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# The Influence of Plain Bearings on Shaft Whirling

*By*

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*Summary.*—Experiments are described which show the effects of bearing length, bearing clearance, and lubricant viscosity on the critical whirling speeds of a single-shaft rotor system supported in plain bearings.

The critical speed of a two-bearing–shaft rotor system is shown to depend upon rotor unbalance for bearings of normal clearance. When the clearance is small relative to the bearing length, unsymmetric stiffness characteristics are obtained and produce two critical speeds instead of one.

The lubricant in a drip-feed bearing is shown to have a stiffening effect, with consequent increase of critical whirling speed. It is also shown that the critical whirl amplitudes with dry bearings can be appreciably larger than those obtained with lubricated bearings.

1. *Introduction.*—Experiments have previously been carried out on model rigs to establish a correct theoretical approach to the problem of shaft whirling[1,2,3,4].

The experiments described in this report show the effects that lubrication and dimensional changes of plain bearings can have on the critical whirling speed of a simple shaft–rotor system.

A shaft supported by two bearings with a rotor midway between the bearings, is shown to have non-linear stiffness characteristics. An approximate method of calculating the critical speed of the non-linear system is given. The critical speed is shown to be dependent on the magnitude of the unbalance forces and damping in the system ; accurate calculations of critical speed can be made only when the critical whirl amplitude is known. A horizontal shaft system with long bearings and small bearing clearances is shown to have complicated non-linear stiffness characteristics due to asymmetry caused by the static deflection of the system in the vertical plane, in which case two critical speeds may be obtained instead of one, as usually.

The effective stiffness of the system is also shown to be appreciably effected by the nature of the lubricant in the bearings, variations of the order of 50 per cent having been measured. Large damping forces also arise from a lubricated bearing. The amplitude of whirl at the critical speed is shown to increase by almost 300 per cent when the bearing is allowed to run dry. Dry friction effects arising from a dry bearing are also demonstrated.

2. *Description of the Experimental Rig.*—Details of the experimental rig are given in Fig. 23 and Appendix I. The whirling system consists of a hollow shaft $1 \cdot 2$ in. O.D. $\times$ 1 in. I.D. and 36 in. long, carrying a rotor rigidly fixed at its centre. The shaft is supported by plain phosphor-bronze bearings at either end, the distance between the inner faces of the bearings being $29\frac{1}{2}$ in. To minimise the flexibility of the supports the bearings are located in housings made from 3 in. thick steel plate, rigidly attached to a channel-section base.

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The bearing lengths vary from 3·25 in. to 1·25 in., and bearing clearances of 0·004 in. and 0·001 in. are used.

Spiral grooves are cut in the bearings for lubricating purposes. Drip-feed lubricators mounted in the bearing housings supply lubricating oil to the bearings.

The shaft is belt-driven from a variable-speed motor, with a flexible coupling between the driving pulley and the driven end of the shaft. The shaft speed is recorded by a moving film camera using a contact breaker on the driving pulley in conjunction with a fifty-cycle timing trace obtained from the A.C. mains.

The motion of the centre of the shaft is measured by means of two linear translation inductance pick-ups, mounted at right angles to each other, on a steel block similar in shape to the bearing housings. This block is rigidly attached to the base of the rig near the rotor and serves also as a guard ring to limit the whirl amplitude of the shaft.

The pick-ups consist of a dust core attached to one end of a spindle ; on the other end of the spindle is a shoe, which is held constantly in contact with the shaft by means of a light spring. The dust core moves inside a coil wound on a hollow insulated core, which is fixed to the body of the pick-up and is hence held rigidly relative to the base plate. Any relative movement between the centre of the shaft and the base plate causes the position of the dust core to change relative to the coil, with a consequent change of inductance of the instrument, which is linear over a range of $\pm$ 0·2 in.

By using the appropriate electronic circuits the variations of inductance are converted to voltage variations, which are applied to the $X$ and $Y$ plates of a C.R.T. The vertical pick-up is connected to the $Y$ plates, and the horizontal pick-up to the $X$ plates ; the overall gain of each circuit is the same.

Using this method, the motion of the shaft at the rotor is magnified and reproduced on the C.R.T. screen, and then photographed by means of an F.24 camera. Calibrations are made by inserting feeler gauges between the pick-up head and the rotor, and measuring the resultant deflections on the camera screen.

Records of shaft speed and whirl path are taken simultaneously.

3. *Details of the Experiments.*—3.1. *The Effects of Rotor Unbalance on the Critical Whirling Speed.*—Bearings of length 3·25 in. and clearance of 0·004 in. were fitted, and a series of experiments carried out to investigate the effect of rotor unbalance on the critical whirling speed of the system.

A load-deflection curve was plotted for static loading at the rotor (Fig. 1). The deflections at the rotor at which the flexibility of the system changed were noted. The unbalance on the rotor was then adjusted by attaching small weights at a radius of 2 in., so that at the critical whirling speed the whirl amplitudes came successively within each of the three stiffness ranges of the system (*see* Appendix II). Amplitude-frequency curves were plotted (Fig. 7) for the different unbalance conditions.

In Table 1 the experimental results are compared with theoretical calculations of critical whirling speed. The calculations were made using both estimated and measured stiffnesses.

3.2. *The Whirling Characteristics of the Shaft–Rotor System Rotating in Dry Bearings.*—Whilst investigating the effects of rotor unbalance on the critical whirling speed, it was observed that when the oil supply to the bearings was cut off, the mode and amplitude of vibration of the shaft near the critical whirling speed were affected. The general change in the whirling characteristics of the system was so marked that it was considered to be of sufficient interest to warrant a separate investigation.

