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Some Additional Notes on the Derivation of Airworthiness Performance Climb Standards

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Summary.—After the publication of a report (R. & M. 2631[1]) on the derivation of airworthiness performance climb standards, various subsidiary points raised in the course of discussions were examined. Some of these have been collected together in the present note. They are in the nature of elaborations of the original method and include a refined method of deriving the take-off climb standard, a method of treating interdependence of engine failure and a method for including the effect of sideslip in the margin allowed for pilotage errors. The main principles set forth in the earlier report remain unaffected.

1. *Introduction.*—After the publication of a report (R. & M. 2631[1]) on the derivation of airworthiness performance climb standards, various subsidiary points raised in the course of discussions were examined. These are collected together in the present note for convenience of reference. They are in the nature of elaborations of the original method, the main principles set forth in the earlier report remaining unaffected.

2. *A Refined Method of Deriving the Take-off Climb Standard.*—2.1. *The Criterion of Climb Performance in a Stage.*—In the method of R. & M. 2631[1], an incident was regarded as occurring in a stage if the climb gradient fell below the prescribed datum value anywhere within the stage. A more precise viewpoint for some purposes would be to regard an incident as occurring only if the average climb gradient throughout the stage falls below the datum; this is particularly applicable to the take-off climb stage where clearance of ground obstacles is the essential consideration. If this new criterion is adopted, the point in the stage at which an engine fails becomes an important factor, and this report gives a simple treatment to illustrate the method and the magnitude of the effects.

2.2. *Statement of the Problem.*—Referring to Fig. 1, suppose OB represents the flight path of an aeroplane with all engines operating and OC the path of the same aircraft with one engine inoperative. Suppose NM represents an obstruction; then if γ_1, the climb gradient with one engine inoperative, is less than γ_c, the slope of ON, it does not necessarily follow that an incident will occur after an engine failure in this stage; for instance the path OPQ for an engine failure at P clears the obstacle. It follows that whether or not an incident takes place may depend on where the engine fails, and the distribution of the point of failure in the stage must be taken into account in finding the incident rate.

* A. & A.E.E. Report Tech. 61, received 7th November, 1950.

Let us assume that engine failures are uniformly distributed in the stage. An incident will in fact occur when the flight path OPQ fails to clear N, but the analysis may be simplified by making the pessimistic assumption that an incident occurs when Q falls below D. If we include the usual datum gradient clearance, γ_d, the condition for an incident will be given by

$$\gamma_a < \gamma_c + \gamma_d$$

where γ_a is the slope of OQ. The problem is then to find the probability distribution of γ_a, from which the climb standard $\bar{\gamma}_1$ corresponding to a prescribed incident probability may be derived.

2.3. *The Probability Distribution of Mean Climb Gradient* γ_a.—Write OP/OB $= \lambda$ and suppose the climb gradients all engines operative, γ_0, and one engine inoperative, γ_1, are related by

$$\gamma_0 = p + q\gamma_1$$

where p and q are constants.

Then, referring to Fig. 1,

$$AQ \simeq \gamma_0 OP + \gamma_1 PQ$$

so that

$$\begin{aligned} \gamma_a &\simeq \lambda\gamma_0 + (1-\lambda)\gamma_1 \\ &= \lambda(p + q\gamma_1) + (1-\lambda)\gamma_1 \\ &= p\lambda + (q-1)\lambda\gamma_1 + \gamma_1 \ldots \end{aligned} \qquad (1)$$

Let $f_1(\gamma_a)$ be the frequency function for the distribution of γ_a,

$f_2(\lambda)$ be the frequency function for the distribution of λ,

$f_3(\gamma_1)$ be the frequency function for the distribution of γ_1.

Then

$$f_1(\gamma_a)\, d\gamma_a = \int f_2(\lambda)\, d\lambda\, f_3(\gamma_1)\, d\gamma_1,$$

the integral being taken over either $d\lambda$ or $d\gamma_1$ subject to the relation (1). Choosing γ_1 as the integration variable,

$$f_1(\gamma_a)\, d\gamma_a = \int_{-\infty}^{\infty} f_2\left(\frac{\gamma_a - \gamma_1}{p + (q-1)\gamma_1}\right) f_3(\gamma_1)\, \frac{d\gamma_a}{d\gamma_a/d\lambda}\, d\gamma_1. \qquad (2)$$

Define the distribution function $F(x)$ of each distribution by[2]

$$F(x) = \int_{-\infty}^{x} f(x)\, dx,$$

so that $F_1(\gamma_a)$, for example, is the probability of the climb gradient falling below γ_a.

