R. & M. No. 3006
(17,967)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

The Three-dimensional Boundary-layer Equations and some Power Series Solutions

By

L. C. SQUIRE, B.Sc.,
Department of Mathematics, University of Bristol

LONDON: HER MAJESTY'S STATIONERY OFFICE
1957
PRICE 5s. 6d. NET

The Three-dimensional Boundary-layer Equations and some Power Series Solutions

By

L. C. SQUIRE, B.Sc.,

Department of Mathematics, University of Bristol

COMMUNICATED BY PROFESSOR L. HOWARTH

Reports and Memoranda No. 3006

October, 1955

Summary.—The boundary-layer equations are derived for a very general co-ordinate system, and various theorems hitherto only proved in more restrictive systems are extended to this general system. The particular case of streamlines of zero geodesic curvature is investigated in detail and a solution of such a flow found by a power series method. Finally Howarth's[4] stagnation-point solution is extended to second-order terms by numerical investigation.

1. *The Boundary-Layer Equations.*—The derivation of the laminar boundary-layer equations in three dimensions has been carried out by various authors (Hayes[1], Howarth[3] and Timman[6]). In these derivations the differences which arise are due to the choice of different curvilinear co-ordinate systems, use of which are necessitated by the surface curvature. The simplest equations are obtained by Howarth[3] but the co-ordinate system used is rather restricted. This system is first described, and it is then shown how these restrictions may be removed whilst retaining the simple equations. Howarth's equations are based on an orthogonal curvilinear co-ordinate system consisting of surfaces parallel to the body and two other families of surfaces to complete the triply orthogonal set. The advantage of this system lies in the use of parallel surfaces which allows the use of a metric in the form :

$$ds^2 = h_1^2\, dx^2 + h_2^2\, dy^2 + dz^2 \qquad \ldots \quad (1)$$

where x, y are curvilinear co-ordinates measured in the surface and z normal to the surface. The length elements h_1 and h_2 within the boundary layer may be regarded as functions of x and y only. The resulting equations for steady flow are:

$$\frac{u}{h_1}\frac{\partial u}{\partial x} + \frac{v}{h_2}\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{vu}{h_1 h_2}\frac{\partial h_1}{\partial y} - \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial x} = -\frac{\partial}{\rho h_1}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \qquad \ldots \quad (2)$$

$$\frac{u}{h_1}\frac{\partial v}{\partial x} + \frac{v}{h_2}\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial y} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial x} = -\frac{1}{\rho h_2}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} \qquad \ldots \quad (3)$$

$$\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{1}{h_2}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x} + \frac{v}{h_1 h_2}\frac{\partial h_1}{\partial y} = 0 \qquad \ldots \quad (4)$$

where u and v are the boundary-layer velocity components. The equations (2) to (4) are restricted in practice by the use of the completely orthogonal system since this implies that the lines $x =$ constant and $y =$ constant are lines of principal curvature of the surface and that the length elements are connected by Lamé's relations for triply orthogonal systems (*see* Howarth's paper, pp. 240, 241). These restrictions may be removed by the use of a curvilinear co-ordinate system which is only orthogonal on the body itself but which retains the parallel surfaces as one set of co-ordinate surfaces. The metric of such a system will be more complicated than equation (1) but will reduce to equation (1) on the body: it will be written

$$ds'^2 = h_1'^2\,dx'^2 + h_2'^2\,dy'^2 + dz'^2 + h_{12}\,dx'\,dy' + h_{23}\,dy'\,dz' + h_{31}\,dz'\,dx' \,. \qquad (5)$$

Then, if the body surface is smooth and the co-ordinate system regular, within the boundary layer h_{12}', h_{13}' and h_{23}' are $O(\delta)$ and $h_1'^2$, $h_2'^2$ are functions of the surface co-ordinates (x_0', y_0') plus terms of order (δ). The boundary-layer equations may now be obtained in exactly the same manner as in Ref. 3 except that it is necessary to use vector expansions of curl **v**, grad v^2, in non-orthogonal co-ordinates. These expansions have been given fully by Weatherburn[8] (pp. 62 to 72) and may be substituted into the vector form of the Navier-Stokes equations. The substitution is very long but straightforward and is not reproduced here. When boundary-layer approximations are carried out, and the relative order of the terms in (5) noted, Howarth's equations (2) to (4) are regained. These equations based on the unrestricted co-ordinate system outlined above will be used throughout this paper.

Prandtl stated that the disappearance of the curvature terms in equations of type (2) to (4) implied that the surface of the body was a developable surface but did not give a proof. This may be given as follows.

The disappearance of the curvature terms implies that $\partial h_1/\partial y$ and $\partial h_2/\partial x$ can, by appropriate choice of co-ordinate system, be made identically zero over the whole surface. Substitution of these values into the Gaussian curvature of the surface:

$$K = \frac{h_2}{h_1}\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial x} + \frac{h_1}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial y} - h_1\frac{\partial^2 h_1}{\partial y^2} - h_2\frac{\partial^2 h_2}{\partial x^2} \qquad (6)^*$$

shows that $K = 0$ and this is the condition of a developable surface†‡.

It should be noted that this result does not imply that the curvature terms always disappear on a developable surface. For example, on a cone the metric may be written (Howarth, p. 242)

$$ds^2 = y^2 \sin^2\alpha\,dx^2 + dy^2 + dz^2 \qquad (7)$$

($x \equiv \phi$ the polar angle) so that

$$\frac{1}{h_1h_2}\frac{\partial h_1}{\partial y} = \frac{1}{y}\,; \qquad \frac{1}{h_1h_2}\frac{\partial h_2}{\partial x} = 0\,. \qquad (8)$$

Substitution of these values in (6) shows, however, that $K = 0$.

It should also be noted that one of the curvature terms can be removed by choosing one set of the surface co-ordinates as geodesics. This follows from the fact that the terms

$$-\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}\,; \qquad -\frac{1}{h_1h_2}\frac{\partial h_1}{\partial y}$$

are in fact the geodesic curvature of the lines $x =$ constant and $y =$ constant respectively§. Further one of the terms will be zero along any single co-ordinate line which is a geodesic (by the above value of the geodesic curvature).

---

* Weatherburn[7], page 93.

† Weatherburn[7], page 76.

‡ Professor Howarth informs me that a similar derivation of the equations and a proof of the developable property are contained in an unpublished thesis by Mr. E. J. Watson written in 1950 (Liverpool University).

§ Weatherburn[7], page 110; note that in our present notation

$$E = h_1^2 \text{ and } G = h_2^2 \text{ while } E_2 = 2h_1\frac{\partial h_1}{\partial y}\,.$$

This property will now be used to prove the following theorem, which states, that on a streamline which is a geodesic of the surface, the boundary-layer flow is in the direction of the main stream. For simplicity let us choose a co-ordinate system such that $y =$ constant coincides with such a streamline, then by the result stated above

$$-\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y} = 0$$

along $y =$ constant. Then equation (3) may be written

$$\frac{u}{h_1}\frac{\partial v}{\partial x} + \frac{v}{h_2}\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial x} = \nu\frac{\partial^2 v}{\partial z^2}, \quad \ldots \quad (9)$$

since by the irrotational condition

$$-\frac{1}{\rho h_2}\frac{\partial p}{\partial y} = \frac{1}{2h_2}\frac{\partial}{\partial y}(U^2 + V^2)$$

$$= \frac{U}{h_1}\frac{\partial V}{\partial x} + \frac{V}{h_2}\frac{\partial V}{\partial y} + \frac{VU}{h_1 h_2}\frac{\partial h_2}{\partial x} - \frac{U^2}{h_1 h_2}\frac{\partial h_1}{\partial y} = 0. \quad \ldots \quad (10)$$

The boundary conditions on v are $v = 0$ at the wall and $v \to 0$ as the main stream is approached. A solution of equation (9) with these boundary conditions is

$$v \equiv 0 \quad \ldots \quad (11)$$

so that on $y =$ constant, v and V are both zero.

