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Summary.—The problem of estimating flutter and stability derivatives for wings of finite span describing simple harmonic oscillations in compressible flow is considered. It is shown that the problem can be reduced to a similar one for an equivalent wing in incompressible flow. The lateral dimensions of the equivalent wing are $\sqrt{(1 - M^2)}$ times those of the original wing and the frequency of oscillation is increased by the factor $(1 - M^2)^{-1}$ where M denotes the Mach number of the compressible flow. The mode of oscillation is different but related to that of the original wing and leads to a more complicated condition for tangential flow. It is suggested, however, that sufficient accuracy might be obtained by representing the boundary condition to first-order accuracy in the frequency and then solving quite generally the integral equation which determines the velocity potential at the surface of the wing. The comparisons made in Table 1 and Figs. 2a to 2d indicate that the above procedure is reasonably satisfactory in the two-dimensional case. For $M = 0{\cdot}7$, the values of the derivatives given by the formulae derived in this report show fair agreement with the 'exact' results of Refs. 1 and 2 over a wide range of frequency parameter values. Since flutter derivatives for wings of finite span are not usually very sensitive to variations in frequency parameter, the scheme of calculation suggested should be sufficiently accurate for all practical purposes when the combined effects of thickness and viscosity are negligible. It does, however, require a reliable method for calculating derivatives for low aspect ratio wings in incompressible flow, since the aspect ratio of the equivalent wing is $\sqrt{(1 - M^2)}$ times that of the original wing and becomes small for the higher values of M.

1. *Introduction.*—The theory for thin aerofoils of infinite span oscillating in inviscid compressible flow[3] has been fully developed, but for wings of finite span the position is less satisfactory. In the case of supersonic flow, analytical solutions can be derived for certain plan-forms but there is little systematic information on aerodynamic derivatives, and there is still less for subsonic flow. In this paper the subsonic case is considered, and it is shown that the problem of determining flutter and stability derivatives can be reduced to a similar problem for a wing of related plan-form in incompressible flow. If A and f denote the aspect ratio and frequency of oscillation of the original wing, the corresponding values for the equivalent reduced wing in incompressible flow are $A\sqrt{(1 - M^2)}$ and $f/(1-M^2)$ respectively, where M represents the Mach number of the flow over the original thin wing. The boundary condition for tangential flow over the equivalent wing depends on the mode of motion of the original wing and also on M (*see* section 2). If the original wing is oscillating as a rigid body, the corresponding equivalent wing must distort in a particular way in the chordwise direction. For instance, if w denotes the downwash distribution for the actual wing, the corresponding downwash W for the equivalent wing at $M = 0$ is proportional to $w \exp(-i\lambda X)$ where X is a non-dimensional chordwise distance and λ is a function of f and M. Known methods[4,5] for calculating derivatives in the case of incompressible flow can then be used and solutions of the compressible problem obtained indirectly.

The exponential term in the downwash complicates matters, but, fortunately, the evidence from two-dimensional theory given in Table 1 of this report suggests that replacing $\exp(-i\lambda X)$ by $1 - i\lambda X$ does not lead to serious error in the derivative values for a wide range of frequency parameter values. In view of this, it is believed that, as far as the calculation of flutter and stability derivatives is concerned, sufficient accuracy might be given by an approximate method in which the downwash distribution is represented to first order only in the frequency. Since λ is usually multiplied by X, neglecting terms of higher order than λ is equivalent to neglecting terms in X of corresponding order, and hence any distortion in the chordwise direction.

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The scheme requires, however, a reliable method for estimating aerodynamic forces on wings of low aspect ratio, since when M tends to unity the aspect ratio of the equivalent wing tends to zero. For oscillating wings of very low aspect ratios, Garrick's extended form of R. T. Jones' steady theory[6] may be used.

2. *Equations of Motion.*—Let U_0, p_0, ρ_0 be the uniform velocity, pressure, and density respectively, of the airstream in the undisturbed state, and let V_s denote the velocity of propagation of small disturbances caused by introducing an oscillating thin wing into the field of flow. If $U_0 + u$, v, w denote the velocity components of the disturbed flow at a point x, y, z at time t, the linearised form of Euler's equations for the motion are

$$\frac{du}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \quad \frac{dv}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \quad \frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \quad \ldots \qquad (1)$$

where $d/dt \equiv \partial/\partial t + U_0\,\partial/\partial x$, and p is the pressure. Furthermore, since $V_s^2 = dp/d\rho$ the equation of continuity is expressible in the form

$$\frac{1}{V_s^2}\frac{dp}{dt} + \rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0\,. \qquad (2)$$

If ϕ denotes the velocity potential of the disturbance superimposed on the steady flow, then $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$ and $w = \partial\phi/\partial z$. Substitution in (1) and integration then yields

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + U_0\frac{\partial\phi}{\partial x} = -\frac{p - p_0}{\rho_0} + f(t)\,. \qquad (3)$$

At infinity, ϕ and $p - p_0$ tend to zero; hence $f(t) = 0$.

The oscillating wing gives rise to a vortex wake across which there is a discontinuity in the velocity potential. Let ϕ_a and ϕ_b represent the values of ϕ above and below the sheet of discontinuity, representing the wing and its wake. Then, if $k \equiv \phi_a - \phi_b$, it follows from (3) that the lift distribution $\bar{l}(x,y)$ is given by

$$\bar{l}(x,y) = p_b - p_a = \rho_0\left(\frac{\partial k}{\partial t} + U_0\frac{\partial k}{\partial x}\right). \qquad (4)$$

Since there is no discontinuity in the pressure field in the wake, the condition

$$\frac{\partial k}{\partial t} + U_0\frac{\partial k}{\partial x} = 0, \quad (x > x_t) \qquad (5)$$

must be satisfied.

By eliminating p from (2) and (3), it may be shown that ϕ must satisfy the equation

$$\frac{d^2\phi}{dt^2} = V_s^2\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right] \qquad (6)$$

over the whole field of flow and the appropriate condition for tangential flow over the wing. Let the downward displacement at time t of any point x,y on the surface of the wing be denoted by $\zeta(x,y,t)$. Then, since there is no flow normal to the wing, the following condition must be satisfied at the surface of the wing

$$w = \frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} + U_0\frac{\partial\zeta}{\partial x}\,. \qquad (7)$$

The problem is then reduced to one of finding a solution of (6) which satisfies (5) in the wake and (7) on the wing.

3. *Method of Solution.*—In the present report, it is assumed that the wing is describing simple harmonic oscillations of constant amplitude, but the analysis is also applicable to oscillations of

growing amplitude provided that at the time t under consideration the amplitude is small - otherwise the linearised equations of section 2 become invalid. Firstly, the variables x, y, z, t are replaced by the non-dimensional variables X, Y, Z, T defined by

$$x = lX, \quad y = \beta^{-1}lY, \quad z = \beta^{-1}lZ, \quad t = lT/U_0, \qquad (8)$$

where l is a convenient length, and where $\beta \equiv \sqrt{(1 - M^2)}$ with $M \equiv U_0/V_s$. This transformation reduces the lateral dimension of the wing by the factor β.

If f is the frequency of the oscillation, the velocity potential ϕ of the disturbance may be conveniently expressed in the form

$$\phi = l\Phi \, e^{i(\lambda X + \omega T)} \qquad (9)$$

where $\omega = 2\pi fl/U_0$, and $\lambda = M^2.\beta^{-2}\omega$.

By substituting the above expression for ϕ in (4) and writing $K = \Phi_a - \Phi_b$, the following equation for the lift distribution is obtained, namely,

$$\tilde{l}(X, Y) = \rho_0 U_0 \left(\frac{\partial}{\partial T} + \frac{\partial}{\partial X}\right) K \, e^{i(\lambda X + \omega T)}, \qquad (10)$$

$$= \rho_0 U_0 \Gamma \, e^{i(\lambda X + \omega T)}, \qquad (11)$$

where $\Gamma \equiv i\nu K + \partial K/\partial X$ and $\nu = \beta^{-2}\omega$. Since $\tilde{l}(X, Y)$ is zero everywhere in the wake, the condition $\Gamma = 0$, which implies that

$$K(X) = K(X_t) e^{-i\nu(X - X_t)}, \qquad (12)$$

must be satisfied when $X \geqslant X_t$, the value of X at the trailing edge.

It also follows from (6) and (9) that Φ must satisfy the equation

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial Z^2} + \varkappa^2 \Phi = 0 \qquad (13)$$

where $\varkappa = M\nu = M\omega\beta^{-2}$. The corresponding boundary condition at the surface of the equivalent wing is

$$w = \frac{\partial \Phi}{\partial Z} = \frac{w}{\beta} e^{-i(\lambda X + \omega T)}, \qquad (14)$$

where w is defined by (7).