With the rotor balanced, within experimental limits, the system was run in dry bearings of 0·004 in. clearance and 3·25 in. long. Whirl amplitudes were measured in the usual way. Records covering the speed range are shown in Fig. 21, and the whirl amplitude-frequency curves are shown plotted in Fig. 11.

In order to clarify some of the effects recorded during these experiments, visual observations using stroboscopic light were made, with the bearings bored out to 1·236 in. diameter, giving a clearance of 0·036 in. between shaft and bearing. It was anticipated that the effects would be more marked when the shaft was running in bearings with excessive clearance.

3.3. *The Effect of Bearing Dimensions and Lubrication on the Critical Whirl Amplitude and Frequency*.—During the experiments described in section 3.1, it was observed that when the rotor unbalance was small, *i.e.*, when the shaft was whirling with support at the inner edges of the bearings only, the shaft did not always have the same critical whirling speed. Further investigations of this phenomena showed that when the temperature of the oil leaving the bearing increased, the critical whirling speed decreased. It appeared, therefore, that the viscosity of the lubricating oil in the bearings had an effect on the whirling of the shaft, and tests were carried out to investigate this effect.

Four grades of oil were obtained of which the specific-gravity (temperature and viscosity) temperature curves are shown in Figs. 9 and 10 respectively. Using the four grades of oil in turn to lubricate the bearings, a series of experiments were then carried out with bearing clearances of 0·004 in. and 0·001 in. and bearing lengths of 3·25 in., 2·25 in. and 1·25 in. After each test the bearings were allowed to cool, so that the temperature of the oil passing through the bearings during each test was approximately constant. This temperature was measured approximately by means of the thermometers placed against the inner faces of the bearings close to the shaft, so that with a liberal supply of oil being fed to the bearings there was a constant flow over the thermometer bulb as the oil left the bearing. The errors arising from the approximate method of measuring oil temperature were minimised to some extent by keeping the temperature gradient of the oil flowing through the bearing as small as possible.

The results of these tests are plotted in Figs. 12 to 19.

4. *Discussion of the Results*.—4.1. *The effect of Rotor Unbalance on the Critical Whirling Speed*.—The load-deflection curves shown in Figs. 1 to 6 were obtained experimentally for the different bearings used. It is apparent, from an examination of the load-deflection curves, that a shaft supported in plain bearings has non-linear stiffness characteristics. This non-linearity is caused by changes in the end conditions of the system. Initially the shaft is simply supported by the inner edges of the bearings and the system is linear, but as the deflection at the rotor increases the shaft becomes supported by the inner edge of one bearing and by both the inner and outer edges of the other. Although the bearings have nominally the same clearance, due to manufacturing tolerances there will be some difference and in this intermediate state the shaft becomes asymmetrically supported. Finally, with a further increase in deflection at the rotor, the shaft will be supported across both bearings and will then be in a semi-encastré condition.

An approximate method of calculating the frequency response of an undamped non-linear system is described in Appendix III, from which the amplitude-frequency curve of Fig. 8 has been obtained for the shaft–rotor system assuming a rotor unbalance (mh) of 25 gm in. In Appendix II the flexibilities are calculated for the three cases of end conditions. The calculated flexibilities are in good agreement with those obtained by experiment (Fig. 1).

Fig. 7 shows the experimental amplitude-frequency curves for rotor unbalances chosen to give critical whirl amplitudes in each of the three ranges of flexibility. The critical speeds obtained from the curves of Fig. 7 are tabulated in Table 1 and compared with the theoretical criticals obtained from Fig. 8. The latter criticals are found by assuming critical whirl amplitudes identical with the measured amplitudes of Fig. 7 and that the peak amplitude occurs along the line $0\ 0'\ 0''$ (Fig. 8).



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The results show that there is good agreement between calculated critical speeds and those obtained experimentally, but accurate estimates of the critical speeds of the non-linear system can only be made when the whirl amplitude at the critical speed is known.

4.2. *The Whirling Characteristics of the Shaft–Rotor System Rotating in Dry Bearings.*—In Fig. 11 the amplitude-frequency curves are plotted for shaft speeds increasing from 32 to 60 r.p.s., and decreasing from 60 to 32 r.p.s. The records from which these curves were plotted are shown in Fig. 21. The critical speed and amplitude of the shaft during these experiments indicate that at all times the shaft was supported on the inner edges of the bearings only.

Figs. 11 and 21 show that for increasing speeds the shaft initially whirls in an elliptic form, the major axis of the ellipse being almost horizontal. As the critical speed is approached the whirl path approaches a more circular form. The critical whirling speed is 44 r.p.s. ; to increase the speed of the shaft above this speed required an appreciable increase in power from the motor, more so than is usually required to take a shaft through its critical speed. However, once the shaft speed increased above the critical speed the whirl amplitudes decreased rapidly. It is significant that these curves, for increasing speeds, are similar in shape to those obtained from the forced vibration of a system having dry friction damping[5].

For speeds decreasing from 60 to 32 r.p.s. the whirl response of the shaft is entirely different from that for increasing speeds. The whirl path is again elliptical but the major elliptic axis is vertical. The critical speed is slightly lower than before. As the shaft passes through its critical, the horizontal axis of the ellipse decreases and for a short period between 42·4 and 40·6 r.p.s. is negative, indicating a reverse whirl. Below 40·6 r.p.s. the shaft suddenly resumes a steady whirl of more circular form. During repeat tests, the shaft always showed the same tendencies of whirl both for increasing and decreasing speeds. It was not always possible to reproduce the narrow band of reverse whirl, due probably, to a variation in the state of dryness of the bearing, although it was obvious from visual observations on a monitor C.R.T. that the shaft always approached this condition up to the point where the horizontal elliptic axis became zero.

