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Forced Flow against a Rotating Disc

By

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1. *Summary.*—The steady motion of an incompressible viscous fluid due to an infinite rotating plane lamina has been considered by Von Kármán[1] and by Cochran[2] : the motion of fluid flowing with axial symmetry towards an infinite stationary plane lamina has been dealt with by Hömann[3]. The present paper deals with the general question of steady irrotational flow with axial symmetry against an infinite rotating lamina, of which the above are two special cases.

2. *Notation.*—Let r, θ, z be cylindrical polar co-ordinates : the lamina is taken to be the plane $z = 0$, and is rotating with constant angular velocity ω about the axis $r = 0$. Let u, v, w be the components of fluid velocity in the directions of r, θ, z, increasing, and let p be the pressure. Axial symmetry is assumed, so that all quantities are independent of θ.

3. *Solution for Non-Viscous Fluid.*—When viscosity is ignored, the solution will be the same for a rotating as for a stationary disc, since the rotation of the disc only affects the fluid velocity through the viscous drag which it exerts. This solution may then be obtained from that for a source in the presence of an infinite plane by letting the distance between source and plane tend to infinity.

If the source is of strength m, at the point (0, 0, a), and the plane is given by $z = 0$, then it is known that the potential of the resulting flow is

$$\phi = m\left(\frac{1}{l} + \frac{1}{l'}\right) \qquad \text{(3.1)}$$

where

$$\left.\begin{aligned} l &= \left\{(a - z)^2 + r^2\right\}^{1/2} \\ l' &= \left\{(a + z)^2 + r^2\right\}^{1/2} \end{aligned}\right\} \qquad \text{(3.2)}$$

Expanding the expression for ϕ in powers of $1/a$, we obtain

$$\phi = \frac{m}{a}\left\{2 + \frac{2z^2 - r^2}{2a^2} + O\left(\frac{1}{a^3}\right)\right\}. \qquad \text{(3.3)}$$

So that if m and a tend to infinity in such a way that m/a^3 remains equal to k, a finite constant, then

$$\phi = k(z^2 - \tfrac{1}{2}r^2) \qquad \text{(3.4)}$$

ignoring the constant term $2ka^2$.

From this,

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial r} = kr \\ w &= -\frac{\partial\phi}{\partial z} = -2kz \end{aligned}\right\} \qquad (3.5)$$

and by Bernouilli's equation

$$\int\frac{dp}{\rho} = -\tfrac{1}{2}q^2 = -\tfrac{1}{2}k^2(r^2 + 4z^2)$$

so that

$$\frac{1}{\rho}(p_0 - p) = \tfrac{1}{2}k^2(r^2 + 4z^2) \qquad (3.6)$$

where p_0 is the pressure at the stagnation point, *i.e.*, at the origin.

This is the complete solution for the non-viscous case ; the result is quoted by Hömann. We shall use it as giving a guide to the form of the solution when viscosity is taken into account.

4. *Equations of Motion for Viscous Fluid.*—The general equations governing the steady motion of the fluid are, assuming axial symmetry

continuity

$$u + r\frac{\partial u}{\partial r} + r\frac{\partial w}{\partial z} = 0 \qquad (4.1)$$

Navier-Stokes

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right) \qquad (4.2)$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right) \qquad (4.3)$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) \qquad (4.4)$$

For boundary conditions, we have that at the disc the velocities of fluid and disc are equal, *i.e.*,

$$u = 0,\ v = \omega r,\ w = 0 \text{ when } z = 0. \qquad (4.5)$$

And far from the disc the velocity will tend to that attained in the non-viscous case, at least for u and v : there is a finite change in w, the axial inflow velocity, due to the radial outflow now caused by the disc's rotation, *i.e.*,

$$u \to kr,\ v \to 0,\ w \to -2kz - c \text{ as } z \to \infty \qquad (4.6)$$

where c is some constant to be determined.

5. We seek a solution in the form

$$u = ru_1(z),\ v = rv_1(z),\ w = w_1(z),$$

$$\frac{1}{\rho}(p_0 - p) = \tfrac{1}{2}k^2r^2 + p_1(z) \qquad (5.1)$$

then the equations become

$$2u_1 + \frac{dw_1}{dz} = 0 \qquad (5.2)$$

$$u_1{}^2 - v_1{}^2 + w_1\frac{du_1}{dz} = k^2 + \nu\frac{d^2u_1}{dz^2} \qquad (5.3)$$

$$2u_1v_1 + w_1\frac{dv_1}{dz} = \nu\frac{d^2v_1}{dz^2} \qquad (5.4)$$

$$w_1\frac{dw_1}{dz} = \frac{dp_1}{dz} + \nu\frac{d^2w_1}{dz^2} \qquad (5.5)$$

with boundary conditions

$$\left.\begin{array}{l} u_1 = 0, \quad v_1 = \omega, \quad w_1 = 0 \text{ when } z = 0 \\ u_1 \rightarrow k, \quad v_1 \rightarrow 0, \quad w_1 \rightarrow -2kz - c \text{ as } z \rightarrow \infty \end{array}\right\} \qquad (5.6)$$

6. The equations may now be made non-dimensional by the substitutions

$$u_1 = \eta N'(\zeta), v_1 = \omega G(\zeta), w_1 = -2(\nu\eta)^{1/2} N(\zeta),$$
$$p_1 = \nu\eta P(\zeta) \qquad (6.1)$$

where $\eta = (k^2 + \omega^2)^{1/2}, \zeta = (\eta/\nu)^{1/2} z$.

Equation (5.2) is then automatically satisfied, and (5.3) (5.4) (5.5), become

$$N'^2 - \Omega^2G^2 - 2NN'' = K^2 + N''' \qquad (6.2)$$

$$2GN' - 2G'N = G'' \qquad (6.3)$$

$$4NN' - 2N'' = P' \qquad (6.4)$$

where $\Omega = \omega/\eta$, $K = k/\eta$ so that $\Omega^2 + K^2 = 1$ and primes denote differentiation with respect to ζ. Boundary conditions now are

$$\left.\begin{array}{l} N = 0, \quad N' = 0, \quad G = 1 \text{ when } \zeta = 0 \\ N \rightarrow K\zeta + \tfrac{1}{2}C, \quad N' \rightarrow K, \quad G \rightarrow 0 \text{ as } \zeta \rightarrow \infty \end{array}\right\} \qquad (6.5)$$

where $C = c/\sqrt{(\nu/\eta)}$

These substitutions, and the resulting equations, reduce with slight modifications to those used by von Kármán and Cochran when $k = 0$, and to those used by Hömann when $\omega = 0$. There is however an error of sign in Cochran's equation (10) which corresponds to (6.4) above; the term corresponding to P' is given a negative sign. This affects the integral of the equation also.