It can also be seen that if v and V are both zero on any line, that line is a geodesic. In this case equation (3) may be written

$$(U^2 - u^2)\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y} = 0 \text{ for all } z, \quad \ldots \quad (12)$$

but $U \neq u$ for all z, therefore $\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y} = 0$ and $y =$ constant is a geodesic.

2. *An Example of the Flow on Streamline Geodesics.*—On such streamlines equations (2) and (4) reduce by the results of the last section to almost two-dimensional form:

$$\frac{u}{h_1}\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \frac{U}{h_1}\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \quad \ldots \quad (13)$$

$$\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{1}{h_2}\frac{\partial v}{\partial y} - \frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}. \quad \ldots \quad (14)$$

These equations are not sufficient to determine the flow on $y =$ constant since they contain the term $\partial v/\partial y$. However, on a body in which U, V and the length elements are regular this solution may be found quite easily. For example if the geodesic is $y = 0$ the boundary-layer velocities u, v, w may be expanded in power series in y with coefficients which are functions of x and z. Substituting these expansions in equations (2) to (4) and retaining lowest order terms in y will then give a series of partial differential equations for u, v, w as function of x and z. These equations can be solved by ordinary methods.

To illustrate this method and to provide some data for Ref. 5, the boundary layer has been computed for the velocity distribution

$$U = x - \tfrac{1}{3}x^3 - \tfrac{1}{4}xy^2 + O(y^2) \; ; \qquad V = \tfrac{1}{4}y(1 - x^2) + O(y^3) \; ,$$

on a flat surface. (The restriction that the surface should be flat is unnecessary if it is assumed that the length elements are of the form

$$h_1 = 1 + y^2 f_1(x) \; ; \qquad h_2 = 1 + y^2 f_2(x) \; .)$$

With a certain co-ordinate system these length elements represent approximately the length elements on a line of symmetry of an ellipsoid. Similarly the velocity distribution used represents approximately the shape of the velocity distribution near such a line. With this velocity the pressure gradients may be written

$$-\frac{1}{\rho h_1}\frac{\partial p}{\partial x} = x - \tfrac{4}{3}x^3 + \tfrac{1}{3}x^5 + O(y^2)$$

$$-\frac{1}{\rho h_2}\frac{\partial p}{\partial y} = y(\tfrac{1}{16} - \tfrac{5}{8}x^2 + \tfrac{11}{48}x^4) \; . \qquad \ldots \qquad (15)$$

To solve the boundary-layer equations, the velocities are written in the form $u = u_0 + yu_1$, $v = yv_1 + y^2v_2$, $w = w_0 + w_1y$, which, substituted into the equations, result in the following three equations for u_0, v_1, w_0:

$$u_0\frac{\partial u_0}{\partial x} + w_0\frac{\partial u_0}{\partial z} = (x - \tfrac{4}{3}x^3 + \tfrac{1}{3}x^5) + \nu\frac{\partial^2 u_0}{\partial z^2} \qquad \ldots \qquad (16)$$

$$u_0\frac{\partial v_1}{\partial x} + v_1^2 + w_0\frac{\partial v_1}{\partial z} = (\tfrac{1}{16} - \tfrac{5}{8}x^2 + \tfrac{11}{48}x^4) + \nu\frac{\partial^2 v_1}{\partial z^2} \qquad \ldots \qquad (17)$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial z} = -\, v_1 \; . \qquad \ldots \qquad (18)$$

It should be noted that (16) and (18) differ only by one term ($-\, v_1$) from the corresponding two-dimensional problem, the solution of which has been given by Howarth[2] and is quoted in *Modern Developments in Fluid Dynamics*, Vol. 1, page 151.

Equations (16) to (18) may be solved in the usual manner by putting

$$u_0 = xf_1' - \tfrac{1}{3}x^3f_3' + x^5f_5' + x^7f_7' \qquad \ldots \qquad (19)$$

$$v_1 = \tfrac{1}{4}g_0' - \tfrac{1}{4}x^2g_2' + x^4g_4' + x^6g_6' \qquad \ldots \qquad (20)$$

where f_i, g_i are functions of $\eta = z/\nu^{1/2}$ and have boundary conditions $f_i = f_i' = g_i = g_i' = 0$ at $\eta = 0$

$$f_1', f_3', g_0'\, g_2' \rightarrow 1 \; ; \qquad \underset{i>3}{f_i'}, \underset{j>2}{g_j'} \rightarrow 0 \text{ as } \eta \rightarrow \infty \; .$$

Substitution into (16) to (18) results in the following pairs of differential equations:

$$\left.\begin{aligned} f_1'^2 - f_1f_1'' - \tfrac{1}{4}g_0f_1'' &= 1 + f_1''' \\ g_0'^2 - 4f_1g_0'' - g_0g_0'' &= 1 + 4g_0''' \end{aligned}\right\} \qquad \ldots \qquad (21)$$

$$\left.\begin{aligned} f_3''' + f_3''(f_1 + \tfrac{1}{4}g_0) - 4f_3'f_1' + 3(f_3 + \tfrac{1}{4}g_2)f_1'' &= -4 \\ g_2''' + g_2''(f_1 + \tfrac{1}{4}g_0) - 2g_2'(f_1' + \tfrac{1}{4}g_0') + (f_3 + \tfrac{1}{4}g_2)g_0'' &= -\tfrac{5}{2} \end{aligned}\right\} \quad \ldots \quad (22)$$

$$\left.\begin{aligned} f_5''' + f_5''(f_1 + \tfrac{1}{4}g_0) - 6f_5'f_1' + (5f_5 + g_4)f_1'' &= \tfrac{1}{3}\{f_3'^2 - 1 - f_3''(f_3 + \tfrac{1}{4}g_2)\} \\ g_4''' + g_4''(f_1 + \tfrac{1}{4}g_0) - g_4'(4f_1' + \tfrac{1}{2}g_0') + (5f_5 + g_4)\tfrac{1}{4}g_0'' & \\ = \tfrac{1}{16}g_2'^2 - \tfrac{11}{48} + \tfrac{1}{6}f_3'g_2' - \tfrac{1}{4}g_2''(f_3 + \tfrac{1}{4}g_2) & \end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned} f_7''' + f_7''(f_1 + \tfrac{1}{4}g_0) - 8f_7'f_1' + (7f_7 + g_6)f_1'' &= \tfrac{1}{3}\{f_3''(5f_5 + g_4) + 3f_5''(f_3 + \tfrac{1}{4}g_2) - 8f_3'f_5'\} \\ g_6''' + g_6''(f_1 + \tfrac{1}{4}g_0) - g_6'(6f_1' + \tfrac{1}{2}g_0') + (7f_7 + g_6)\tfrac{1}{4}g_0'' & \\ = \tfrac{1}{4}g_2(5f_5 + g_4) + g_4''(f_3 + \tfrac{1}{4}g_2) - \tfrac{4}{3}g_4'f_3' - \tfrac{1}{2}g_2'f_5' - \tfrac{1}{2}g_2'g_4' & \end{aligned}\right\} . \quad (24)$$

Equation (21) has been solved by Howarth[4] to 3 figures; for the present investigation this solution has been extended to 4 figures by relaxation and equations (22), (23) and (24) solved by normal central-difference techniques. The functions are tabulated in Table 1 which shows corresponding values of $f_1''(0)$, $f_3''(0)$, $f_5''(0)$ and $f_7''(0)$ for the two-dimensional problem. The skin friction is plotted in Fig. 1 where it is seen that the three-dimensional effect is almost negligible.

