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Summary.—A theory is developed for estimating the effect of wind-tunnel walls on the air forces acting on an aerofoil oscillating in a subsonic airstream. It can only be applied for a range of frequencies well below the frequency at which transverse vibrations of the air stream may be induced. The possibility of resonance occurring for certain combinations of tunnel height, frequency of oscillation of aerofoil, wind speed and Mach number was first pointed out by Runyan and Watkins, and the present paper confirms their conclusions.

The method is applied to calculate aerodynamic derivatives for an oscillating flat plate in a wind tunnel of height equal to 4·75 aerofoil chord and a Mach number $M = 0{\cdot}7$. Results obtained are tabulated for comparison with the known theoretical free-stream values. It is shown that the influence of the walls is considerable even at frequencies of oscillation well below that of resonance.

Measurements of the pitching-moment damping coefficient for the RAE 104 aerofoil of 2-in. chord in the 9·5-in. × 9·5-in. Wind Tunnel have been made by Bratt and his results for $M = 0{\cdot}7$ differ appreciably from the corresponding estimated values given in this note. However, by the use of the equivalent profile method much better agreement may be obtained. This method is used to estimate the pitching-moment damping for a range of Mach numbers and low-frequency parameter values corresponding to those used in the tests. Fairly good agreement between estimated and measured values is obtained up to $M = 0{\cdot}8$, and the calculations indicate, in accordance with experiment, a loss of damping at the highest Mach numbers considered.

In the Appendix the properties of the series of Hankel functions

$$\Sigma_0 = \tfrac{1}{2} H_0^{(2)}(\mu|a|) + \sum_{n=1}^{\infty} (-1)^n H_0^{(2)}[\mu\sqrt{(a^2 + n^2\pi^2)}] ,$$

and

$$\Sigma_1 = \frac{H_1^{(2)}(\mu|a|)}{2|a|} + \sum_{n=1}^{\infty} (-1)^n \frac{H_1^{(2)}[\mu\sqrt{(a^2 + n^2\pi^2)}]}{\sqrt{(a^2 + n^2\pi^2)}} ,$$

which arise in the theory are discussed. Some results of general mathematical interest are obtained. In particular it is proved that the real parts of Σ_0 and Σ_1 are zero when $0 < \mu < 1$. When $|a| = 0$, these real parts degenerate to the well-known null series considered by Schlömilch.

1. *Introduction.*—The problem of a two-dimensional aerofoil oscillating between two parallel walls in incompressible flow has been considered by many writers[1,2,3], but little is known about the corresponding problem for subsonic compressible flow. Measurements of pitching-moment damping made by Bratt[4] at the National Physical Laboratory† differ considerably from the theoretical values for free-stream conditions. It is suggested in Ref. 1 that the discrepancy is mainly due to wall interference effects. In Ref. 5 Runyan and Watkins draw attention to the possibility of transverse vibrations of the flow in the tunnel at certain critical frequencies and

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†A full report on this work has not yet been issued.

suggest that at such frequencies tunnel corrections may be large. The present paper confirms their conclusion as to the existence of critical frequencies, but no attempt is made to estimate the interference effects under such critical conditions. The experimental results obtained by Bratt relate to frequencies well below the first critical resonance frequency, and the derivative values estimated in this report correspond to similar conditions. The method of calculation used is not sufficiently accurate at frequencies approaching resonance.

Over the limited range of frequencies considered it is found that the lift and moment damping derivatives are very sensitive to interference effects. The experimental values for the pitching-moment damping for the RAE 104 aerofoil agree in trend with theory up to $M = 0 \cdot 8$, when the aerofoil is represented by a flat plate, but better agreement with the actual values is obtained when the equivalent profile method of Ref. 6 is used. This method makes use of the measured steady motion characteristics of the aerofoil section and indirectly allows for thickness and boundary-layer effects. To some extent it also appears to take the effect of shock-wave—boundary-layer interaction into account.

Notation and Basic Formulae

$c(= 2l)$	Chord
U_0	Main stream velocity
ρ_0	Air density
Ox, Oz	Axes of co-ordinates (*see* Fig. 1)
$X(= lX = -l\cos\vartheta)$	Distance along Ox of Point P
$\mathbf{z}(= lZ/\beta)$	Distance normal to Ox
$z(= lz'e^{ipt})$	Downward displacement at mid-chord
$\alpha(= \alpha' e^{ipt})$	Angular displacement
$t(= lT/U_0)$	Time
$f\,(= p/2\pi)$	Frequency of oscillation
ζ	Downward displacement of P
Hl	Tunnel height
$M(\equiv U_0/U_s)$	Mach number
U_s	Velocity of sound
$\phi(= l\Phi e^{i(\lambda X + \omega T)})$	Velocity potential of disturbed motion
$w(= w' e^{i\omega T})$	Downwash distribution, $\partial\phi/\partial z$
$\bar{\omega}(= 2\omega = pc/U_0)$	Frequency parameter

$$\nu = \omega/\beta^2 ; \quad \varkappa = M\nu ; \quad \lambda = M^2\nu ; \quad \beta = \sqrt{(1 - M^2)} ; \quad h = H\beta$$

$k(= lKe^{i(\lambda X + \omega T)})$	Discontinuity in ϕ
$K(= \Phi_a - \Phi_b)$	Discontinuity in Φ
$W(=(w'/\beta)\,e^{-i\lambda X})$	Downwash $\partial\Phi/\partial Z$

K_n distributions

$$K_0 = 2\left\{\sin\vartheta + e^{i\nu\cos\vartheta}\left[X_0(\nu)\vartheta + 2\sum_{n=1}^{\infty}(-1)^n X_n(\nu)\frac{\sin n\vartheta}{n}\right]\right\}$$

$$= 2\pi X_0(\nu)\, e^{-i\nu X} \ldots X \geqslant 1$$

$$K_1 = \sin\vartheta + \frac{\sin 2\vartheta}{2}$$

$$K_n = \frac{\sin(n+1)\vartheta}{n+1} - \frac{\sin(n-1)\vartheta}{n-1} \ldots n \geqslant 2.$$

Γ_n distributions

$$\Gamma_n = i\nu K_n + \frac{\partial K_n}{\partial X}$$

$$\Gamma_0 = 2\left[C(\nu)\cot\frac{\vartheta}{2} + i\nu\sin\vartheta\right]$$

$$\Gamma_1 = -2\sin\vartheta + \cot\frac{\vartheta}{2} + i\nu\left(\sin\vartheta + \frac{\sin 2\vartheta}{2}\right)$$

$$\Gamma_n = -2\sin n\vartheta + i\nu\left[\frac{\sin(n+1)\vartheta}{n+1} - \frac{\sin(n-1)\vartheta}{n+1}\right] \ldots n \geqslant 2$$

$$C(\nu) = \frac{H_1^{(2)}(\nu)}{H_1^{(2)}(\nu) + iH_0^{(2)}(\nu)}$$

$$X_0(\nu) = C(\nu)J_0(\nu) + i[1 - C(\nu)]J_1(\nu)$$

$$X_n(\nu) = C(\nu)J_n(\nu) - i[1 - C(\nu)]J_n'(\nu)$$

$J_n(\nu) \equiv$ Bessel functions

$H_n^{(2)}(\nu) \equiv$ Hänkel functions.

Lift and Moment Integrals

$$R_n = \int_{-1}^{1} \Gamma_n\, e^{i\lambda X}\, dX$$

$$R_n' = \frac{\partial R_n}{\partial\lambda} = i\int_{-1}^{1} \Gamma_n\, e^{i\lambda X}\, X\, dX$$

$$R_0 = 2\pi\left\{C(\nu)[J_0(\lambda) - iJ_1(\lambda)] + \frac{i\nu}{2}[J_0(\lambda) + J_2(\lambda)]\right\}$$

$$R_1 = -\pi\left(1 - \frac{\nu}{\lambda}\right)[J_2(\lambda) + iJ_1(\lambda)]$$

$$R_n = (-i)^{n+1}\pi\left(1 - \frac{\nu}{\lambda}\right)\left[J_{n+1}(\lambda) + J_{n-1}(\lambda)\right] \ldots n \geqslant 2$$

$$R_0' = 2\pi\left\{C(\nu)[J_0'(\lambda) - iJ_1'(\lambda)] + \frac{i\nu}{2}[J_0'(\lambda) + J_2'(\lambda)]\right\}$$

$$R_1' = -\pi\left(1 - \frac{\nu}{\lambda}\right)[J_2'(\lambda) + iJ_1'(\lambda)] - \frac{\nu\pi}{\lambda^2}[J_2(\lambda) + iJ_1(\lambda)]$$

$$R_n' = (-i)^{n+1}\pi\left\{\left(1 - \frac{\nu}{\lambda}\right)\left[J_{n+1}'(\lambda) + J_{n-1}'(\lambda)\right] + \frac{\nu}{\lambda^2}\left[J_{n+1}(\lambda) + J_{n-1}(\lambda)\right]\right. \ldots n \geqslant 2.$$

2. *Basic Theory*.—An aerofoil of chord $c(= 2l)$ is assumed to be describing simple harmonic pitching and plunging oscillations in a wind tunnel of height Hl as illustrated by the following diagram :