A particular solution of (13) which represents an outward radiating disturbance is $r^{-1} e^{-i\varkappa r}$. By the use of Green's theorem[7], it may then be shown that the general solution for oscillatory motion is given by

$$4\pi\Phi(X_1, Y_1, Z_1) = \iint K(X, Y) \frac{\partial}{\partial Z_1}\left(\frac{e^{-i\varkappa r}}{r}\right) dX \, dY, \qquad (15)$$

where $r \equiv \sqrt{[(X - X_1)^2 + (Y - Y_1)^2 + Z_1^2]}$. The integral is taken over the part of the plane $Z = 0$ representing the equivalent wing and the wake. On differentiation, (15) yields

$$4\pi W(X_1, Y_1, 0) = \iint_{Z_1 \to 0} K \frac{\partial^2}{\partial Z_1^2}\left(\frac{e^{-i\varkappa r}}{r}\right) dX \, dY, \qquad (16)$$

where W on the aerofoil, $Z_1 = 0$, is given by (14) and K is unknown. To determine K, it is convenient to write (16) in the form

$$4\pi(W + I_0) = \iint_{Z_1 \to 0} K \frac{\partial^2}{\partial Z_1^2}\left(\frac{1}{r}\right) dX \, dY, \qquad (17)$$

where

$$4\pi I_0 = \iint_{Z_1 \to 0} K \frac{\partial^2}{\partial Z_1^2} . f(r) \, dX \, dY \qquad (18)$$

and $$f(r) = \frac{1 - e^{-i\varkappa r}}{r}. \qquad (19)$$

It is shown in Appendix I that I_0 is of order $\varkappa^2 \log_e \varkappa$. Hence, to first-order accuracy in frequency, (16) may be replaced by

$$4\pi W = \iint_{Z_1 \to 0} K \frac{\partial^2}{\partial Z_1^2}\left(\frac{1}{r}\right) dX\, dY \qquad (20)$$

with the wake condition

$$\Gamma = i\nu K + \frac{\partial K}{\partial X} = 0. \qquad (21)$$

Equations (20) and (21) are precisely what one would have to satisfy in the case of the wing of 'reduced' plan-form oscillating in an incompressible fluid with a frequency parameter ν (frequency f/β^2) and a complex downwash amplitude $W = \beta^{-1} w' e^{-i\lambda X}$, where $w = w' e^{i\omega T}$ represents the downwash distribution for the original wing. Hence, if the general incompressible flow solution for the oscillating wing of reduced plan-form were known, the solution for the original wing in compressible flow could be derived.

Numerical methods for solving the problem of the oscillating wing of any plan-form in incompressible flow already exist, but it is difficult to assess their accuracy. Comparisons with the few experimental results available are useful but, since the measured value of any aerodynamic coefficient is dependent on thickness and boundary-layer effects, one has no control on accuracy.

For the oscillating aerofoil of infinite span solutions which are claimed to be reasonably accurate are given in Refs. 1 and 2. These solutions are compared with solution C of this paper in Table 1 and Figs. 2a to 2d. Since, however, the numerical values of the derivatives for $M = 0{\cdot}7$ given in Refs. 1 and 2 differ appreciably for the higher values of the frequency parameter in some cases, there still appears to be some uncertainty in the derivative values even for two dimensions.

4. *Two-dimensional Theory.*—For an aerofoil of infinite span and chord $c(= 2l)$, equation (16) gives

$$2\pi W(X_1, 0) = \int_0^\infty \int_{-1}^\infty K(X) \frac{\partial^2}{\partial Z_1^2} \cdot \frac{-i\varkappa\sqrt{(Y^2+\alpha^2)}}{\sqrt{(Y^2 + \alpha^2)}} \,. \, dX\, dY, \qquad (22)$$

where $\alpha^2 = (X - X_1)^2 + Z_1^2$ and $Z_1 \to 0$. Let $\alpha^2 t^2 \equiv Y^2 + \alpha^2$ and substitute in (22). Then, since

$$\int_1^\infty \frac{e^{-i\varkappa|\alpha|t}}{\sqrt{(t^2 - 1)}} dt = -\frac{\pi}{2} i H_0^{(2)}(\varkappa|\alpha|), \qquad (23)$$

where $H_0^{(2)} \equiv J_0 - iY_0$ is Hankel's function of zero order, it follows that

$$2\pi W(X_1) = -\int_{-1}^\infty K(X) \frac{\partial^2}{\partial Z_1^2}\left[\frac{\pi}{2} i H_0^{(2)}(\varkappa|\alpha|)\right] dX. \qquad (24)$$

When $Z_1 \to 0$, (24) yields, after differentiation and integration by parts, the expression

$$2\pi W(X_1) = \int_{-1}^\infty \frac{1}{X - X_1} \frac{\partial}{\partial X}\left[K(X) \cdot \frac{\pi}{2} \varkappa |X - X_1| (Y_1 + iJ_1)\right] dX \qquad (25)$$

from which $K(X)$ may be determined when $W(X_1)$ is known. This relation corresponds to the more complicated integral equation first derived by Possio for an aerofoil oscillating in a compressible fluid[3].

The solution of (25) may be derived by iteration as follows.

$$\psi(\varkappa|X - X_1|) = 1 + \tfrac{1}{2}\pi\varkappa|X - X_1|(Y_1 + iJ_1). \qquad (26)$$

Then (25) may be written in the form

$$2\pi(W + I) = \int_{-1}^{\infty} \frac{1}{X_1 - X} \frac{\partial K}{\partial X} dX, \qquad (27)$$

where

$$2\pi I = \int_{-1}^{\infty} \frac{1}{X_1 - X} \frac{\partial}{\partial X} [K(X).\psi(\varkappa | X - X_1 |)] dX. \qquad (28)$$

Furthermore, by (12),

$$K(X) = K(1)e^{-i\nu(X-1)} \qquad (29)$$

in the wake. Equation (27) corresponds in form to the integral equation which arises in the theory of unsteady motion in incompressible flow, and can be treated by the methods developed to solve the latter problem[8,9]. It should also be noted that I tends to zero linearly as the frequency is decreased (see Appendix I). If the integral I were known, (27) could be solved exactly by the method of Ref. 8 or by the use of the general formula given by Küssner and Schwarz[9]. According to Ref. 9 the solution of (27) when $W + I$ is known is given by

$$\Gamma = \frac{2}{\pi}\int_0^{\pi} (W + I)\left\{\left[C(\nu)(1 - \cos\vartheta) + \cos\vartheta\right] \cot\frac{\theta}{2} + i\nu L(\theta,\vartheta)\sin\vartheta + \frac{\sin\theta}{\cos\theta - \cos\vartheta}\right\} d\vartheta \qquad (30)$$

and

$$\left.\begin{aligned} K(X) &= e^{-i\nu X}\int_{-1}^{X} \Gamma\, e^{i\nu X}\, dX, \quad -1 \leqslant X \leqslant 1 \\ &= e^{-i\nu X}\int_{-1}^{1} \Gamma\, e^{i\nu X}\, dX, \quad X \geqslant 1 \end{aligned}\right\} \qquad (31)$$

In (30), $X = -\cos\vartheta$,

$$L(\theta,\vartheta) = \tfrac{1}{2}\log_e \frac{1 - \cos(\theta + \vartheta)}{1 - \cos(\theta - \vartheta)} = 2\sum_{n=1}^{\infty} \frac{\sin n\theta\ \sin n\vartheta}{n} \qquad (32)$$

and $C(\nu)$ is the usual oscillatory lift function. By the use of (28), (30) and (31) it should be possible to determine $\Gamma(X)$ and $K(X)$ corresponding to any $W(X)$ by successive approximation as follows. Let $I = 0$ initially in (30). Then determine Γ_1 and by the use of (31) derive K_1 the first approximation to K. Substitute $K = K_1$ in (28) to obtain I_1 the first approximation to I, and substitute $I = I_1$ in (30) to give Γ_2, and hence K_2 a second approximation to K. This process could be continued until $K_n = K_{n+1}$ when further iteration would be unnecessary. The corresponding lift distribution would be given by (11).