Then from (2), by integrating γ_a from $-\infty$ to γ_a,

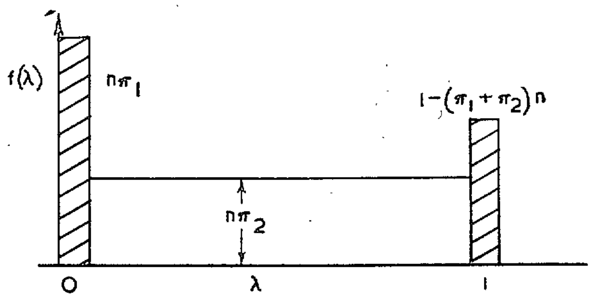
$$F_1(\gamma_a) = \int_{-\infty}^{\infty} F_2\left(\frac{\gamma_a - \gamma_1}{p + (q-1)\gamma_1}\right) f_3(\gamma_1)\, d\gamma_1. \qquad (3)$$

To evaluate $F_1(\gamma_a)$ further, we need to know the distribution function F_2 of λ. If $n\pi_1$ is the probability of one engine having failed before reaching O and $n\pi_2$ the probability of one engine failing along OB (assumed equally likely to happen at any point of OB), the distribution of λ will be:—

between $\lambda = \infty$ and $\lambda = 1$, $\quad F_2(\lambda) = 1$

$\lambda = 1$ and $\lambda = 0$, $\quad F_2(\lambda) = n(\pi_1 + \lambda\pi_2)$

$\lambda = 0$ and $\lambda = -\infty$, $F_2(\lambda) = 0$.



The frequency function of λ.

The corresponding values of γ_1 for a given γ_a give, using (1):—

$$\left.\begin{array}{llll} \text{between } \gamma_1 = -\infty & \text{and } \gamma_1 = \dfrac{\gamma_a - p}{q}, & F_2 = 1, \\ \gamma_1 = \dfrac{\gamma_a - p}{q} & \text{and } \gamma_1 = \gamma_a, & F_2 = n(\pi_1 + \lambda\pi_2), \\ \gamma_1 = \gamma_a & \text{and } \gamma_1 = \infty, & F_2 = 0. \end{array}\right\} \quad \ldots \quad (4)$$

Hence, using (4) we can divide the right-hand side of (3) into the three sections, giving

$$F_1(\gamma_a) = \int_{-\infty}^{(\gamma_a - p)/q} f_3(\gamma_1)\,d\gamma_1 + \int_{(\gamma_a-p)/q}^{\gamma_a} n(\pi_1 + \lambda\pi_2) f_3(\gamma_1)\,d\gamma_1 + \int_{\gamma_a}^{\infty} 0 \,.\, f_3(\gamma_1)\,d\gamma_1$$

$$= F_3\left(\frac{\gamma_a - p}{q}\right) + \int_{(\gamma_a-p)/q}^{\gamma_a} n\left\{\pi_1 + \frac{\gamma_a - \gamma_1}{p + (q-1)\gamma_1}\pi_2\right\} f_3(\gamma_1)\,d\gamma_1$$

i.e.,

$$F_1(\gamma_a) = F_3\left(\frac{\gamma_a - p}{q}\right) + n\pi_1\left\{F_3(\gamma_a) - F_3\left(\frac{\gamma_a - p}{q}\right)\right\}$$

$$+ n\pi_2\int_{(\gamma_a-p)/q}^{\gamma_a} \frac{\gamma_a - \gamma_1}{p + (q-1)\gamma_1} f_3(\gamma_1)\,d\gamma_1$$

$$= (1 - n\pi_1)F_3\left(\frac{\gamma_a - p}{q}\right) + n\pi_1 F_3(\gamma_a)$$

$$+ n\pi_2\int_{(\gamma_a-p)/q}^{\gamma_a} \frac{\gamma_a - \gamma_1}{p + (q-1)\gamma_1} f_3(\gamma_1)\,d\gamma_1 \,. \quad \ldots \quad (5)$$

Now γ_1 is normally distributed with known standard deviation, so that F_3 can be evaluated for any mean $\bar{\gamma}_1$ at once from tables. The third term must be evaluated by graphical integration.