The effects described above were also obtained when the bearings were bored out to give 0·036 in. clearance on the diameter. For these tests visual observations were made using stroboscopic light. At shaft speeds around 22 r.p.s. there was a pronounced hammering of the shaft in the bearings, the motion was one of rocking of the shaft in the bottom of the bearing, the vertical amplitude of the rotor being small and the horizontal amplitude being relatively large. At about 35 r.p.s. there appeared to be a resonant condition of vibration indicative of a critical whirling speed. Finally, at about 38 r.p.s. a further hammering of the shaft in the bearings occurred. This was best reproduced by driving the shaft at higher speeds and reducing speed until the hammering occurred. Under these conditions the motion of the shaft in the bearings was predominantly vertical with a similar motion at the rotor.

During the investigations in connection with the bearings of large clearance, another rather disconcerting phenomena was found to occur intermittently. As the shaft speed was decreased slowly in the neighbourhood of the critical whirling speed of 38 r.p.s. the shaft became violently unstable. There was a loud bang and the amplitude at the rotor became large enough to cause the shaft to run round the inside of the guard ring, the shaft speed being greatly reduced due to an increased load on the driving motor. It was found by experiment that by further reducing the motor speed the vibratory level could be reduced until it was finally possible to withdraw the guard ring, leaving the shaft vibrating in the bearings with an amplitude of approximately 0·125 in. With the guard ring withdrawn, the shaft speed was about 1·5 r.p.s. and the rotor was whirling in the reverse direction to the rotation at approximately 18 c.p.s. The application of a vertical force (by hand) of about 10 lb was sufficient to overcome the friction forces driving the whirl. The shaft would then run steadily in the bearings when the load was removed.

The explanation of this phenomena is given in Appendix III.

From the results of these experiments with dry bearings it is apparent that the dry friction forces arising from the bearings can have an appreciable effect on the nature of the shaft vibrations. There appear to be three types of motion of the system : at speeds below this critical there is a rocking of the shaft in the bottom of the bearings resulting in a predominantly lateral vibration of the rotor ; as the shaft speed increases the rotor whirls in a normal manner ; and as the shaft speed is decreased from a speed above the critical, the rotor vibrates predominantly in a vertical plane, due to bouncing of the shaft in the bearings. An appreciable increase in bearing clearance appears to result in the spreading of these phenomena over a larger frequency range. With a more normal bearing clearance of 0·004 in. on 1·2 in. shaft diameter, vertical vibration for decreasing shaft speeds takes the place of the normal circular whirl obtained for increasing shaft speeds. The tendency for the shaft to whirl in the reverse direction is probably due to partial running round the inside of the bearings, although this is very intermittent as the shaft usually slips in the bearings.

These experiments have been described in detail to illustrate the effects of dry friction forces on the whirling of the system. However, the most important practical effect of lack of lubrication in plain bearings may be found from a comparison between Figs. 11 and 12, which give amplitude-frequency curves for dry and for well lubricated bearings. For the dry bearing the amplitude of whirl at the critical speed is 0·014 in., whereas with a thin oil for lubrication the amplitude is seen to be reduced to 0·005 in. In this case the lubrication was effected by drip feed and the shaft unloaded. It is probable that this reduction of amplitude of whirl when the bearings are lubricated would be greater in the case of a heavily loaded shaft supported in pressure fed bearings. In many large turbine installations it is common practice to design the shaft–rotor system with a critical whirling speed lower than the operating speed, and to rely on balancing of the rotating parts and structural damping to enable the shaft to run through its critical without excessive vibration. If, therefore, the lubrication of the bearings is inadequate, a large increase in amplitude at the critical speed may be experienced with consequent danger of bearing failures.

4.3. *The Effect of Bearing Dimensions and Lubrication on the Critical Whirl Amplitude and Frequency.*—4.3.1. *Experiments with bearing clearances of* 0·004 *in.*—The amplitude-frequency curves pertaining to these experiments for bearing lengths of 3·25 in., 2·25 in. and 1·25 in. are plotted in Figs. 12, 13, 14. In each case curves are plotted for different grades of lubricant in the bearings. For the 3·25 in. bearings the rotor was balanced, but in the other two cases the unbalance on the rotor was increased by 20 gm in. because an increase in the rotor unbalance was found to improve the consistency of the results. In all three cases, however, the whirl amplitude at the rotor was small enough to ensure that the shaft would be supported by the inner edges of the bearings only.

These experiments show that the oil in the bearings has an appreciable stiffening effect on the shaft, the stiffening increasing with the oil viscosity. In Table 3 the results have been tabulated, including the effective flexibility of the shaft at the rotor, determined from the measured critical speed. From this table it is seen that the effective flexibility of the shaft at the rotor is reduced by as much as 49 per cent of the measured static flexibility, the effect being greatest for the longer bearings. A plot of log log viscosity against (critical speed)$^2$ is shown in Fig. 15. Over the range of lubricating oils used, the (critical speed)$^2$ may be assumed to be proportional to log log viscosity. Theoretically, if the viscosity of the oil could be increased indefinitely and efficient lubrication maintained, the end conditions of the shaft would approach encastrè conditions and the critical speed would approach a corresponding theoretical value of 80 r.p.s. In practice, it was found that with drip feed efficient lubrication was not possible with oils of a higher viscosity than those used ; consequently, when these thicker oils were used the critical speed was found to fall due to the bearing being only partially lubricated.