7. *Solution of the Equations.*—Equation (6.4) integrates immediately to give

$$P = 2(N^2 + N') \qquad (7.1)$$

since $P = 0$ when N, N' vanish.

Equations (6.2), (6.3) have been integrated numerically for various given values of $\mu = \omega/k$. The method was to find a series solution for N, G which held for small values of ζ: this involves two unknown parameters a, b. By successive use of forward integration formulæ on values calculated from these series using trial values of a, b, tables of values of the functions were built up for increasing ζ. Then by adjusting the trial values, solutions could be found which satisfied the boundary conditions for large ζ. Finally these solutions were keyed in with values computed from an asymptotic expansion for large ζ, and the three parameters of this expansion were thereby determined.

8. For small ζ the series expansions are

$$N = a\zeta^2 - \tfrac{1}{6}\zeta^3 - \frac{1}{12}\left(\frac{\mu^2}{1+\mu^2}\right)b\zeta^4 - \frac{1}{60}\left(\frac{\mu^2}{1+\mu^2}\right)b^2\zeta^5 \quad \ldots\ldots$$

$$N' = 2a\zeta - \tfrac{1}{2}\zeta^2 - \frac{1}{3}\left(\frac{\mu^2}{1+\mu^2}\right)b\zeta^3 - \frac{1}{12}\left(\frac{\mu^2}{1+\mu^2}\right)b^2\zeta^4 + \frac{1}{30}\left(\frac{1-\mu^2}{1+\mu^2}\right)a\zeta^5 \quad \ldots\ldots$$

$$N'' = 2a - \zeta - \left(\frac{\mu^2}{1+\mu^2}\right)b\zeta^2 - \frac{1}{3}\left(\frac{\mu^2}{1+\mu^2}\right)b^2\zeta^3 + \frac{1}{6}\left(\frac{1-\mu^2}{1+\mu^2}\right)a\zeta^4 + \frac{1}{60}\left(\frac{\mu^2 - 1 - 8\mu^2 ab}{1+\mu^2}\right)\zeta^5$$

$$G = 1 + b\zeta + \tfrac{2}{3}a\zeta^3 + \tfrac{1}{12}(2ab - 1)\zeta^4 - \frac{1}{30}\left(\frac{1+2\mu^2}{1+\mu^2}\right)\zeta^5 \quad \ldots\ldots$$

$$G' = b + 2a\zeta^2 + \tfrac{1}{3}(2ab - 1)\zeta^3 - \frac{1}{6}\left(\frac{1+2\mu^2}{1+\mu^2}\right)\zeta^4 - \frac{2}{15}\left\{2a^2 + \left(\frac{\mu^2}{1+\mu^2}\right)b^2\right\}\zeta^5 \qquad (8.1)$$

The asymptotic solution for large ζ is found by putting

$$\left.\begin{aligned} N &= K\zeta + \tfrac{1}{2}C + x \quad i.e., N' = K + x',\ N'' = x'',\ N''' = x''' \\ G &= y \quad i.e.,\ G' = y',\ G'' = y'' \end{aligned}\right\} \qquad (8.2)$$

where x and y are small and powers higher than the first are neglected. Substituting these in (6.2), (6.3)

$$x''' + 2(K\zeta + \tfrac{1}{2}C)x'' - 2Kx' = 0$$

$$y'' + 2(K\zeta + \tfrac{1}{2}C)y' - 2Ky = 0$$

i.e., x', y both satisfy

$$f'' + 2(K\zeta + \tfrac{1}{2}C)f' - 2Kf = 0. \qquad (8.3)$$

If now $K \neq 0$, write $K\zeta + \tfrac{1}{2}C = t$ and (8.3) becomes

$$K\frac{d^2f}{dt^2} + 2t\frac{df}{dt} - 2f = 0. \qquad (8.4)$$

The complete solution of this is

$$f = R_1 t + R_2 t\int^t u^{-2}\, e^{-u^2/K}\, du \qquad (8.5)$$

where R_1, R_2 are arbitrary constants. To satisfy the condition that f tends to zero as t tends to infinity, R_1 must vanish, and so the solutions for x' and y are

$$\left.\begin{aligned} x' &= At\int_t^\infty u^{-2}\,\mathrm{e}^{-u^2/K}\,du \\ y &= Bt\int_t^\infty u^{-2}\,\mathrm{e}^{-u^2/K}\,du \end{aligned}\right\} \qquad \ldots \quad (8.6)$$

where A, B are arbitrary constants. Since

$$N = K\zeta + \tfrac{1}{2}C + \text{(terms of 1st order)}$$

we may replace $K\zeta + \frac{1}{2}C$ $(= t)$ by N in the small terms, and finally, simplifying the integral, write the asymptotic solution for N', G as

$$\left.\begin{aligned} N' &\sim K - A\left\{\mathrm{e}^{-N^2/K} - 2N/\sqrt{K}\int_{N/\sqrt{K}}^\infty \mathrm{e}^{-u^2}\,du\right\} \\ G &\sim \quad B\left\{\mathrm{e}^{-N^2/K} - 2N/\sqrt{K}\int_{N/\sqrt{K}}^\infty \mathrm{e}^{-u^2}\,du\right\} \end{aligned}\right\} \qquad \ldots \quad (8.7)$$

If now $K = 0$, (8.3) becomes

$$f'' + Cf' = 0$$

so that x', y are both proportional to $\mathrm{e}^{-C\zeta}$, and the asymptotic solutions are

$$\left.\begin{aligned} N' &\sim A_0\mathrm{e}^{-C\zeta} \\ G &\sim B_0\mathrm{e}^{-C\zeta} \end{aligned}\right\} \qquad \ldots \quad (8.8)$$

9. It is of interest to consider the relationship between the two solutions (8.7), (8.8). Both satisfy equation (8.3)

$$f'' + 2(K\zeta + \tfrac{1}{2}C)f' - 2Kf = 0$$

with formal boundary conditions

$$f(K, \zeta) = \phi_0(K) \text{ when } \zeta = \zeta_0$$

$$f(K, \zeta) \to 0 \qquad \text{when } \zeta \to \infty$$

where $\phi_0(K)$ is the value obtained from the integrated solution with which a join is to be made at $\zeta = \zeta_0$. Solutions of (8.3) are

$$f(K, \zeta) = K\zeta + \tfrac{1}{2}C$$

$$f(K, \zeta) = \left(K\zeta + \tfrac{1}{2}C\right)\int_{K\zeta+\frac{1}{2}C}^\infty u^{-2}\,\mathrm{e}^{-u^2/K}\,du$$

so that the characteristic equation is

$$\mathop{\mathrm{Lt}}_{\zeta \to \infty} \begin{vmatrix} K\zeta + \tfrac{1}{2}C & K\zeta_0 + \tfrac{1}{2}C \\ 0 & (K\zeta_0 + \tfrac{1}{2}C) \displaystyle\int_{K\zeta_0+\frac{1}{2}C}^{\infty} u^{-2}\, \mathrm{e}^{-u^2/K}\, du \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad (9.1)$$

