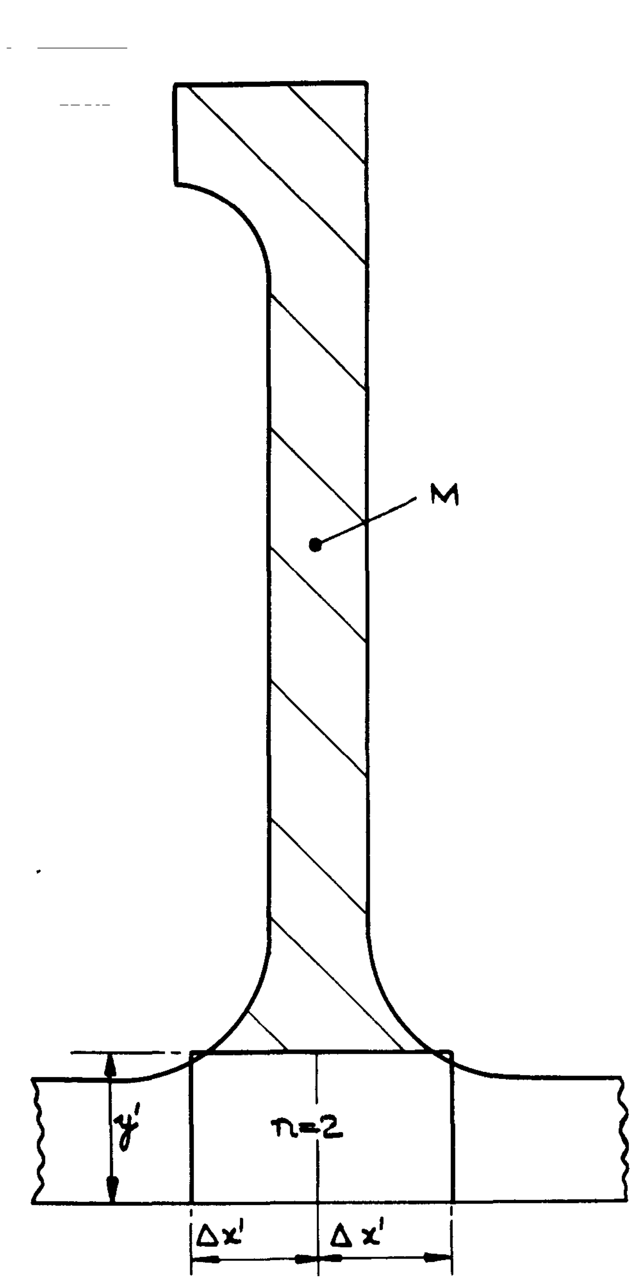


FIG.3 ENLARGED VIEW OF FLANGE

FIG 4

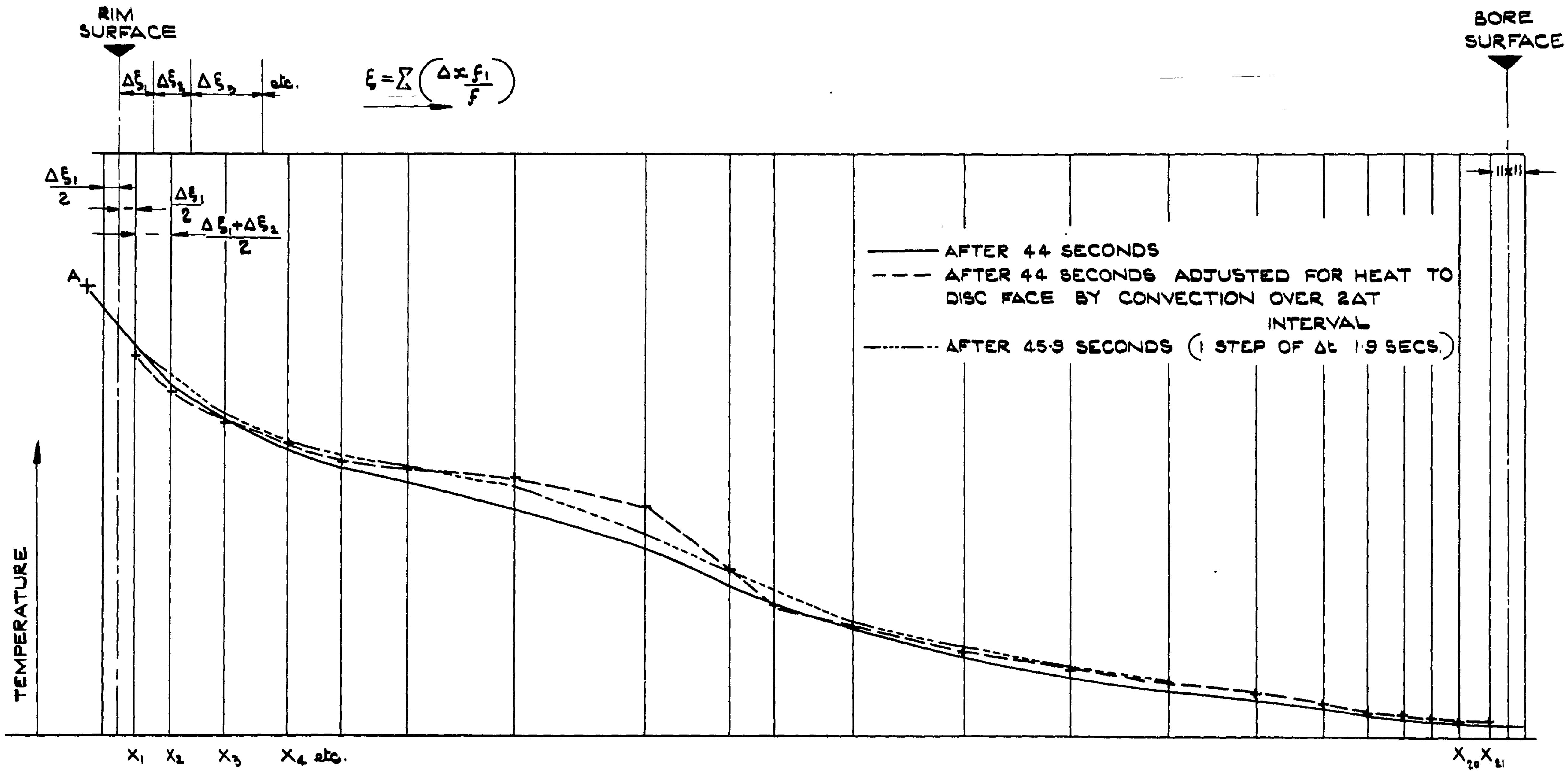


DIAGRAM FOR CARRYING OUT STEP BY STEP CONSTRUCTION