TABLE 1

z	f_1'	g_0'	f_3'	g_2'	f_5'	g_4'
0	0·0000	0·0000	0·000	0·000	0·000	0·000
0·2	0·2296	0·1560	0·513	0·387	0·009	0·011
0·4	0·4203	0·3013	0·875	0·679	0·007	0·014
0·6	0·5745	0·4346	1·107	0·886	−0·003	0·011
0·8	0·6959	0·5541	1·233	1·023	−0·016	0·004
1·0	0·7887	0·6583	1·281	1·102	−0·028	−0·003
1·2	0·8575	0·7463	1·277	1·140	−0·036	−0·011
1·4	0·9069	0·8178	1·244	1·147	−0·040	−0·016
1·6	0·9412	0·8738	1·197	1·135	−0·040	−0·019
1·8	0·9641	0·9158	1·149	1·114	−0·036	−0·020
2·0	0·9788	0·9460	1·106	1·089	−0·030	−0·019
2·2	0·9880	0·9667	1·071	1·065	−0·024	−0·016
2·4	0·9934	0·9803	1·046	1·045	−0·018	−0·013
2·6	0·9966	0·9888	1·028	1·030	−0·012	−0·010
2·8	0·9983	0·9939	1·016	1·018	−0·008	−0·007
3·0	0·9992	0·9968	1·009	1·011	−0·005	−0·005
3·2	0·9996	0·9984	1·004	1·006	−0·003	−0·003
3·4	0·9998	0·9992	1·002	1·004	−0·002	−0·002
3·6	0·9999	0·9996	1·001	1·002	−0·002	−0·002
3·8	1·0000	0·9998	1·000	1·001	−0·001	−0·001
4·0		0·9999		1·000	−0·001	−0·001
						−0·001
derivative at $\eta = 0$	1·2476	0·8051	2·961	2·183	0·078	0·077
Two-dimensional value	1·2326		2·898		0·0801	

$$f_7''(0) \simeq -0{\cdot}002$$

$$g_6''(0) \simeq 0{\cdot}004$$

The results of this section show that on a smooth body with an external flow free of singularities the boundary-layer equations can be integrated along streamlines of zero geodesic curvature without influence from the rest of the boundary layer. A similar state is shown in the approximate method of solution developed in Ref. 5.

3. *Flow at the Stagnation Point.*—Howarth[4] has obtained a solution of the equations at a stagnation point. This solution is based on the fact that at such a point the external-flow velocity components can, with a suitable rotation of axes, be expressed in the form

$$U = ax\,; \qquad V = by\,. \qquad \qquad (25)$$

Then a solution of the boundary-layer equations can be found by writing

$$u = axf_0'(\eta)\,; \qquad v = byg_0'(\eta) \text{ where } \eta = z\sqrt{\frac{a}{\nu}}\,. \qquad \qquad (26)$$

The resulting non-linear equations depend on the ratio b/a which can be restricted to the range $0 < b/a < 1$ and solutions are given for $b/a = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. However in approximate methods based on the momentum-integral relations it has been found that this solution, although giving boundary conditions, does not provide enough information for the integration of the approximate equations away from the point. This lack is due both to the difficulty of integrating partial differential equations away from a point and to the fact that the velocity *profiles* as given by Howarth's solution are in general indeterminate at the origin.

This indeterminacy appears when it is required to find the velocity profiles in directions not corresponding with Howarth's axes. For example to find the profile for the component of velocity in the direction making an angle α to the line $x = 0$; the boundary-layer velocity component in this direction is $u' = axf_0'(\eta)\cos\alpha + byg_0'(\eta)\sin\alpha$ and the mainstream component $U' = ax\cos\alpha + by\sin\alpha$ so that the profile is given by

$$\frac{u'}{U'} = \frac{f_0' + \dfrac{by}{ax}g_0'\tan\alpha}{1 + \dfrac{by}{ax}\tan\alpha} \qquad \qquad (27)$$

and this, for $\alpha \neq (0 \text{ or } \frac{1}{2}\pi)$, depends on the ratio y/x as the origin is approached.

Thus for general use the next order terms of the boundary-layer velocities are needed so that it is not necessary to start at the origin. These are now found.

The external flow at the origin is in the form:

$$U = Ax + Ky + A_{11}x^2 + A_{12}xy + A_{22}y^2 \qquad \qquad (28)$$

$$V = Kx + By + B_{11}x^2 + B_{12}xy + B_{22}y^2 \qquad \qquad (29)$$

and the length elements may be written:

$$h_1 = 1 + C_1x + C_2y\,; \qquad h_2 = 1 + D_1x + D_2y\,. \qquad \qquad (30)$$

The coefficients of y in U, and x in V, are equal by the irrotational condition and this will give two further conditions between the coefficients of (28), (29) and (30) if required*. Under Howarth's linear transformation

---

* Irrotational condition is: $\frac{\partial}{\partial x}(h_2V) - \frac{\partial}{\partial y}(h_1U) = 0$.

Therefore $\quad 2B_{11} + 2D_1K = A_{12} + C_1K + C_2A$

and $\quad 2A_{22} + 2C_2K = B_{12} + D_1B + D_2K$.

$$x = x' \cos \alpha - y' \sin \alpha$$
$$y = x' \sin \alpha + y' \cos \alpha ; \qquad \alpha = \tfrac{1}{2} \tan^{-1} \{2K/(A - B)\} \quad \ldots \quad \ldots \quad (31)$$

the metric $ds^2 = h_1^2\, dx^2 + h_2^2\, dy^2$ becomes

$$ds^2 = (h_1^2 \cos^2 \alpha + h_2^2 \sin^2 \alpha)\, dx'^2 + (h_2^2 - h_1^2) \cos \alpha \sin \alpha\, dx'\, dy' + (h_1^2 \sin^2 \alpha + h_2^2 \cos^2 \alpha)\, dy'^2 \quad \ldots \quad (32)$$

so that the system is no longer orthogonal, except at the origin where $h_1 = h_2 = 1$ and the cross-term zero. This means that the second-order terms cannot be found directly by use of Howarth's transformation. There are now three methods of proceeding and these will be discussed.

The first is to solve the boundary-layer equations in the original co-ordinates with no transformation since the first-order solution can be found from Howarth's solution. This has been rejected since it was found that the second-order flow depended on the solution of sets of four simultaneous differential equations. The second method is to add quadratic terms to the transformation (31) so that the metric (32) is orthogonal to order x and y. However, the transformation is no longer (1 : 1) and the reverse transformation is difficult to find explicitly. The third alternative, which is the one chosen, consists of using the transformation (31) and solving the resulting non-orthogonal equations.