A variation of the above procedure would be to expand the left-hand side of (27) in the form

$$W + I_n = U_0[C_0 + C_1(\tfrac{1}{2} + \cos\vartheta) + \sum_{n=2}^{\infty} C_n \cos n\vartheta] \qquad (33)$$

where I_n is the nth approximation to I. It then follows from (27) and (30) that

$$\Gamma = U_0 \sum_{n=0}^{\infty} C_n\Gamma_n \qquad (34)$$

where

$$\left.\begin{aligned} \Gamma_0 &= 2[C(\nu)\cot\tfrac{1}{2}\theta + i\nu\sin\theta] \\ \Gamma_1 &= -2\sin\theta + \cot\tfrac{1}{2}\theta + i\nu\left(\sin\theta + \frac{\sin 2\theta}{2}\right) \\ n \geqslant 2,\ \Gamma_n &= -2\sin n\theta + i\nu\left(\frac{\sin(n+1)\theta}{n+1} = \frac{\sin(n-1)\theta}{n-1}\right) \end{aligned}\right\} \qquad (35)$$

5. *Applications.*—Consider an aerofoil describing pitching and translational oscillations as indicated in the following diagram:—

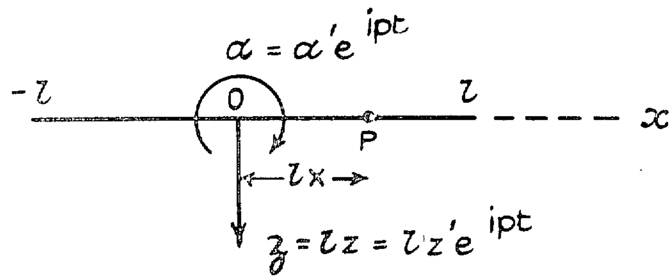

FIG. 1

The downward displacement ζ of the point P is then given by

$$\zeta = l(z' + X\alpha')e^{i\omega T} \qquad (36)$$

and the downwash amplitude

$$w' = U_0[i\omega(z' + X\alpha') + \alpha']. \qquad (37)$$

It follows from (27), since $\beta W = w'e^{-i\lambda X}$, that

$$\frac{2\pi}{\beta}\left[w'e^{-i\lambda X} + \beta I\right] = \int_{-1}^{\infty} \frac{1}{X_1 - X}\frac{\partial K}{\partial X}\, dX. \qquad (38)$$

It is shown in Ref. 10 that

$$e^{i\lambda\cos\theta} = J_0(\lambda) + 2\sum_{n=1}^{\infty} i^n J_n(\lambda)\cos n\theta, \qquad (39)$$

and, since $X = -\cos\theta$,

$$W = \frac{w\,e^{-i\lambda X}}{\beta} = \frac{U_0}{\beta}\left[\alpha' + i\omega z' - \omega\alpha'\frac{\partial}{\partial\lambda}\right]e^{i\lambda\cos\theta}$$

$$= U_0\left[C_0 + C_1(\tfrac{1}{2} + \cos\theta) + \sum_{n=2}^{\infty} C_n\cos n\theta\right] \qquad (40)$$

where

$$\beta C_0 = (\alpha' + i\omega z')(J_0 - iJ_1) - \omega\alpha'(J_0' - iJ_1')$$

$$n \geqslant 1,\ \beta C_n = 2i^n[(\alpha' + i\omega z')J_n - \omega\alpha' J_n'] \qquad (41)$$

with $J_n' \equiv \dfrac{\partial J_n(\lambda)}{\partial\lambda}$.

(*a*) *Boundary Condition A.*—When $I = 0$ is assumed, the solution of (38) is readily obtained by using the results of incompressible flow theory. It follows that Γ, in this case, is given by (34), where the C_n's and Γ_n's are defined by (41) and (35) respectively. The corresponding lift distribution for compressible flow is then given by (11) in the form

$$\tilde{l}(X) = \rho_0 U_0^2\left[\ \Sigma\, C_n\Gamma_n\right]e^{i(\lambda X + \omega T)}. \qquad (42)$$

Since $X = -\cos\theta$, it follows that

$$e^{i\lambda X} = e^{-i\lambda\cos\theta} = J_0(\lambda) + 2\sum_{n=1}^{\infty}(-i)^n J_n(\lambda)\cos n\theta, \qquad (43)$$

and that

$$X\,\mathrm{e}^{i\lambda X} = -\,i\,\frac{\partial}{\partial\lambda}\,\mathrm{e}^{i\lambda X} = -\,i\Big[J_0{}' + 2\sum_{n=1}^{\infty}(-\,i)^n J_n{}'(\lambda)\cos n\theta\Big]. \qquad (44)$$

It may then be proved that

$$\left.\begin{aligned}
&\int_{-1}^{1}\Gamma_0\,\mathrm{e}^{i\lambda X}\,dX = 2\pi\Big[\Big(C(\nu) + \frac{i\nu}{2}\Big)J_0(\lambda) - iC(\nu)J_1(\lambda) + \frac{i\nu}{2}\,J_2(\lambda)\Big]\\
&\int_{-1}^{1}\Gamma_0\,\mathrm{e}^{i\lambda X}X\,dX = -\,2\pi i\Big[\Big(C + \frac{i\nu}{2}\Big)J_0{}' - iCJ_1{}' + \frac{i\nu}{2}J_2{}'\Big]\\
&\int_{-1}^{1}\Gamma_1\,\mathrm{e}^{i\lambda X}\,dX = -\,\pi\Big(1 - \frac{\nu}{\lambda}\Big)(J_2 + iJ_1)\\
&\int_{-1}^{1}\Gamma_1\,\mathrm{e}^{i\lambda X}X\,dX = \pi i\Big(1 - \frac{\nu}{\lambda}\Big)(J_2{}' + iJ_1{}') + \frac{\pi i\nu}{\lambda^2}(J_2 + iJ_1)\\
&\text{and for } n \geqslant 2\\
&\int_{-1}^{1}\Gamma_n\,\mathrm{e}^{i\lambda X}\,dX = (-\,i)^{n+1}\,\pi\Big(1 - \frac{\nu}{\lambda}\Big)(J_{n+1} + J_{n-1})\\
&\int_{-1}^{1}\Gamma_n\,\mathrm{e}^{i\lambda X}X\,dX = (-\,i)^{n+2}\,\pi\Big[\Big(1 - \frac{\nu}{\lambda}\Big)(J_{n+1}{}' + J_{n-1}{}' + \frac{\nu}{\lambda^2}(J_{n+1} + J_{n-1})\Big]
\end{aligned}\right\} \qquad (45)$$

Hence, by the use of (41), (42) and (45) it may be shown that the total lift L is given by

$$\begin{aligned}
\frac{L\,\mathrm{e}^{-ipt}}{\pi\rho_0 l U_0{}^2} &= C_0[2C(\nu)(J_0 - iJ_1) + i\nu(J_0 + J_2)] - C_1\Big(1 - \frac{\nu}{\lambda}\Big)(J_2 + iJ_1)\\
&\quad + \sum_{n=2}(-\,i)^{n+1}C_n\Big(1 - \frac{\nu}{\lambda}\Big)(J_{n+1} + J_{n-1}).
\end{aligned} \qquad (46)$$

Similarly, the moment M about the mid-chord axis* is given by

$$\begin{aligned}
\frac{M\,\mathrm{e}^{-ipt}}{\pi\rho_0 l^2 U_0{}^2} &= C_0[2C(\nu)(J_1{}' + iJ_0{}') - \nu(J_0{}' + J_2{}')]\\
&\quad + C_1\Big[\Big(1 - \frac{\nu}{\lambda}\Big)(J_1{}' - iJ_2{}') + \frac{\nu}{\lambda^2}(J_1 - iJ_2)\Big]\\
&\quad + \sum_{n=2}^{\infty}(-\,i)^n C_n\Big[\Big(1 - \frac{\nu}{\lambda}\Big)(J_{n+1}{}' + J_{n-1}{}') + \frac{\nu}{\lambda^2}(J_{n+1} + J_{n-1})\Big].
\end{aligned} \qquad (47)$$

It is shown in Appendix II that

$$\left.\begin{aligned}
\Sigma_1 &= \sum_{n=2}^{\infty}J_n(J_{n-1}{}' + J_{n+1}{}') = \frac{J_0J_2 + J_1J_3}{2}\\
\Sigma_2 &= \sum_{n=2}^{\infty}J_n{}'(J_{n-1} + J_{n+1}) = \frac{J_1{}^2 + J_2{}^2}{2}\\
\Sigma_3 &= \sum_{n=2}^{\infty}J_n(J_{n-1} + J_{n+1})\\
&= \frac{\lambda}{2}\Big[J_1{}'^2 + J_2{}'^2 + \Big(1 - \frac{1}{\lambda^2}\Big)J_1{}^2 + \Big(1 - \frac{4}{\lambda^2}\Big)J_2{}^2\Big]\\
\Sigma_4 &= \sum_{n=2}^{\infty}J_n{}'(J_{n-1}{}' + J_{n+1}{}')\\
&= \frac{1}{4}\frac{d}{d\lambda}(J_1{}^2 + J_2{}^2) + \frac{1}{\lambda^3}\int_0^{\lambda}\lambda_1\int_0^{\lambda_1}(J_1{}^2 + J_2{}^2)\lambda_0\,d\lambda_0\,d\lambda_1\\
&\text{where}\\
&\int_0^{\lambda_1}(J_1{}^2 + J_2{}^2)\,\lambda_0\,d\lambda_0 = 2\sum_{n=2}^{\infty}nJ_n(\lambda_1)^2.
\end{aligned}\right\} \qquad (48)$$

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* For the mid-chord axis, $M = il\,dL/d\lambda$, where L is given by (46) and $C_0, C_1, \ldots C_n$ are regarded as being independent of λ.