This has no root for $0 \leqslant K \leqslant 1$, *i.e.*, there is no characteristic number for K in this range. Then, by a known existence theorem, since the coefficients of equation (8.3) and the boundary conditions are all uniformly continuous functions of ζ and analytic in K the solution is also uniformly continuous in and analytic in K for $0 \leqslant K \leqslant 1$. In particular

$$f(K, \zeta) \to f(0, \zeta) \text{ as } K \to 0$$

so that

$$A \left\{ \mathrm{e}^{-N^2/K} - \frac{2N}{\sqrt{K}} \int_{N/\sqrt{K}}^{\infty} \mathrm{e}^{-u^2}\, du \right\} \to A_0 \mathrm{e}^{-C\zeta} \text{ as } K \to 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9.2)$$

For small K

$$\frac{2N}{\sqrt{K}} \int_{N/\sqrt{K}}^{\infty} \mathrm{e}^{-u^2} du \sim \mathrm{e}^{-N^2/K} \left(1 - \frac{K}{2N^2} + \ldots \right).$$

Substituting in (9.2)

$$A\ \mathrm{e}^{-N^2/K} \frac{K}{2N^2} \to A_0 \mathrm{e}^{-C\zeta}$$

which gives finally

$$A \to A_0 \frac{C^2}{2K} \mathrm{e}^{C^2/4K} (1 + O(K)) . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9.3)$$

This equation shows the variation of A with K when K is small. A similar expression holds for B.

10. *Details of the Numerical Solution.*—The solution was obtained for five values of μ : $\mu = 0, \frac{1}{2}, 1, 2, \infty$. These correspond respectively to $K = 1, \Omega = 0$; $K = 0{\cdot}894$, $\Omega = 0{\cdot}447$; $K = 0{\cdot}707$, $\Omega = 0{\cdot}707$; $K = 0{\cdot}447$, $\Omega = 0{\cdot}894$; $K = 0$, $\Omega = 1$. The case $\mu = 0$ had already been treated by Hömann, but his numerical results were inaccurate and have been corrected. Accurate results for the case $\mu = \infty$ had been produced by Cochran,and are quoted here for completeness.

For a given value of μ, trial values α, β of a, b could be guessed from the evidence of previous solutions or by rough approximations. Using α, β, the first few entries in the tables of N, N', N'', N''', G, G', G'' were computed from the series expansions (8.1) and the equations (6.2), (6.3). The ζ interval was taken as $0{\cdot}1$. Forward numerical integration was then performed using the three-strip formula of Milne

$$y_4 - y_0 = \frac{4h}{3} (2q_1 - q_2 + 2q_3)$$

checked by Simpson's rule

$$y_4 - y_0 = \frac{2h}{3} (q_0 + 4q_2 + q_4)$$

where $q = y'$ and h is the interval. The errors of these two formulæ are almost equal and in opposite senses ($+\ 14/45\ q^{\mathrm{iv}}\ h^5$, $-\ 16/45\ q^{\mathrm{iv}}\ h^5$) so that where the values obtained from them differ, the mean, or more accurately the weighted mean (in the ratio 8 : 7) is a more accurate estimate of $y_4 - y_0$ than either separately.

A running check on alternate values using the Simpson two-strip rule in the form

$$y_{2n} - y_0 = \frac{h}{3}(q_0 + 4q_1 + 2q_2 + 4q_3 + \ldots\ldots + 4q_{2n-1} + q_{2n}$$

$$y_{2n+1} - y_1 = \frac{h}{3}(q_1 + 4q_2 + 2q_3 + 4q_4 + \ldots\ldots + 4q_{2n} + q_{2n+1})$$

was carried out on every entry in the N'' and G' columns for $\zeta \geqslant 1{\cdot}5$; by linking the values obtained with the two entries y_0, y_1 at the very beginning of the table, this check smoothes out most of the tendency to periodic fluctuation which may be introduced by the ordinary method. This fluctuation first occurs in N'' and in G', and if controlled there does not affect the columns produced by integrating these.

The integration was carried on until the functions N', G showed signs of becoming constant, generally at about $\zeta = 2{\cdot}0$. By extrapolation an estimate of their limiting values n_1, g_1, could be obtained. Then similar integrations were carried out using trial values $(\alpha + \varepsilon_1, \beta)$, $(\alpha, \beta + \varepsilon_1)$, $(\alpha + \varepsilon_1, \beta + \varepsilon_1)$ where $|\varepsilon_1|$ was taken generally as 0·1 (in one case 0·01) and the sign of ε_1 was determined by inspection so as to decrease the errors of n_1, g_1. Limits (n_2, g_2), (n_3, g_3), (n_4, g_4) were obtained for these three sets of trial values, and bi-linear interpolation for the given boundary conditions then led to a new trial starting point α_{11}, β_{11}. This gave rise to another set of trial values $(\alpha_{11} + \varepsilon_{11}, \beta_{11})$ $(\alpha_{11}, \beta_{11} + \varepsilon_{11})$, $(\alpha_{11} + \varepsilon_{11}, \beta_{11} + \varepsilon_{11})$ where $|\varepsilon_{11}|$ was $0{\cdot}1\ |\varepsilon_1|$. From the corresponding set of limits better trial values α_{111}, β_{111} could be determined, and in general these were now sufficiently near the true for the required adjustment to find a, b to be obvious.

The process of finding the integrated solutions corresponding to various trial values of the initial parameters could in some cases be shortened. Where numerical solutions N_1, $N_1' \ldots\ldots\ldots G_1''$; N_2, $N_2' \ldots\ldots\ldots G_2''$ had been worked out for initial values α_1, β_1, and α_2, β_2 respectively, then the solution for initial values $L\alpha_1 + M\alpha_2$, $L\beta_1 + M\beta_2$ could be found approximately by

$$N = LN_1 + MN_2$$

$$N' = LN_1' + MN_2'$$

$$G'' = LG_1'' + MG_2''$$

The first few entries for each function were then corrected by accurate calculation from the series expansions ; successive approximations to the true values for all ζ could then be obtained from these by integrating up the approximate values using the Simpson two-strip rule. The successive approximation process converged very rapidly if α_1 β_1 and α_2 β_2 were fairly near points.