The boundary-layer equations in non-orthogonal co-ordinates may be found in the same way as described in section 1; with the general metric

$$ds^2 = h_1^2\, dx^2 + h_2^2\, dy^2 + 2g\, dx\, dy + dx^2 \quad \ldots \quad (33)$$

the equations are:

$$\frac{u}{h_1}\frac{\partial u}{\partial x} + \frac{v}{h_2}\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u^2 \frac{h_1 g}{p^2}\left[\frac{1}{h_1}\frac{\partial h_1}{\partial y} + \frac{g}{h_1^3}\frac{\partial h_1}{\partial x} - \frac{1}{h_1^2}\frac{\partial g}{\partial x}\right] + \frac{v^2 h_1}{p^2}\left[\frac{\partial g}{\partial y} - h_2\frac{\partial h_2}{\partial x} - \frac{g}{h_2}\frac{\partial h_2}{\partial y}\right]$$
$$+ \frac{uv}{p^2}\left[h_1 h_2\left(1 + \frac{g^2}{h_1^2 h_2^2}\right)\frac{\partial h_1}{\partial y} - 2g\frac{\partial h_2}{\partial x}\right] = -\frac{h_2^2 h_1}{p^2}\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{g h_1}{p^2}\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\frac{\partial^2 u}{\partial z^2}. \quad \ldots \quad (34)$$

$$\frac{u}{h_1}\frac{\partial v}{\partial x} + \frac{v}{h_2}\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{u^2 h_2}{p^2}\left[\frac{\partial g}{\partial x} - h_1\frac{\partial h_1}{\partial y} - \frac{g}{h_1}\frac{\partial h_1}{\partial x}\right] + \frac{v^2 g h_2}{p^2}\left[\frac{1}{h_2}\frac{\partial h_2}{\partial x} + \frac{g}{h_2^3}\frac{\partial h_2}{\partial y} - \frac{1}{h_2^2}\frac{\partial g}{\partial y}\right]$$
$$+ \frac{uv}{p^2}\left[h_1 h_2\left(1 + \frac{g^2}{h_1^2 h_2^2}\right)\frac{\partial h_2}{\partial x} - 2g\frac{\partial h_1}{\partial y}\right] = -\frac{h_1^2 h_2}{p^2}\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{g h_2}{p^2}\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 v}{\partial z^2}. \quad \ldots \quad (35)$$

$$\frac{\partial}{\partial x}\left(\frac{p}{h_1}u\right) + \frac{\partial}{\partial y}\left(\frac{p}{h_2}v\right) + p\frac{\partial w}{\partial z} = 0 \quad \ldots \quad (36)$$

where $p^2 = h_1^2 h_2^2 - g^2$, and P is the pressure.

The main-stream velocity components in the new co-ordinate system are now required. These may be found by noting that if $\mathbf{r}$ is the position vector of a point on a surface $(1/h_1)\,(\partial \mathbf{r}/\partial x)$ is a unit vector in the direction x increasing. Using this fact and the transformation (31) the new velocity components are:

$$U' = \left(U\frac{\cos\alpha}{h_1} + V\frac{\sin\alpha}{h_2}\right)(h_1^2 \cos^2\alpha + h_2^2 \sin^2\alpha)^{1/2} \quad \ldots \quad (37)$$

$$V' = \left(-U\frac{\sin\alpha}{h_1} + V\frac{\cos\alpha}{h_2}\right)(h_1^2 \sin^2\alpha + h_2^2 \cos^2\alpha)^{1/2} \quad \ldots \quad (38)$$

where dashes denote the new system. Then the velocity components may be written:

$$U = ax + a_{11}x^2 + a_{12}xy + a_{22}y^2 \quad \ldots \qquad (39)$$

$$V = by + b_{11}x^2 + b_{12}xy + b_{22}y^2 \quad \ldots \qquad (40)$$

and the length elements become:

$$h_1 = 1 + c_1x + c_2y; \quad h_2 = 1 + d_1x + d_2y; \quad g = e_1x + e_2y \quad \ldots \qquad (41)$$

where dashes have now been dropped and where the relations between the coefficients of (28) to (30) and those of (37) to (41) are given in the Appendix. Equations (34) to (36) are now solved by putting:

$$u = af_0' + (a_{11}f_{11}' + ac_1f_{12}' + ad_1f_{13}' + b_{12}f_{14}')x^2 + (a_{12}f_{21}' + ac_2f_{22}' + ad_2f_{23}' + b_{22}f_{34}')xy + (a_{22}f_{31}' + ab^2(d_1 - e_2)f_{32}')y^2 \quad \ldots \qquad (42)$$

$$v = byg_0' + (b_{11}g_{11}' + a(c_2 - e_1)g_{12}')x^2 + (b_{12}g_{24}' + ac_1g_{22}' + ad_1g_{23}' + a_{11}g_{11}')xy + (b_{22}g_{34}' + ac_2g_{32}' + ad_2g_{33}' + a_{12}g_{31}')y^2, \quad \ldots \qquad (43)$$

where f' and g' are functions of $\eta = z\sqrt{(a/\nu)}$. The boundary conditions are all $f = f' = g = g' = 0$ at $\eta = 0$ and f_0', f_{11}', f_{21}', f_{31}', g_0', g_{11}', g_{24}', $g_{34}' = 1$ as $\eta \to \infty$ while the remainder $\to 0$. Substitution in (34) to (36) then produces the following sets of differential equations:—

$$f_{1i}''' + f_{1i}''(f_0 + kg_0) - 3f_0'f_{1i}' + f_0''(2f_{1i} + g_{2i}) = F_{1i}$$

$$g_{2i}''' + g_{2i}''(f_0 + kg_0) - g_{2i}'(f_0' + 2kg_0') + kg_0''(2f_{1i} + g_{2i}) = G_{2i}$$

where:

$F_{11} = -3$ $\qquad$ $F_{12} = 1 - f_0'^2 + f_0f_0''$

$G_{21} = 0$ $\qquad$ $G_{22} = kf_0g_0''$

$F_{13} = (kg_0 - f_0)f_0''$ $\qquad$ $F_{14} = 0$

$G_{23} = k[f_0'g_0' - 1 + g_0''(f_0 - kg_0) + k(1 - g_0'^2)]$ $\qquad$ $G_{24} = -(1 + 2k)$

and $\quad k = b/a. \quad \ldots \qquad (44)$

$$f_{2i}''' + f_{2i}''(f_0 + kg_0) - f_{2i}'(2f_0' + kg_0') + f_0''(f_{2i} + 2g_{3i}) = F_{2i}$$

$$g_{3i}''' + g_{3i}''(f_0 + kg_0) - 3kg_0'g_{3i}' + kg_0''(f_{2i} + 2g_{3i}) = G_{3i}$$

where

$F_{21} = -(2 + k)$ $\qquad$ $F_{22} = f_0f_0'' - kg_0f_0'' + 1 - f_0'^2 - k(1 - g_0'f_0')$

$G_{31} = 0$ $\qquad$ $G_{32} = k(f_0 - kg_0)g_0''$

$F_{23} = kg_0f_0''$ $\qquad$ $F_{24} = 0$

$G_{33} = k[1 - g_0'^2 + g_0g_0'']$ $\qquad$ $G_{34} = -3k. \quad \ldots \qquad (45)$

$$f_{3i}''' + f_{3i}''(f_0 + kg_0) - f_{3i}'(f_0' + 2kg_0') = F_{3i}$$

where

$$F_{32} = (1 - g_0'^2)k; \qquad F_{31} = -(1 + 2k). \quad \ldots \qquad (46)$$

$$g_{1i}''' + g_{1i}''(f_0 + kg_0) - g_{1i}'(2f_0' + kg_0') = G_{1i}$$

where

$$G_{11} = -(2 + k)\,; \qquad G_{12} = 1 - f_0'^2\,. \qquad (47)$$

These equations have been solved for $k(= b/a) = 0, \frac{1}{2}, 1$ and the results are tabulated in Tables 3 to 12. At $k = 0$ some of the solutions are identically zero, while others have been solved in different problems or are equal to f_0, g_0. The sources of these solutions are given in Table 2 where the wall derivative is tabulated. Similarly at $k = 1$ some of the equations are equal and these are also indicated in Table 2. For convenience in interpolation solutions of each function are given in Tables 3 to 12 for the three values of k, there is thus some duplication of results as indicated in Table 2. It is hoped that the solutions of this section will enable all boundary-layer solutions to be started from the stagnation point with accuracy.