2.4. *Determination of Climb Standard* $\bar{\gamma}_1$.—We have to solve (5) to give the value of the mean climb gradient $\bar{\gamma}_1$ that corresponds to a given prescribed value of the incident probability $F_1(\gamma_a)$ for $\gamma_a = \gamma_c + \gamma_d$. The only method is successive approximation, *i.e.*, choose a $\bar{\gamma}_1$ and compute the value of $F_1(\gamma_a)$ corresponding to it and then repeat the process with a better estimate. The final value can be obtained by interpolation. A worked example for a four-engined aircraft is given in Appendix I to illustrate the method. The resulting climb standard, using the values of R. & M. 2631[1] is

$$(0\cdot5 + 8\cdot2\bar{D}/\bar{W}) \text{ per cent}$$

additional to the clearance gradient. This compares with the standard

$$(0\cdot5 + 13\cdot6\bar{D}/\bar{W}) \text{ per cent}$$

resulting from the assumptions made in R. & M. 2631[1].

2.5. *Discussion.*—Comparison of the two standards shows that the use of the more refined method relaxes the standard by about $4 \cdot 4\bar{D}/\bar{W}$ per cent. In this simple treatment, cases with two engines dead have not been considered, because from previous work it can be shown that their effect is very small in the take-off climb case. At most the above requirement would be increased by $0 \cdot 2\bar{D}/\bar{W}$ per cent. Another simplification we may here note is that a common starting (50 ft height) point has been assumed irrespective of whether an engine has failed during take-off or not. Since the main contribution to incidents occurs from such cases, it would be most reasonable to assume the 50 ft height point appropriate to an engine cut take-off and this would be somewhat pessimistic.

3. *Interdependence of Engine Failure.*—In the method of R. & M. 2631[1], one of the assumptions made was that engine failures were independent, and the probabilities of two or more engines being inoperative were deduced from the single engine failure rate on this basis. It has been asked whether the method could make allowance for some degree of interdependence of engine failures. Such allowance would be required if evidence were found that the failure of a second engine was sometimes a direct consequence of the failure of the first, within the scope of events for which the requirements provide. This section indicates the general method of treating this problem.

The general case of an aircraft with n engines would lead to some heavy algebra and in the absence of experimental evidence there is little to guide us as to what laws of interdependence to assume. It might well be that the form of the correlation between engines would be dependent on the particular installation, or alternatively one might assume that only adjacent engines interact. As an illustration of the method of treatment, an example is given of a four-engined aircraft where the failure of one engine may cause the failure of a second engine on the same side (with probability q), but has no influence on the failure of engines on the opposite side.

This example is worked out in detail in Appendix II, where expressions are derived for the π_{rs}, the probability of s or more engines failing during a stage when r engines are inoperative on entering the stage. These expressions give π_{rs} in terms of π, the engine failure rate and q, the interdependence probability defined above, and would for this particular case be substituted in the appropriate equations of R. & M. 2631[1] as indicated in Appendix II. It will be noted that if $q = 0$ (*i.e.*, engine failures independent), they reduce to the expressions for π_{rs} derived in R. & M. 2631[1]. If $q = 1$ (complete interdependence), the expressions approximate closely to those for a twin engined aircraft with independence and twice the failure rate. In the general case, the expressions can be simplified somewhat by neglecting third-order terms and higher as in section 7.2 of Part II of R. & M. 2631[1], but further approximation cannot be made in the absence of any evidence as to the order of q if it differs from zero.

4. *Allowance in the Pilotage Margin for the Effect of Sideslip.*—4.1. *The Problem.*—In R. & M. 2631[1], allowance for the following sources of pilotage error was included:—

(a) Forward speed

(b) Flap settings

(c) Engine control settings.

It has been suggested that allowance should also be made for the additional drag incurred by inaccurate flying resulting in sideslipping.

To include this effect, let us assume a normal distribution of angle of sideslip, β, and determine the relationship between β and $\Delta D/D$. Since positive and negative values of β give positive values of $\Delta D/D$ the resulting distribution of climb gradient will not, of course, be normal.

4.2. *The Effect of Sideslip on Climb Gradient.*—If we assume

$$\frac{\Delta C_{DZ}}{C_{DZ}} = K\beta^2 ,$$

where K is a constant and C_{DZ} is the profile-drag coefficient, then the second and third terms in the expression for the climb gradient given in equation (4) (Part II) of R. & M. 2631[1] become

$$-\frac{\bar{D}}{\bar{W}}\left[\left(\frac{D_{\bar{m}}}{\bar{D}}\right)\left(\frac{V_i}{\bar{V}_i}\right)^2\left(\frac{W}{\bar{W}}\right)^{-1}\right](1 + K\beta^2)$$

and $$-\frac{\bar{C}_{L\,\max}}{\pi A r}\left[\left(\frac{V_i}{\bar{V}_i}\right)^{-2}\left(\frac{W}{\bar{W}}\right) - \left(\frac{V_i}{\bar{V}_i}\right)^2\left(\frac{W}{\bar{W}}\right)^{-1}(1 + K\beta^2)\right].$$