4.3.2. *Experiments with bearing clearances of* 0·001 *in.*—Amplitude-frequency curves for the bearing with 0·001 in. clearances are plotted in Figs. 16, 17 and 18. The whirling characteristics of the system supported in bearings of length 3·25 in. and 2·25 in. are more complex than would be expected. Fig. 22 shows a series of records taken over the appropriate frequency range for the 2·25 in. bearing. The shaft has a critical whirl at 57·4 r.p.s., the whirl form being elliptical the major axis of the ellipse in the horizontal plane. Above this speed the whirl amplitudes decrease but the major axis of the ellipse is re-orientated as the shaft speed increases. Above 64 r.p.s. the whirl amplitude increases until at 65·2 r.p.s. there appears to be a second critical whirl of an elliptic form with the major axis almost vertical, the minor axis having small amplitude. The curves of Figs. 16 and 17 show the variation of the horizontal and vertical axes of the elliptic whirl against shaft rotational speed.

The viscosity of the lubricating oil was found to affect the stiffness of the system in a similar way to that observed with the bearing of 0·004 in. clearance. Both the critical speeds are affected in a similar manner (Fig. 19).

The explanation of the two critical speeds is found by considering the end conditions of the stationary shaft. The effective weight of the shaft and rotor (which is concentrated at the centre of the shaft) is approximately 17 lb. The flexibility of the shaft (at the centre) assuming support at the inner edges of the bearings is 0·00035 in./lb. (Fig. 5). This static load on the simply supported system would cause an initial deflection of about 0·006 in. at the centre of the shaft. This is sufficient to cause one end of the shaft to make contact with the top of the outer edge of the bearing. For a superimposed vertical load directed downwards, the shaft flexibility will be appropriate to the case of a shaft simply supported at one end and supported across the bearing at the other. For a superimposed vertical load directed upwards, the flexibility will be appropriate to the case of simple support at both bearings. Finally, for a horizontal load in either direction the shaft flexibility will be appropriate to conditions between the two for vertical loading. For the static case there are therefore three different flexibilities to consider.

Consider next the flexibility of the system when whirling. Due to small bearing clearances, at a critical speed the ends of the shaft will whirl around the bore of the bearing. For the condition where the shaft end, which was initially in contact with the top of the bearing, is in contact with a point at 90 deg to its initial position, the two vertical flexibilities will now apply to the horizontal axis and the horizontal flexibility will apply to the vertical axis. The rotating system has therefore asymmetrical stiffness as the flexibilities in the direction of the major axes will fluctuate four times per revolution of the shaft. For this reason it is to be expected that the whirl characteristics of the system will be complex.

In Fig. 20 a theoretical curve of critical speed against effective flexibility of the system (*i.e.*, the flexibility of a simple system of similar dimensions which would have the same critical speed), has been plotted. The measured flexibilities for the 3·25 in. and 2·25 in. bearings cases have been superimposed on the curve, as also have the critical speed obtained from Figs. 16 and 17. A consideration of the relative positions of these latter points and the measured flexibilities shows that the first critical occurs at speeds somewhat lower than the critical appropriate to the second flexibility (simple support at one end and support across the bearing at the other), although in the case of the 3·25 in. bearing the stiffening effect of lubrication brings the critical up to a speed slightly higher than this. The second critical appears to be associated with the third flexibility (support across both bearings).

There is no measurable difference in the critical speed with change of lubricant when the bearings are shortened to 1·25 in., and the whirl form is circular as would be expected.

4.3.3. *The damping effect of lubricating oil in the bearings*.—As mentioned previously there is a marked decrease in whirl amplitude at the critical whirling speed when the bearings are adequately lubricated. In the case of bearings with 0·004 in. clearance there is no consistent effect on the whirl amplitude as the viscosity of the lubricant is increased, although Fig. 12 shows a tendency for the amplitude at the critical speed to increase as the viscosity of the lubricant increases. The increase in unbalance for the tests of the 2·25 in. and 1·25 in. bearings appears to have masked this tendency. For bearings of 0·001 in. clearance and 3·25 in. and 2·25 in. long there is again a tendency for the amplitude at the critical speed to increase with increase of viscosity of the lubricant (Figs. 16 and 17).

5. *Conclusions.*—Tests with dry bearings show that lack of lubrication in plain bearings can have a marked effect on the form of whirl at the critical speed. The most important effect is the large increase in amplitude at the critical speed that would result from failure of the lubricating system. This is an important factor, especially in cases where a system is designed to have a critical speed below the operating speed, in which case a large amplitude of whirl may build up whilst running through the critical.

The length of bearing has no effect on the critical speed (except through the stiffening effect of the lubricant) unless the clearance is small or the unbalance forces are large, in which case the system will have non-linear stiffness characteristics and the critical speeds can only be estimated on the assumption of a known critical whirl amplitude or an accurate knowledge of the unbalance force and the damping in the system. In the design stage, therefore, it would be necessary to calculate critical speeds assuming first a critical whirl amplitude within the first range of linear stiffness (support at the inner edges of the bearings only) and then a critical whirl amplitude which would be the maximum tolerable for smooth running of the system. Operating speeds should then be avoided between the two calculated criticals.

Due to initial bending under static load, a horizontal shaft–rotor system supported in long bearings (say $L/D$ ratio greater than 2) is shown to be capable of having two critical speeds, near to, but less than, the speeds obtained by assuming linear stiffness with three or four-point support by the bearings. Ideally, a system of this type should be designed to operate at speeds below the critical speed calculated on the assumption of linear stiffness with simple support at the inner edges of the bearings, or above that critical speed calculated on the assumption of linear stiffness with four-point support across the bearings.

An appreciable increase in effective stiffness can be achieved by increasing the viscosity of the lubricant, which acts as an auxiliary spring inside the bearings. The tests on variation of lubricant viscosity also show a tendency for the amplitude at the critical speed to increase as the viscosity increases, particularly when the unbalance forces are small.