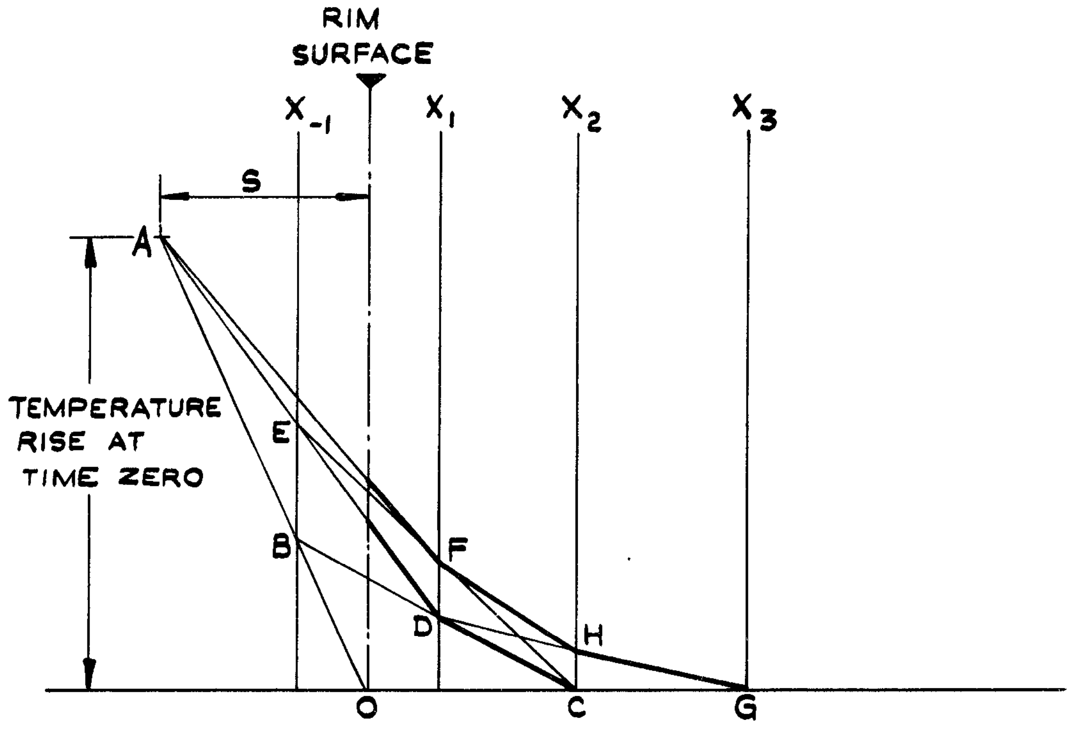


FIG. 5 START OF CONSTRUCTION AT RIM SURFACE.

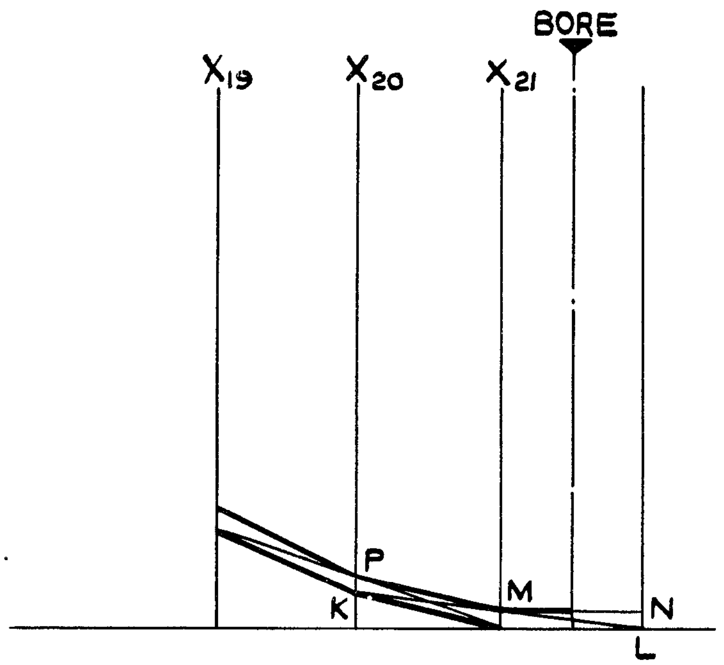


FIG. 6 CONSTRUCTION AT BORE SURFACE.