Having found a, b and computed the solution based on them, the parameters of the asymptotic series were found simply by substitution. The asymptotic series hold when $(K - N')^2$, G^2 are small enough to be ignored ; *i.e.*, for three decimal accuracy, for ζ such that $K - N'$, $G < 0{\cdot}023$. All values of N', G for $\zeta > \zeta_0$, where ζ_0 defines this limiting condition, provide estimates of the parameters A, B, C ; using the means of these estimates, the functions N', G were computed from the asymptotic series, and their values compared with the integrated solution. In no case did the difference exceed 0·001, so that this method appears to be satisfactory.

11. The case $\mu = 0$ (*i.e.*, $K = 1$, $\Omega = 0$), a stationary disc with forced flow, had already been dealt with by Hömann, who had solved equation (6.2) and obtained the result $a = 0{\cdot}6586$—there is of course no angular velocity. However his method is not satisfactory ; he produces a series expansion up to terms in ζ^{20} and then equates this to an asymptotic solution similar to (8.7) at an apparently arbitrary point $\zeta = 1{\cdot}8$. The equations thus produced determine the five unknown parameters. The solution thereby obtained has a discontinuity at $\zeta = 1{\cdot}8$, immediately obvious on differencing, and indeed to be expected from the method. In fact this value of a leads to a terminal value for N' of 1·015 instead of the required 1. However the correct

value of a could be determined in this case by interpolation for a single variable only (by use of equation (6.2) which now contained only functions of N and its differential coefficients) ; it proved to be $a = 0{\cdot}656$. Using this solution, equation (6.3) was then solved ; for given N, N' (6.3) is a linear differential equation for G, and so determination of b was by linear interpolation. This value of b (1·075) is meaningless in itself, since $G \equiv 0$ for this case, but it is a limiting value for the other solutions. Coefficients for the asymptotic series were obtained as previously described.

$\mu = \infty$ (*i.e.* $K = 0$, $\Omega = 1$) had already been treated by Cochran ; he obtained the values $a = 0{\cdot}255$, $b = -0{\cdot}616$, $A_0 = 0{\cdot}934$, $B_0 = 1{\cdot}208$, $C = 0{\cdot}886$. The asymptotic series here takes the form (8.8) and more terms of it can be derived, as higher powers of the exponential $e^{-c\zeta}$.

$\mu = \frac{1}{2}$, $\mu = 1$, $\mu = 2$ were treated by the process described above, using trial values suggested by the solutions already obtained. Tables and graphs of the values of a, b, A, B, C and of the variation of N, N', G, and P as functions of ζ, are given at the end of this paper.

[Values of the constants and functions were actually obtained using the equations involving the parameters in a less convenient form ; the above choice was only made later, and the results obtained from the previous process have been converted to those which would have been obtained in this way. The converted tables have been checked by integrating up using the Simpson rule].

12. *Frictional Moment Coefficient.*—The constant b is of especial interest since it is proportional to the viscous torque exerted by the liquid on the disc. The shearing stress opposing motion is

$$\rho\nu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \rho\nu\omega r\left(\frac{\eta}{\nu}\right)^{1/2} b$$

so that the total retarding moment on one side of a disc of radius s, neglecting edge effects, is

$$M = \int_0^s \rho\nu\omega\left(\frac{\eta}{\nu}\right)^{1/2} b.\ 2\pi r^3\,dr = \tfrac{1}{2}\pi\rho\,(\nu\eta)^{1/2}\,\omega b s^4. \qquad (12.1)$$

Writing $S = \pi s^2$, the disc area, and $R = s^2\omega^2/\nu\eta$, the Reynolds number, (12.1) becomes

$$M = \tfrac{1}{2}\rho s^3\omega^2 S R^{-1/2}\,b \qquad (12.2)$$

so that we can take b as the torque coefficient

$$b = \frac{R^{1/2}}{\rho S s^3\omega^2}M. \qquad (12.3)$$

Von Kármán and Cochran use a similar form for their special case $K = 0$. The variation of this coefficient with μ is shown in Table 6 and in graph 5.

13. *Inflow Velocity.*—The constant C has also direct physical significance. The disc has two distinct effects on the velocity of the fluid along the axis ; it acts as a centrifugal fan by its rotation and so sucks the fluid towards it, but it also acts as an infinite plane barrier opposing by viscosity the radial flow due to an externally imposed fluid motion. C is a measure of the resultant of these two effects on the fluid's velocity far from the disc ; when C is positive, this resultant effect is a suction, indicating that the ' fan ' effect is predominant. When C is negative the barrier effect dominates. As would be expected, when the forced flow is of small importance relative to the rotation (*i.e.*, for K zero or small) C is positive ; it decreases to zero as K increases, vanishing for $K = 1/4$ approximately, and then decreases further to its final negative value when there is no rotation at all (*i.e.* $K = 1$, $\Omega = 0$). This variation is shown in Table 6 and graph 6.

14. *Boundary Layer Analogue : Displacement and Momentum Thickness.*—The solution found is a solution of the exact Navier-Stokes equations with no boundary layer approximations. Cochran has pointed out that, in the case with which he deals, the solution does in fact bear

out the usual boundary layer assumptions, in that the effect of the motion of the disc on the velocity of the fluid is perceptible only in a layer of thickness $O\{(\nu/\omega)^{1/2}\}$ and the pressure change is $O(\rho\nu\omega)$ through that thickness. The general case also fulfils these conditions. It is convenient for this purpose to consider a stationary disc with a fluid given a rotation as well as a forced motion against it ; it follows immediately that the solution for this case is given by

$$\bar{u} = u, \bar{v} = r\omega - v, \bar{w} = w, \frac{1}{\rho}(\bar{p}_0 - \bar{p}) = \frac{1}{\rho}(p_0 - p) + \tfrac{1}{2}\omega^2 r^2$$

where u, v, w, p, are the velocity and pressure components of the solution already given. Then the 'main stream' velocity tangential to the disc is the resultant of components kr, ωr and so is ηr ; and the fluid velocity becomes indistinguishably different from this for some finite ζ, that is, since $z = (\nu/\eta)^{1/2}\zeta$, in a distance $O\{(\nu/\eta)^{1/2}\}$. The pressure change is $O(\rho\nu\eta)$, so that the assumptions of boundary layer theory are again satisfied.