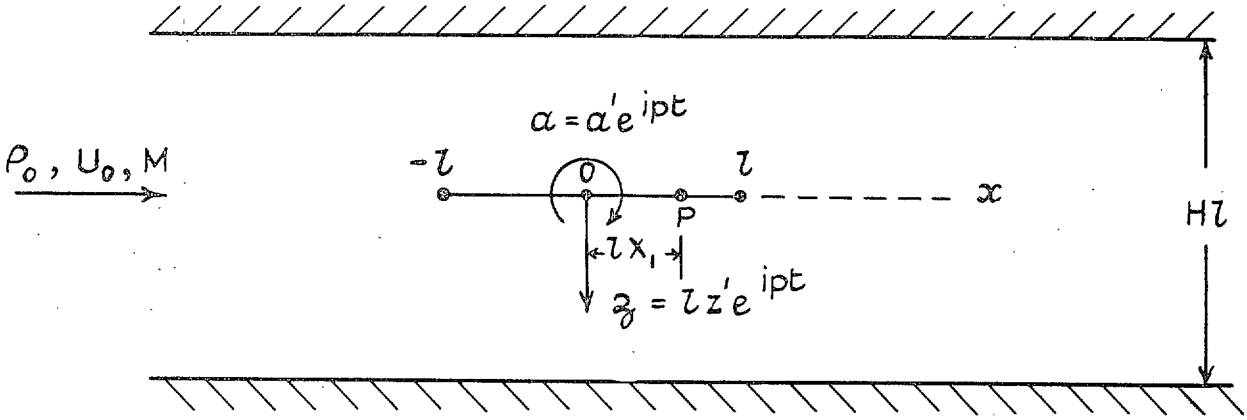

FIG. 1.

In the usual complex notation the downward displacement of the mid-chord point O is denoted by $z(\equiv lz'e^{ipt})$, and the pitching oscillation about the mid-chord axis is $\alpha(\equiv \alpha' e^{ipt})$, where $p/2\pi$ represents the frequency and t denotes the time. The downward displacement ζ of the point P on the aerofoil is then defined by

$$\zeta = l(z' + X_1\alpha')e^{ipt} . \qquad \ldots \qquad (1)$$

The corresponding downwash $w(\equiv w' e^{ipt})$ is given by

$$w = U_0[i\omega(z' + X_1\alpha') + \alpha']e^{ipt} \qquad \ldots \qquad (2)$$

where $\omega \equiv pl/U_0$ and U_0 is the velocity of the undisturbed air stream.

Let $\qquad x = lX, \quad \mathbf{z} = lZ/\beta, \quad t = lT/U_0 \qquad \ldots \qquad (3)$

where $\beta = \sqrt{(1 - M^2)}$ and M is the Mach number. Then, if ϕ is the velocity potential of the disturbance caused by the oscillating aerofoil, the downwash is

$$w = \frac{\partial\phi}{\partial \mathbf{z}} = \frac{\beta}{l}\frac{\partial\phi}{\partial Z} . \qquad \ldots \qquad (4)$$

Let us now write

$$\phi = l\Phi\, e^{i(\lambda X + \omega T)} , \qquad \ldots \qquad (5)$$

where $\lambda = M^2\nu$ and $\nu = \omega\beta^{-2}$. As in Ref. 1, it may then be shown that Φ satisfies the wave equation

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Z^2} + \varkappa^2\Phi = 0, \qquad \ldots \qquad (6)$$

where $\varkappa = M\nu$. It also follows from (2), (4) and (5) that the corresponding boundary condition on the aerofoil in the new co-ordinates is

$$W = \frac{\partial \Phi}{\partial Z} = \frac{w' e^{-i\lambda X}}{\beta} \quad \ldots \quad (7)$$

From Euler's equations of motion it may also be deduced that the lift distribution $\bar{l}(X)$ is given by

$$\bar{l}(X) = \rho_0 U_0 \Gamma \, e^{i(\lambda X + \omega T)} , \quad \ldots \quad (8)$$

where

$$\Gamma = i\nu K + \frac{\partial K}{\partial X} \quad \ldots \quad (9)$$

and $K(\equiv \Phi_a - \Phi_b)$ is the jump in the modified velocity potential across the sheet of discontinuity representing the aerofoil and its wake. Since there is no pressure discontinuity in the wake

$$i\nu K + \frac{\partial K}{\partial X} = 0. \quad \ldots \quad (10)$$

The solution of (6) for the boundary conditions specified by (2) and (4) has already been obtained for free-stream conditions. In Ref. 7 it is shown that the problem reduces to that of solving the integral equation

$$2\pi W(X_1, Z_1) = -\int_{-1}^{\infty} K(X) \frac{\partial^2}{\partial Z_1^2} \left\{ \frac{\pi i}{2} \left\{ H_0^{(2)}[\varkappa\{(X - X_1)^2 + Z_1^2\}^{1/2}] \right\} dX \quad \ldots \quad (11)$$

where W is known on the aerofoil. From (10) and (11) the required distribution $K(X)$ may be determined.

In the case of an aerofoil in a wind tunnel, the presence of the tunnel walls must be taken into account. This is done by the introduction of a system of image distributions at $z = \pm nHl$ where $n = 1, 2, \ldots \infty$. In view of (3) the image positions in the new co-ordinates would be defined by $Z = \pm nh$, where $h \equiv H\beta$. It then follows that $K(X)$ must satisfy the integral equation

$$2\pi W(X_1, Z_1) = -\int_{-1}^{\infty} K(X) \frac{\partial^2}{\partial Z_1^2} S_0(X - X_1, Z_1)\, dX, \quad \ldots \quad (12)$$

where

$$S_0(X - X_1, Z_1) = \frac{\pi i}{2}\left\{ H_0^{(2)}\left[\varkappa\{(X - X_1)^2 + Z_1^2\}^{1/2}\right] + \sum_{n=1}^{\infty} (-1)^n H_0^{(2)}\left[\varkappa\{(X - X_1)^2 + (nh - Z_1)^2\}^{1/2}\right] \right.$$
$$\left. + \sum_{n=1}^{\infty} (-1)^n H_0^{(2)}\left[\varkappa\{(X - X_2)^2 + (nh + Z_1)^2\}^{1/2}\right] \right\} \quad (13)$$

includes the effect of the image system. Since

$$\frac{\partial^2 S_0}{\partial Z_1^2} + \frac{\partial^2 S_0}{\partial X^2} + \varkappa^2 S_0 = 0, \quad \ldots \quad (14)$$

it may be deduced from (12) by integration by parts that

$$2\pi W(X_1, Z_1) = \int_{-1}^{\infty} \left[\varkappa^2 K S_0 - \frac{\partial K}{\partial X}\frac{\partial S_0}{\partial X} \right] dX . \quad \ldots \quad (15)$$

When $Z_1 = 0$, W on the aerofoil is given by (7) ; the problem is then to find a distribution K which satisfies the wake condition (10) and the following equation, namely

$$2\pi \frac{w' e^{-i\lambda X_1}}{\beta} = \int_{-1}^{\infty} \left[\varkappa^2 K S_0 - \frac{\partial K}{\partial X}\frac{\partial S_0}{\partial X} \right] dX , \quad \ldots \quad (16)$$

where now

$$S_0 = \frac{\pi i}{2}\left\{H_0^{(2)}(\varkappa|X - X_1|) + 2\sum_{n=1}^{\infty}(-1)^n H_0^{(2)}(\varkappa\sqrt{\{(X - X_1)^2 + n^2h^2\}})\right\}$$

$$= \pi i \Sigma_0 \qquad \ldots \qquad (17)$$

and

$$\frac{\partial S_0}{\partial X} = -\pi i\varkappa(X - X_1)\left\{\frac{H_1^{(2)}(\varkappa|X - X_1|)}{2|X - X_1|} + \sum_{n=1}^{\infty}(-1)^n \frac{H_1^{(2)}(\varkappa\sqrt{\{(X - X_1)^2 + n^2h^2\}})}{\sqrt{\{(X - X_1)^2 + n^2h^2\}}}\right\}$$

$$\equiv -\pi i\varkappa(X - X_1)\frac{\pi}{h}\Sigma_1 \qquad \ldots \qquad (18)$$

The properties of the series represented by Σ_0 and Σ_1 are discussed in the Appendix. It is shown that, for given values of $X - X_1$, the series Σ_0 becomes divergent when $\varkappa h = (2m - 1)\pi$ where $m = 1, 2, 3$, etc., and that Σ_1 has discontinuities in slope at these critical values. The series Σ_0 has also been considered by Runyan and Watkins[5] and they suggest that some kind of 'resonance' effect should arise when Σ_0 becomes infinite.

The parameter $\varkappa h/\pi$ is independent of aerofoil chord and depends only on the frequency, the tunnel height, the velocity of sound, U_s, and the speed of flow in the tunnel. In terms of these variables the first critical condition occurs when the frequency

$$f = U_s\frac{\sqrt{(1 - M^2)}}{2Hl} = \frac{U_s\beta}{2Hl} \qquad \ldots \qquad (19)$$

where Hl is the tunnel height.