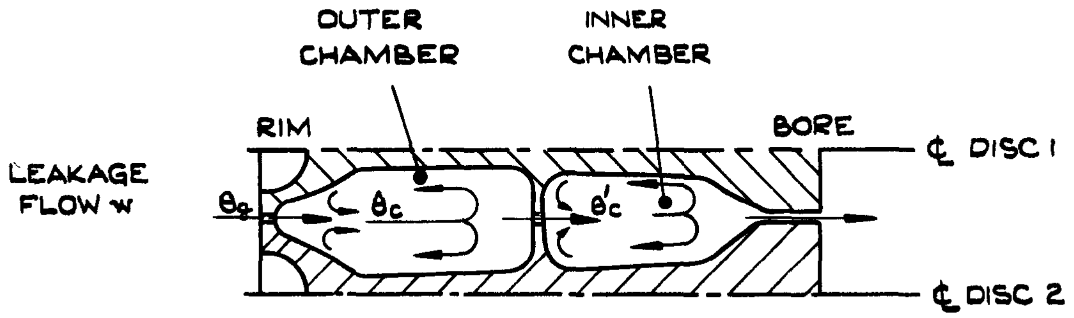


FIG.7 ILLUSTRATING LEAKAGE AIR FLOW PATH BETWEEN ADJACENT DISCS

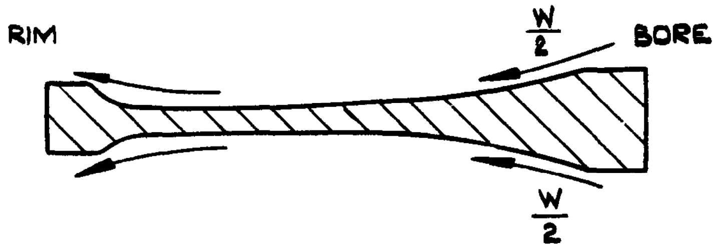


FIG.8 ILLUSTRATING COOLING AIR FLOW PATH OVER TURBINE DISC

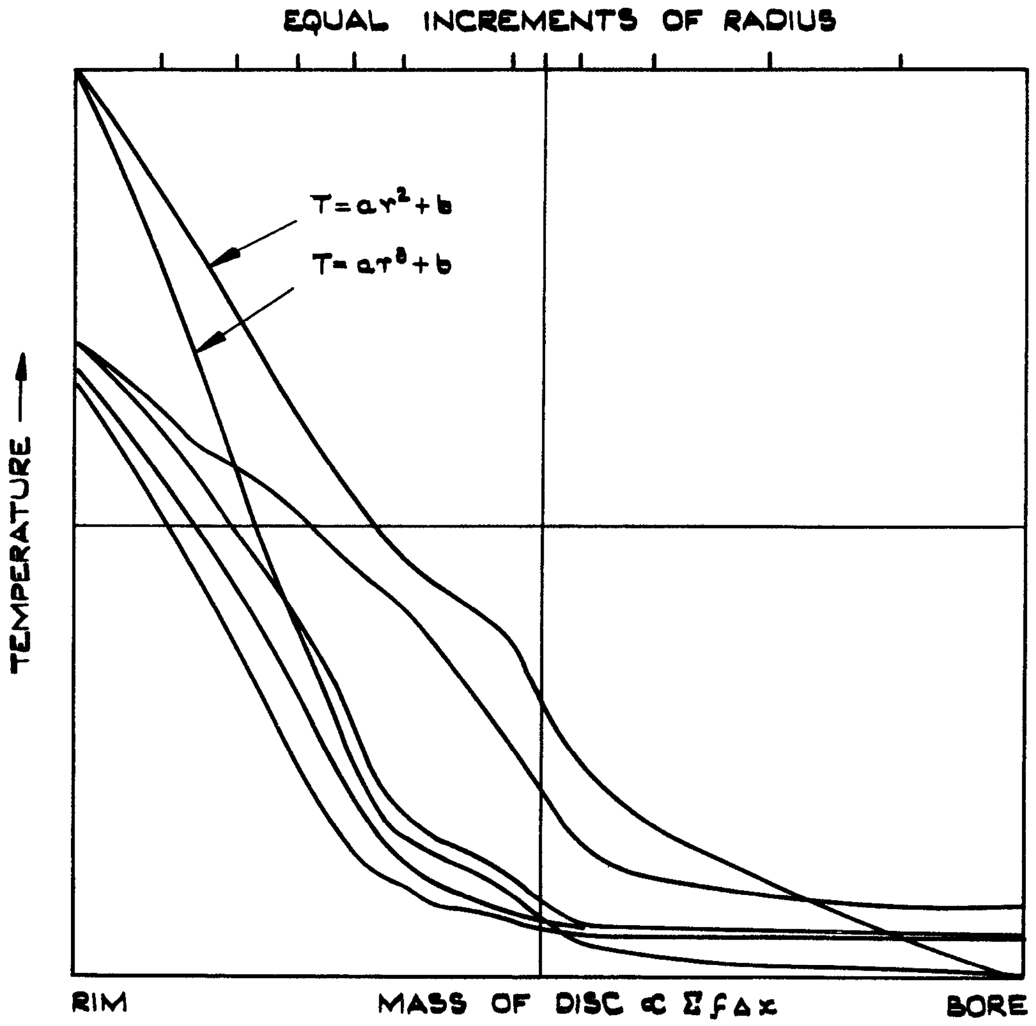


FIG. 9 TEMPERATURE DISTRIBUTIONS IN DISC FOR WHICH BORE THERMAL STRESSES HAVE BEEN CALCULATED

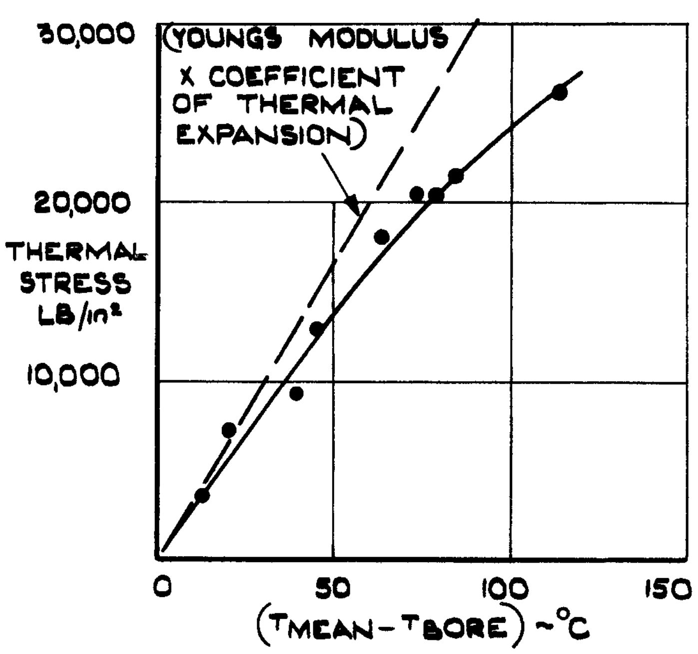


FIG. 10 THERMAL STRESS AT DISC BORE VERSUS TEMPERATURE DIFFERENCE ($T_{MEAN} - T_{BORE}$)