Finally it should be pointed out that although equations (44) to (47) were obtained from the non-orthogonal boundary-layer equations, the same equations would have been obtained if an orthogonal transformation could have been found. This is best seen in equations (42) and (43), where the only point the non-orthogonal terms enter are in $a(c_2 - e_1)g_{21}'x^2$ and $ab^2(d_1 - e_2)f_{32}'y^2$. In the orthogonal case these terms would be $ac_2g_{21}'x^2$ and $ab^2d_1f_{32}'y^2$ with the same equations for f_{32}' and g_{21}'.

4. *Acknowledgements*.—The author would like to thank Professor L. Howarth for much help and encouragement in the preparation of this paper. Also the Department of Scientific and Industrial Research for a Maintenance Grant.

REFERENCES

No.	*Author*	*Title, etc.*
1	W. D. Hayes	The three-dimensional boundary layer. NAVORD Report No. 1313. 1951
2	L. Howarth	On the calculation of steady flow in the boundary layer near the surface of a cylinder in a stream. R. & M. 1632. July, 1934.
3	L. Howarth	The boundary layer in three-dimensional flow. I. Derivation of the equations for flow along a general curved surface. *Phil. Mag.*, Series 7, Vol. 42, pp. 239 to 243. 1951.
4	L. Howarth	The boundary layer in three-dimensional flow. II. The flow near a stagnation point. *Phil. Mag.*, Series 7, Vol. 42, pp. 1433 to 1440. 1951.
5	L. C. Squire	An approximate method for three-dimensional boundary layers. Thesis, University of Bristol. 1956. (To be published.)
6	R. Timman	A calculation method for three-dimensional laminar boundary layers. N.L.L. Report F.66. 1950.
7	C. E. Weatherburn	*Differential Geometry*, Vol. I. Cambridge University Press. 1927.
8	C. E. Weatherburn	*Differential Geometry*, Vol. II. Cambridge University Press. 1930.

(4995) A*

APPENDIX

Using the results (37) and (38) together with the transformation (31) it can be shown that:

$$a_1 = A \cos^2 \alpha + B \sin^2 \alpha + 2K \cos \alpha \sin \alpha .$$

$$a_{11} = \cos \alpha \{A_{11} \cos^2 \alpha + A_{12} \cos \alpha \sin \alpha + A_{22} \sin^2 \alpha + (A \cos \alpha + K \sin \alpha) \times (c_1 - C_1 \cos \alpha - C_2 \sin \alpha)\} + \sin \alpha \{B_{11} \cos^2 \alpha + B_{12} \sin \alpha \cos \alpha + B_{22} \sin^2 \alpha + (K \cos \alpha + B \sin \alpha)(c_1 - D_1 \cos a - D_2 \sin \alpha)\} .$$

$$a_{12} = \cos \alpha \{A_{12} \cos 2\alpha + (A_{22} - A_{11}) \sin 2\alpha + (A \cos \alpha + K \sin \alpha) \times (c_2 - C_2 \cos \alpha + C_1 \sin \alpha) - (A \sin \alpha - K \cos \alpha)(c_1 - C_1 \cos \alpha - C_2 \sin \alpha)\} + \sin \alpha \{B_{12} \cos 2\alpha + (B_{22} - B_{11}) \sin 2\alpha + (K \cos \alpha + B \sin \alpha)(c_2 - D_2 \cos \alpha + D_1 \sin \alpha) - (K \sin \alpha - B \cos \alpha) \times (c_1 - D_1 \cos \alpha - D_2 \sin \alpha)\} .$$

$$a_{22} = \cos \alpha \{A_{11} \sin^2 \alpha - A_{12} \cos \alpha \sin \alpha + A_{22} \cos^2 \alpha - (A \sin \alpha - K \cos \alpha) \times (c_2 - C_2 \cos \alpha + C_1 \sin \alpha)\} + \sin \alpha \{B_{11} \sin^2 \alpha - B_{12} \cos \alpha \sin \alpha + B_{22} \cos^2 \alpha - (K \sin \alpha - B \cos \alpha)(c_2 - D_2 \cos \alpha + D_1 \sin \alpha)\} .$$

$$b_1 = A \sin^2 \alpha + B \cos^2 \alpha - 2K(\sin \alpha \cos \alpha) .$$

$$b_{11} = - \sin \alpha \{A_{11} \cos^2 \alpha + A_{12} \sin \alpha \cos \alpha + A_{22} \sin^2 \alpha + (A \cos \alpha + K \sin \alpha)(d_1 - C_1 \cos \alpha - C_2 \sin \alpha)\} + \cos \alpha \{B_{11} \cos^2 \alpha + B_{12} \sin \alpha \cos \alpha + B_{22} \sin^2 \alpha + (K \cos \alpha + B \sin \alpha)(d_1 - D_1 \cos \alpha - D_2 \sin \alpha)\} .$$

$$b_{12} = - \sin \alpha \{A_{12} \cos 2\alpha + (A_{22} - A_{11}) \sin 2\alpha + (A \cos \alpha + K \sin \alpha) \times (d_2 - C_2 \cos \alpha + C_1 \sin \alpha) - (A \sin \alpha - K \cos \alpha)(d_1 - C_1 \cos \alpha - C_2 \sin \alpha)\} + \cos \alpha \{B_{12} \cos 2\alpha + (B_{22} - B_{11}) \sin 2\alpha + (K \cos \alpha + B \sin \alpha)(d_2 - D_2 \cos \alpha + D_1 \sin \alpha) - (K \sin \alpha - B \cos \alpha) \times (d_1 - D_1 \cos \alpha - D_2 \sin \alpha) .$$

$$b_{22} = - \sin \alpha \{A_{11} \sin^2 \alpha - A_{12} \sin \alpha \cos \alpha + A_{22} \cos^2 \alpha - (A \sin \alpha - K \cos \alpha)(d_2 - C_2 \cos \alpha + C_1 \sin \alpha)\} + \cos \alpha \{B_{11} \sin^2 \alpha - B_{12} \sin \alpha \cos \alpha + B_{22} \cos^2 \alpha - (K \sin \alpha - B \cos \alpha)(d_2 - D_2 \cos \alpha + D_1 \sin \alpha)\} .$$

$$c_1 = 2[(C_1 \cos \alpha + C_2 \sin \alpha) \cos^2 \alpha + (D_1 \cos \alpha + D_2 \sin \alpha) \sin^2 \alpha] .$$

$$c_2 = 2[(C_2 \cos \alpha - C_1 \sin \alpha) \cos^2 \alpha + (D_2 \cos \alpha - D_1 \sin \alpha) \sin^2 \alpha] .$$

$$d_1 = 2[(C_1 \cos \alpha + C_2 \sin \alpha) \sin^2 \alpha + (D_1 \cos \alpha + D_2 \sin \alpha) \cos^2 \alpha] .$$

$$d_2 = 2[(C_2 \cos \alpha - C_1 \sin \alpha) \sin^2 \alpha + (D_2 \cos \alpha - D_1 \sin \alpha) \sin^2 \alpha] .$$

$$e_1 = - 2[C_1 \cos \alpha + C_2 \sin \alpha - D_1 \cos \alpha - D_2 \sin \alpha] \sin \alpha \cos \alpha .$$

$$e_2 = - 2[C_2 \cos \alpha - C_1 \sin \alpha - D_2 \cos \alpha + D_1 \sin \alpha] \sin \alpha \cos \alpha .$$

TABLE 2

Wall Derivative of Functions of Equations (44) *to* (47)