We can calculate also from this solution analogues to the boundary layer quantities, displacement thickness (δ) and momentum thickness (θ). They are defined by the relations

$$U\delta = \int (U - u)dz$$

$$U^2\theta = \int u(U - u)\,dz$$

where U is the main stream velocity, u the velocity in the boundary layer. In our case these relations become

$$\eta r\delta = \int [\eta r - \{u^2 + (\omega r - v)^2\}^{1/2}]\,dz \qquad (14.1)$$

$$\eta^2 r^2\theta = \int \{u^2 + (\omega r - v)^2\}^{1/2}[\eta r - \{u^2 + (\omega r - v)^2\}^{1/2}]\,dz \qquad (14.2)$$

or since $u = \eta r N'(\zeta), v = \omega r\, G(\zeta), \zeta = \left(\frac{\eta}{\nu}\right)^{1/2} z$

$$\delta = \left(\frac{\nu}{\eta}\right)^{1/2} \int [1 - \{N'^2 + \Omega^2(1 - G)^2\}^{1/2}]\,d\zeta \qquad (14.3)$$

$$\theta = \left(\frac{\nu}{\eta}\right)^{1/2} \int \{N'^2 + \Omega^2(1 - G)^2\}^{1/2}[1 - \{N'^2 + \Omega^2(1 - G)^2\}^{1/2}]d\zeta \qquad (14.4)$$

$(\eta/\nu)^{1/2}\delta$, $(\eta/\nu)^{1/2}\theta$, and the ratio $\delta/\theta = H$ have been calculated for the various μ, and are shown in Table 7 and graph 7. H is usually assumed constant and taken as 2·4 in turbulence calculations; it may be seen that the value here varies from 2·294 to 2·153 as μ increases from zero to infinity.

15. My thanks are due to Prof. W. G. Bickley, who suggested this problem to me and has helped at every stage of the work.

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TABLE 1

Values of G, N, N', P

When $\mu = \infty$
$K = 0$
$\Omega = 1$

ζ	G	N	N'	P
0	1·000	0	0	0
0·1	939	0·003	0·046	0·092
0·2	878	009	084	167
0·3	819	019	114	228
0·4	762	032	136	275
0·5	708	046	154	312
0·6	656	062	166	340
0·7	607	079	174	361
0·8	561	097	179	377
0·9	517	115	181	388
1·0	468	133	180	395
1·1	439	151	177	400
1·2	404	168	173	403
1·3	371	186	168	405
1·4	341	202	162	406
1·5	313	218	156	406
1·6	288	233	148	405
1·7	264	248	141	404
1·8	242	261	133	403
1·9	222	274	126	402
2·0	203	286	118	401
2·1	186	298	111	399
2·2	171	309	104	398
2·3	156	319	097	397
2·4	143	328	091	396
2·5	131	337	084	395
2·6	120	345	078	395
2·8	101	361	068	395
3·0	083	373	058	395
3·2	071	384	050	395
3·4	059	393	042	394
3·6	050	401	036	394
3·8	042	408	031	393
4·0	035	413	026	393
4·2	029	418	022	393
4·4	024	422	018	393

TABLE 2

Values of G, N, N', P

When $\mu = 2$
i.e. $K = 0{\cdot}447$
$\Omega = 0{\cdot}894$

ζ	G	N	N'	P
0	1·000	0	0	0
0·1	920	0·003	0·064	0·130
0·2	842	012	119	238
0·3	765	027	166	333
0·4	692	046	207	418
0·5	622	068	242	493
0·6	556	094	272	562
0·7	494	122	298	626
0·8	437	153	320	687
0·9	383	186	339	747
1·0	335	221	355	808
1·1	291	257	370	873
1·2	251	295	382	938
1·3	216	334	392	1·007
1·4	184	374	401	1·082
1·5	156	414	408	1·159
1·6	131	455	415	1·244
1·7	109	498	420	1·336
1·8	091	540	425	1·433
1·9	076	583	429	1·538
2·0	062	625	433	1·647
2·1	051	668	436	1·764
2·2	041	711	438	1·887
2·3	033	755	440	2·020
2·4	026	800	441	2·162
2·5	021	844	443	2·311
2·6	017	888	444	2·465
2·7	013	932	445	2·627
2·8	010	977	445	2·799
2·9	008	1·022	446	2·981
3·0	006	1·066	447	3·167

TABLE 3

Values of G, N, N', P

When $\mu = 1$
$K = 0{\cdot}707$
$\Omega = 0{\cdot}707$

ζ	G	N	N'	P
0	1	0	0	0
0·1	0·907	0·005	0·089	0·178
0·2	816	017	168	337
0·3	727	038	240	483
0·4	643	065	303	614
0·5	564	098	360	739
0·6	490	137	411	860
0·7	422	180	455	975
0·8	361	228	495	1·094
0·9	305	279	529	1·214
1·0	256	334	559	1·341
1·1	213	391	586	1·478
1·2	175	451	607	1·621
1·3	143	512	625	1·774
1·4	116	576	642	1·948
1·5	093	640	655	2·129
1·6	073	706	665	2·327
1·7	057	774	675	2·548
1·8	044	841	683	2·781
1·9	033	910	689	3·034
2·0	026	979	693	3·303
2·1	019	1·049	697	3·595
2·2	015	1·119	700	3·904
2·3	011	1·189	702	4·231
2·4	007	1·259	703	4·576
2·5	005	1·329	704	4·940
2·6	004	1·400	705	5·330
2·7	003	1·470	706	5·734
2·8	002	1·541	706	6·161
2·9	002	1·611	707	6·605
3·0	001	1·681	707	7·066

TABLE 4

Values of G, N, N', P

When $\mu = \frac{1}{2}$
$K = 0{\cdot}894$
$\Omega = 0{\cdot}447$

ζ	G	N	N'	P
0	1	0	0	0
0·1	0·898	0·006	0·112	0·224
0·2	798	022	214	429
0·3	702	048	307	619
0·4	611	083	391	796
0·5	526	126	466	964
0·6	448	176	533	1·128
0·7	377	232	592	1·292
0·8	314	294	643	1·459
0·9	251	360	688	1·636
1·0	210	433	727	1·829
1·1	169	507	760	2·034
1·2	134	584	788	2·258
1·3	105	664	812	2·506
1·4	082	746	830	2·773
1·5	063	830	846	3·070
1·6	047	916	858	3·394
1·7	035	1·002	868	3·744
1·8	026	1·089	875	4·122
1·9	018	1·176	880	4·526
2·0	013	1·265	885	4·970
2·1	010	1·354	888	5·443
2·2	007	1·443	890	5·944
2·3	005	1·532	892	6·478
2·4	003	1·621	893	7·041
2·5	002	1·710	894	7·636
2·6	002	1·800	894	8·268