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# APPENDIX I

*Details of Experimental Rig*

| | |
|---|---|
| Length of shaft between inner edges of bearings | 29·5 in. |
| Outer diameter of shaft | 1·200 in. |
| Inner diameter of shaft | 1·00 in. |
| Diameter of rotor | 6 in. |
| Weight of rotor | 15·41 lb |
| Weight of shaft | 4·41 lb |
| Equivalent weight of shaft simply supported | 1·77 lb |
| Equivalent weight of shaft supported across bearing at one end and simply supported at the other | 1·74 lb |
| Equivalent weight of shaft supported across both bearings | 1·44 lb |

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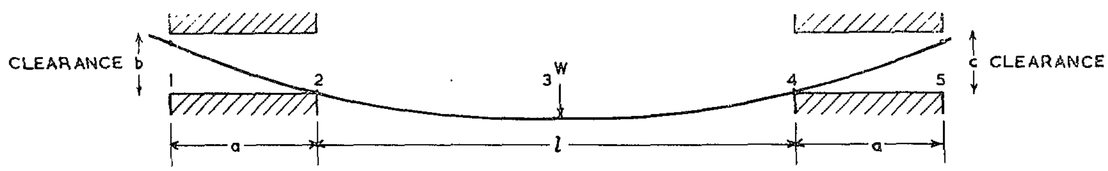
# APPENDIX II

*Calculated Flexibilities of a Shaft in Plain Bearings*




Consider a shaft supported in two plain bearings as shown diagrammatically above. The length of shaft between bearings is $l$, and the length of the bearings is $a$. The bearing clearances are $b$ and $c$ respectively ($b < c$).

If the shaft is subjected to a gradually increasing load at its centre (point 3), the end conditions will change as the load increases. From zero to a load $W_b$ the shaft will be simply supported at points 2 and 4. For a load $W_b$ the point 1 is assumed to make contact with the top of the bearing and the shaft is then constrained at points 1, 2 and 4. Finally, when a load $W_c$ is reached, point 5 also makes contact with the top of the bearing and the shaft is then constrained at points 1, 2, 4 and 5.

For the three conditions mentioned above the stiffness of the shaft will be as follows :

*Case I. Simple support at points* 2 *and* 4.

$$y_{33} = \text{deflection at point 3 due to unit load there}$$

$$= \frac{l^3}{48EI} = 0 \cdot 000356 \text{ in./lb.}$$

*Case II. Shaft supported at points* 1, 2 *and* 4.

When the shaft is supported at points 2 and 4 the flexibility coefficients are assumed to be as follows :

$y_{13}$ = deflection at 1 due to unit load at 3 = deflection at 3 due to unit load at 1
$y_{33}$ = ,, ,, 3 ,, ,, ,, ,, ,, 3
$y_{11}$ = ,, ,, 1 ,, ,, ,, ,, ,, 1.

With no constraint at point 1 a load of $(W_b + 1)$ at 3 would cause deflections $Y_1 = (W_b + 1)y_{13}$ at 1 and $Y_3 = (W_b + 1)y_{33}$ at 3.

The constraint provided by the bearing at point 1 reduces $Y_1$ to $b$, i.e., by an amount equal to $(W_b + 1)y_{13} - b$, and $b = W_b y_{13}$.

Reduction in deflection at point 1 $= (W_b + 1)y_{13} - W_b y_{13}$

$$= y_{13} \,.$$

The effective reaction at 1 $= y_{13}/y_{11}$ which results in a decrease in deflection at 3 of $(y_{13}/y_{11}) \times y_{31} = (y_{13}^2/y_{11})$ .

Let $y_{33}$ = deflection at 3 due to unit load at 3 for the case where the shaft is constrained at points 1, 2 and 4, *i.e.*, when the shaft load increases from $W_b$ to $W_b + 1$,

then $$y_{33} = (W_b + 1)y_{33} - \frac{y_{13}^2}{y_{11}} - W_b y_{33} = y_{33} - \frac{y_{13}^2}{y_{11}} \,,$$

where $$y_{11} = \frac{a^2(l + a)}{3EI} \,, \quad y_{33} = \frac{l^3}{48EI} \,, \quad y_{13} = \frac{al^2}{16EI} \,,$$

*i.e.*, $y_{11} = 0{\cdot}000076$ in./lb, $y_{33} = 0{\cdot}000356$ in./lb and $y_{13} = 0{\cdot}000118$ in./lb

for a length of shaft of $29\frac{1}{2}$ in. between bearings and $3\frac{1}{4}$ in. bearing length.

Therefore $y_{33} = 0{\cdot}000172$ in./lb.

*Case III. Shaft supported at points* 1, 2, 4 *and* 5.

Consider a load $(W_c + 1)$ at 3 and let $b = c$. The latter assumption will simplify the analysis without changing the result.

For simple support at points 2 and 4 the points 1 and 5 would deflect by $(W_c + 1)y_{13}$. This deflection is reduced by bearing reactions at points 1 and 5 to a value $C = W_c y_{13}$, i.e., by an amount $y_{13}$ as in Case II.

Consider unit loads simultaneously placed at points 1 and 5. Then deflections at points 1 and 5 will be equal to $y_{11} + y_{15} = Y_{11}$ say.

Therefore $$Y_{11} = \frac{a^2(l + a)}{3EI} + \frac{a^2 l}{6EI} = \frac{a^2(3l + 2a)}{6EI}$$

$$= 0{\cdot}000111 \text{ in./lb.}$$

The reactions at points 1 and 5 are given by $y_{13}/Y_{11}$.