Hence, by use of (41) and (48), the following expressions for the lift and pitching moment may be derived

$$\left.\begin{aligned}\frac{\beta L}{\pi\rho_0 l U_0^2} &= [(\alpha + i\omega z)(J_0 - iJ_1) - \omega\alpha(J_0' - iJ_1')][2C(\nu)(J_0 - iJ_1) + i\nu(J_0 + J_2)] \\ &\quad + 2[(\alpha + i\omega z)J_1 - \omega\alpha J_1']\left(1 - \frac{\nu}{\lambda}\right)(J_1 - iJ_2) \\ &\quad - 2i\left(1 - \frac{\nu}{\lambda}\right)[(\alpha + i\omega z)\Sigma_3 - \omega\alpha\Sigma_2]. \\ \frac{\beta M}{\pi\rho_0 l^2 U_0^2} &= [(\alpha + i\omega z)(J_0 - iJ_1) - \omega\alpha(J_0' - iJ_1')][2C(\nu)(J_1' + iJ_0') - \nu(J_0' + J_2')] \\ &\quad + 2[(\alpha + i\omega z)J_1 - \omega\alpha J_1']\left[\left(1 - \frac{\nu}{\lambda}\right)(J_2' + iJ_1') + \frac{\nu}{\lambda^2}(J_2 + iJ_1)\right] \\ &\quad + 2(\alpha + i\omega z)\left[\left(1 - \frac{\nu}{\lambda}\right)\Sigma_1 + \frac{\nu}{\lambda^2}\Sigma_3\right] - 2\omega\alpha\left[\left(1 - \frac{\nu}{\lambda}\right)\Sigma_4 + \frac{\nu}{\lambda^2}\Sigma_2\right].\end{aligned}\right\} \quad (49)$$

In the notation generally used in flutter theory, the aerodynamic coefficients are defined by the relations

$$\left.\begin{aligned}\frac{L}{c\rho_0 U_0^2} &= (l_z + i\tilde{\omega}l_{\dot{z}})\,\frac{\mathbf{z}}{c} + (l_\alpha + i\tilde{\omega}_{\dot{\alpha}})\,\alpha \\ \frac{M}{\rho_0 c^2 U_0^2} &= (m_z + i\tilde{\omega}m_{\dot{z}})\,\frac{\mathbf{z}}{c} + (m_\alpha + (i\tilde{\omega}m_{\dot{\alpha}})\,\alpha\end{aligned}\right\} \quad \ldots \quad (50)$$

where $\tilde{\omega} = 2\omega = pc/U_0$ and $\mathbf{z} = \dfrac{c}{2}z$.

Let

$$\left.\begin{aligned}R &\equiv 2C(\nu)[J_0(\lambda) - iJ_1(\lambda)] + i\nu[J_0(\lambda) + J_2(\lambda)], \\ S &\equiv \left(1 - \frac{\nu}{\lambda}\right)[J_2(\lambda) + iJ_1(\lambda)], \\ R' &\equiv \partial R/\partial\lambda, \qquad S' \equiv \partial S/\partial\lambda,\end{aligned}\right\} \quad \ldots \quad (51)$$

where ν is regarded as being independent of λ (actually $\lambda = M^2\nu$). From (49) and (50) it then follows that

$$\left.\begin{aligned}l_z + i\tilde{\omega}l_{\dot{z}} &= \frac{\pi i\omega}{\beta}\left[(J_0 - iJ_1)R - 2iJ_1S - 2i\left(1 - \frac{\nu}{\lambda}\right)\Sigma_3\right] \\ l_\alpha + i\tilde{\omega}l_{\dot{\alpha}} &= \frac{\pi}{2\beta}\Big\{[J_0 - iJ_1 - \omega(J_0' - iJ_1')]R - 2i(J_1 - \omega J_1')S \\ &\qquad - 2i\left(1 - \frac{\nu}{\lambda}\right)[\Sigma_3 - \omega\Sigma_2]\Big\} \\ m_z + i\tilde{\omega}m_{\dot{z}} &= -\frac{\pi\omega}{2\beta}\left\{(J_0 - iJ_1)R' - 2iJ_1S' - 2i\left[\left(1 - \frac{\nu}{\lambda}\right)\Sigma_1 + \frac{\nu}{\lambda^2}\Sigma_3\right]\right\} \\ m_\alpha + i\tilde{\omega}m_{\dot{\alpha}} &= \frac{\pi i}{4\beta}\Big\{[J_0 - iJ_1 - \omega(J_0' - iJ_1')]R' - 2i(J_1 - \omega J_1')S' \\ &\qquad - 2i\left[\left(1 - \frac{\nu}{\lambda}\right)(\Sigma_1 - \omega\Sigma_4) + \frac{\nu}{\lambda^2}(\Sigma_3 - \omega\Sigma_2)\right]\Big\}\end{aligned}\right\} \quad \ldots \quad (52)$$

and from these formulae the individual aerodynamic coefficients for the boundary condition assumed can be determined exactly.

(*b*) *Boundary Condition B*.—To obtain this boundary condition, $I = 0$ is again assumed, and (40) is replaced by

$$W = \beta^{-1}U_0[(\alpha' + i\omega z')(1 + i\lambda \cos\theta) - i\omega\alpha' \cos\theta], \quad (53)$$

which yields

$$\left.\begin{aligned} \beta C_0 &= (\alpha' + i\omega z')\left(1 - \frac{i\lambda}{2}\right) + \frac{i\omega}{2}\alpha' \\ \beta C_1 &= i\lambda(\alpha' + i\omega z') - i\omega\alpha' \end{aligned}\right\} \quad (54)$$

$n \geqslant 2, \quad C_n = 0.$

Condition (53) is obtained by substituting $1 + i\lambda \cos\theta$ for $\exp(i\lambda \cos\theta)$ and neglecting terms of second and higher order in the frequency*. The same condition could be derived by regarding the equivalent aerofoil as rigid and neglecting distortion terms in X^2 and higher powers. Once the boundary condition has been chosen, the general solution of (38) for any value of ω can be obtained as already shown. For condition B, the formulae derived for the mid-chord aerodynamic coefficients are

$$\left.\begin{aligned} l_z + i\tilde{\omega}l_{\dot{z}} &= \frac{\pi i\omega}{\beta}\left[\left(1 - \frac{i\lambda}{2}\right)R - i\lambda S\right] \\ l_\alpha + i\tilde{\omega}l_{\dot{\alpha}} &= \frac{\pi}{2\beta}\left\{\left[1 + \frac{i(\omega - \lambda)}{2}\right]R + i(\omega - \lambda)S\right\} \\ m_z + i\tilde{\omega}m_{\dot{z}} &= -\frac{\pi\omega}{2\beta}\left[\left(1 - \frac{i\lambda}{2}\right)R' - i\lambda S'\right] \\ m_\alpha + i\tilde{\omega}m_{\dot{\alpha}} &= \frac{\pi i}{4\beta}\left\{\left[1 + \frac{i(\omega - \lambda)}{2}\right]R' + i(\omega - \lambda)S'\right\}. \end{aligned}\right\} \quad (55)$$

A comparison of the results for boundary conditions A and B revealed that the neglect of second-order terms in frequency in the boundary condition did not lead to serious discrepancies over the practical range of values of the frequency parameter (*see* Table 1). For both A and B, it is assumed that $I = 0$, and the approximate solutions of (38) obtained can only be regarded as a rough approximation to the true solution. A better approximation to the exact solution can, however, be made by including the I term.

(*c*) *Boundary Condition C*.—It is shown in Appendix I that, when $\omega \to 0$,

$$2\pi I \to i\nu\delta K(1), \quad (56)$$

where $K(1)$ is the value of K at the trailing edge, and where

$$\delta = \log_e \frac{M}{2} + \sqrt{(1 - M^2)}\log_e\left(\frac{1 + \sqrt{(1 - M^2)}}{M}\right). \quad (57)$$

Since in the limit

$$K(1) = \frac{2\pi}{\beta}(\alpha' + i\omega z')U_0, \quad (58)$$

the left-hand side of (38) may be written in the form

$$2\pi(W + I) = \frac{2\pi U_0}{\beta}[(\alpha' + i\omega z')(1 + i\lambda\cos\theta) - i\omega\alpha\cos\theta + i\nu\delta(\alpha' + i\omega z')] \quad (59)$$

to first-order accuracy in frequency. With this modified boundary condition, (38) can be solved exactly for any value of the frequency parameter. The solution thus obtained may be regarded as a first approximation to the exact solution of (25).