The numerical results given in this paper correspond to values of frequency well below the first critical and refer to the range $O < \varkappa h \ll \pi$ and small values of $\varkappa$. It is shown in the Appendix that

$$\left.\begin{aligned} S_0 &\rightarrow \log_e \tanh\frac{\pi|X - X_1|}{2h} + O(\varkappa^2) \\ \frac{\partial S_0}{\partial X} &\rightarrow \frac{\pi}{h}\operatorname{cosech}\frac{\pi(X - X_1)}{h} - \frac{\varkappa^2(X - X_1)}{2}\log_e \tanh\frac{\pi|X - X_1|}{2h} + O(\varkappa^4)\end{aligned}\right\} . \qquad (20)$$

Hence, if $\varkappa$ is such that terms of order $\varkappa^4$ and greater can be neglected, equation (16) may be expressed in the approximate form

$$\frac{2\pi w' e^{-i\lambda X_1}}{\beta} = \frac{\varkappa^2}{2}\left(\frac{\partial}{\partial X_1} + 1\right)\int_{-1}^{\infty} K\bar{S}_0\,dX - \int_{-1}^{\infty}\frac{\partial K}{\partial X}\frac{\partial \bar{S}_0}{\partial X}\,dX \qquad \ldots \qquad (21)$$

where

$$\left.\begin{aligned} \bar{S}_0 &= \log_e \tanh\frac{\pi|X - X_1|}{2h} \\ \frac{\partial \bar{S}_0}{\partial X} &= \frac{\pi}{h}\operatorname{cosech}\frac{\pi(X - X_1)}{h}\end{aligned}\right\} . \qquad \ldots \qquad (22)$$

If terms of order $\varkappa^2$ are also neglected, equation (21) reduces to

$$\frac{2\pi w' e^{-i\lambda X_1}}{\beta} = -\frac{\pi}{h}\int_{-1}^{\infty}\frac{\partial K}{\partial X}\operatorname{cosech}\frac{\pi(X - X_1)}{h}\,dX \qquad \ldots \qquad (23)$$

which corresponds in form to the integral equation which arises in the incompressible flow problem, namely

$$2\pi w' = -\frac{\pi}{H}\int_{-1}^{\infty} \frac{\partial K}{\partial X} \operatorname{cosech} \frac{\pi(X - X_1)}{H}\, dX\,. \quad \ldots \quad (24)$$

The solution of this equation is given in Ref. 1, and by using similar methods the solution of (23) may readily be found. In the present paper equation (21) is treated similarly to obtain a solution which includes terms of order $\varkappa^2$. It is thought that solutions of (21) obtained for small values of $\varkappa$ would approximate closely to those given by equation (16) provided $\varkappa h$ is much less than π.

3. *Method of Solution.*—As in the case of incompressible flow, the integral equation (21) is reduced to a system of linear simultaneous equations. It is assumed that the distributions of Γ and K can be represented in the form

$$\left.\begin{aligned} \Gamma &= U_0[C_0\Gamma_0 + C_1\Gamma_1 + \text{etc.}] \\ \text{and} \qquad K &= U_0[C_0K_0 + C_1K_1 + \text{etc.}] \end{aligned}\right\} \quad \ldots \quad (25)$$

where Γ_n, K_n are defined in the list of symbols.

The corresponding downwash W is given similarly by

$$W = U_0[C_0W_0 + C_1W_1 \ldots], \quad \ldots \quad (26)$$

where W_n is the downwash distribution corresponding to K_n as defined by (21), which is regarded as being equivalent to (16) for small values of $\varkappa$.

By the use of (9) and integration by parts, it may be shown that

$$\int_{-1}^{\infty} K\bar{S}_0\, dX = \frac{1}{\nu^2}\left(\frac{\partial}{\partial X_1} - i\nu\right)\int_{-1}^{1} \Gamma\bar{S}_0\, dX + \frac{1}{\nu^2}\int_{-1}^{\infty} \frac{\partial K}{\partial X}\frac{\partial \bar{S}_0}{\partial X}\, dX\,, \quad \ldots \quad (27)$$

where the first integral on the right, namely

$$\begin{aligned} \int_{-1}^{1} \Gamma\bar{S}_0\, dX &= \int_{-1}^{1} \Gamma \log_e \tanh \frac{\pi|X - X_1|}{2h}\, dX \\ &= \int_{-1}^{1} \Gamma\left[\log_e \frac{\pi|X - X_1|}{2h} - \frac{\pi^2}{12h^2}(X - X_1)^2 + \ldots\right] dX\,. \quad \ldots \quad (28) \end{aligned}$$

The second integral occurs in the incompressible flow problem and has been dealt with in detail in Ref. 1. From (21), (27) and (28), and by the use of previous work, it may then be deduced that

$$\left.\begin{aligned} \left(\frac{\partial}{\partial X_1} - i\nu\right)\int_{-1}^{1} \Gamma_0\bar{S}_0\, dX &= 2\pi(B_0 + B_1\cos\vartheta_1 + B_2\cos 2\vartheta_1) \\ \int_{-1}^{\infty} \frac{\partial K_0}{\partial X}\frac{\pi}{h} \operatorname{cosech} \frac{\pi(X_1 - X)}{h}\, dX &= 2\pi(A_0 + A_1\cos\vartheta_1 + \text{etc.}) \end{aligned}\right\} \quad (29)$$

where, if only first-order terms in $g(\equiv \pi^2/12h^2)$ are retained,

$$\left.\begin{aligned}
A_0 &= 1 + J_0(\nu)F(\nu) - 2g\left(\frac{1}{2} + \frac{C(\nu)}{i\nu}\right) \\
n \geqslant 1 \ldots A_n &= 2i^n J_n(\nu)F(\nu) \\
B_0 &= C(\nu)[1 - g - i\nu(L - g)] + \frac{\nu^2}{2}(L - g) + \frac{\nu^2 g}{8} \\
B_1 &= i\nu C(\nu)(1 - g) - i\nu(1 - g) + 2gC(\nu) \\
B_2 &= -\frac{i\nu}{4}[i\nu(1 - g) - 2gC(\nu)] \\
F(\nu) &= 2gX_0(\nu)e^{-i\nu}\left(1 - \frac{i}{\nu}\right) - i\nu X_0(\nu)[P - Q] \\
P &= \int_1^\infty \frac{e^{-i\nu\xi}}{\xi}\,d\xi \\
Q &= \int_1^\infty \frac{\pi}{h} e^{-i\nu\xi} \operatorname{cosech} \frac{\pi\xi}{h}\,d\xi \\
L &\equiv \log_e(\pi/4h)\,.
\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

It follows from equations (21) and (27) that

$$\begin{aligned}
W_0 = \sum_{n=0} A_n \cos n\vartheta_1 \\
&+ \frac{\varkappa^2}{2\nu^2}\Big\{B_0 - A_0 - B_1 + A_1 - 3A_3 + (B_1 - A_1 - 4B_2 + 4A_2)\cos\vartheta_1 \\
&+ (B_2 - A_2 - 6A_3)\cos 2\vartheta_1\Big\} + \text{etc.} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)
\end{aligned}$$

Terms of higher order than $\cos 2\vartheta_1$ are neglected in (31) and in the subsequent analysis.

Since the distributions K_1, K_2 are independent of frequency (21) yields simpler expressions for $W_1, W_2, \ldots W_n$. Thus

$$W_1 = \frac{1}{2} - \frac{g}{2} + \cos\vartheta_1 + \frac{\varkappa^2}{8}\left[L - \frac{5g}{4} - \frac{5}{2}(1 - g)\cos\vartheta_1 - \frac{1 + g}{2}\cos 2\vartheta_1\right] \quad (32)$$

$$W_2 = \frac{g}{2} - \frac{\varkappa^2}{8}\left(L - \frac{2g}{3}\right) + \frac{\varkappa^2}{4}(1 - g)\cos\vartheta_1 + \left[1 - \frac{\varkappa^2}{12}\left(1 - \frac{3g}{4}\right)\right]\cos 2\vartheta_1 \quad (33)$$

where $L \equiv \log_e(\pi/4h)$.

It is assumed that sufficient accuracy is obtained by the use of only three terms in (25) and (26), and that therefore

$$\begin{aligned}
W &= U_0[C_0W_0 + C_1W_1 + C_2W_2] \\
&= U_0[P_0 + P_1\cos\vartheta_1 + P_2\cos 2\vartheta_1] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)
\end{aligned}$$

where

$$P_0 = C_0\left[A_0 + \frac{\varkappa^2}{2\nu^2}(B_0 - A_0 - B_1 + A_1 + 3A_3)\right]$$

$$+ C_1\left[\frac{1}{2} - \frac{g}{2} + \frac{\varkappa^2}{8}\left(L - \frac{5g}{4}\right)\right]$$

$$+ C_2\left[\frac{g}{2} - \frac{\varkappa^2}{8}\left(L - \frac{2g}{3}\right)\right]$$

$$P_1 = C_0\left[A_1 + \frac{\varkappa^2}{2\nu^2}(B_1 - A_1 - 4B_2 + 4A_2)\right]$$

$$+ C_1\left[1 - \frac{5\varkappa^2}{16}(1 - g)\right] + C_2\frac{(1 - g)k^2}{4}$$

$$P_2 = C_0\left[A_2 + \frac{\varkappa^2}{2\nu^2}(B_2 - A_2 + 6A_3)\right]$$

$$- C_1\frac{\varkappa^2}{16}(1 + g) + C_2\left[1 - \frac{\varkappa^2}{12}\left(1 - \frac{3g}{4}\right)\right]. \quad \ldots \ldots \ldots \quad (35)$$

It should be noted that higher powers of g than the first are neglected in the above formulae.