The additional term due to sideslip in the expression for the climb gradient is thus

$$\gamma_q = K\beta^2\left(\frac{V_i}{\bar{V}_i}\right)^2\left(\frac{W}{\bar{W}}\right)^{-1}\left\{\frac{\bar{C}_{L\,\max}}{\pi A r} - \frac{\bar{D}}{\bar{W}}\frac{D_{\bar{m}}}{\bar{D}}\right\}.$$

If we make the simplifying assumptions of R. & M. 2631[1], Part II, section 4.4, this contribution from β alone may be written

$$\gamma_q = -K\beta^2\left\{\frac{\bar{D}}{\bar{W}} - \frac{\bar{C}_{L\,\max}}{\pi A r}\right\}$$

$$= -K\beta^2\frac{\bar{D}}{\bar{W}}(1 - k') \text{ , in the notation of R. \& M. } 2631^1 \quad \ldots \quad (6)$$

4.3. *The Distribution of Climb Gradient.*—Write the climb gradient as

$$\gamma = \gamma_p + \gamma_q$$

where γ_p is the climb gradient in the absence of sideslip and is distributed normally about the mean, $\bar{\gamma}$, as in R. & M. 2631[1]. To obtain the distribution of γ, we have to combine the normal distribution of γ_p with the non-normal distribution of γ_q and we may proceed as in R. & M. 2631[1], Part II, section 4.4. Writing F_p and F_q as the distribution functions of γ_p and γ_q respectively, the distribution function F of γ is given by[2]

$$F(\gamma) = \int_{-\infty}^{\infty} F_p(\gamma - y)F_q'(y)\,dy \quad \ldots \quad (7)$$

where F_q' is the frequency function of γ_q, *i.e.*, the derivative of F_q.

For any $\bar{\gamma}$, the normal distribution function F_p can be obtained from tables, using the expression for the standard deviation derived in R. & M. 2631. The distribution function F_q of γ_q is derived in Appendix III from the expression for γ_q of equation (6) and the assumed normal distribution of β. It is of the form

$$F_q'(\gamma_q) = \sqrt{\left(\frac{a}{\pi}\right)}\, e^{a\gamma_q}/\sqrt{(-\gamma_q)}, \text{ for } \gamma_q < 0 \quad \ldots \quad (8)$$

which is a Pearson Type III distribution, where a is a parameter given by

$$a = \frac{1}{2K(1 - k')\sigma_\beta^2\bar{D}/\bar{W}} , \quad \ldots \quad (9)$$

σ_β being the standard deviation of the normal distribution of β.

For any selected $\bar{\gamma}$, the distribution of γ can then be evaluated by graphical integration of the right-hand side of equation (7).

4.4. *Determination of the Climb Standard* $\bar{\gamma}_1$.—By definition, $F(\gamma)$ is the probability of the climb gradient falling below γ. Hence if in equation (7) we take $\gamma = \gamma_d$, the datum performance, the value of $\bar{\gamma}$ that gives $F(\gamma_d)$ equal to the required incident probability is the climb standard $\bar{\gamma}_1$. We must proceed by successive approximation as in section 2.4. We may try first the value $\bar{\gamma}$ equal to the climb standard excluding sideslip (as worked out in R. & M. 2631[1]) and obtain the probability $F(\gamma_d)$: then try a second $\bar{\gamma}$ to bracket the desired value of $F(\gamma_d)$ and obtain the new climb standard $\bar{\gamma}_1$ by interpolation.

A worked example is given in Appendix IV, for the approach case (four-engined aircraft) with $\bar{D}/\bar{W} = 0{\cdot}14$. Equation (6) shows that this is a fairly unfavourable case, as the large $\bar{D}/\bar{W}$ and smaller k' gives a large effect of β on γ. Taking $K = 0{\cdot}0025$ and $\sigma_\beta = 2\frac{1}{2}$ (degrees), the climb margin is increased from $0{\cdot}0133$ to $0{\cdot}0149$, *i.e.*, the standard must be increased from $0{\cdot}137\bar{D}/\bar{W}$ to $0{\cdot}144\bar{D}/\bar{W}$.

REFERENCES

No.	*Author*	*Title, etc.*
1	F. G. R. Cook and A. K. Weaver ..	The Derivation of Airworthiness Performance Climb Standards. R. & M. 2631. July, 1948.
2	M. G. Kendall	*The Advanced Theory of Statistics*, Vol. I (in particular Chapters 1, 6, 10). C. Griffin & Co. Ltd. 1945.