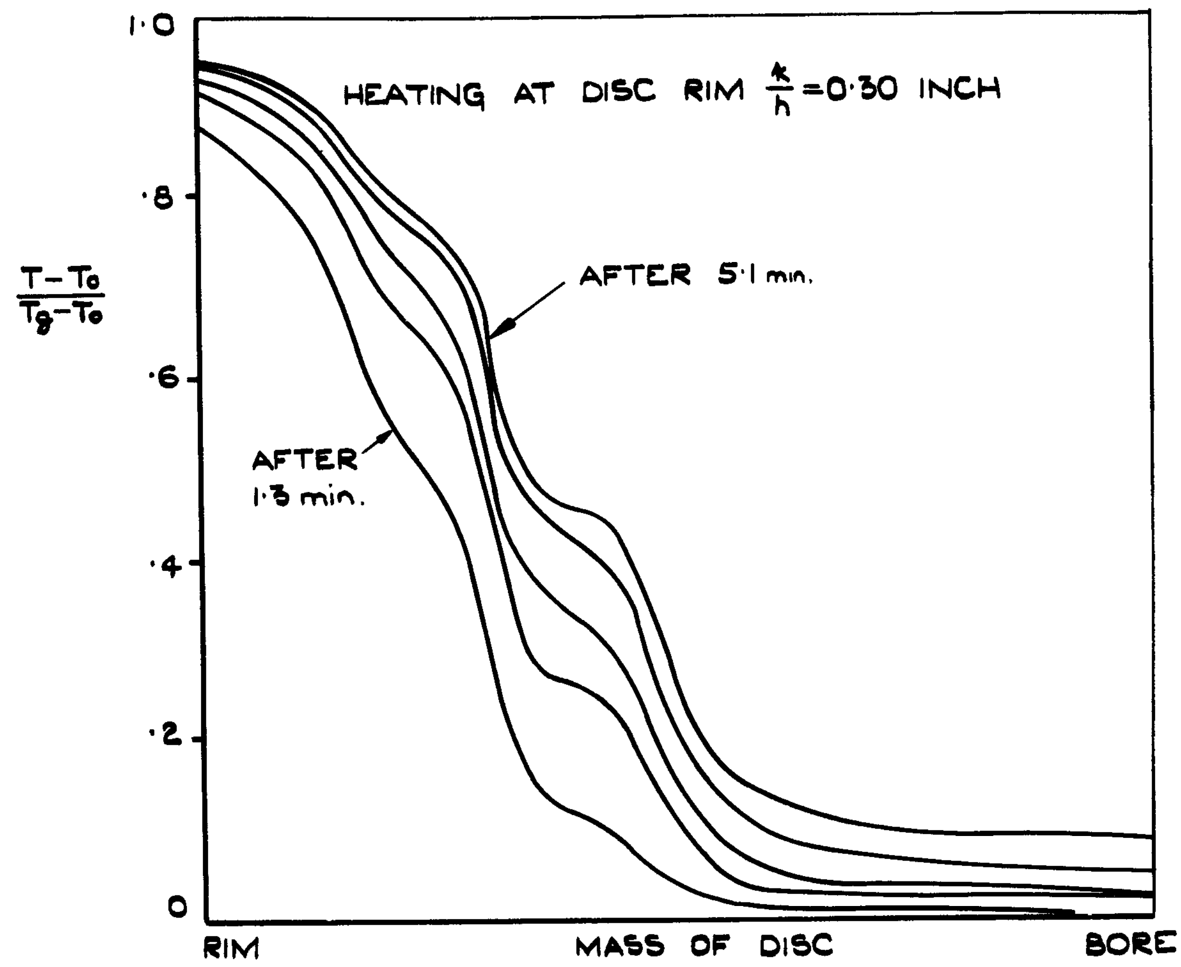


FIG. 11 TEMPERATURE DISTRIBUTIONS IN DISC AT VARIOUS STAGES OF TRANSIENT