For the oscillating flat plate shown in Fig. 1, the downwash distribution w is given by (2). It then follows from (7) that, on the aerofoil,

$$W = \frac{U_0}{\beta}\left(\bar{a} - \omega\alpha'\frac{\partial}{\partial\lambda}\right)e^{i\lambda}\cos\vartheta_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (36)$$

where $X_1 = -\cos\vartheta_1$ and $\bar{a} \equiv \alpha' + i\omega z'$. Now

$$e^{i\lambda\cos\vartheta_1} = J_0(\lambda) + 2\sum_{n=1} i^n J_n \cos n\,\vartheta_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (37)$$

and, by substituting (37) in (36), it follows, since (34) and (36) must be identical, that

$$\left.\begin{aligned} P_0 &= \frac{1}{\beta}\left(\bar{a} - \omega\alpha'\frac{\partial}{\partial\lambda}\right)J_0(\lambda) \\ P_n &= \frac{2i^n}{\beta}\left[\bar{a} - \omega\alpha'\frac{\partial}{\partial\lambda}\right]J_n(\lambda) \ldots n \geqslant 1 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (38)$$

By combining (35) and (38), a set of equations is obtained from which the coefficients C_0, C_1, C_n may be determined in the form

$$C_n = a_n z' + b_n \alpha'$$

where a_n and b_n are numerical coefficients.

For the relatively low frequencies considered in this report sufficient accuracy is obtained when only three equations are retained; $C_3 \ldots C_n$ being assumed zero. When C_0, C_1 and C_2 have been determined the lift distribution $\bar{l}(X)$ is given by (8) and (25) in the form

$$\bar{l}(X) = \rho U_0^2 [C_0 \Gamma_0 + C_1 \Gamma_1 + C_2 \Gamma_2] \, e^{i(\lambda X + \omega T)} \quad \ldots \quad (39)$$

Let
$$\left.\begin{aligned} R_n &= \int_{-1}^{1} \Gamma_n \, e^{i\lambda X} \, dX \\ \text{and} \quad R_n' &= i \int_{-1}^{1} \Gamma_n \, e^{i\lambda X} \, X \, dX \end{aligned}\right\} , \quad \ldots \quad (40)$$

where $R_n' = \partial R_n / \partial \lambda$ when Γ_n is assumed to be independent of λ.

The total lift L and the pitching moment M about the half-chord axis are then defined by

$$\left.\begin{aligned} L &= \rho_0 l U_0^2 \sum_{n=0}^{n=2} C_n R_n \, e^{i\omega T} \\ M &= \rho_0 l^2 U_0^2 \sum_{n=0}^{n=2} C_n i R_n' \, e^{i\omega T} \end{aligned}\right\} \quad \ldots \quad (41)$$

where R_n, R_n' are the known functions of Mach number and frequency listed in section 1. The coefficients C_n are linearly dependent on z' and α' and given by (35) and (38). In terms of the chord $c(= 2l)$ as standard length, formulae (41) are expressed in the usual non-dimensional form

$$\frac{L}{\rho_0 c U_0^2} = (l_z + i\bar{\omega} l_{\dot{z}}) \frac{z}{c} + (l_\alpha + i\bar{\omega} l_{\dot{\alpha}})\alpha$$

$$\frac{M}{\rho_0 c^2 U_0^2} = (m_z + i\bar{\omega} m_{\dot{z}}) \frac{z}{c} + (m_\alpha + i\bar{\omega} m_{\dot{\alpha}})\alpha \quad \ldots \quad (42)$$

where $\bar{\omega} = 2\omega = pc/U_0$. When C_0, C_1 and C_2 are known, the numerical values of the aerodynamic coefficients l_z, $l_{\dot{z}}$, etc., may be derived by a comparison of formulae (41) and (42).

For low values of the frequency parameter, approximate formulae for the derivatives may be obtained by neglecting terms of second order in frequency. Equations (35) and (38) then yield

$$\left.\begin{aligned} \frac{\bar{a}}{\beta} &= C_0 A_0 + C_1 \left(\frac{1}{2} - \frac{g}{2}\right) + \frac{C_2 g}{2} \\ \frac{i(\lambda \bar{a} - \omega \alpha')}{\beta} &= C_0 A_1 + C_1 \\ 0 &= C_0 A_2 + C_2 \end{aligned}\right\} \quad \ldots \quad (43)$$

from which the limiting forms of C_0, C_1 and C_2 may be determined. After substitution in (41), and by the use of (42), the following approximate formulae may be derived :—

$$
\left.
\begin{aligned}
l_z &= m_z = 0 \\
l_{\dot{z}} &= l_\alpha = \frac{\pi}{\beta}(1 + 2g) \\
l_{\dot{\alpha}} &= \frac{\pi}{2\beta^3}\left[\frac{(3\beta^2 - 1)(1 + g)}{2} - (1 + 4g)\bar{E}\right] \\
m_{\dot{z}} &= m_\alpha = \frac{\pi}{4\beta}(1 + g) \\
m_{\dot{\alpha}} &= \frac{\pi}{8\beta^3}\left[(1 + 3g)\,\bar{E} + (1 - \beta^2)\left(1 + \frac{3g}{2}\right)\right] \\
\text{where} \quad \bar{E} &= \log_e\left[\frac{2\left(1 + \cosh\frac{\pi}{h}\right)}{\sinh\frac{\pi}{h}}\right]
\end{aligned}
\right\} \quad \ldots \quad (44)
$$

It should be noted that these limiting values of $l_{\dot{\alpha}}$ and $m_{\dot{\alpha}}$ are finite and not infinite as for free-stream conditions.

The values of the derivatives obtained for $M = 0 \cdot 7$, and $H = 9 \cdot 5$ are given in Table 2 and plotted in Figs. 4a, 4b, 4c and 4d. This particular value of H was chosen to correspond to Bratt's tests in the $9\frac{1}{2} \times 9\frac{1}{2}$-in. Wind Tunnel on an aerofoil of 2-in. chord. It is thought that for this case the values obtained are fairly reliable up to $\bar{\omega} = 0 \cdot 4$; the first critical value of $\bar{\omega}$ for resonance being $\bar{\omega} = 0 \cdot 67$. The method of calculation used cannot be applied for $\bar{\omega}$ values near or higher than the critical value for the first resonance.

4. *Further Applications.*—In Figs. 5 and 6, Bratt's experimental values for the pitching-moment damping derivative for an RAE 104 aerofoil oscillating about an axis at $0 \cdot 445$ chord are compared with the estimated values. It was found that agreement between theory and experiment, even when in allowance was made for wall interference, still remained unsatisfactory. This was not unexpected as similar discrepancies had been obtained for incompressible flow[1]. In the latter case satisfactory agreement was obtained by the use of the equivalent profile method of Ref. 6 which makes an approximate allowance for thickness and viscous effects. The same method was therefore used to obtain the final results given in this paper.

The basis of the scheme is to represent the aerofoil by a thin line which gives, according to the linearized theory for steady flow, the measured lift distribution (steady derivatives if the distribution is unknown). Let the measured lift distribution for an incidence α be represented by

$$l(\vartheta) = \rho_0 U_0^2 [A(\alpha)\bar{\Gamma}_0 + B(\alpha)\bar{\Gamma}_1 + \text{etc.}] \quad \ldots \quad (45)$$

where

$$\bar{\Gamma}_0 = 2\cot\frac{\vartheta}{2}, \quad \bar{\Gamma}_1 = -2\sin\vartheta + \cot\frac{\vartheta}{2}, \quad \bar{\Gamma}_n = -2\sin\vartheta. \quad \ldots \quad (46)$$

Then, if only the first two terms are used, the equivalent thin surface would be defined by

$$\frac{2\mathbf{z}}{c} = \beta\left[A + B + X\left(A + \frac{B}{2}\right) - \frac{BX^2}{2}\right] - 2\bar{h}\alpha \quad \ldots \quad (47)$$

where $\bar{h}c$ is the distance of the axis of rotation behind the leading edge (*see* Fig. 2). It also follows from (45) that the lift and pitching-moment coefficients referred to quarter-chord are

$$\left.\begin{aligned} C_L &= 2\pi A(\alpha) = 2\pi A'\alpha \\ C_M(\tfrac{1}{4}) &= \frac{\pi B(\alpha)}{4} = \frac{\pi B'\alpha}{4} \end{aligned}\right\} . \qquad (48)$$

Hence from a knowledge of C_L and C_M, the equivalent profile defined by (47) could be determined. For the purposes of this illustration the lift distribution is represented by two terms only, and in the more general case of an aerofoil with a flap it seems that one more term would be sufficient provided the frequency parameters considered are not too high.