Function	$k = 0$	Remarks ($k = 0$)	$k = \frac{1}{2}$	$k = 1$	Remarks ($k = 1$)
f_{11}''	+2·395	(*)	+2·394	+2·385	
g_{21}''	0·000	($\equiv 0$)	+0·131	+0·194	
f_{12}''	−0·580		−0·565	−0·536	
g_{22}''	0·0000	($\equiv 0$)	−0·069	−0·097	
f_{13}''	+0·070		+0·044	0·000	($\equiv 0$)
g_{23}''	0·000	($\equiv 0$)	+0·105	0·000	($\equiv 0$)
f_{14}''	+0·070	($= f_{13}$)	+0·090	+0·099	
g_{24}''	+1·233	($= f_0$)	+1·847	+2·287	
f_{21}''	+1·853		+2·000	+2·287	($= g_{24}$)
g_{31}''	0·000	($\equiv 0$)	+0·073	+0·099	($= f_{14}$)
f_{22}''	−0·617		−0·282	0·000	($\equiv 0$)
g_{32}''	0·000	($\equiv 0$)	−0·035	0·000	($\equiv 0$)
f_{23}''	0·000	($\equiv 0$)	−0·060	−0·097	($= g_{22}$)
g_{33}''	0·000	($\equiv 0$)	−0·343	−0·536	($= f_{12}$)
f_{24}''	+0·096	(**)	+0·177	+0·194	($= g_{21}$)
g_{34}''	+0·570	($= g_0$)	+1·684	+2·385	($= f_{11}$)
f_{31}''	+1·233	($= f_0$)	+1·791	+2·188	
f_{32}''	0·000	($\equiv 0$)	−0·134	−0·438	
g_{11}''	+1·747		1·994	+2·188	($= f_{31}$)
g_{12}''	−0·515		−0·479	−0·438	($= f_{32}$)

N.B.—Functions are equal at corresponding values of k only.

* This function has been obtained by Howarth in a two-dimensional problem. It is given in *Modern Developments in Fluid Dynamics*, page 153, where it is denoted by f_2'.

** Since $g_{31}' = g_0'$ the equation for f_{24} is similar to an equation solved by Howarth as an investigation of equations f_0', g_0' when k is small. The solution here is equal to $2f_1$ where f_1 is tabulated in Table 1 of Howarth's paper[4].

TABLE 3

η	f_{11}' (0)*	$f_{11}'(\frac{1}{2})$	f_{11}' (1)	g_{21}' (0)	g_{21}' ($\frac{1}{2}$)	g_{21}' (1)
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·420	0·419	0·417	0·000	0·026	0·038
0·4	0·726	0·724	0·715	0·000	0·052	0·073
0·6	0·933	0·930	0·920	0·000	0·073	0·099
0·8	1·059	1·053	1·035	0·000	0·087	0·111
1·0	1·126	1·113	1·087	0·000	0·094	0·108
1·2	1·152	1·130	1·097	0·000	0·091	0·094
1·4	1·147	1·120	1·084	0·000	0·081	0·074
1·6	1·131	1·098	1·063	0·000	0·067	0·052
1·8	1·104	1·073	1·041	0·000	0·052	0·033
2·0	1·083	1·049	1·024	0·000	0·037	0·019
2·2	1·059	1·030	1·012	0·000	0·026	0·009
2·4	1·038	1·017	1·006	0·000	0·015	0·003
2·6	1·026	1·007	1·002	0·000	0·009	0·000
2·8	1·014	1·003	1·001	0·000	0·004	0·000
3·0	1·008	1·001	1·000	0·000	0·002	0·000
3·2	1·003	1·000	1·000	0·000	0·000	0·000
3·4	1·000					
3·6	1·000					

TABLE 4

η	f_{12}' (0)	f_{12}' ($\frac{1}{2}$)	f_{12}' (1)	g_{22}' (0)	g_{22}' ($\frac{1}{2}$)	g_{22}' (1)
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	—0·096	—0·093	—0·087	0·000	—0·014	—0·019
0·4	—0·155	—0·149	—0·137	0·000	—0·026	—0·037
0·6	—0·182	—0·173	—0·155	0·000	—0·037	—0·050
0·8	—0·186	—0·172	—0·149	0·000	—0·044	—0·056
1·0	—0·173	—0·156	—0·128	0·000	—0·047	—0·055
1·2	—0·151	—0·130	—0·100	0·000	—0·045	—0·048
1·4	—0·124	—0·101	—0·072	0·000	—0·040	—0·037
1·6	—0·098	—0·074	—0·047	0·000	—0·033	—0·026
1·8	—0·074	—0·051	—0·029	0·000	—0·026	—0·017
2·0	—0·053	—0·033	—0·016	0·000	—0·018	—0·012
2·2	—0·037	—0·020	—0·008	0·000	—0·012	—0·005
2·4	—0·024	—0·011	—0·003	0·000	—0·009	—0·002
2·6	—0·016	—0·005	—0·001	0·000	—0·004	—0·001
2·8	—0·010	—0·002	0·000	0·000	—0·002	0·000
3·0	—0·005	—0·001	0·000	0·000	—0·001	0·000
3·2	—0·002	0·000	0·000	0·000	0·000	0·000
3·4	—0·001	0·000	0·000	0·000	0·000	0·000
3·6	0·000	0·000	0·000	0·000	0·000	0·000

* The bracket throughout these tables (3 to 12) gives the value of k.

TABLE 5

η	$f_{13}'(0)$	$f_{13}'(\frac{1}{2})$	$f_{13}'(1)$	$g_{23}'(0)$	$g_{23}'(\frac{1}{2})$	$g_{23}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·014	0·009	0·000	0·000	+0·016	0·000
0·4	0·027	0·017	0·000	0·000	0·023	0·000
0·6	0·037	0·023	0·000	0·000	0·023	0·000
0·8	0·045	0·026	0·000	0·000	0·019	0·000
1·0	0·048	0·029	0·000	0·000	0·014	0·000
1·2	0·047	0·030	0·000	0·000	0·008	0·000
1·4	0·044	0·022	0·000	0·000	+0·003	0·000
1·6	0·038	0·017	0·000	0·000	0·000	0·000
1·8	0·031	0·013	0·000	0·000	−0·001	0·000
2·0	0·024	0·009	0·000	0·000	−0·002	0·000
2·2	0·018	0·006	0·000	0·000	−0·002	0·000
2·4	0·013	0·003	0·000	0·000	−0·001	0·000
2·6	0·009	0·002	0·000	0·000	−0·001	0·000
2·8	0·006	0·001	0·000	0·000	0·000	0·000
3·0	0·004	0·000	0·000	0·000	0·000	0·000
3·2	0·002	0·000	0·000	0·000	0·000	0·000
3·4	0·002	0·000	0·000	0·000	0·000	0·000
3·6	0·001	0·000	0·000	0·000	0·000	0·000

TABLE 6

η	$f_{14}'(0)$	$f_{14}'(\frac{1}{2})$	$f_{14}'(1)$	$g_{24}'(0)$	$g_{24}'(\frac{1}{2})$	$g_{24}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·014	0·018	0·019	0·227	0·330	0·398
0·4	0·027	0·034	0·037	0·415	0·583	0·683
0·6	0·037	0·048	0·050	0·566	0·766	0·870
0·8	0·045	0·056	0·056	0·686	0·890	0·979
1·0	0·048	0·057	0·055	0·778	0·968	1·032
1·2	0·047	0·054	0·048	0·847	1·011	1·049
1·4	0·044	0·046	0·038	0·897	1·030	1·047
1·6	0·038	0·037	0·027	0·980	1·034	1·036
1·8	0·031	0·027	0·018	0·957	1·031	1·024
2·0	0·024	0·019	0·010	0·973	1·024	1·014
2·2	0·018	0·012	0·005	0·984	1·017	1·008
2·4	0·013	0·007	0·003	0·990	1·009	1·004
2·6	0·009	0·004	0·001	0·995	1·005	1·002
2·8	0·006	0·002	0·000	0·997	1·002	1·001
3·0	0·004	0·001	0·000	0·998	1·001	1·000
3·2	0·002	0·000	0·000	0·999	1·000	
3·4	0·002	0·000	0·000	1·000		
3·6	0·001	0·000	0·000			