TABLE 5

Values of N, N', P When $\mu = 0$, $K = 1$, $\Omega = 0$

ζ	N	N'	P
0	0	0	0
0·1	0·006	0·126	0·253
0·2	025	242	486
0·3	055	349	703
0·4	094	445	908
0·5	143	532	1·104
0·6	200	609	1·298
0·7	265	677	1·494
0·8	335	736	1·697
0·9	412	787	1·912
1·0	492	830	2·145
1·1	577	865	2·397
1·2	665	896	2·677
1·3	756	920	2·985
1·4	849	940	3·322
1·5	944	955	3·693
1·6	1·040	967	4·099
1·7	1·138	975	4·538
1·8	1·235	983	5·019
1·9	1·334	988	5·536
2·0	1·433	992	6·092
2·1	1·533	995	6·686
2·2	1·632	997	7·320
2·3	1·732	998	7·994
2·4	1·832	999	8·706
2·5	1·932	999	9·460

TABLE 6

Values of a, b, A, B, C For various μ, K

μ	∞	2	1	$\frac{1}{2}$	0
a	0·255	0·343	0·468	0·583	0·656
b	−0·616	−0·800	−0·933	−1·025	−1·075
A	A_0=0·934	0·133	0·269	0·358	0·408
B	B_0=1·208	0·567	0·485	0·466	0·421
C	0·886	−0·275	−0·440	−0·526	−0·569
K	0	0·447	0·707	0·894	1

TABLE 7

Values of Displacement and Momentum Thickness Coefficients $(\eta/\nu)^{1/2}\delta$, $(\eta/\nu)^{1/2}\theta$ and $H = \delta/\theta$ for various μ, K

μ	∞	2	1	$\frac{1}{2}$	0
$(\eta/\nu)^{1/2}\delta$	1·143	0·787	0·661	0·596	0·569
$(\eta/\nu)^{1/2}\theta$	0·531	0·348	0·289	0·260	0·248
$H = \delta/\theta$	2·153	2·261	2·284	2·292	2·294
K	0	0·447	0·707	0·894	1

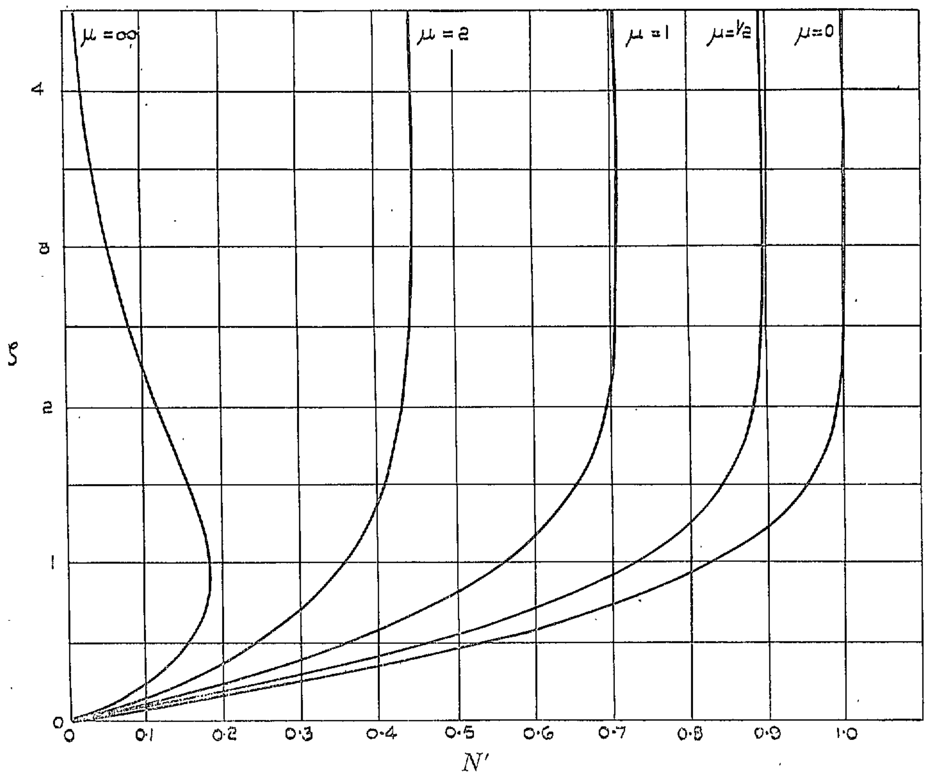

FIG. 1. Variation of N'.

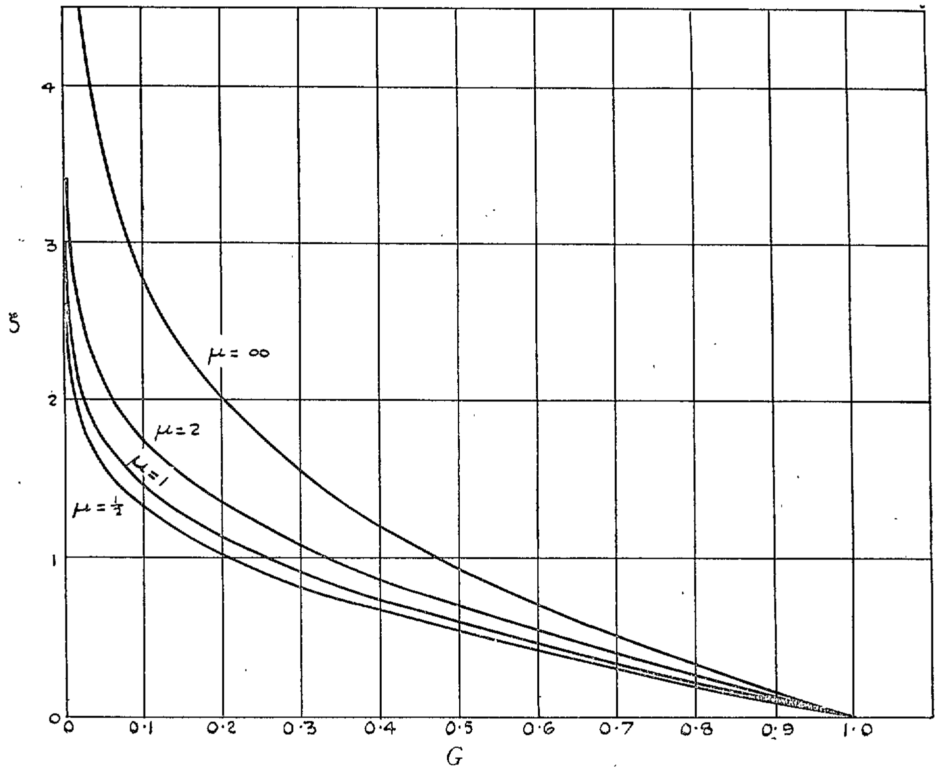

FIG. 2. Variation of G.