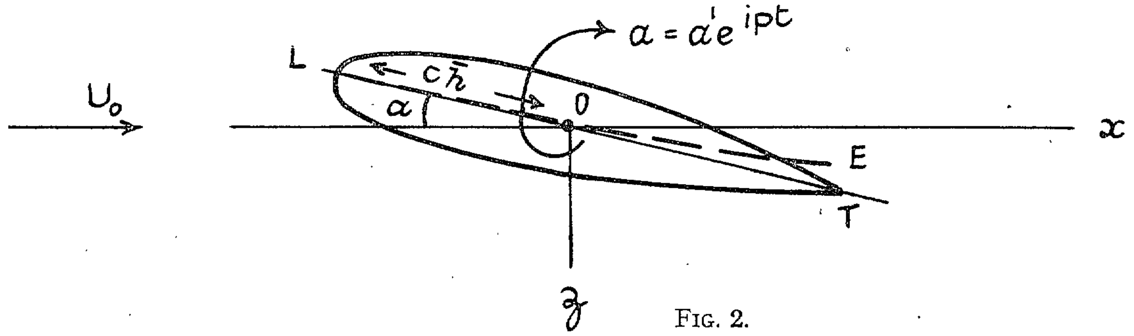

FIG. 2.

For an aerofoil at an incidence α, as shown in Fig. 2 above, the equivalent profile is represented by the line LE. As the aerofoil oscillates, LE is assumed to change shape in phase with incidence. The corresponding downwash is then given generally by

$$\begin{aligned} w &= \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right) \mathbf{z} \\ &= \left(\dot{\alpha}\frac{\partial}{\partial \alpha} + U_0 \frac{\partial}{\partial x}\right) \mathbf{z} \end{aligned}$$

and, hence,

$$w(x_1) = \alpha\beta U_0[p_0 + p_1 \cos \vartheta_1 + p_2 \cos 2\vartheta_1] \qquad (49)$$

where

$$\left.\begin{aligned} p_0 &= A' + \frac{B'}{2} + i\omega\left(A' + \frac{3B'}{4} - \frac{2\bar{h}}{\beta}\right) \\ p_1 &= B' - i\omega\left(A' + \frac{B'}{2}\right) \\ p_2 &= -\frac{i\omega B'}{4} \end{aligned}\right\} ; \qquad (50)$$

and the symbols A' and B' represent the slopes of A and B respectively at zero incidence. It follows that W would be given by (34) where now

$$\begin{aligned} P_0 &= \alpha[p_0 J_0(\lambda) + ip_1 J_1(\lambda) - p_2 J_2(\lambda)] \\ P_1 &= 2i\alpha\left[p_0 J_1(\lambda) - \frac{ip_1}{2}\Big(J_0(\lambda) - J_2(\lambda)\Big) + \frac{p_2}{2}\Big(J_1(\lambda) - J_3(\lambda)\Big)\right] \\ P_2 &= -2\alpha\left[p_0 J_2(\lambda) - \frac{ip_1}{2}\Big(J_1(\lambda) - J_3(\lambda)\Big) - \frac{p_2}{2}\Big(J_0(\lambda) + J_4(\lambda)\Big)\right] \end{aligned}$$

and so on. $\qquad$ (51)

The above formulae replace those obtained for the flat plate in section 3. By combining (35) with (51), a set of equations is obtained from which the coefficients C_0, C_1 and C_2 can be determined. The values of A' used were obtained from pressure measurements given in Ref. 8* for a 5-in. chord aerofoil of the same section, and the values of B' were chosen to give the measured value of the steady pitching moment about the $0 \cdot 445c$ axis obtained from Bratt's results by extrapolation to zero frequency. Unfortunately his apparatus could not be used to measure steady loads and so the true values of A' and B' for the oscillated aerofoil are unknown. However, it was thought that it would be worth while to attempt calculation of the derivatives with the steady data available, in order to illustrate the method. Calculations were done for several Mach numbers and the values of A' and B' used are given in Table 1 below

TABLE 1

Values of A' and B'

M	A'	B'
0·7	1·062	0·2786
0·8	1·359	0·1051
0·825	1·254	0·1687
0·85	1·108	−0·1615
0·875	0·926	0·0309
0·9	0·730	0·5809

Unfortunately it is not certain that the values for a 5-in. chord aerofoil given in Ref. 8 are directly applicable to the 2-in. chord aerofoil used in the oscillatory tests. However, the agreement between the estimated values and the measured pitching-moment damping is fairly good (*see* Fig. 6). Even the experimental drop in damping at high M is indicated by the method of calculation suggested. The above comparison illustrates the possibilities of the scheme, but, in view of the uncertainties mentioned above, further calculations for an aerofoil with accurately known steady characteristics are required to test the validity of the method.

Concluding Remarks.—This paper draws attention to the importance of wind-tunnel interference in oscillatory tests on two-dimensional aerofoils and emphasizes the difficulty of interpreting wind-tunnel data for free-flight conditions. The effect of the tunnel walls on the derivatives $l_{\dot{\alpha}}$ and $m_{\dot{\alpha}}$ is very important at low frequency parameters in the range of interest in stability research. To estimate the effect on flutter derivatives the present theory would have to be extended to higher frequencies and a method would have to be developed for obtaining solutions near the critical frequencies for 'resonance.' Near such frequencies one would expect interference effects to be large. In the three-dimensional case, similarly, interference effects would probably be important near the critical frequencies for transverse vibrations in the wind tunnel.

To obtain realistic estimates of flutter and stability derivatives at very high subsonic speeds, it appears that thickness and viscous effects must be taken into account. The equivalent profile method allows for such effects and seems to be fairly reliable at high as well as low speeds (*see* Fig. 6). It may also be used to estimate control-surface derivatives for high speeds, but as yet it has only been shown to be satisfactory at low speeds[9]. If the method is to be applied to the

*The results for a 2-in. chord aerofoil given in Ref. 8, thought to be less reliable than those given for the 5-in. chord aerofoil, were not used. The values of A' for $M = 0 \cdot 875$ and $M = 0 \cdot 9$ given in Table 1 may also be inaccurate as they were estimated by extrapolation.

best advantage, then it is essential that the characteristics of the aerofoil control system in steady motion should first be accurately determined experimentally to provide reliable values of A', B', etc., for use in the calculation of the oscillatory derivatives.

Acknowledgement.—The numerical results given in this paper were computed by Miss Sylvia W. Skan of the Aerodynamics Division, N.P.L.

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APPENDIX

Summation of Series

(i) *Series* Σ_1.—The series Σ_1 defined by (18) is summed by the use of the theorem of residues. Write $\mu = \varkappa h/\pi$ and $a = \pi(X - X_1)/h$. Then consider the integral of the function $F(Z)$, where

$$F(Z) = \frac{e^{2m\pi i Z} H_1^{(2)} \{\mu\sqrt{(a^2 + Z^2)}\}}{\sqrt{(a^2 + Z^2)} \,.\, \sin Z}, \quad \ldots \quad (52)$$

round the semi-circular contour shown in Fig. 3.

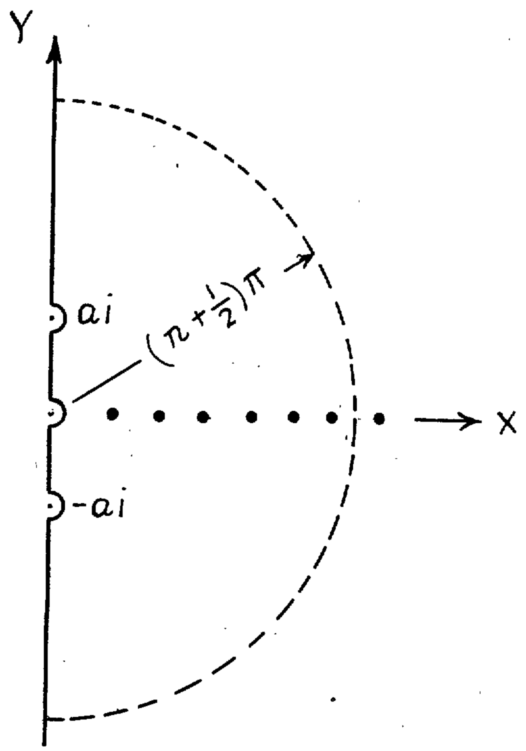

FIG. 3.

Since $\sin Z = 0$ when $Z = 0, \pi, 2\pi$, etc., $F(Z)$ will have poles at these points. The radius of he semi-circle is such that the contour passes between two of these singularities.