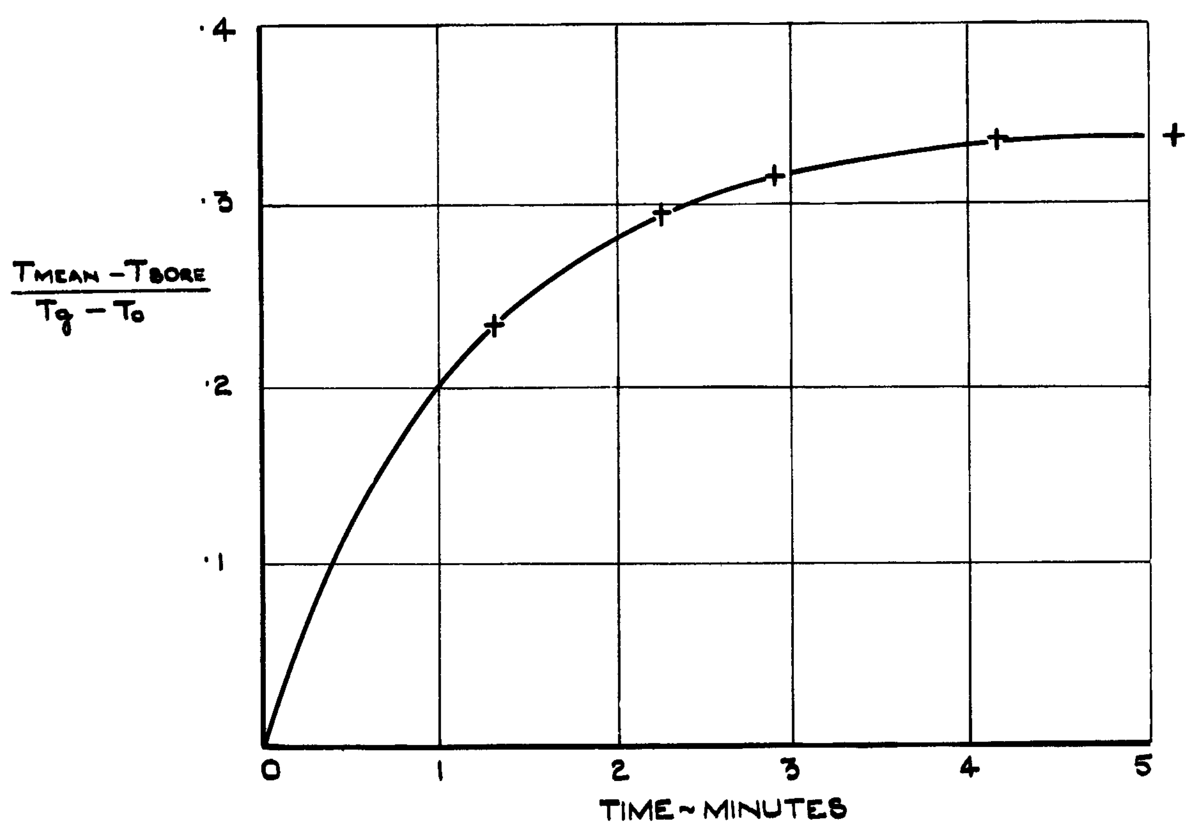
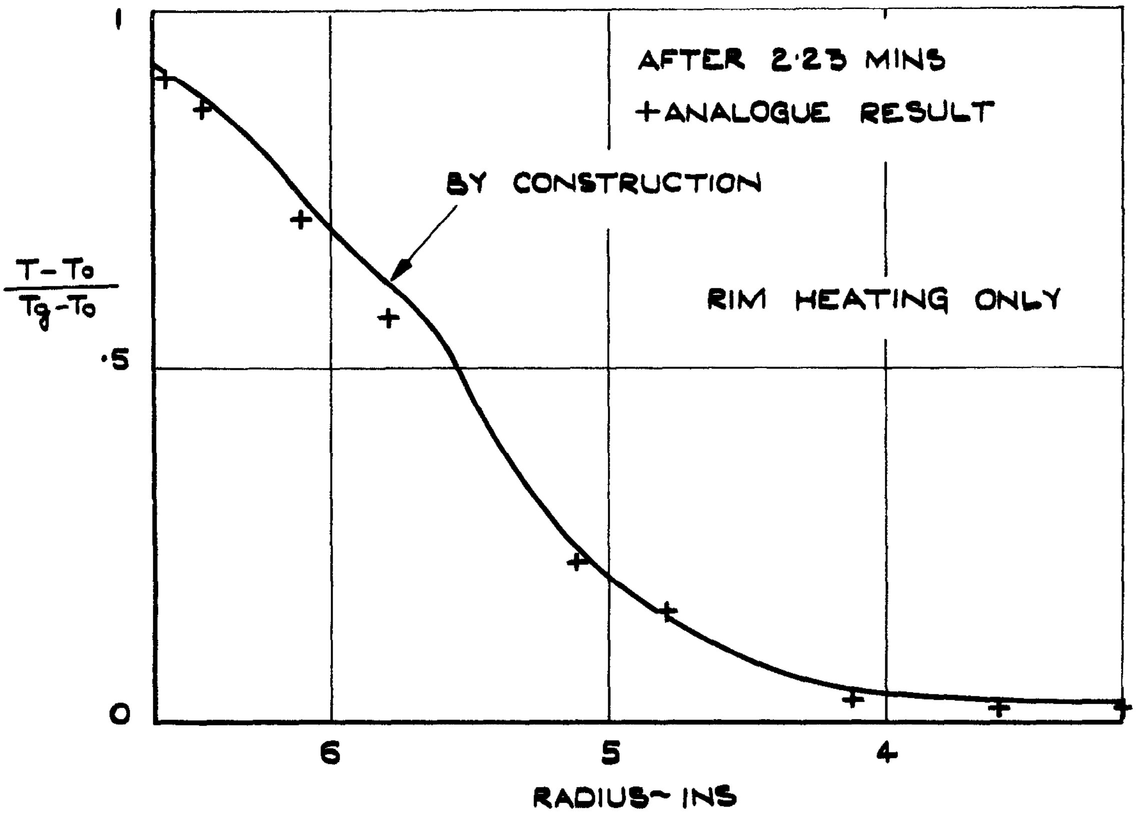
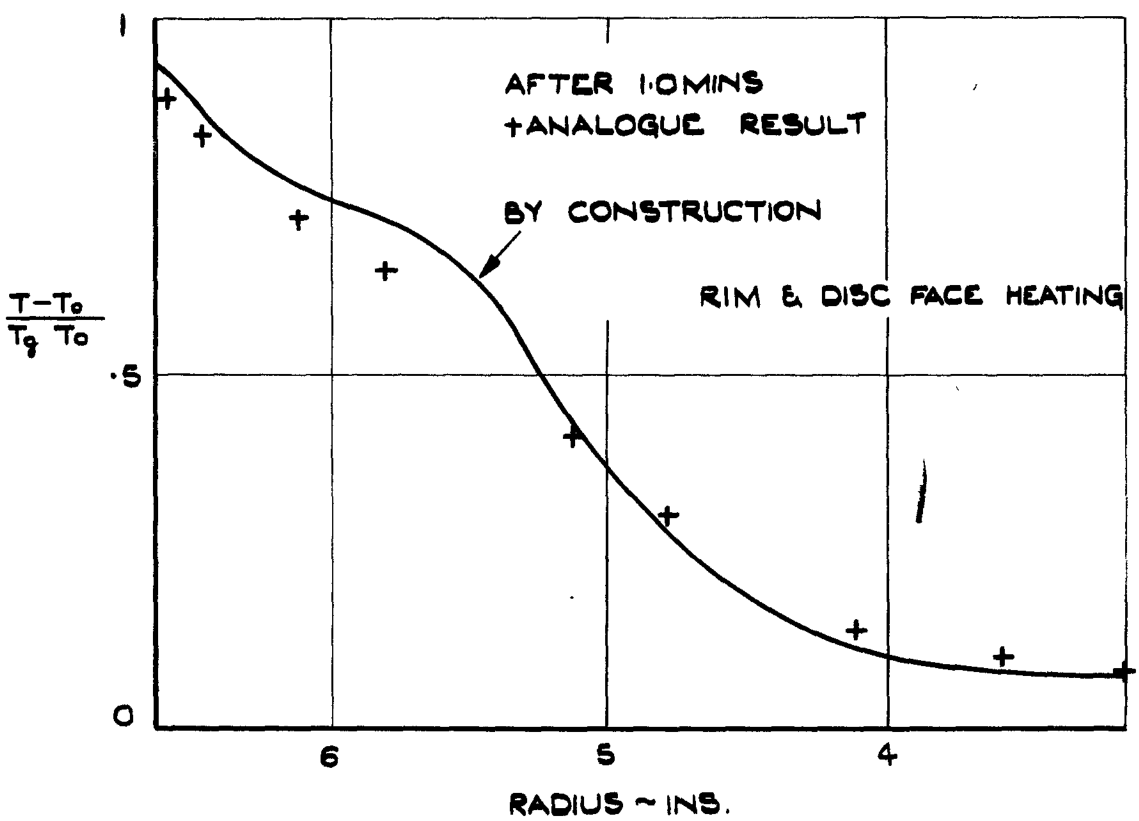