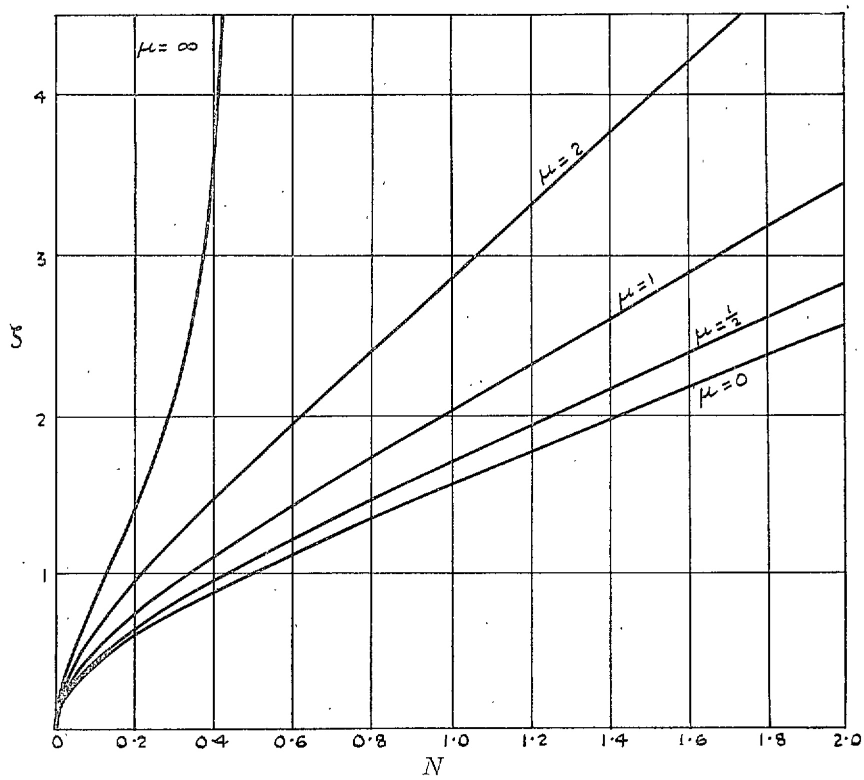

FIG. 3. Vibration of N.

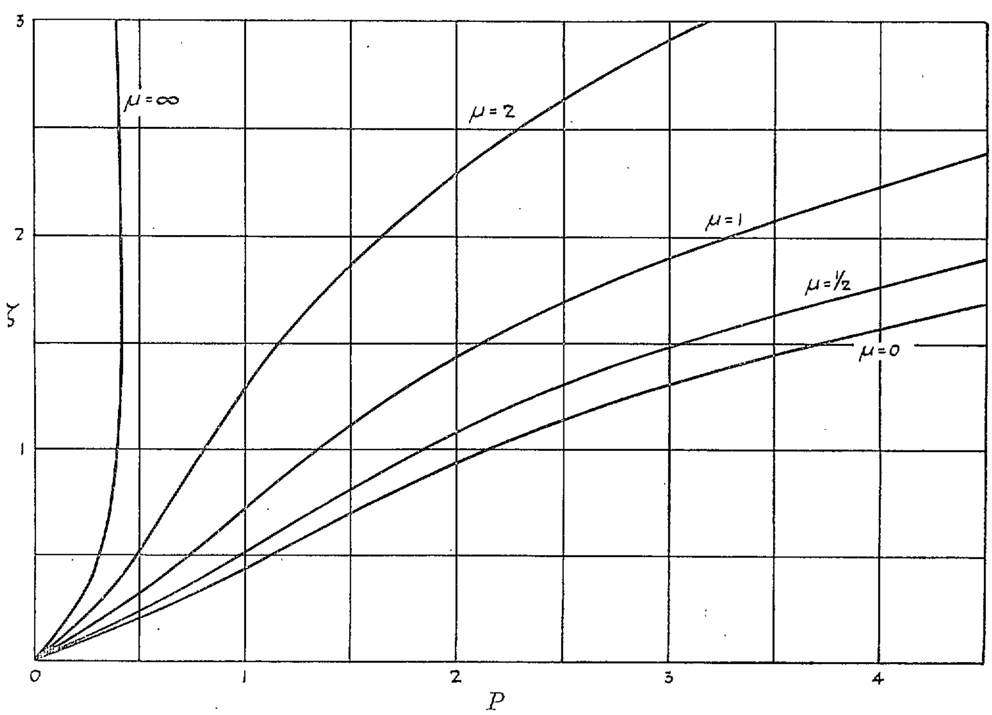

FIG. 4. Variation of P.

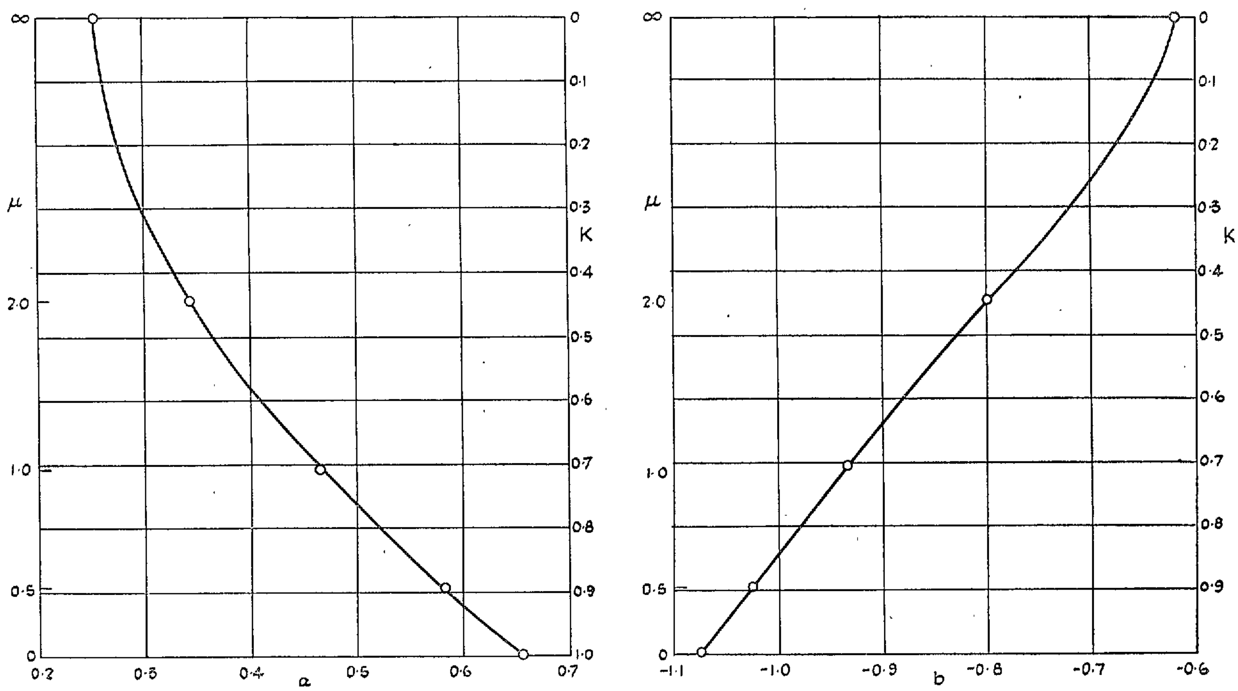

FIG. 5. Variation of a and b.