NOTATION

The notation is chosen as far as possible so as to be consistent with that of R. & M. 2631[1].

a	A parameter defined by equation (9)
A	Equivalent aspect ratio
C_{DZ}	Profile-drag coefficient
C_L	Lift coefficient
D	Drag
f	Frequency function (with appropriate suffixes)
F	Distribution function (with appropriate suffixes) defined by $$F(x) = \int_{-\infty}^{x} f(t)\, dt$$
K	A constant
k'	Ratio of induced to total drag
n	Number of engines
p, q	Constants relating climb performance all engines operating and one engine inoperative; also used as suffixes in section 4.
r	$(\bar{V}_i/\bar{V}_{is})^2$
V_i	Aircraft equivalent air speed (with suffix s, the stalling speed)
W	Aircraft weight
β	Angle of sideslip
γ_0	Climb gradient all engines operative
γ_1	Climb gradient one engine inoperative

$\bar{\gamma}_1$ Climb standard one engine inoperative

γ_a Average climb gradient throughout stage

γ_c Average climb gradient required to clear obstacles

γ_d Datum climb gradient

γ_p Climb gradient in absence of sideslip

γ_q Contribution of sideslip to climb gradient

λ Parameter defining the point at which an engine fails

π Probability of an engine failing

π_{rs} The probability of s or more engines failing during a stage when r engines are inoperative on entering the stage

π_t The probability of any one engine failing in stage t

σ Standard deviation (with appropriate suffixes)

APPENDIX I

Worked Example Illustrating the Refined Method of Deriving the Take-off Climb Standard Described Section 2

1. *Data Assumed.*—We shall take the take-off climb case for a four-engined aircraft. Following the notation and taking the values used in R. & M. 2631[1], we have

$$\pi_1 = 0\cdot1116 \times 10^{-4}$$

$$\pi_2 = 0\cdot238 \times 10^{-3} - 0\cdot1116 \times 10^{-4} = 0\cdot227 \times 10^{-3}$$

$$\gamma_d = 0\cdot005 + 0\cdot049\bar{D}/\bar{W}$$

$$\left.\begin{aligned} p &= 0\cdot33\bar{D}/\bar{W} \\ q &= 1\cdot33 \end{aligned}\right\}$$ These values correspond to the assumption of $\Delta D/D = 0$ for the propeller of the inoperative engine and are pessimistic.

A clearance gradient (γ_c) of 3 per cent will be considered, with $\bar{D}/\bar{W} = 0\cdot10$ and incident rate of $1\cdot0 \times 10^{-5}$.

2. *First guess at the standard $\bar{\gamma}_1$.*—As suggested in section 2.4, we make a first guess at $\bar{\gamma}_1$. Try $\bar{\gamma}_1 = 0\cdot048$.

By the method of Appendix VII of R. & M. 2631[1], we find the corresponding standard deviation of γ_1,

$$\sigma = 0\cdot00382\,.$$

We have
$$\gamma_a = \gamma_c + \gamma_d = 0\cdot040,$$

and
$$\frac{\gamma_a - p}{q} = \frac{0\cdot040 - 0\cdot033}{1\cdot33} = \frac{0\cdot007}{1\cdot33} = 0\cdot0053\,.$$

We have to evaluate $F_1(\gamma_a)$ given by equation (5) of section 2.3, giving

$$F_1(0\cdot04) \simeq F_3(0\cdot0053) + 0\cdot4464 \times 10^{-4}\, F_3(0\cdot04)$$

$$+ \int_{0\cdot0053}^{0\cdot040} \frac{0\cdot040 - \gamma_1}{0\cdot033 - 0\cdot33\gamma_1} f_3(\gamma_1)\, d\gamma_1 \times 0\cdot908 \times 10^{-3}$$

$$= T_1 + T_2 + T_3 \text{ say.} \qquad (1)$$

$$\text{For } T_1, \quad t = -\frac{0\cdot0053 + 0\cdot048}{0\cdot00382} \simeq \frac{0\cdot043}{0\cdot0038} > 10\,,$$

so that from tables of the normal distribution,

$$F_3(0\cdot0053) < 10^{-15} \text{ and } T_1 \text{ may be neglected.}$$

$$\text{For } T_2, \quad t = \frac{-0\cdot040 + 0\cdot048}{0\cdot00382} = \frac{0\cdot008}{0\cdot00382} = 2\cdot10\,,$$

giving $F_3(0\cdot04) = 0\cdot018$, from tables,

so that $T_2 = n\pi_1 F_3 = 8\cdot03 \times 10^{-7}$.