FIG. 12 TEMPERATURE DIFFERENCE ($T_{MEAN} - T_{BORE}$) DURING HEATING TRANSIENT





COMPARISON OF TRANSIENT TEMPERATURE OBTAINED BY CONSTRUCTION AND ANALOGUE

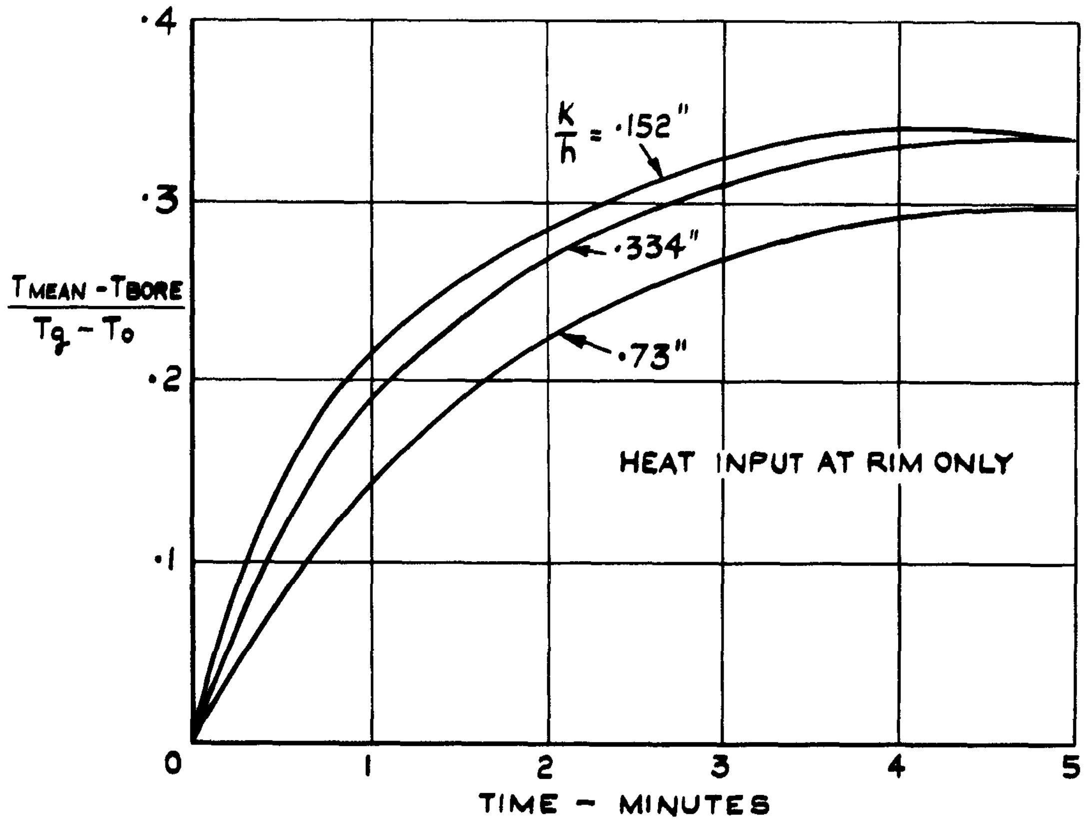


FIG.14. EFFECT OF CHANGING HEAT TRANSFER COEFFICIENT AT RIM ON TEMPERATURE DIFFERENCE.