Therefore the resultant deflection at 3 $= 2y_{13}^2/Y_{11}$

and $$Y_{33} = y_{33} - (2y_{13}^2/Y_{11}) = 0{\cdot}000126 \text{ in./lb.}$$

In this case $Y_{33}$ = the deflection at 3 due to unit load at 3, *i.e.*, when the load at 3 increases from $W_c$ to $W_c + 1$.

# APPENDIX III

*The Effects of Non-linear Stiffness on the Critical Whirling Speed*

Consider a shaft–rotor system supported in long bearings at each end of the shaft. The shaft is assumed to have no mass and the rotor to have a mass $m$. The clearances are assumed to be different in the two bearings.

The analysis is confined to a consideration of displacements ($x$) in the vertical plane where $x$ is the displacement of the elastic centre of the shaft at the point of attachment of the rotor. The c.g. of the rotor is assumed to be offset from the elastic centre of the shaft by an amount $h$. In the following analysis damping is neglected as this simplifies the analysis without appreciably affecting the result.

The equation to the motion of a linear system has been shown[2] to be :

$$m\ddot{x} + b\dot{x} + Sx = mh\omega^2 \cos \omega t \quad \ldots \quad (1)$$

where $S$ is the linear stiffness of the shaft measured at the rotor. For the non-linear system the equation of motion must be modified to take account of the non-linear stiffness and hence becomes (neglecting damping) :

$$m\ddot{x} + F(x) = mh\omega^2 \cos \omega t \text{ where } F(x) \text{ is the restoring force exerted by the shaft for a given displacement } x. \quad \ldots \quad (2)$$

A first approximation to the solution of equation (2) will be sufficient, it is thought, to indicate the nature of the resultant oscillatory motion. Such an approximation may be obtained by assuming that the motion $x = f(t)$ is sinusoidal and of frequency equal to the forced frequency. Experimental results show that the above assumptions are justified. The approximate solution is then obtained by assuming that $x = \bar{x} \cos \omega t$ where $\bar{x}$ is the resultant whirl amplitude ($y = \bar{x}$ sin $\omega t$, $\bar{x}^2 = x^2 + y^2)^2$.

Equation (2) is a condition of equilibrium of three forces at any time during the non-harmonic motion. The response of the system is obtained by satisfying the equilibrium condition for the assumed harmonic motion (by a proper choice of $x$) at the instant when $x = \bar{x}$.

Therefore $$- m\omega^2\bar{x} + f(\bar{x}) = mh\omega^2 \quad \ldots \quad (3)$$

or $$f(\bar{x}) = mh\omega^2 + m\omega^2\bar{x}. \quad \ldots \quad (4)$$

When $x = 0$ the three forces are zero (provided $f(o) = 0$) so that the equilibrium condition is again satisfied. For most values of $x$ between 0 and $\bar{x}$ the equilibrium condition is violated but this is of little consequence as it is the maximum value, $\bar{x}$, of $x$ that is significant in the analysis. Values of $\bar{x}$ for various values of $\omega$ are obtained graphically as shown in Fig. 8a. $F(\bar{x})$ is first plotted for positive and negative values of $\bar{x}$, this represents the left-hand side of equation (4). For a particular value of $\omega$ (say $\omega_1$) the right-hand side of equation (4) expresses a straight line with the ordinate intercept $mh\omega_1^2$ and the slope $\tan^{-1}(m\omega_1^2)$. Where the two curves intersect, the left-hand force of equation (4) equals the right-hand force, so that equilibrium exists. This determines $\bar{x}$ as the abscissa of the point of intersection. When $\omega$ is small there is only one point of intersection between the two curves, but as the frequency increases two or three points of intersection are obtained. Fig. 8 shows the frequency-response curve obtained by the graphical method described above assuming an unbalance force of 25 gm in. on the rotor.

Referring to Fig. 8, the line *abc* represents whirl amplitudes obtained below the critical speed and the line *defg* represents whirl amplitudes above the critical speed. The part *de* of the latter curve represents an unstable motion and gives rise to the ' jump ' phenomenon which is well known in connection with certain non-linear electrical circuits. It may be assumed that the inclusion of damping in the equation of motion will have little effect on the response curve except near the critical speed and will tend to round off the peak of the curve as shown by the dashed curve connecting *c* and *d*. Therefore for speeds increasing from zero r.p.m. the response of the system is given by the line *abcd* ; at *d*, instead of following the unstable part of the curve, *de*, the whirl amplitude drops to the point *f*, and for further increase of speed is represented by *fg*. If the speed is now reduced the response of the system follows the stable curve *gfe* to *e*, at which point a jump to the line *abc*, at *b*, occurs and this latter line is followed down to *a*.

For the undamped system the critical speed is obtained when $\tan^{-1} m\omega^2$ is equal to the slope $S_3$ of the steepest portions of the stiffness curve (Fig. 8a); *i.e.*, when $\tan^{-1} S_3 = \tan^{-1} m\omega^2$ or $S_3 = m\omega^2$, where $S_3$ is the stiffness obtained by assuming support of the shaft across both bearings. For a linear system, damping has a negligible effect on the critical speed but this is not so for the non-linear system. As the response curve (Fig. 8) is leaning to the right and because it can be assumed that in the damped case the peak amplitude will lie on the line $0\ 0'\ 0''$, it is obvious that the critical speed will in general be a function of the peak amplitude, which in turn is a function of damping and rotor unbalance. However, for the case where the critical whirl amplitude is small enough for operation within the first range of linear stiffness, $S_1$, at all times, the system is linear and the critical speed is given by $\omega^2 = S_1/m$.