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* The terms α' and $i\omega z'$ are assumed to be of the same order.

Let

$$\left.\begin{aligned} a &= 1 + i\nu\delta - \frac{i\lambda}{2} \\ \text{and} \qquad b &= 1 + i\nu\delta + \frac{i(\omega - \lambda)}{2}. \end{aligned}\right\} \quad \ldots \quad (60)$$

It follows from (72) that

$$\left.\begin{aligned} \beta C_0 &= i\omega z'.a + \alpha' b \\ \beta C_1 &= i\lambda(\alpha' + i\omega z') - i\omega\alpha' \\ n \geqslant 2, \qquad C_n &= 0. \end{aligned}\right\} \quad \ldots \quad (61)$$

The lift distribution is then given by (42) and the corresponding lift and pitching moment may be calculated. In this case the formulae for the aerodynamic coefficients are

$$\left.\begin{aligned} l_z + i\tilde{\omega} l_{\dot{z}} &= \frac{\pi i\omega}{\beta}(aR - i\lambda S) \\ l_\alpha + i\tilde{\omega} l_{\dot{\alpha}} &= \frac{\pi}{2\beta}[bR + i(\omega - \lambda)S] \\ m_z + i\tilde{\omega} m_{\dot{z}} &= -\frac{\pi\omega}{2\beta}(aR' - i\lambda S') \\ m_\alpha + i\tilde{\omega} m_{\dot{\alpha}} &= \frac{\pi i}{4\beta}[bR' + i(\omega - \lambda)S']. \end{aligned}\right\} \quad \ldots \quad (62)$$

A comparison of the values given by (62) and those given by the 'exact' solution show fair agreement for a wide range of $\tilde{\omega}$ values and good agreement for $\tilde{\omega} \leqslant 0{\cdot}4$. From this, it appears that if solution C, which is essentially a first approximation, were extended to include second order terms in frequency on the left-hand side of (38), reliable estimates of the derivatives would be obtained even for higher values of $\tilde{\omega}$.

6. *Concluding Remarks*.—In view of the fact that solutions A and B agree closely, it appears that the left-hand side of (38) need not be known to great accuracy and that the assumption $\exp(-i\lambda X) = 1 - i\lambda X$ can be made at an early stage in the calculations. This is likely to be true also in the three-dimensional case, and it seems probable that solutions of sufficient accuracy would be obtained if the left-hand side of (17) were replaced by its limiting form for low frequencies. Present methods of treating the equivalent incompressible flow problem could then be applied to derive a general solution of (17) for any frequency parameter value. There is no need to limit the solution to low frequencies once the approximate form of the left-hand side of (17) has been chosen. In the two-dimensional case, approximation C is quite satisfactory for $M = 0{\cdot}7$ and $\tilde{\omega} \leqslant 0{\cdot}4$, and it seems likely that the corresponding approximation in three dimensions would give reliable results. Flutter derivatives for wings of finite span are not usually very sensitive to variations of frequency parameter and the method of calculation suggested may give good accuracy for a wider range of $\tilde{\omega}$ values than that given by approximation C in the two-dimensional case. Terms of order $\varkappa^2 \log_e \varkappa$ are neglected in this approximation but they could be included in the corresponding three-dimensional solution if necessary. It is, however, doubtful whether the accuracy of solution for the higher values of $\varkappa$ would be improved since terms of order $\varkappa^2$ then become equally important.

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APPENDIX I

Limiting Forms of Integrals

(*i*) *Three-dimensional Case.*—The integral I_0 defined by (18) may be expressed in the form

$$\begin{aligned} 4\pi I_0 &= \int_{-s}^{s}\int_{X_L}^{X_t} K(X,Y)\,\frac{\partial^2 f}{\partial z_1{}^2}\,dX\,dY \\ &\quad + \int_{-s}^{s}\int_{X_t}^{\infty} K(X_t,Y)\,\mathrm{e}^{-i\nu(X-X_t)}\,\frac{\partial^2 f}{\partial z_1{}^2}\,dX\,dY \\ &= \int_{-s}^{s}\int_{XL}^{X_t} K(X,Y)\,\frac{\partial^2 f}{\partial z_1{}^2}\,dX\,dY \\ &\quad - \int_{-s}^{s}\int_{X_1}^{X_t} K(X_t,Y)\,\mathrm{e}^{-i\nu(X-X_t)}\frac{\partial^2 f}{\partial z_1{}^2}\,dX\,dY \\ &\quad + \int_{-s}^{s}\int_{0}^{\infty} K(X_t,Y)\,\mathrm{e}^{(X_1-X_t+\xi)}\,\frac{\partial^2 f}{\partial z_1{}^2}\,d\xi\,dY \quad \ldots \quad (63) \end{aligned}$$

where $\xi = X - X_1$. When $\varkappa$ is small, the first two integrals are of order $\varkappa^2$ since the approximation

$$f = i\varkappa + \frac{\varkappa^2 \gamma}{2} \quad \ldots \quad (64)$$

is valid within the ranges of integration. This approximation cannot be used in the third integral since ξ varies for zero to infinity and $\varkappa r$ may be large for large values of r. However, when $z \to 0$, it may be proved that

$$\frac{\partial^2 f}{\partial {z_1}^2} = \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1 - e^{-i\varkappa R}}{R}\right), \qquad (65)$$

where $R^2 = \xi^2 + a^2$, and $a = |Y - Y_1|$. Let

$$\sigma(Y) \equiv K(X_t, Y) e^{-iv(X_1 - X_t)}.$$

Then the third integral in (63) reduces to

$$I_3 = \int_{-s}^{s}\int_0^{\infty} \frac{\sigma(Y) e^{-iv\xi}}{R}\frac{\partial}{\partial R}\left(\frac{1 - e^{-i\varkappa R}}{R}\right) d\xi\, dY$$

$$= \int_{-s}^{s} \frac{\sigma(Y)}{Y - Y_1}\frac{\partial}{\partial Y}\int_0^{\infty}\frac{e^{-iv\xi}(1 - e^{-\varkappa R})}{R}\, d\xi\, dY. \qquad (66)$$

It is shown in Ref. 11 that

$$\int_0^{\infty}\frac{e^{-iv\xi} d\xi}{\sqrt{(\xi^2 + a^2)}} = K_0(\nu a) - iT_0(\nu a)$$

$$= -\left(\gamma + \log_e \frac{\nu a}{2}\right) - \frac{\pi i}{2} + i\nu a - \frac{a^2 \nu^2}{4}\log_e \nu + O(\nu^2) \qquad (67)$$

and, since $\varkappa = M\nu$,

$$\int_0^{\infty}\frac{e^{-i(\nu\xi + \varkappa R)}}{R}\, d\xi = \int_0^{\infty}\frac{e^{-i\nu(\xi + MR)}}{\sqrt{(\xi^2 + a^2)}}\, d\xi$$

$$= \int_{Ma}^{\infty}\frac{e^{-i\nu t}\, dt}{\sqrt{(t^2 + a^2\beta^2)}} \qquad (68)$$

where $t \equiv \xi + M\sqrt{(\xi^2 + a^2)}$. Hence, by (67) and (68),

$$\int_0^{\infty}\frac{e^{-iv\xi}(1 - e^{-i\varkappa R})}{R}\, d\xi = \int_0^{\infty} e^{-iv\xi}\left(\frac{1}{\sqrt{(\xi^2 + a^2)}} - \frac{1}{\sqrt{(\xi^2 + a^2\beta^2)}}\right) d\xi$$

$$+ \int_0^{Ma}\frac{e^{-iv\xi} d\xi}{\sqrt{(\xi^2 + a^2\beta^2)}} \qquad (69)$$

$$= \log_e (1 + M) - \frac{M^2 a^2}{4}\,.\,\nu^2 \log_e \nu + O(\nu^2)$$

and, therefore, when $\nu \to 0$,

$$I_3 = \int_{-s}^{s}\frac{\sigma(Y)}{Y - Y_1}\frac{\partial}{\partial Y}\left[\log_e (1 + M) - \frac{a^2\varkappa^2}{4}\log_e \varkappa + O(\varkappa^2)\right] dY$$

$$\sim -\frac{\varkappa^2}{2}\log_e \varkappa \int_{-s}^{s}\sigma(Y)\, dY \qquad (70)$$

where the integral has its steady value.

From (18), (63) and (70) it then follows that the integral I_0 is of order $\varkappa^2 \log_e \varkappa$.

(*ii*) *Two-dimensional Case.*—The integral I defined by (28) may be expressed in the form

$$2\pi I = \int_{-1}^{1}\frac{1}{X_1 - X}\frac{\partial}{\partial X}[K(X)\psi(\varkappa|X - X_1|)]\, dX$$

$$+ \int_1^{\infty}\frac{K(1)}{X_1 - X}\frac{\partial}{\partial X}\left[\psi\, e^{-iv(X-1)}\right] dX \qquad (71)$$

since $K(X) = K(1) e^{-iv(X-1)}$ in the wake.