T_3 is evaluated by graphical integration to give

$$T_3 = 1\cdot892 \times 10^{-9}\,.$$

Hence from equation (1) of this Appendix,

$$F_1(0\cdot04) \simeq T_1 + T_2 + T_3$$

$$= 8\cdot05 \times 10^{-7}\,.$$

Comparing with the desired value of $1\cdot0 \times 10^{-5}$, we see that the assumed value of $\bar{\gamma}_1$ was too high.

3. *Second guess at* $\bar{\gamma}_1$.—Try $\bar{\gamma}_1 = 0\cdot044$.

For this value, we find that $\sigma = 0\cdot00378$.

T_1 will be small as before.

$$\text{For } T_2, \quad t = \frac{-0\cdot048 + 0\cdot044}{0\cdot00378} = \frac{0\cdot004}{0\cdot00378} = 1\cdot057$$

giving $F_3(0\cdot04) = 0\cdot145$,

and $T_2 = 6\cdot49 \times 10^{-6}$.

By graphical integration

$$T_3 = 2\cdot38 \times 10^{-8}\,.$$

Hence from equation (1) of this Appendix,

$$F_1(0\cdot04) \simeq 6\cdot51 \times 10^{-6}$$

so that the standard is still slightly high.

4. *The Final Result.*—By extrapolation $\bar{\gamma}_1 = 0\cdot0432$ corresponds to the required incident probability of $1\cdot0 \times 10^{-5}$. Thus the climb gradient needed, additional to the clearance gradient, is $0\cdot0132$ which, separating the datum gradient not dependent on $\bar{D}/\bar{W}$, gives a standard (above the clearance gradient) of

$$(0\cdot5 + 8\cdot2\bar{D}/\bar{W}) \text{ per cent.}$$

APPENDIX II

An Example of the Interdependence of Engine Failure

In deriving the expressions for the π_{rs}, *i.e.*, the probability of s or more engines failing during a stage when r engines are inoperative on entering the stage, section 6.3 of Part II of R. & M. 2631[1] included the assumption that the failures of individual engines were independent. This appendix illustrates the effect on the π_{rs} of a linkage between failures of engines on the same side.

For simplicity, we shall take the case of a four-engined aircraft, where the failure of one engine may cause the failure of the second engine on the same side.

As in R. & M. 2631[1], let π represent the probability of any one engine failing in the stage, *i.e.*, failure arising from a fault in itself and not from an adjacent engine. Suppose q is the probability of a second engine on the same side failing because of a failure in the first.

First, suppose the aircraft enters the stage with all four engines operating. Then failure of two engines on one side will arise through separate failures of each engine (probability π^2) and failure of the second arising from the first; the probability of one failing of its own accord without the other being $2\pi(1-\pi)$, the probability of one causing the other to fail is $2\pi q(1-\pi)$. Hence, the probability of failure of two engines on one side is

$$2\pi q(1-\pi) + \pi^2 .$$

The probability of a single failure on one side is the probability of one failing of its own accord without the other, $2\pi(1-\pi)$, provided that the one does not cause the other to fail (probability q), *i.e.*,

$$2\pi(1-\pi)(1-q) .$$

The probability of no failures on one side is

$$(1-\pi)^2 .$$

Hence the probability of s engines failing (considering both sides), *i.e.*, $\pi_{0s} - \pi_{0\,s+1}$, is the coefficient of x^s in the expression

$$[\{2\pi(1-\pi)q + \pi^2\}x^2 + 2\pi(1-\pi)(1-q)x + (1-\pi)^2]^2 .$$

Using the π_{rs} notation, this gives

$$\pi_{04} = \{2\pi(1-\pi)q + \pi^2\}^2 \quad \text{(4 engines failing)}$$

$$\pi_{03} - \pi_{04} = 4\pi(1-\pi)(1-q)\{2\pi(1-\pi)q + \pi^2\} \quad \text{(3 engines failing)}$$

$$\left.\begin{aligned}\pi_{02} - \pi_{03} &= 4\pi^2(1-\pi)^2(1-q)^2 \quad \text{(one on each side)}\\ &\quad + 2(1-\pi)^2\{2\pi(1-\pi)q + \pi^2\} \quad \text{(two on one side)}\end{aligned}\right\} \quad \text{(2 engines failing)}$$

$$\pi_{01} - \pi_{02} = 4\pi(1-\pi)^3(1-q) \quad \text{(1 engine failing)}$$

$$1 - \pi_{01} = (1-\pi)^4. \quad \text{(no failures)}$$

Now suppose the aircraft enters the stage with one engine inoperative. For the side with two engines operative, the argument above applies. For the other side, the probability of the remaining engine failing is simply π.