Since

$$H_1^{(2)}\{\mu\sqrt{(a^2 + Z^2)}\} \rightarrow \frac{2i}{\pi\mu\sqrt{(a^2 + Z^2)}} \quad \ldots \quad (53)$$

hen $Z \rightarrow \pm ai$, the function F will also have poles at these points. Furthermore, it may be hown that the integral round the semi-circle will vanish when $2m - 1 < \mu < 2m + 1$. For tegral values of m the residue R_n at the pole $Z = n\pi$ is given by

$$R_n = \frac{(-1)^n H_1^{(2)}\{\mu\sqrt{(a^2 + n^2\pi^2)}\}}{\sqrt{(a^2 + n^2\pi^2)}}. \quad \ldots \quad (54)$$

ence it follows that

$$2\pi i \sum_{n=1}^{\infty} R_n + \int_{-i\infty}^{i\infty} F\, dZ = 0 \quad \ldots \quad (55)$$

here the integral along the y-axis is taken round the poles at $Z = -ai, 0, ai$. On integration, and after some reduction, it may then be shown that

$$\Sigma_1 = \frac{H_1^{(2)}(\mu|a|)}{2|a|} + \sum_{n=1}^{\infty} \frac{(-1)^n H_1^{(2)}\{\mu\sqrt{(a^2+n^2\pi^2)}\}}{\sqrt{(a^2+n^2\pi^2)}}$$

$$= \frac{i \cosh 2\,ma}{\pi a\mu \sinh a} + \frac{iI}{\pi} \quad \ldots \quad (56)$$

where

$$I = \int_0^1 \frac{e^{-\frac{2ma}{y}} I_1\left(\frac{\mu a}{y}\sqrt{(1-y^2)}\right) dy}{y\sqrt{(1-y^2)}\,.\,\sinh\frac{a}{y}}$$

$$+ \frac{2i}{\pi}\int_0^1 \frac{\sinh\frac{2ma}{y}\,.\,K_1\left(\frac{\mu a}{y}\sqrt{(1-y^2)}\right) dy}{y\sqrt{(1-y^2)}\,.\,\sinh\frac{a}{y}}$$

$$- \int_0^1 \frac{\sinh 2may\,.\,H_1^{(2)}\{\mu a\sqrt{(1-y^2)}\}\,dy}{\sqrt{(1-y^2)}\,.\,\sinh ay} \quad \ldots \quad (57)$$

and μ is assumed to be positive.

When $m = 0$, (56) and (57) yield

$$\Sigma_1 = \frac{i}{\pi a\mu \sinh a} + \frac{i}{\pi}\int_1^{\infty} \frac{I_1\{\mu a\sqrt{(t^2-1)}\}\,dt}{\sqrt{(t^2-1)} \sinh at}$$

$$= \frac{i}{\pi a\mu \sinh a} - \frac{i\mu}{2\pi}\log_e \tanh\frac{|a|}{2} + O(\mu^3). \quad \ldots \quad (58)$$

It should be noted that the real part of Σ_1 is zero when $m = 0$ and $0 < \mu < 1$. Hence

$$\frac{J_1(\mu|a|)}{2|a|} + \sum_{n=1}^{\infty}(-1)^n \frac{J_1\{\mu\sqrt{(a^2+n^2\pi^2)}\}}{\sqrt{(a^2+n^2\pi^2)}} = 0. \quad \ldots \quad (59)$$

When a tends to zero (59) reduces to

$$\frac{1}{4} + \sum_{n=1}^{\infty}(-1)^n \frac{J_1(\mu n\pi)}{\mu n\pi} = 0. \quad \ldots \quad (60)$$

This is the well-known result given by Watson[10] of which (59) appears to be a generalization.

(ii) *Series* Σ_0.—Let us next consider the series Σ_0 defined by

$$\Sigma_0 = \frac{H_0^{(2)}(\mu|a|)}{2} + \sum_{n=1}^{\infty}(-1)^n H_0^{(2)}\{\mu\sqrt{(a^2+n^2\pi^2)}\}. \quad \ldots \quad (61)$$

By differentiation, it follows that

$$-\frac{1}{\mu a}\frac{d\Sigma_0}{da} = \frac{H_1^{(2)}(\mu|a|)}{2|a|} + \sum_{n=1}^{\infty}(-1)^n \frac{H_1^{(2)}\{\mu\sqrt{(a^2+n^2\pi^2)}\}}{\sqrt{(a^2+n^2\pi^2)}}$$

$$= \Sigma_1. \quad \ldots \quad (62)$$

Hence

$$\Sigma_0 = \int_a^\infty \mu a \, \Sigma_1 \, da, \qquad (63)$$

where Σ_1 is given by (62). For small values of μ, (62) and (63) yield

$$\Sigma_0 = \frac{i}{\pi}\int_a^\infty \left[\frac{1}{\sinh a} - \frac{\mu^2 a}{\pi}\log_e \tanh\frac{|a|}{2} + O(\mu^4)\right] da$$

$$= -\frac{i}{\pi}\log_e \tanh\frac{a}{2} + O(\mu^2) . \qquad (64)$$

It is clear from (64) that the real part of Σ_0 vanishes when $0 < \mu < 1$; hence

$$\frac{J_0(\mu|a|)}{2} + \sum_{n=1}^{\infty}(-1)^n J_0\left\{\mu\sqrt{(a^2 + n^2\pi^2)}\right\} = 0. \qquad (65)$$

This is again a generalization of a null series discovered by Schlömilch, namely,

$$\frac{1}{2} + \sum_{n=1}^{\infty}(-1)J_0(\mu n\pi) = 0 \qquad (66)$$

when $0 < \mu < 1$.

Furthermore, in Ref. 11, it is proved that

$$\Sigma_0 = + \frac{2i}{\pi}\sum_{n=1}^{\infty}\frac{\exp\{-a\sqrt{[(2n-1)^2 - \mu^2]}\}}{\sqrt{[(2n-1)^2 - \mu^2]}} \qquad (67)$$

and hence, by differentiation and use of (62)

$$\Sigma_1 = -\frac{1}{\mu a}\frac{d\Sigma_0}{da}$$

$$= + \frac{2i}{\pi a\mu}\sum_{n=1}^{\infty}\exp\{-a\sqrt{[(2n-1)^2 - \mu^2]}\}. \qquad (68)$$

Since formulae (62) and (68) correspond, it follows that,

$$\frac{J_0(\mu|a|)}{2} + \sum_{n=1}^{\infty}(-1)^n J_0\{\mu\sqrt{(a^2 + n^2\pi^2)}\} = + \frac{2}{\pi}\sum_{n=1}^{m}\frac{\cos a\sqrt{[\mu^2 - (2n-1)^2]}}{\sqrt{[\mu^2 - (2n-1)^2]}} \qquad (69)$$

when $2n - 1 < \mu < 2m + 1$. Similarly

$$\frac{J_1(\mu|a|)}{2|a|} + \sum_{n=1}^{\infty}(-1)^n\frac{J_1\{\mu\sqrt{(a^2 + n^2\pi^2)}\}}{\sqrt{(a^2 + n^2\pi^2)}} = \frac{2}{\pi\mu a}\sum_{n=1}^{m}\sin a\sqrt{[\mu^2 - (2n-1)^2]} \qquad (70)$$

and it follows that

$$\int_0^1 \frac{\sinh\dfrac{2ma}{y} . K_1\left(\dfrac{\mu a}{y}\sqrt{(1-y^2)}\right) dy}{y\sqrt{(1-y^2)} . \sinh\dfrac{a}{y}} + \frac{\pi}{2}\int_0^1 \frac{\sinh 2may . Y_1\{\mu a\sqrt{(1-y^2)}\}\, dy}{\sqrt{(1-y^2)}\sinh ay}$$

$$= \frac{\pi}{a\mu}\sum_{n=1}^{m}\sin a\sqrt{[\mu^2 - (2n-1)^2]}. \qquad (71)$$

Formulae (69) and (70) are generalizations of those given by Watson for the case $|a| = 0$.

TABLE 2

Mid-Chord Derivatives for Flat Plate

$M = 0 \cdot 7$; Tunnel height $= 4 \cdot 75$ chord

$\bar{\omega}$	l_z		$l_{\dot{z}}$		l_α		$l_{\dot{\alpha}}$	
	free stream	wind tunnel	free stream	wind tunnel	free stream	wind tunnel	free stream	wind tunnel
0	0	0	4·399	4·556	4·399	4·556	—∞	—8·882
0·04	0·022	0·016	4·061	4·506	4·066	4·510	—12·981	—8·715
0·08	0·063	0·058	3·740	4·321	3·757	4·339	—8·903	—7·979
0·20	0·185	0·238	3·054	3·579	3·117	3·657	—3·877	—5·084
0·40	0·297	0·427	2·504	2·799	2·638	2·975	—1·274	—2·026
0·60	0·311	—	2·269	—	2·471	—	—0·367	—

$M = 0 \cdot 7$; Tunnel height $= 4 \cdot 75$ chord

$\bar{\omega}$	$-m_z$		$-m_{\dot{z}}$		$-m_\alpha$		$-m_{\dot{\alpha}}$	
	free stream	wind tunnel	free stream	wind tunnel	free stream	wind tunnel	free stream	wind tunnel
0	0	0	—1·100	—1·119	—1·100	—1·119	∞	3·012
0·04	—0·006	—0·005	—1·014	—1·104	—1·015	—1·106	4·030	2·969
0·08	—0·019	—0·018	—0·928	—1·056	—0·933	—1·061	2·981	2·778
0·20	—0·063	—0·078	—0·743	—0·856	—0·759	—0·880	1·669	2·023
0·40	—0·133	—0·176	—0·581	—0·645	—0·617	—0·694	0·976	1·236
0·60	—0·201	—	—0·496	—	—0·548	—	0·735	—