Since, when $\varkappa \to 0$,

$$\psi(\varkappa\,|X - X_1|) \sim \frac{\varkappa^2}{2}(X - X_1)^2 \log_e \frac{\varkappa}{2} + O(\varkappa^2), \quad \ldots \quad \ldots \quad \ldots \quad (72)$$

the first integral in (71) is of order $\varkappa^2 \log_e \varkappa$. The second integral in (71) is denoted by I_2, where

$$I_2 = -K(1)e^{-i\nu(X_1-1)}Q - \int_{X_1}^{1} \frac{K(1)}{X_1 - X} \frac{\partial}{\partial X}(\psi\, e^{-i\nu(X-1)})\, dX \quad \ldots \quad \ldots \quad \ldots \quad (73)$$

and

$$Q = \int_{X_1}^{\infty} \frac{1}{X - X_1} \frac{\partial}{\partial X}\Big[\psi(\varkappa|X - X_1|)e^{-i\nu(X-X_1)}\Big]\, dX$$

$$= \int_0^{\infty} \frac{\nu}{x} \frac{\partial}{\partial x}\Big[\psi(Mx)e^{-ix}\Big]\, dx\,. \quad \ldots \quad \ldots \quad \ldots \quad (74)$$

Next consider the limiting form of $Q(\varepsilon)$ as $\varepsilon \to 0$, where

$$Q(\varepsilon) = \int_{\varepsilon}^{\infty} \frac{\nu}{x} \frac{\partial}{\partial x}[\psi(Mx)e^{-ix}]\, dx\,. \quad \ldots \quad \ldots \quad \ldots \quad (75)$$

The function ψ is defined by (26) and is expressible in the form

$$\psi(Mx) = 1 + \frac{\pi M x}{2}[Y_1(Mx) + iJ_1(Mx)]\,. \quad \ldots \quad \ldots \quad \ldots \quad (76)$$

By substituting for ψ in (75), it may be shown that

$$\frac{Q(\varepsilon)}{\nu} = \int_{\varepsilon}^{\infty}\Big[\frac{\pi}{2}M^2(Y_0 + iJ_0)e^{-ix} - \frac{\pi i M}{2}(Y_1 + iJ_1)e^{-ix} - \frac{i\, e^{-ix}}{x}\Big]\, dx$$

$$= \int_{\varepsilon}^{\infty}\Big\{\frac{\pi i}{2}\frac{\partial}{\partial x}[e^{-ix}(Y_0 + iJ_0)] - \frac{\pi}{2}\beta(Y_0 + iJ_0)e^{-ix} - \frac{i\, e^{-ix}}{x}\Big\}\, dx\,. \quad \ldots \quad (77)$$

Now

$$\int_{\varepsilon}^{\infty} \frac{e^{-ix}}{x}\, dx \sim -\gamma - \log \varepsilon - \frac{i\pi}{2} + O(\varepsilon), \ldots \quad \ldots \quad \ldots \quad \ldots \quad (78)$$

$$\frac{\pi i}{2}[(Y_0 + iJ_0)e^{-ix}]_{\varepsilon}^{\infty} = -\frac{\pi i}{2}e^{-i\varepsilon}[Y_0(M\varepsilon) + iJ_0(M\varepsilon)] \quad \ldots \quad \ldots \quad \ldots \quad (79)$$

$$\sim -i\Big(\gamma + \log_e \frac{M\varepsilon}{2} + \frac{i\pi}{2} + O(\varepsilon)\Big),$$

and it is proved in Ref. 12 that

$$\int_0^{\infty}[Y_0(Mx) + iJ_0(Mx)]e^{-ix}\, dx = -\frac{2i}{\pi\sqrt{(1 - M^2)}} \log_e \frac{1 - \sqrt{(1 - M^2)}}{M}\,. \quad (80)$$

Hence, when $\varepsilon \to 0$,

$$\frac{Q}{\nu} \sim i\Big[\sqrt{(1 - M^2)} \log_e \frac{1 - \sqrt{(1 - M^2)}}{M} - \log_e \frac{M}{2}\Big] \quad \ldots \quad \ldots \quad \ldots \quad (81)$$

and it follows from (71) and (73) that

$$2\pi I \sim i\nu\delta K(1) + O(\nu^2 \log_e \nu) \quad \ldots \quad \ldots \quad \ldots \quad (82)$$

where

$$\delta = \log_e \frac{M}{2} + \sqrt{(1 - M^2)} \log_e \frac{1 + \sqrt{(1 - M^2)}}{M}\,. \quad \ldots \quad \ldots \quad \ldots \quad (83)$$

APPENDIX II

Summations

(i) *Evaluation of* Σ_1, Σ_2, Σ_3 *and* Σ_4.—By the use of the well-known relations

$$\left.\begin{aligned} 2J'_n(\lambda) &= J_{n-1}(\lambda) - J_{n+1}(\lambda) \\ \frac{2nJ_n}{\lambda} &= J_{n-1} + J_{n+1} \\ \int_0^\lambda J_n^{\,2}\lambda\, d\lambda &= \frac{\lambda^2}{2}\left[J_n'^{\,2} + \left(1 - \frac{n^2}{\lambda^2}\right)J_n^{\,2}\right] \end{aligned}\right\} \qquad (84)$$

it may be shown that

$$\begin{aligned} \Sigma_1 &= \sum_{n=2}^{\infty} J_n(J_{n-1}' + J_{n+1}') \\ &= \tfrac{1}{2}\sum_{n=2}^{\infty} J_n(J_{n-2} - J_{n+2}) \\ &= \tfrac{1}{2}(J_0J_2 + J_1J_3) \end{aligned} \qquad (85)$$

and that

$$\begin{aligned} \Sigma_2 &= \sum_{n=2}^{\infty} J_n'(J_{n-1} + J_{n+1}) = \sum_{n=2}^{\infty} \frac{2nJ_nJ_n'}{\lambda} \\ &= \tfrac{1}{2}\sum_{n=2}^{\infty} (J_{n-1}^{\,2} - J_{n+1}^{\,2}) \\ &= \tfrac{1}{2}(J_1^{\,2} + J_2^{\,2})\,. \end{aligned} \qquad (86)$$

Since, by (86),

$$4\sum_{n=2}^{\infty} nJ_nJ_n' = \lambda(J_1^{\,2} + J_2^{\,2})\,. \qquad (87)$$

it follows by integration that

$$2\sum_{n=2}^{\infty} nJ_n^{\,2} = \int_0^\lambda \lambda(J_1^{\,2} + J_2^{\,2})\, d\lambda\,. \qquad (88)$$

Hence, by (84) and (88),

$$\begin{aligned} \Sigma_3 &= \sum_{n=2}^{\infty} J_n(J_{n-1} + J_{n+1}) = \frac{2}{\lambda}\Sigma nJ_n^{\,2} \\ &= \frac{1}{\lambda}\int_0^\lambda \lambda(J_1^{\,2} + J_2^{\,2})d\lambda \\ &= \frac{\lambda}{2}\left[J_1'^{\,2} + J_2'^{\,2} + \left(1 - \frac{1}{\lambda^2}\right)J_1^{\,2} + \left(1 - \frac{4}{\lambda^2}\right)J_2^{\,2}\right]. \end{aligned} \qquad (89)$$

The remaining series

$$\begin{aligned} \Sigma_4 &= \sum_{n=2}^{\infty} J_n'(J_{n-1}' + J_{n+1}') \\ &= \frac{d\Sigma_2}{d\lambda} - \frac{2}{\lambda}\sum_{n=2}^{\infty} nJ_nJ_n'' \\ &= \frac{d\Sigma_2}{d\lambda} - \frac{2}{\lambda}\frac{d}{d\lambda}\sum_{n=2}^{\infty} nJ_nJ_n' + \frac{1}{\lambda}\sum_{n=2}^{\infty} 2nJ_n'^{\,2} \\ &= \frac{d\Sigma_2}{d\lambda} - \frac{2}{\lambda}\frac{d}{d\lambda}\left[\frac{\lambda(J_1^{\,2} + J_2^{\,2})}{4}\right] + \frac{1}{\lambda}\sum_{n=2}^{\infty} 2nJ_n'^{\,2} \\ &= -\frac{J_1^{\,2} + J_2^{\,2}}{2\lambda} + \frac{1}{\lambda}\sum_{n=2}^{\infty} 2nJ_n'^{\,2}\,. \end{aligned} \qquad (90)$$