TABLE 7

η	$f_{21}'(0)$	$f_{21}'(\frac{1}{2})$	$f_{21}'(1)$	$g_{31}'(0)$	$g_{31}'(\frac{1}{2})$	$g_{31}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·331	0·370	0·398	0·000	0·014	0·019
0·4	0·587	0·645	0·683	0·000	0·028	0·037
0·6	0·771	0·836	0·870	0·000	0·040	0·050
0·8	0·894	0·957	0·979	0·000	0·048	0·056
1·0	0·976	1·025	1·032	0·000	0·051	0·055
1·2	1·022	1·055	1·049	0·000	0·050	0·048
1·4	1·044	1·061	1·047	0·000	0·045	0·038
1·6	1·050	1·055	1·036	0·000	0·038	0·027
1·8	1·047	1·042	1·024	0·000	0·029	0·018
2·0	1·039	1·030	1·014	0·000	0·021	0·010
2·2	1·030	1·019	1·008	0·000	0·014	0·005
2·4	1·021	1·011	1·004	0·000	0·009	0·003
2·6	1·014	1·006	1·002	0·000	0·005	0·001
2·8	1·009	1·003	1·001	0·000	0·003	0·000
3·0	1·005	1·001	1·000	0·000	0·001	0·000
3·2	1·003	1·000		0·000	0·000	0·000
3·4	1·001					
3·6	1·000					

TABLE 8

η	$f_{22}'(0)$	$f_{22}'(\frac{1}{2})$	$f_{22}'(1)$	$g_{32}'(0)$	$g_{32}'(\frac{1}{2})$	$g_{32}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	−0·104	−0·046	0·000	0·000	−0·007	0·000
0·4	−0·169	−0·074	0·000	0·000	−0·014	0·000
0·6	−0·203	−0·086	0·000	0·000	−0·019	0·000
0·8	−0·210	−0·086	0·000	0·000	−0·023	0·000
1·0	−0·199	−0·078	0·000	0·000	−0·024	0·000
1·2	−0·176	−0·065	0·000	0·000	−0·023	0·000
1·4	−0·148	−0·051	0·000	0·000	−0·021	0·000
1·6	−0·118	−0·037	0·000	0·000	−0·017	0·000
1·8	−0·090	−0·026	0·000	0·000	−0·014	0·000
2·0	−0·066	−0·017	0·000	0·000	−0·010	0·000
2·2	−0·046	−0·010	0·000	0·000	−0·007	0·000
2·4	−0·031	−0·006	0·000	0·000	−0·004	0·000
2·6	−0·020	−0·003	0·000	0·000	−0·002	0·000
2·8	−0·012	−0·001	0·000	0·000	−0·001	0·000
3·0	−0·007	0·000	0·000	0·000	0·000	0·000
3·2	−0·004	0·000	0·000	0·000	0·000	0·000
3·4	−0·002	0·000	0·000	0·000	0·000	0·000
3·6	−0·001	0·000	0·000	0·000	0·000	0·000

TABLE 9

η	$f_{23}'(0)$	$f_{23}'(\frac{1}{2})$	$f_{23}'(1)$	$g_{33}'(0)$	$g_{33}'(\frac{1}{2})$	$g_{33}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·000	−0·012	−0·019	0·000	−0·059	−0·087
0·4	0·000	−0·023	−0·037	0·000	−0·098	−0·137
0·6	0·000	−0·031	−0·050	0·000	−0·119	−0·155
0·8	0·000	−0·037	−0·056	0·000	−0·125	−0·149
1·0	0·000	−0·038	−0·055	0·000	−0·119	−0·128
1·2	0·000	−0·035	−0·048	0·000	−0·104	−0·100
1·4	0·000	−0·030	−0·037	0·000	−0·085	−0·072
1·6	0·000	−0·024	−0·026	0·000	−0·065	−0·047
1·8	0·000	−0·018	−0·017	0·000	−0·047	−0·029
2·0	0·000	−0·012	−0·012	0·000	−0·032	−0·016
2·2	0·000	−0·008	−0·005	0·000	−0·020	−0·008
2·4	0·000	−0·004	−0·002	0·000	−0·012	−0·003
2·6	0·000	−0·002	−0·001	0·000	−0·007	−0·001
2·8	0·000	−0·001	0·000	0·000	−0·003	0·000
3·0	0·000	0·000	0·000	0·000	−0·001	0·000

TABLE 10

η	$f_{24}'(0)$	$f_{24}'(\frac{1}{2})$	$f_{24}'(1)$	$g_{34}'(0)$	$g_{34}'(\frac{1}{2})$	$g_{34}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·020	0·035	0·038	0·114	0·307	0·417
0·4	0·038	0·067	0·073	0·228	0·554	0·715
0·6	0·054	0·093	0·099	0·339	0·744	0·920
0·8	0·066	0·109	0·111	0·446	0·880	1·035
1·0	0·072	0·113	0·108	0·547	0·973	1·087
1·2	0·074	0·107	0·094	0·639	1·025	1·097
1·4	0·070	0·093	0·074	0·720	1·049	1·084
1·6	0·062	0·075	0·052	0·789	1·054	1·063
1·8	0·052	0·056	0·033	0·846	1·048	1·041
2·0	0·042	0·038	0·019	0·891	1·037	1·024
2·2	0·032	0·024	0·009	0·926	1·026	1·012
2·4	0·024	0·014	0·003	0·951	1·017	1·006
2·6	0·016	0·007	0·001	0·969	1·009	1·002
2·8	0·010	0·003	0·000	0·981	1·005	1·001
3·0	0·006	0·000	0·000	0·988	1·003	1·000
3·2	0·004	0·000	0·000	0·996	1·001	

TABLE 11

η	$f_{31}'(0)$	$f_{31}'(\frac{1}{2})$	$f_{31}'(1)$	$f_{32}'(0)$	$f_{32}'(\frac{1}{2})$	$f_{32}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·227	0·318	0·378	0·000	—0·022	—0·068
0·4	0·415	0·561	0·646	0·000	—0·034	—0·100
0·6	0·566	0·735	0·819	0·000	—0·039	—0·105
0·8	0·686	0·855	0·922	0·000	—0·038	—0·093
1·0	0·778	0·930	0·976	0·000	—0·033	—0·074
1·2	0·847	0·975	1·000	0·000	—0·027	—0·053
1·4	0·897	0·989	1·008	0·000	—0·020	—0·034
1·6	0·932	1·000	1·008	0·000	—0·014	—0·021
1·8	0·957	1·004	1·006	0·000	—0·009	—0·012
2·0	0·973	1·005	1·003	0·000	—0·005	—0·006
2·2	0·984	1·004	1·001	0·000	—0·003	—0·003
2·4	0·991	1·003	1·000	0·000	—0·001	—0·001
2·6	0·995	1·002		0·000	0·000	0·000
2·8	0·997	1·000		0·000	0·000	0·000
3·0	0·998					
3·2	0·999					