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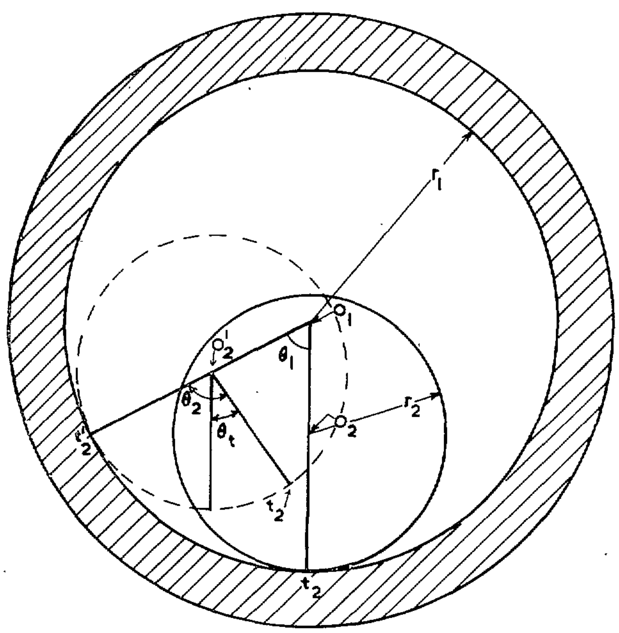
## APPENDIX IV

*The Dry Friction Whirl*



The figure above represents a section through a plain bearing of céntre $O_1$, radius $r_1$. In the bearing is a shaft of radius $r_2$, centre $O_2$, which is initially resting in contact at the point $t_2$ at the bottom of the bearing. The coefficient of friction between the shaft and bearing is assumed to be sufficient to prevent slip so that if the shaft is turned anti-clockwise it will roll round the inside of the bearing. Assuming that the point of contact between the shaft and bearing moves from $t_2$ to $t_2'$ the centre of the shaft from $O_2$ to $O_2'$ through an angle $\theta_1$ about at centre $O_1$, in a clockwise direction. The radius $O_2t_2$ which was originally vertical is now represented by $O_2't_2$ at an angle $\theta_t$ to the vertical. Let angle $t_2O_2't_2' = \theta_2$. From the geometry of the system :

$$\theta_t = \theta_2 - \theta_1$$

and $$r_1\theta_1 = r_2\theta_2.$$

Therefore $$\theta_2 = \frac{r_1}{r_2}\theta_1 .$$

Therefore $$\theta_t = \left(\frac{r_1}{r_2} - 1\right)\theta_1 = \frac{r_1 - r_2}{r_2}\theta_1 ,$$

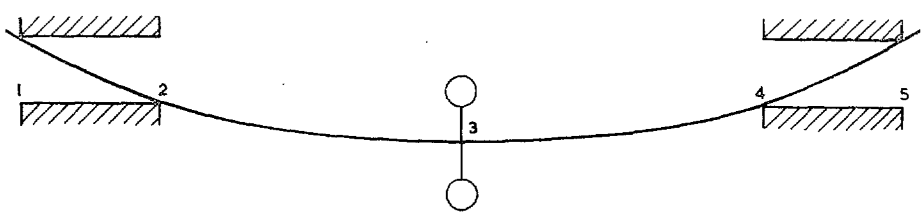
*i.e.*, when the shaft rotates through one revolution about its centre, the centre of the shaft rotates through $r_2/(r_1 - r_2)$ revolutions about the centre of the bearing in the opposite direction to the initial rotation of the shaft. In the case of the bearing with large clearance $r_2 = 0{\cdot}6$ in. $(r_1 - r_2) = 0{\cdot}018$ in. and the frequency ratio is $33{\cdot}4 : 1$. Measurements were taken on the rig at varying shaft speeds and the results tabulated in Table 2.

It is seen that the ratio between shaft speed and frequency of vibration approaches $33{\cdot}4 : 1$ as the shaft speed becomes small but that, probably due to slipping between shaft and bearing, as the shaft speed increases this ratio decreases progressively.

---

## TABLE 1

*The Effect of Rotor Unbalance on the Critical Speed*




| Rotor balance | End conditions | Whirl amplitude at critical speed | Measured critical speed r.p.m. (Fig. 7) | Calculated critical speed r.p.s. (Fig. 8) |
|---|---|---|---|---|
| Rotor balanced | Shaft supported at points 2 and 4 | 0·005 in. | 42 | 40 |
| 20 gm in. of unbalanced added | Shaft supported at points 1, 2 and 4 | 0·030 in. | 52 | 49 |
| 40 gm in. of unbalance added | Shaft supported at points 1, 2 4 and 5 | 0·070 in. | 58 | 57 |

## TABLE 2

*The Critical Frequencies of a Shaft Rotating in Bearings of Excessive Clearance*

| Shaft rotational speed (r.p.m.) | Whirl frequency C.P.M. | Frequency ratio |
|---|---|---|
| 92 | 2290 | 24·9 |
| 90 | 2270 | 25·2 |
| 85 | 2250 | 26·4 |
| 80 | 2240 | 26·7 |
| 75 | 2040 | 27·2 |
| 65 | 1900 | 29·2 |
| 60 | 1850 | 30·8 |

## TABLE 3

*The Effect of Lubricant Viscosity on the Critical Whirling Speed*

Bearing Clearance = 0·004 in.