Since

$$\int_0^\lambda 2J_n^2\lambda\,d\lambda = \lambda^2\left[J_n'^2 + \left(1-\frac{n^2}{\lambda^2}\right)J_n^2\right]$$

$$= \lambda^2\left[2J_n'^2 - \frac{d}{d\lambda}(J_nJ_n')\right] - \lambda J_nJ_n' , \qquad \ldots \quad (91)$$

it follows that

$$\int_0^\lambda \sum_{n=2}^\infty 2nJ_n^2\lambda\,d\lambda = \lambda^2\sum_{n=2}^\infty 2nJ_n'^2 - \left(\lambda^2\frac{d}{d\lambda}+\lambda\right)\left(\frac{\lambda\Sigma_2}{2}\right)$$

and, therefore, by (89),

$$\lambda^2\sum_{n=2}^\infty 2nJ_n'^2 = \frac{\lambda^2}{2}\frac{d}{d\lambda}(\lambda\Sigma_2) + \frac{\lambda^2\Sigma_2}{2} + \int_0^\lambda \lambda^2\Sigma_3\,d\lambda . \qquad \ldots \quad (92)$$

Hence, by (90) and (92),

$$\Sigma_4 = \tfrac14\frac{d}{d\lambda}(J_1^2+J_2^2) + \frac{1}{\lambda^3}\int_0^\lambda\lambda\int_0^\lambda(J_1^2+J_2^2)\lambda_1\,d\lambda_1\,d\lambda . \qquad \ldots \quad (93)$$

TABLE 1

Comparison of Results for $M = 0{\cdot}7$ (Mid-chord Axis)

	l				$l_{\dot z}$			
$\bar{\omega}$	Approximations			Refs.* 1 and 2	Approximations			Refs.* 1 and 2
	A	B	C		A	B	C	
0	0	0	0	0	4·399	4·399	4·399	4·399
0·04	0·0199	0·0199	0·0225	0·0223	4·072$_5$	4·073	4·065$_5$	4·061
0·08	0·0538	0·0539	0·0636	0·0629	3·766	3·767	3·748	3·740
0·2	0·1404	0·1407	0·1917	0·1849 (0·193)	3·123	3·130	3·086	3·054 (3·05)
0·4	0·1517	0·1534	0·3251	0·2975 (0·313)	2·587	2·611	2·590	2·504 (2·51)
0·6	0·0127	0·0149	0·3647	0·3120 (0·360)	2·317	2·366	2·442$_5$	2·269 (2·25)
0·8	−0·2493	−0·2518	0·3294	0·2613 (0·317)	2·150	2·232	2·467$_5$	2·172 (2·12)

TABLE 1—*continued*

	l_α				$l_{\dot\alpha}$			
$\bar{\omega}$	Approximations			Refs.* 1 and 2	Approximations			Refs.* 1 and 2
	A	B	C		A	B	C	
0	4·399	4·399	4·399	4·399	−∞	−∞	−∞	−∞
0·04	4·078	4·078	4·071	4·066	−11·402	−11·400	−13·041	−12·981
0·08	3·783	3·781	3·762	3·757	−7·475$_5$	−7·472	−8·993	−8·903
0·2	3·177	3·169	3·124	3·117 (3·11)	−2·738	−2·7275	−4·002	−3·881 (−3·85)
0·4	2·693$_5$	2·667$_5$	2·647	2·637 (2·63)	−0·3128	−0·2979	−1·3705	−1·2775 (−1·45)
0·6	2·469	2·417	2·493	2·471 (2·435)	0·5374	0·5503	−0·4213	−0·370$_5$ (−0·46)
0·8	2·345	2·258	2·494	2·448 (2·38)	0·9311	0·9369	0·0287	0·032 (−0·07)

TABLE 1—*continued*

$\bar{\omega}$	$-m_z$				$-m_{\dot{z}}$			
	Approximations			Refs.* 1 and 2	Approximations			Refs.* 1 and 2
	A	B	C		A	B	C	
0	0	0	0	0	−1·0998	−1·0998	−1·0998	−1·099
0·04	−0·0058	−0·0058	−0·0065	−0·0064	−1·0168	−1·0170	−1·0146	−1·0135
0·08	−0·0167	−0·0167	−0·0191	−0·0188	−0·9383	−0·9384	−0·9316	−0·9280
0·2	−0·0536	−0·0538	−0·0664	−0·0629	−0·7719	−0·7724	−0·7496	−0·743
				(−0·063)				(−0·745)
0·4	−0·1069	−0·1080	−0·1509	−0·1330	−0·6352	−0·6362	−0·5859	−0·5808
				(−0·147)				(−0·582)
0·6	−0·1523	−0·1559	−0·2460	−0·2016	−0·5727	−0·5736	−0·4938	−0·4960
				(−0·212)				(−0·487)
0·8	−0·1951	−0·2040	−0·3612	−0·2768	−0·5416	−0·5423	−0·4286	−0·4406
				(−0·288)				(−0·427)

$\bar{\omega}$	$-m_\alpha$				$-m_{\dot{\alpha}}$			
	Approximations			Refs.* 1 and 2	Approximations			Refs.* 1 and 2
	A	B	C		A	B	C	
0	−1·0998	−1·0998	−1·0998	−1·0995	∞	∞	∞	∞
0·04	−1·0185	−1·0184	−1·0161	−1·0148	3·6453	3·6450	4·0547	4·0297
0·08	−0·9428	−0·9426	−0·9358	−0·9333	2·6463	2·6453	3·0245	2·9808
0·2	−0·7872	−0·7867	−0·7638	−0·7595	1·4261	1·4228	1·7391	1·6690
				−0·755)				(1·670)
0·4	−0·6684	−0·6676	−0·6173	−(0·6166	0·7915	0·7848	1·0529	0·9761
				−(0·617)				(1·01)
0·6	−0·6240	−0·6235	−0·5437	−0·5474	0·5654	0·5557	0·8058	0·7350
				(−0·532)				(0·770)
0·8	−0·6129	−0·6133	−0·4996	−0·5040	0·4589	0·4465	0·6921	0·6301
				(−0·488)				(0·648)

* Values in brackets were obtained from tables given in Ref. 2.