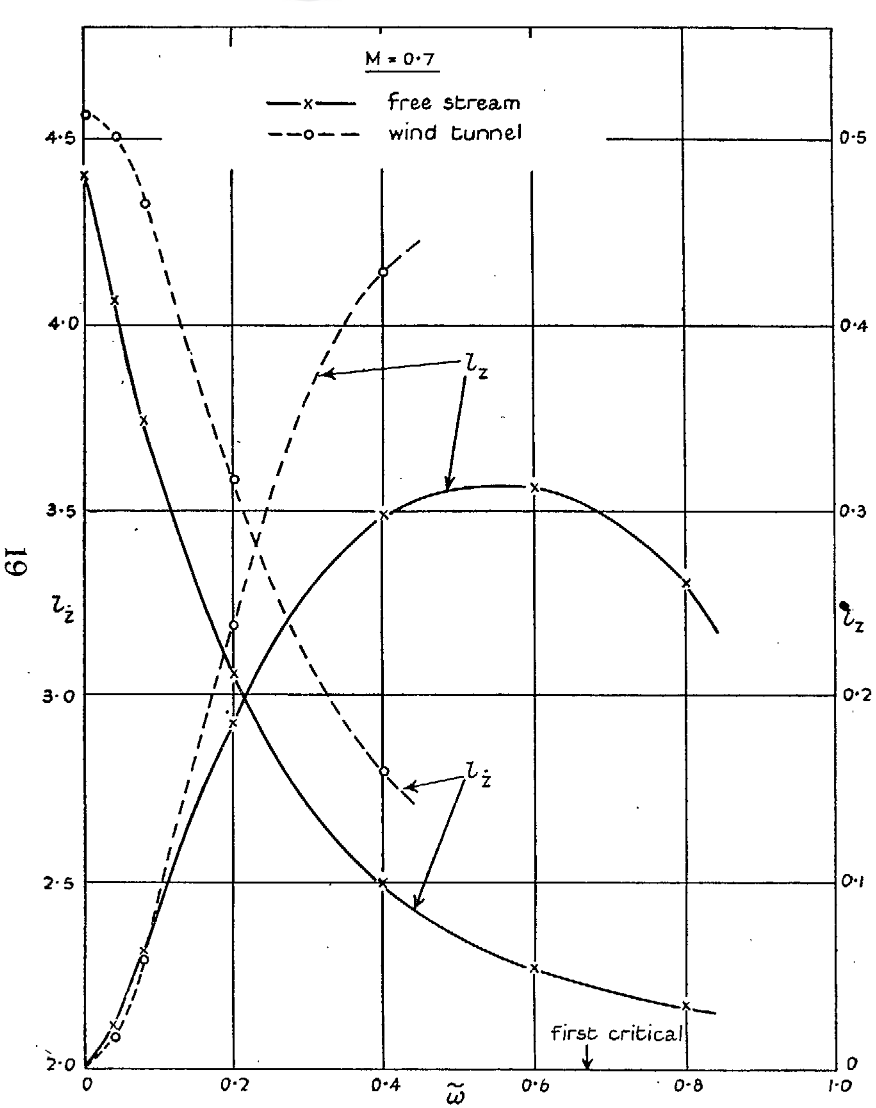

Fig. 4a. Mid-chord derivatives for $M = 0 \cdot 7$.

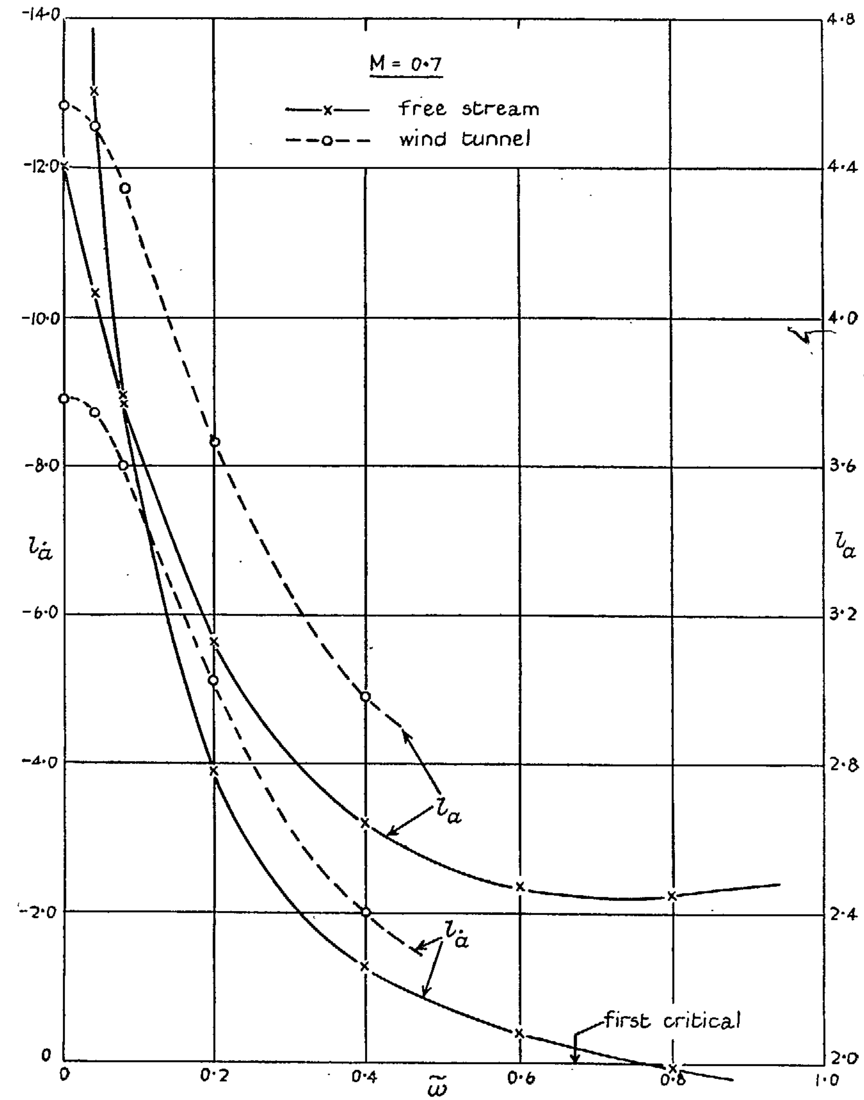

Fig. 4b. Mid-chord derivatives for $M = 0 \cdot 7$.

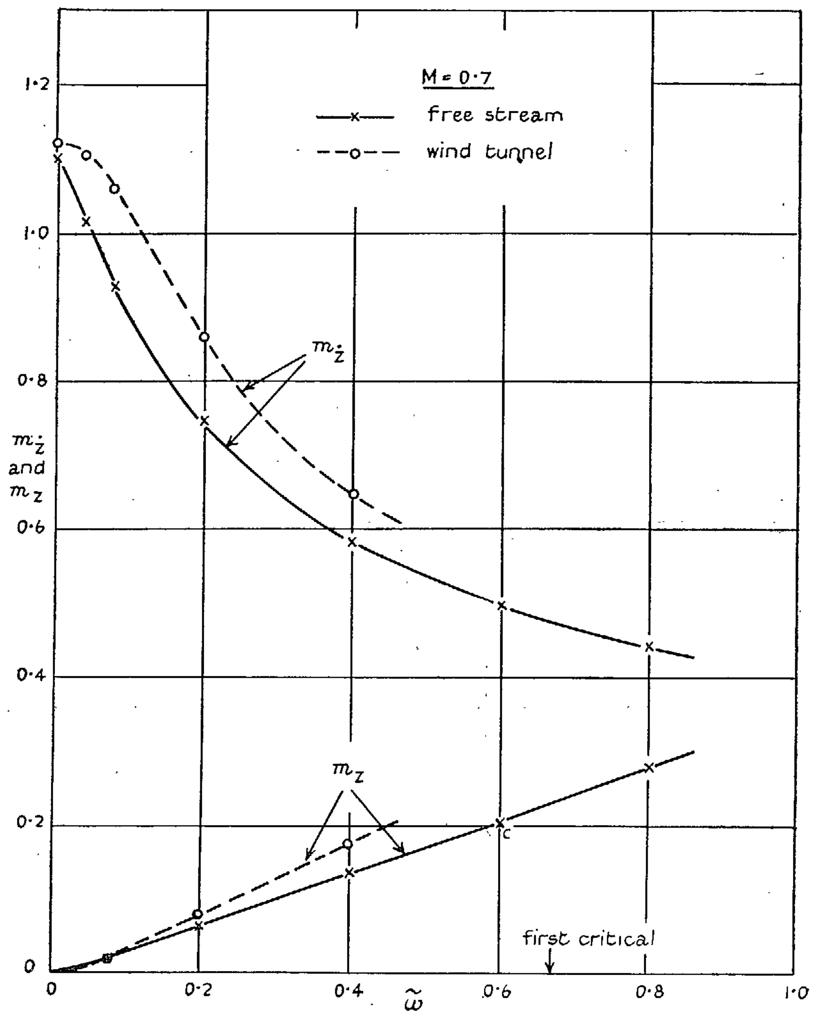

FIG. 4c. Mid-chord derivatives for $M = 0{\cdot}7$.