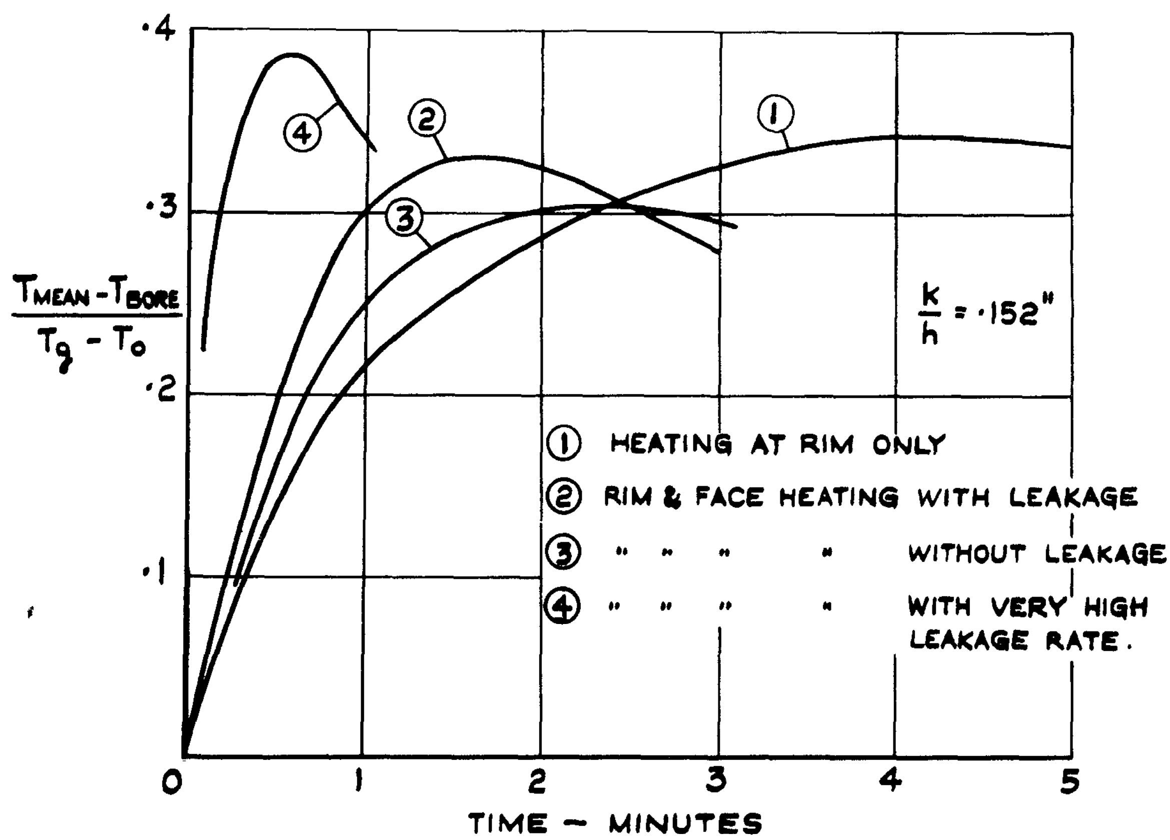
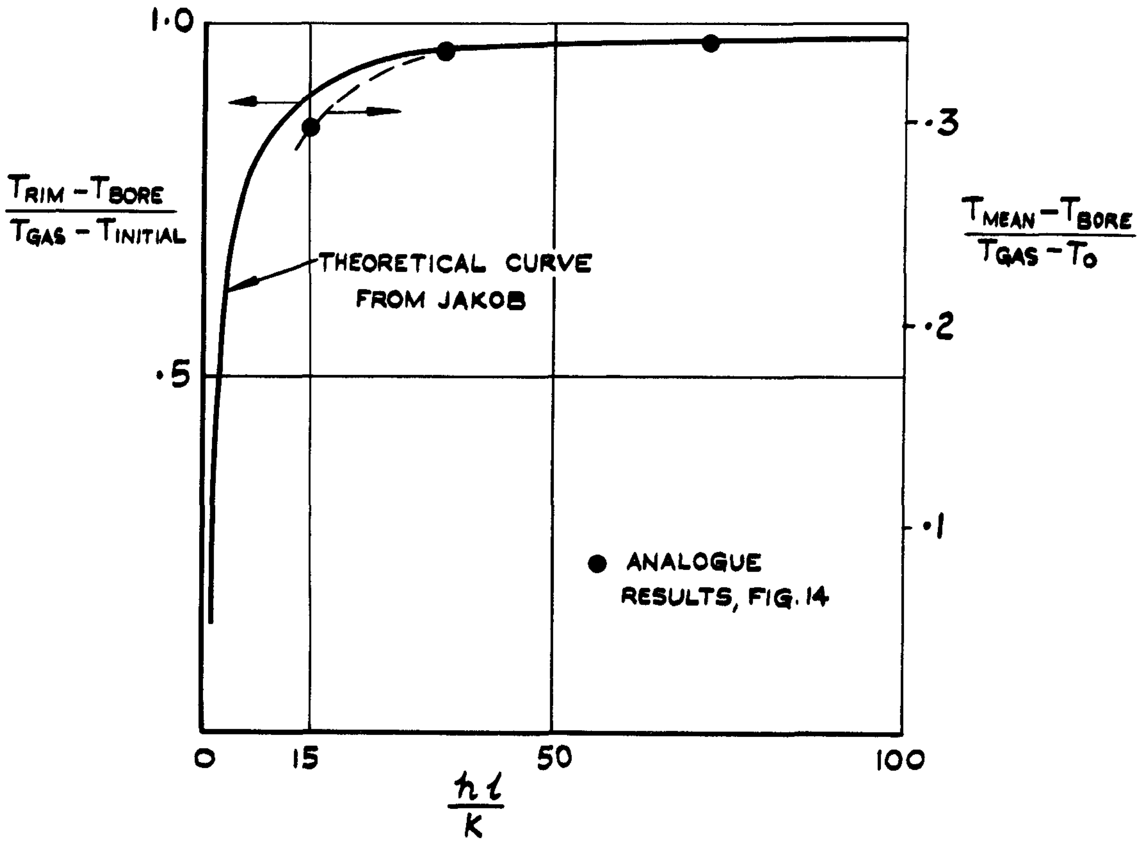


FIG.15 EFFECT OF HEATING AT FACE OF DISC ON TEMPERATURE DIFFERENCE.



FOR COMPRESSOR DISC

l = 2·5 INCHES

h = (h AT RIM) × 4·35

MAXIMUM TEMPERATURE DIFFERENCE AS FUNCTION OF DISC RIM HEAT TRANSFER PARAMETER.