Hence $\pi_{1s} - \pi_{1s+1}$ is the coefficient of x^s in the expansion

$$(\pi x + 1 - \pi)[\{2\pi(1-\pi)q + \pi^2\}x^2 + 2\pi(1-\pi)(1-q)x + (1-\pi)^2]$$

giving

$$\pi_{13} = \pi\{2\pi(1-\pi)q + \pi^2\}$$

$$\begin{aligned}\pi_{12} - \pi_{13} &= 2\pi^2(1-\pi)(1-q) \quad \text{(one on each side)}\\ &\quad + (1-\pi)\{2\pi(1-\pi)q + \pi^2\} \quad \text{(two on one side)}\end{aligned}$$

$$\pi_{11} - \pi_{12} = \pi(1-\pi)^2 + 2\pi(1-\pi)^2(1-q)$$
$$= \pi(1-\pi)^2(3-2q)$$
$$1 - \pi_{11} = (1-\pi)^3 .$$

If the aircraft enters the stage with one engine inoperative on each side, $\pi_{2s} - \pi_{2s+1}$ is the coefficient of x^s in the expansion of

$$(\pi x + 1 - \pi)^2 .$$

If the aircraft enters the stage with two engines on one side inoperative, $\pi_{2s} - \pi_{2s+1}$ is the coefficient of x^s in

$$\{2\pi(1-\pi)q + \pi^2\}x^2 + 2\pi(1-\pi)(1-q)x + (1-\pi)^2 .$$

Finally on entering the stage with three engines inoperative, $\pi_{31} = \pi$.

The π_{rs} derived from these expressions would then replace, in this particular case, the expressions (37) given in R. & M. 2631, Part II, section 6.3. The π_{rs} differences will be substituted directly in equation (39) for ${}_ip_r$ and the derived π_{rs} in equation (36) for Q.

APPENDIX III

The Distribution of the Contribution of Sideslip to Climb Gradient

Equation (6) of section 4.2 gives the contribution of sideslip angle β to climb gradient as

$$\gamma_q = -K\beta^2(1-k')\bar{D}/\bar{W}.$$

If β is normally distributed about zero mean with standard deviation σ_β,

$$f(\beta) = \frac{1}{(2\pi\sigma_\beta{}^2)^{1/2}} \exp(-\beta^2/\sigma_\beta{}^2)$$

is the distribution function of β.

Write $\gamma_q = -x = -\beta^2/2a\sigma_\beta{}^2$ say,

where $a = 1/\{2K(1-k')\sigma_\beta{}^2\bar{D}/\bar{W}\}$.

Then the distribution function f_x of x is given by

$$f_x(x)\,dx = f(\beta)\,d\beta = \frac{1}{(2\pi\sigma_\beta{}^2)^{1/2}}\, e^{-ax}\, d\beta$$

$$= \frac{1}{(2\pi\sigma_\beta{}^2)^{1/2}}\, e^{-ax}\, (2a\sigma_\beta{}^2)^{1/2}\, \tfrac{1}{2}x^{-1/2}\, dx \text{ for positive } \beta ,$$

since for $\beta > 0$, $\beta = (2a\sigma_\beta{}^2)^{1/2}x^{1/2}$.

Similarly for $\beta < 0$, $\beta = -(2a\sigma_\beta{}^2)^{1/2}x^{1/2}$, and $f_x(x)\,dx = -f(\beta)\,d\beta$, and we get a similar result.

Hence, combining the two contributions from $-\beta$ and β to x,

$$f_x(x)\,dx = 2\frac{1}{\pi^{1/2}}\, e^{-ax}a^{1/2}\, \tfrac{1}{2}x^{-1/2}\, dx$$

or the distribution function

$$f_x(x) = \left(\frac{a}{\pi}\right)^{1/2} e^{-ax}/x^{1/2},\ x > 0,$$

which is a Pearson Type III distribution[2].

Reverting to the variable $\gamma_q = -x$, the distribution function f_q of γ_q is given by

$$f_q(\gamma_q) = \left(\frac{a}{\pi}\right)^{1/2} e^{a\gamma_q}/(-\gamma_q)^{1/2}, \text{ for } \gamma_q < 0 .$$

APPENDIX IV

An Example of the Effect on Climb Gradient Standard of Including Sideslip

1. *Distribution of* γ_q.—We shall take the approach case for a four-engined aircraft, with $\bar{D}/\bar{W} = 0\cdot14$. For the approach, $k' = 0\cdot6$ (*see* R. & M. 2631[1], Part III, section 8.4).