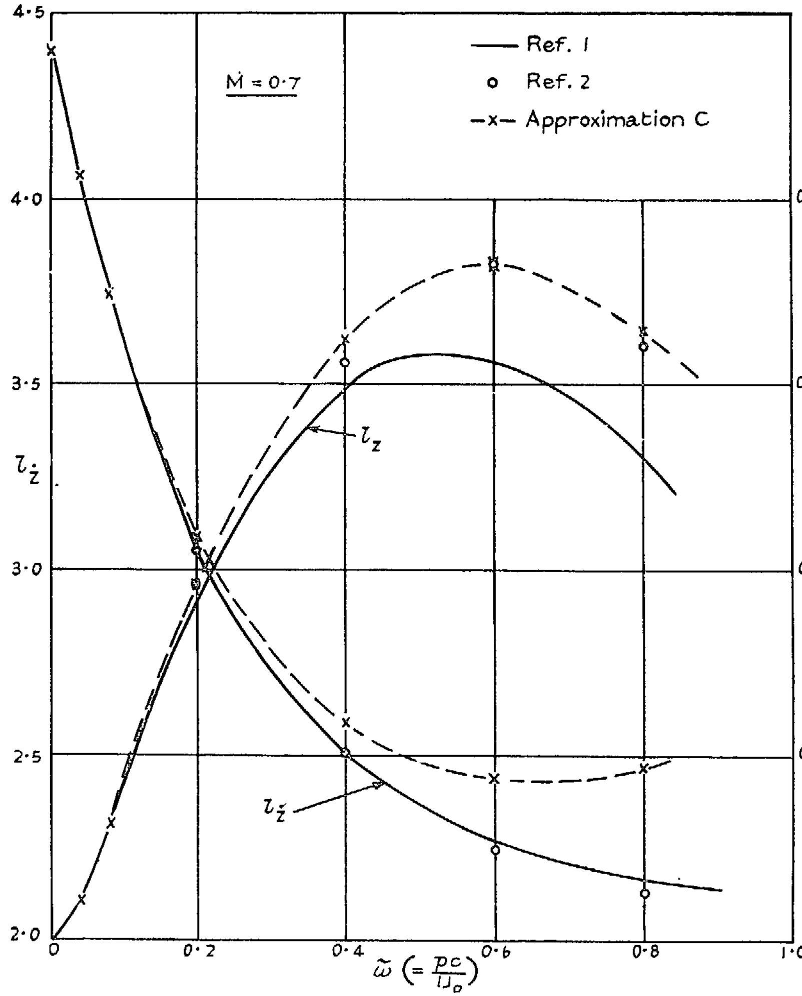

FIG. 2a. Mid-chord derivatives for $M = 0 \cdot 7$

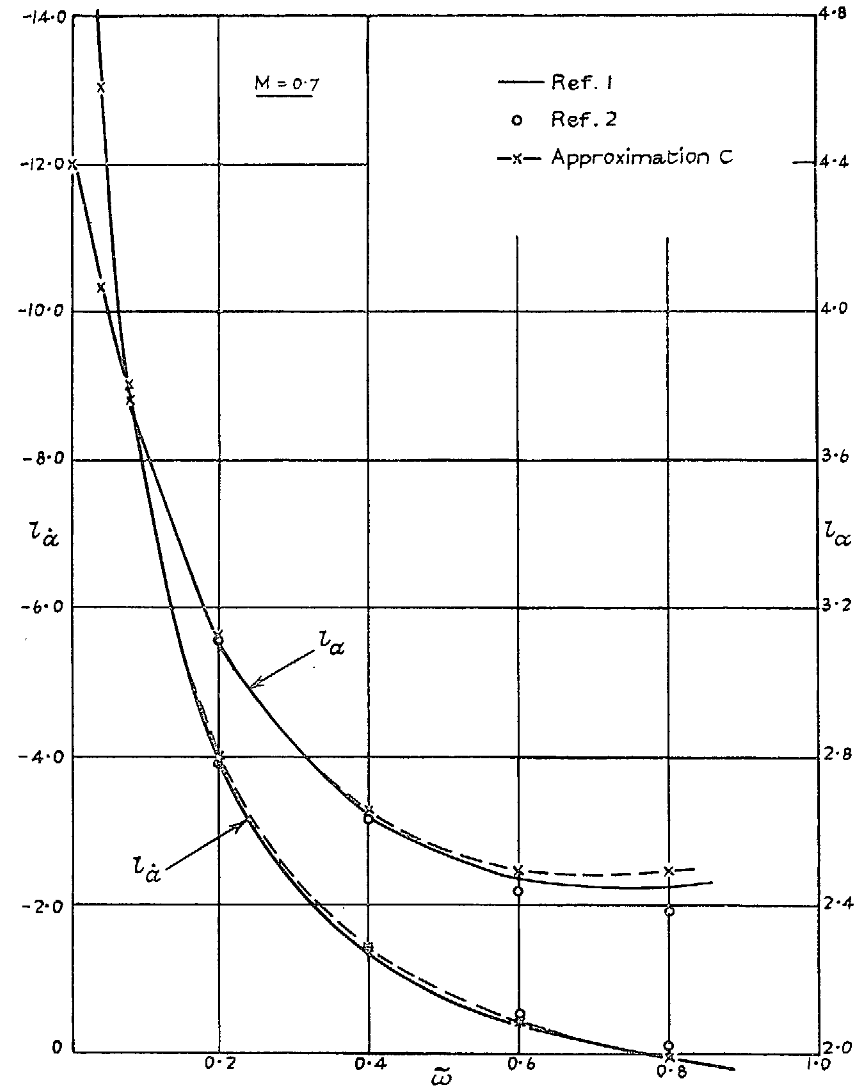

FIG. 2b. Mid-chord derivatives for $M = 0 \cdot 7$

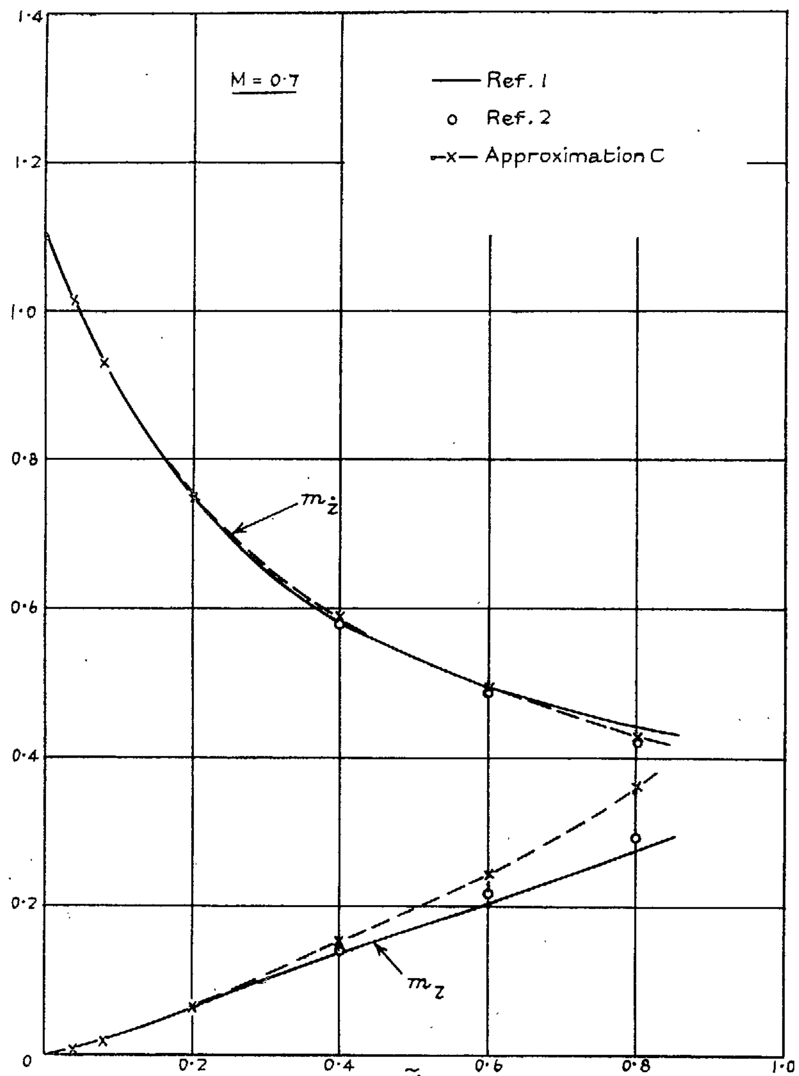

FIG. 2c. Mid-chord derivatives for $M = 0 \cdot 7$.

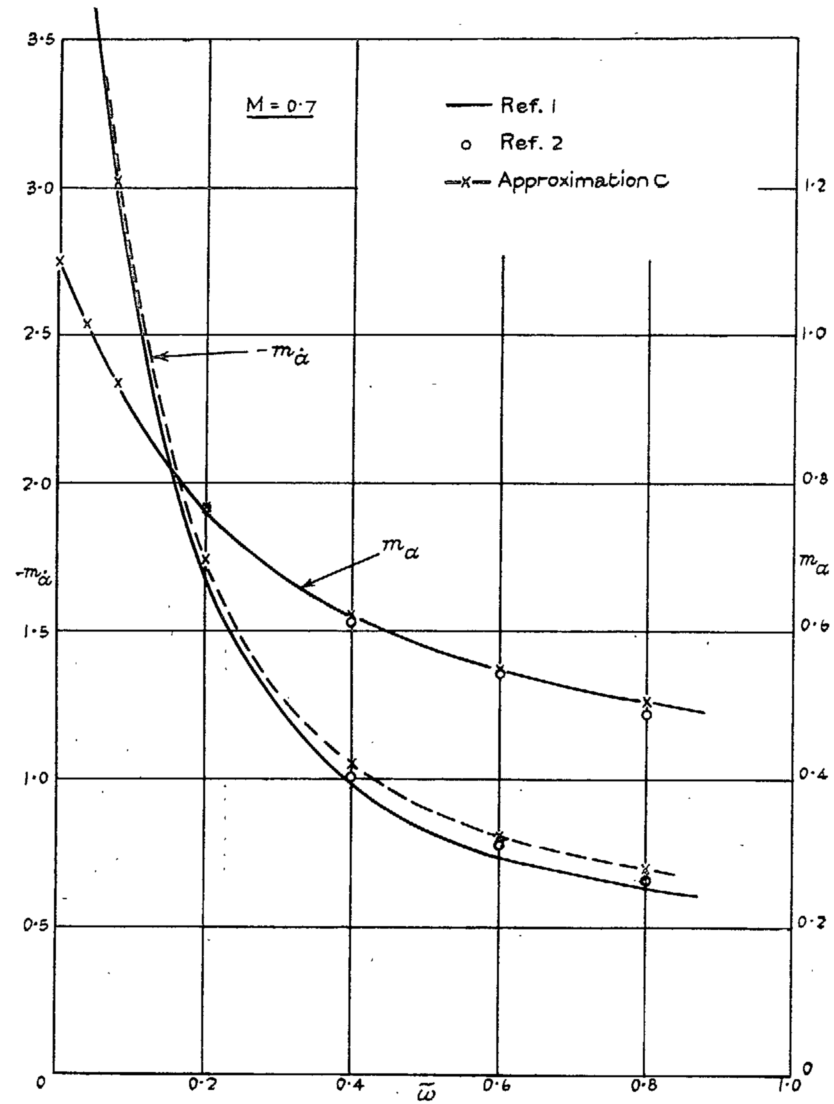

FIG. 2d. Mid-chord derivatives for $M = 0 \cdot 7$.

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