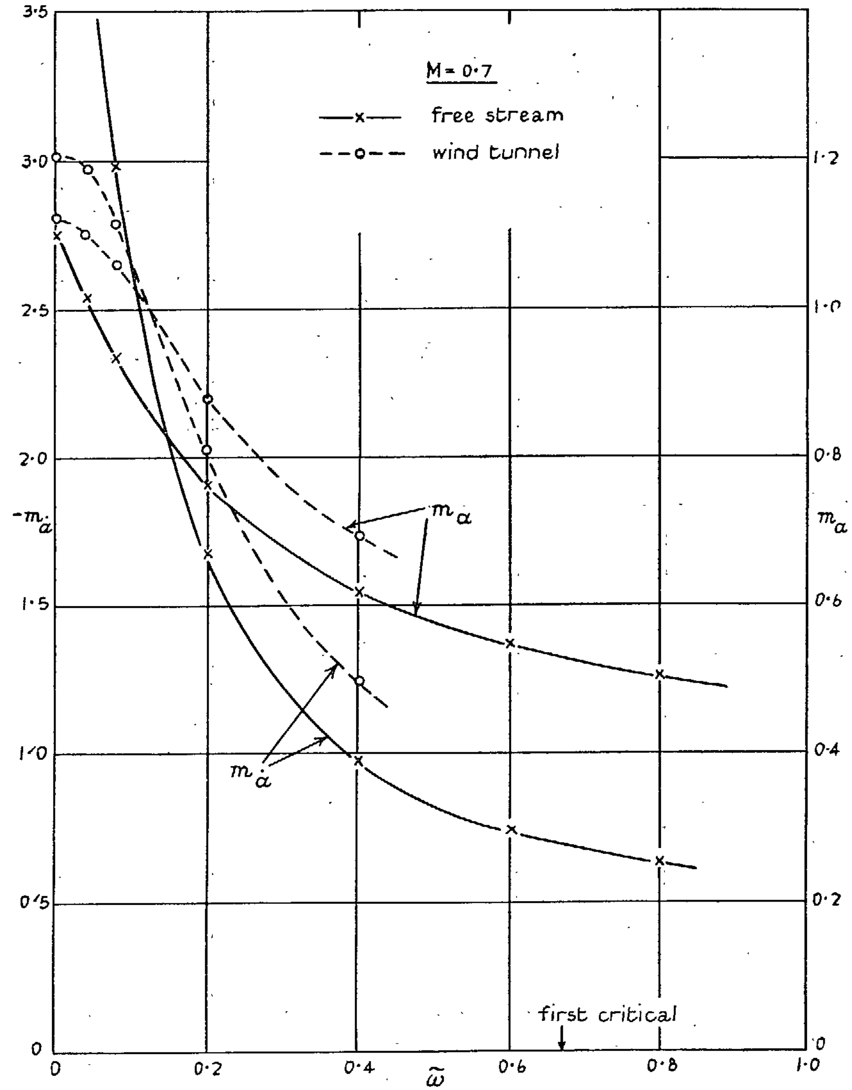

FIG. 4d. Mid-chord derivatives for $M = 0{\cdot}7$.

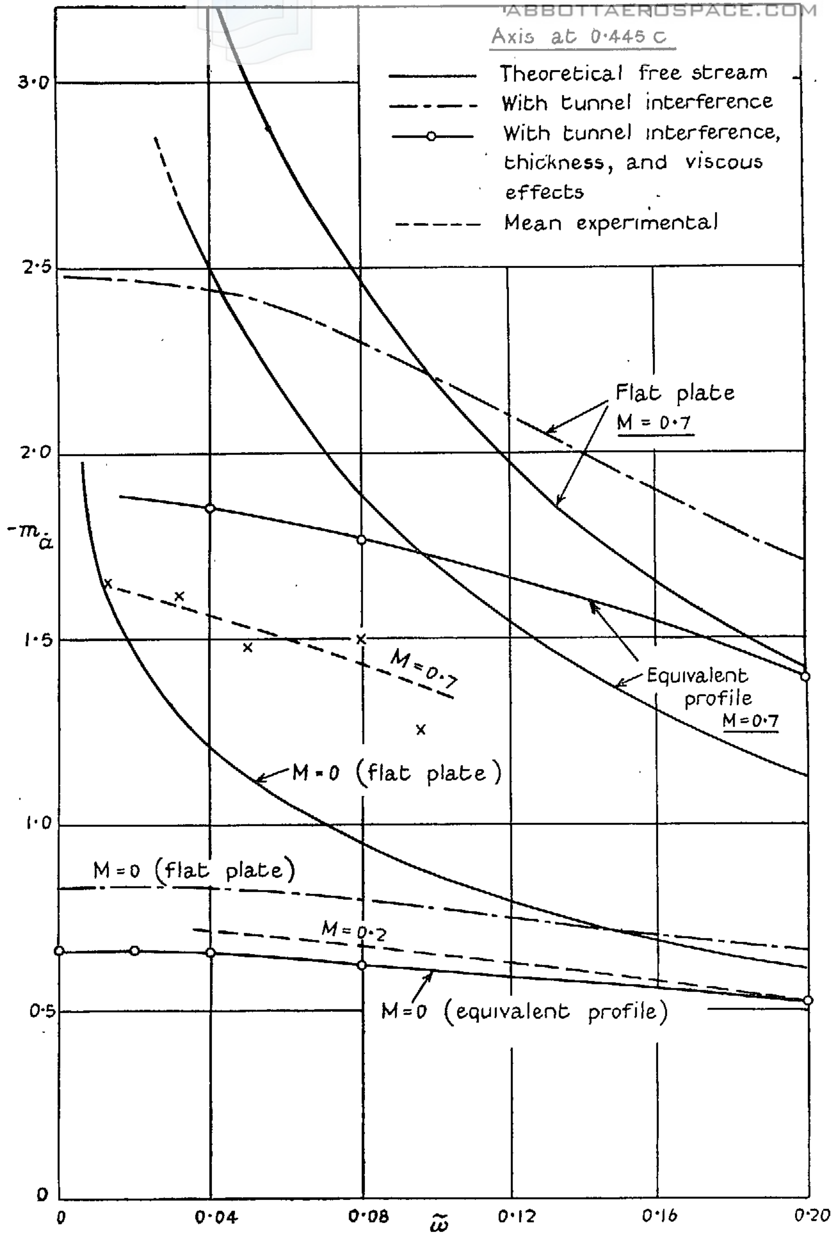

FIG. 5. Pitching-moment damping coefficient for RAE 104 aerofoil.

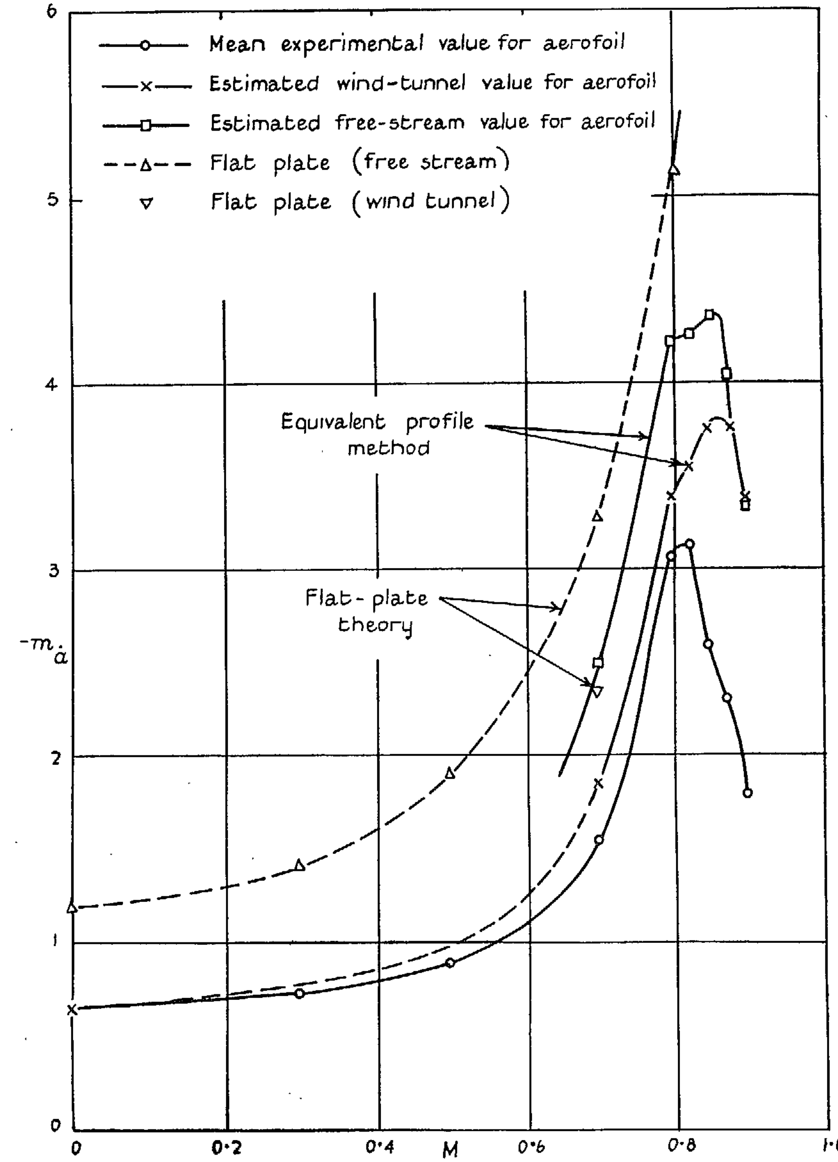

FIG. 6. Pitching-moment damping coefficient for the RAE 104 aerofoil at various Mach numbers. (Axis at $0 \cdot 445c$; $\tilde{\omega} = 0 \cdot 04$.)

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