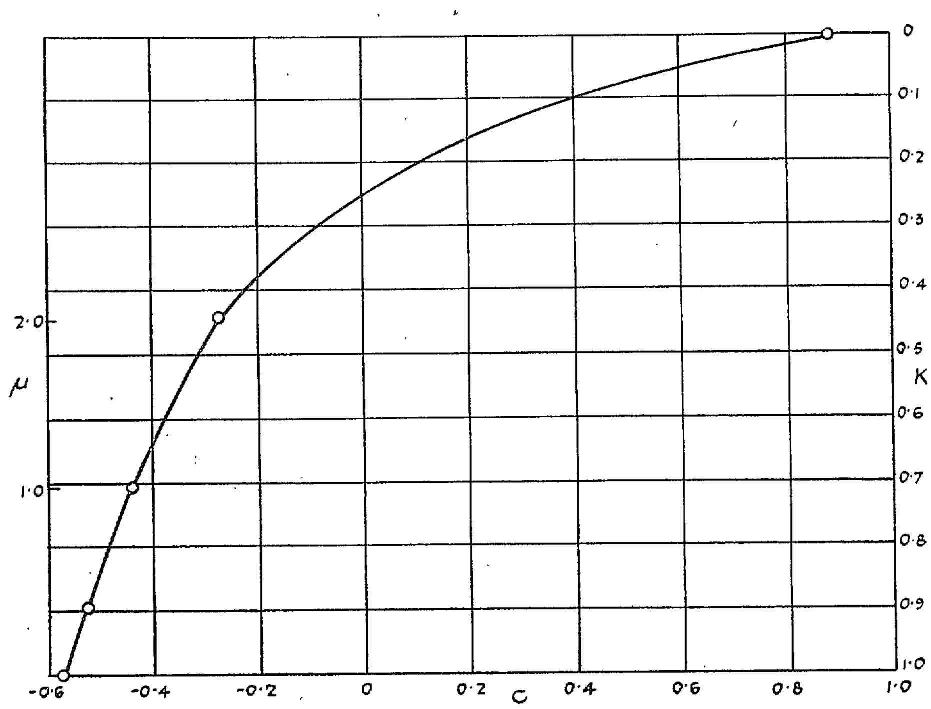

FIG. 6. Variation of c.

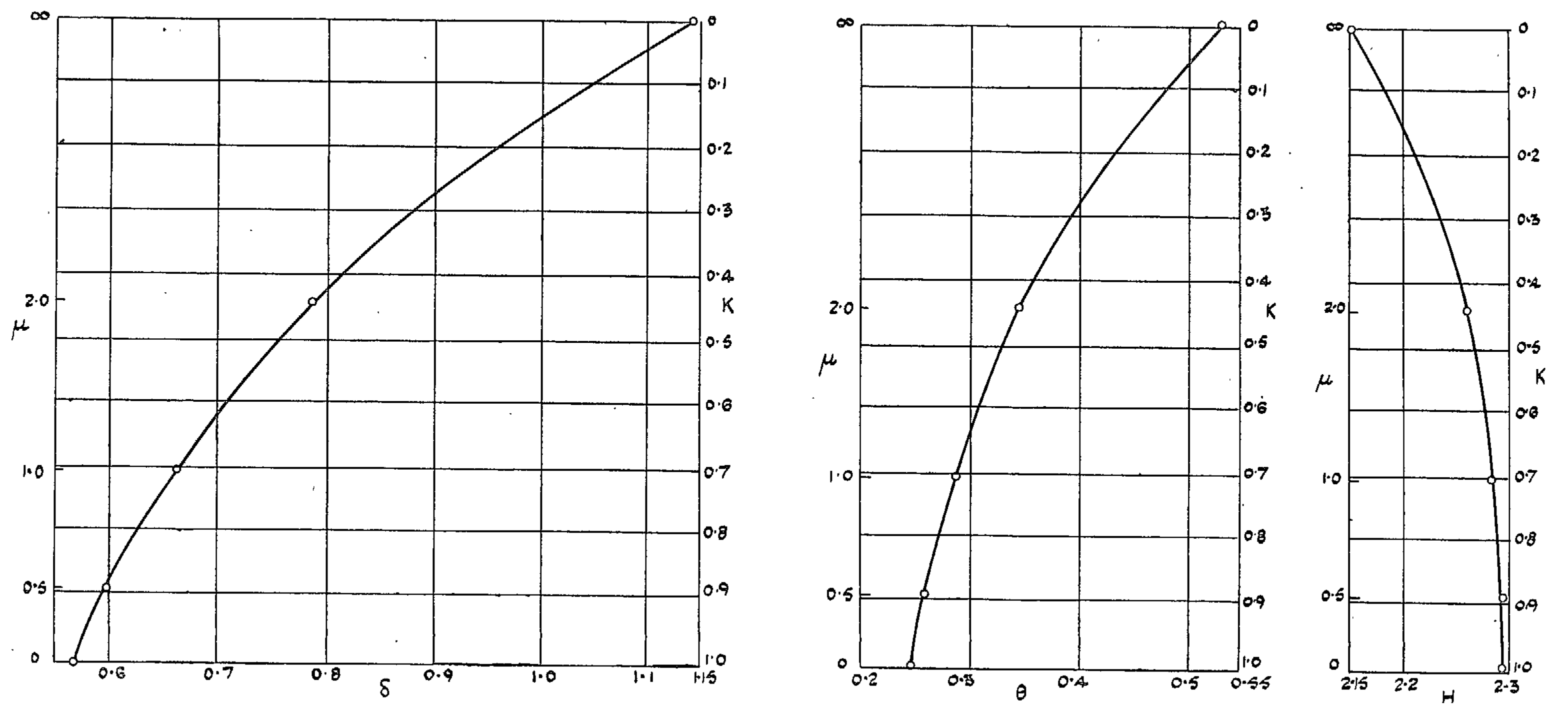

FIG. 7. Variation of δ, θ and H.

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