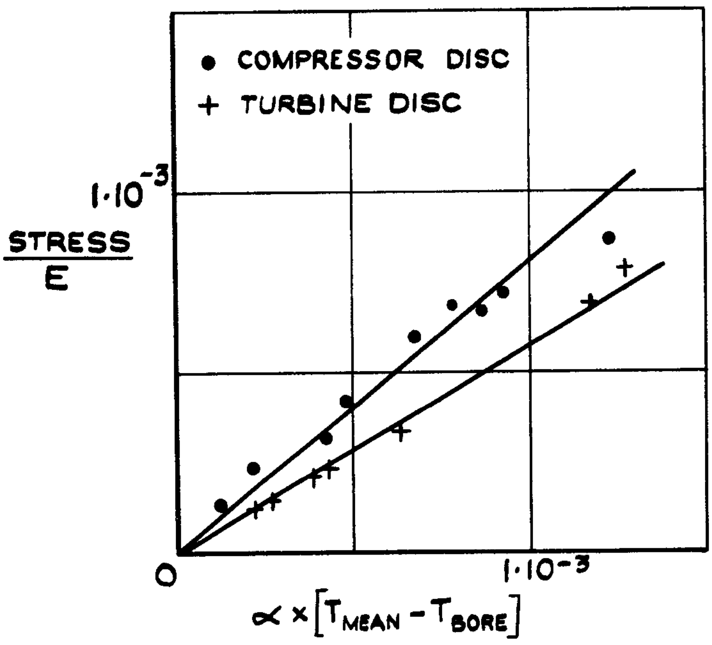


FIG. 17 RELATIONSHIP BEWEEN THERMAL STRESS PRODUCED AND TEMPERATURE DIFFERENCE.

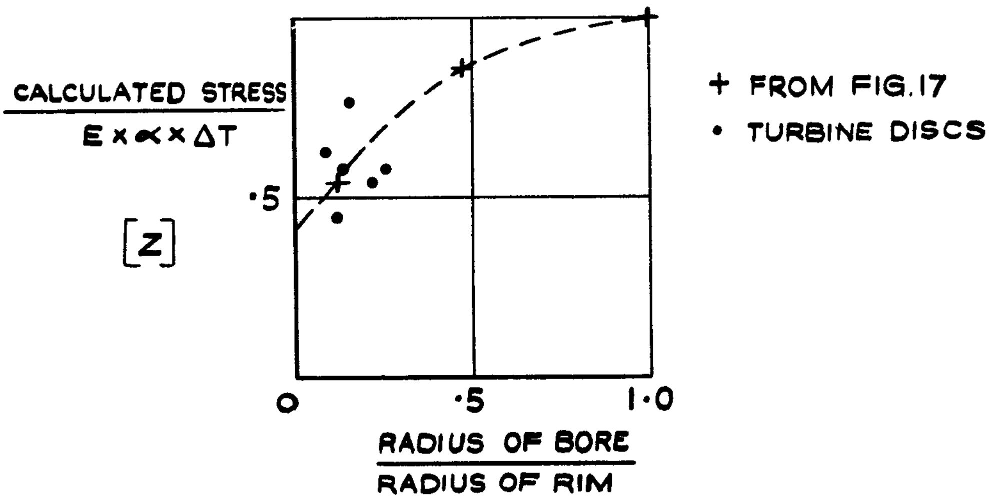
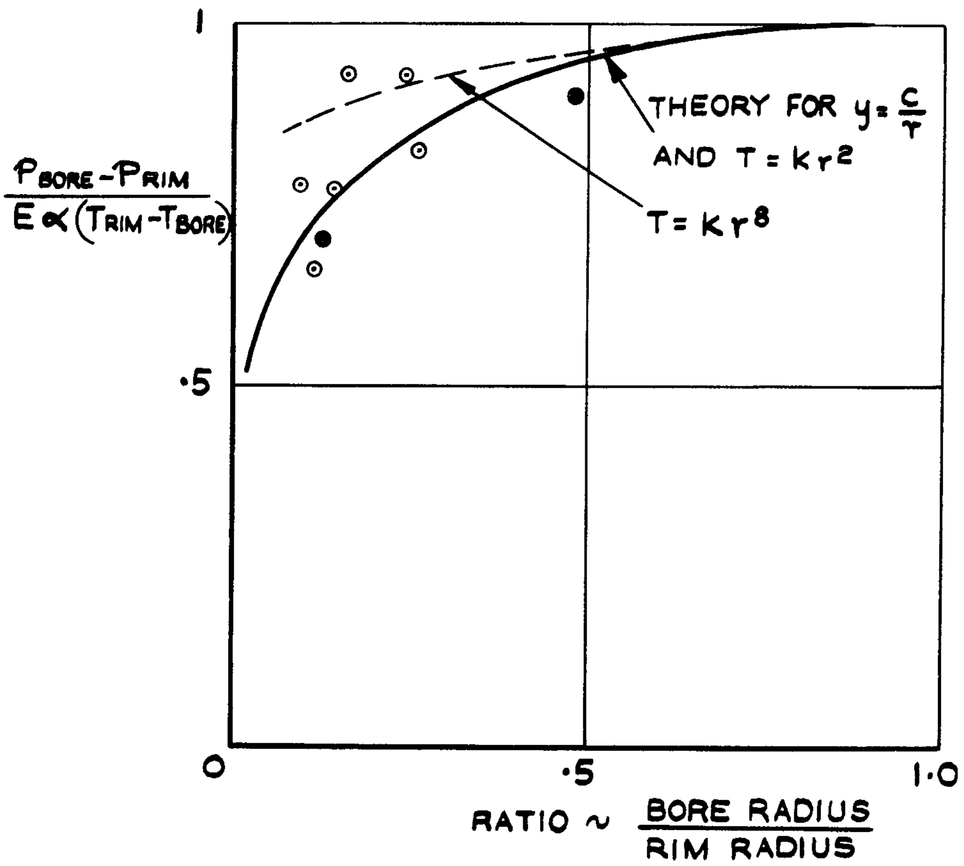


FIG. 18 PROPORTIONALITY BETWEEN THERMAL STRESS AND STRESS BASED ON TEMPERATURE DIFFERENCE.

FIG. 19



RANGE OF THERMAL STRESS IN A DISC OF VARIABLE THICKNESS

A.R.C. C.P. No. 586. May, 1961
621.438-253:539.319
Cox, M.

THE ESTIMATION OF TRANSIENT TEMPERATURE DISTRIBUTIONS AND THERMAL STRESSES IN TURBINE AND COMPRESSOR DISCS

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It is concluded, for the particular case evaluated, that the temperature difference producing thermal stress at the disc bore is about one third of the difference between the initial and final steady-state disc temperatures. Some results obtained by an electrical analogue are presented which indicate the effects of a change in heat transfer coefficient and in the manner in which the disc receives heat over the disc face.

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