Taking $K = 0\cdot0025$ and $\sigma_\beta = 2\cdot5$ (degrees), equation (6) of section 4.2 gives

$$\gamma_q = -\, 0\cdot00014\beta^2 .$$

Hence the distribution of γ_q is, from equation (8) of section 4.2,

$$F_q'(\gamma_q) = \left(\frac{571\cdot4}{\pi}\right)^{1/2} \exp 571\cdot4\gamma_q/(-\gamma_q)^{1/2} ,$$

since $a = 1/\{2K(1-k')\sigma_\beta{}^2\, \bar{D}/\bar{W}\} = 571\cdot4$.

Hence $F_q'(\gamma_q) = 13\cdot47 \exp 571\cdot4\gamma_q/(-\gamma_q)^{1/2}$.

2. *Derivation of the Climb Standard* $\bar{\gamma}_1$.—For the approach case, using the figures of R. & M. 2631[1], the required probability of an incident, one engine inoperative, is $0\cdot00265$. Taking the datum performance $\gamma_d = 0\cdot0059$, it follows from section 4.4 that we have to find $\bar{\gamma}_1$ such that

$$F(0\cdot0059) = 0\cdot00265 \qquad \ldots \qquad (1)$$

Try first the standard $\bar{\gamma}_1 = 0\cdot0192$ derived in Appendix VII of R. & M. 2631[1] for the case excluding sideslip. For this value, $\sigma_p = 0\cdot00475$. By graphical integration of equation (7) of section 4.3, we obtain

$$F(0\cdot0059) = \int_{-\infty}^{\infty} F_p(0\cdot0059 - y) F_q'(y)\, dy$$
$$= 0\cdot00585 ,$$

so that the effect of including sideslip is to increase the probability from $0\cdot00265$ to $0\cdot00585$.

Try now $\bar{\gamma}_1 = 0\cdot0209$; we find that $\sigma_p = 0\cdot00477$ and by graphical integration we get

$$F(0\cdot0059) = 0\cdot00232 .$$

Hence by interpolation, the solution of equation (1) is

$$\bar{\gamma}_1 = 0\cdot0208 \text{ or } 0\cdot148\bar{D}/\bar{W}.$$

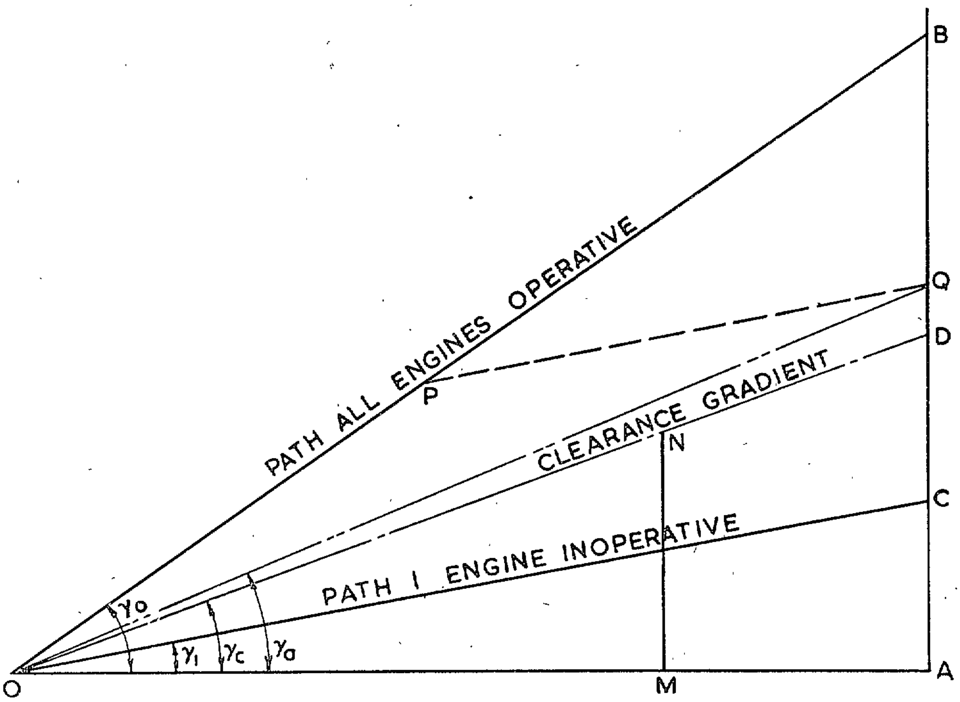


FIG. 1. Illustration of the take-off climb stage.

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