TABLE 12

η	$g_{11}'(0)$	$g_{11}'(\frac{1}{2})$	$g_{11}'(1)$	$g_{12}'(0)$	$g_{12}'(\frac{1}{2})$	$g_{12}'(1)$
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·310	0·349	0·378	—0·083	—0·076	—0·068
0·4	0·545	0·605	0·646	—0·131	—0·115	—0·100
0·6	0·715	0·780	0·819	—0·148	—0·126	—0·105
0·8	0·831	0·892	0·922	—0·146	—0·118	—0·093
1·0	0·908	0·957	0·976	—0·130	—0·100	—0·074
1·2	0·955	0·992	1·000	—0·108	—0·077	—0·053
1·4	0·982	1·007	1·008	—0·085	—0·056	—0·034
1·6	0·996	1·010	1·008	—0·063	—0·038	—0·021
1·8	1·002	1·010	1·006	—0·045	—0·024	—0·012
2·0	1·005	1·008	1·003	—0·031	—0·015	—0·006
2·2	1·005	1·005	1·001	—0·013	—0·008	—0·003
2·4	1·004	1·003	1·000	—0·008	—0·004	—0·001
2·6	1·003	1·002		—0·004	—0·002	0·000
2·8	1·002	1·001		—0·002	—0·001	0·000
3·0	1·001	1·000		—0·001	—0·000	0·000
3·2	1·001			0·000	0·000	0·000

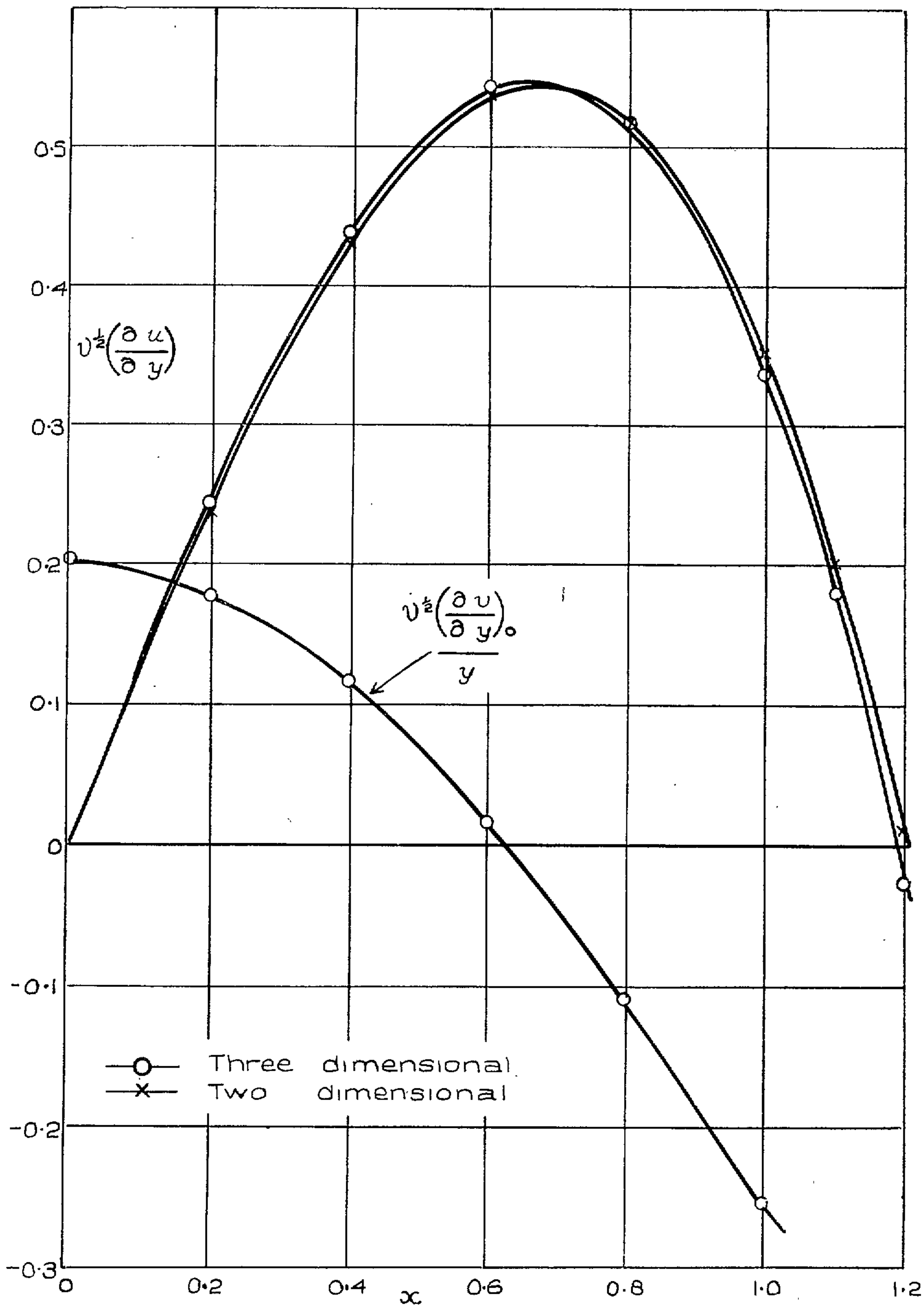

FIG. 1. Skin friction in two and three dimensions.

(4995) Wt. 20/9036 K.7 4/57 Hw. PRINTED IN GREAT BRITAIN

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50s. (51s. 9d.).
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63s. (64s. 9d.)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control Structures, and a miscellaneous section. 50s. (51s. 9d.)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control Structures. 63s. (64s. 9d.)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75s. (76s. 9d.)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47s. 6d. (49s. 3d.)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80s. (81s. 9d.)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90s. (92s. 6d.)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84s. (86s. 3d.)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84s. (86s. 3d.)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130s. (132s. 6d.)
Vol. II. Aircraft, Airscrews, Controls. 130s. (132s. 6d.)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130s. (132s. 3d.)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130s. (132s. 3d.)

Annual Reports of the Aeronautical Research Council—

1937 2s. (2s. 2d.) 1938 1s. 6d. (1s. 8d.) 1939–48 3s. (3s. 3d.)

Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—

April, 1950 - - - - R. & M. 2600 2s. 6d. (2s. 8d.)

Author Index to all Reports and Memoranda of the Aeronautical Research Council—

1909—January, 1954 R. & M. No. 2570 15s. (15s. 6d.)

Indexes to the Technical Reports of the Aeronautical Research Council—

December 1, 1936—June 30, 1939	R. & M. No. 1850	1s. 3d. (1s. 5d.)
July 1, 1939—June 30, 1945	R. & M. No. 1950	1s. (1s. 2d.)
July 1, 1945—June 30, 1946	R. & M. No. 2050	1s. (1s. 2d.)
July 1, 1946—December 31, 1946	R. & M. No. 2150	1s. 3d. (1s. 5d.)
January 1, 1947—June 30, 1947	R. & M. No. 2250	1s. 3d. (1s. 5d.)

Published Reports and Memoranda of the Aeronautical Research Council—

Between Nos. 2251–2349	R. & M. No. 2350	1s. 9d. (1s. 11d.)
Between Nos. 2351–2449	R. & M. No. 2450	2s. (2s. 2d.)
Between Nos. 2451–2549	R. & M. No. 2550	2s. 6d. (2s. 8d.)
Between Nos. 2551–2649	R. & M. No. 2650	2s. 6d. (2s. 8d.)

Prices in brackets include postage

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London W.C.2; 423 Oxford Street, London W.1 (Post Orders: P.O. Box 569, London S.E.1) 13a Castle Street, Edinburgh 2; 39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; Tower Lane, Bristol, 1; 80 Chichester Street, Belfast, *or through any bookseller.*

S.O. Code No. 23-3006