| Bearing length (in.) | Fig. No. | Log Log viscosity of oil | Critical speed r.p.s. | Effective flexibility at rotor in./lb | Measured static flex. at rotor in./lb | Per cent change in flexibility |
|---|---|---|---|---|---|---|
| 3·25 | 12 | 0·105 | 44·5 | $0{\cdot}286 \times 10^{-3}$ | $0{\cdot}370 \times 10^{-3}$ | −23 |
| 3·25 | 12 | 0·275 | 51·0 | $0{\cdot}218 \times 10^{-3}$ | $0{\cdot}370 \times 10^{-3}$ | −41 |
| 3·25 | 12 | 0·405 | 54·0 | $0{\cdot}198 \times 10^{-3}$ | $0{\cdot}370 \times 10^{-3}$ | −46 |
| 3·25 | 12 | 0·43 | 55·0 | $0{\cdot}188 \times 10^{-3}$ | $0{\cdot}370 \times 10^{-3}$ | −49 |
| 2·25 | 13 | 0·137 | 41·0 | $0{\cdot}337 \times 10^{-3}$ | $0{\cdot}368 \times 10^{-3}$ | − 8 |
| 2·25 | 13 | 0·27 | 43·0 | $0{\cdot}307 \times 10^{-3}$ | $0{\cdot}368 \times 10^{-3}$ | −17 |
| 2·25 | 13 | 0·32 | 43·75 | $0{\cdot}296 \times 10^{-3}$ | $0{\cdot}368 \times 10^{-3}$ | −20 |
| 2·25 | 13 | 0·41 | 44·75 | $0{\cdot}283 \times 10^{-3}$ | $0{\cdot}368 \times 10^{-3}$ | −23 |
| 2·25 | 13 | 0·44 | 46·75 | $0{\cdot}266 \times 10^{-3}$ | $0{\cdot}368 \times 10^{-3}$ | −28 |
| 1·25 | 14 | 0·135 | 41·0 | $0{\cdot}337 \times 10^{-3}$ | $0{\cdot}333 \times 10^{-3}$ | + 1 |
| 1·25 | 14 | 0·41 | 42·0 | $0{\cdot}323 \times 10^{-3}$ | $0{\cdot}333 \times 10^{-3}$ | − 3 |
| 1·25 | 14 | 0·43 | 43·0 | $0{\cdot}306 \times 10^{-3}$ | $0{\cdot}333 \times 10^{-3}$ | − 9 |

Bearing Clearance = 0·001 in.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | $\times 10^{-3}$ | | $\times 10^{-3}$ | |
| 3·25 | 16 | 0·105 | 55·5 | 65 | 0·199 | 0·135 | 0·175 | 0·107 |
| 3·25 | 16 | 0·275 | 56·5 | 66·5 | 0·178 | 0·129 | 0·175 | 0·107 |
| 3·25 | 16 | 0·400 | 60 | 67·3 | 0·156 | 0·126 | 0·175 | 0·107 |
| 2·25 | 17 | 0·195 | 54 | 64 | 0·194 | 0·139 | 0·160 | 0·114 |
| 2·25 | 17 | 0·330 | 55·5 | 65 | 0·184 | 0·134 | 0·160 | 0·114 |
| 2·25 | 17 | 0·430 | 57 | 65·75 | 0·175 | 0·127 | 0·160 | 0·114 |
| 1·25 | 18 | 0·105 | 45 | | 0·280 | | 0·358 | |
| 1·25 | 18 | 0·400 | 45 | | 0·280 | | 0·358 | |

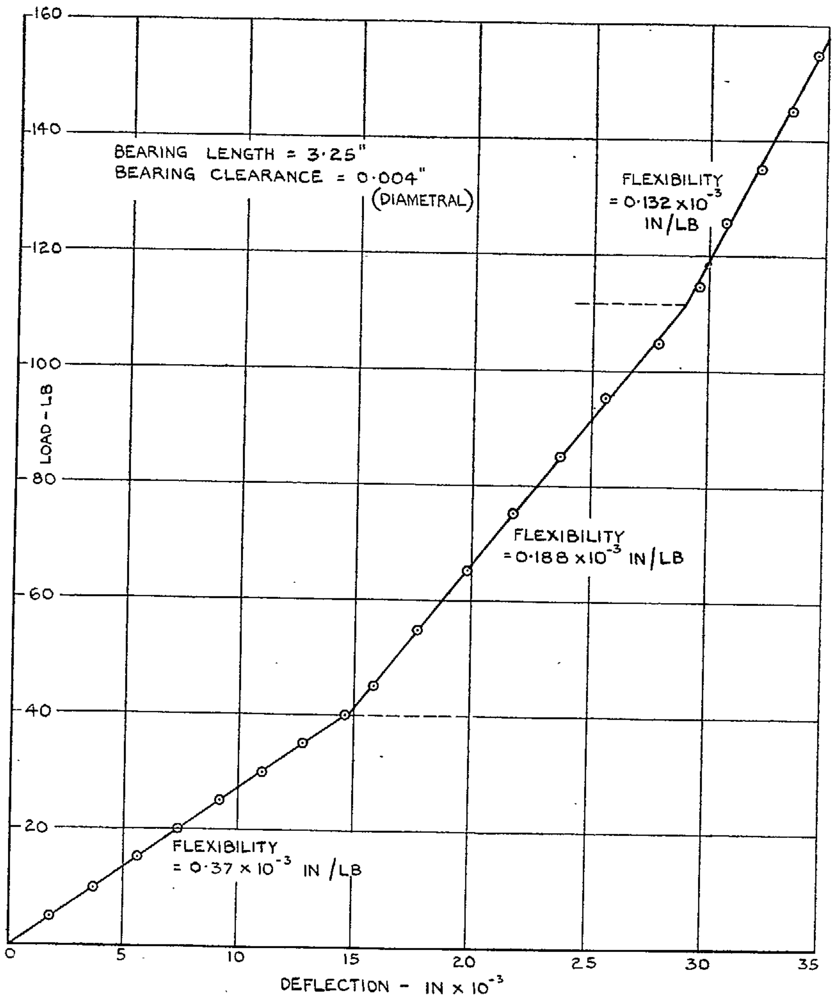



FIG. 1. Static load–deflection curve for a shaft supported in plain bearings.

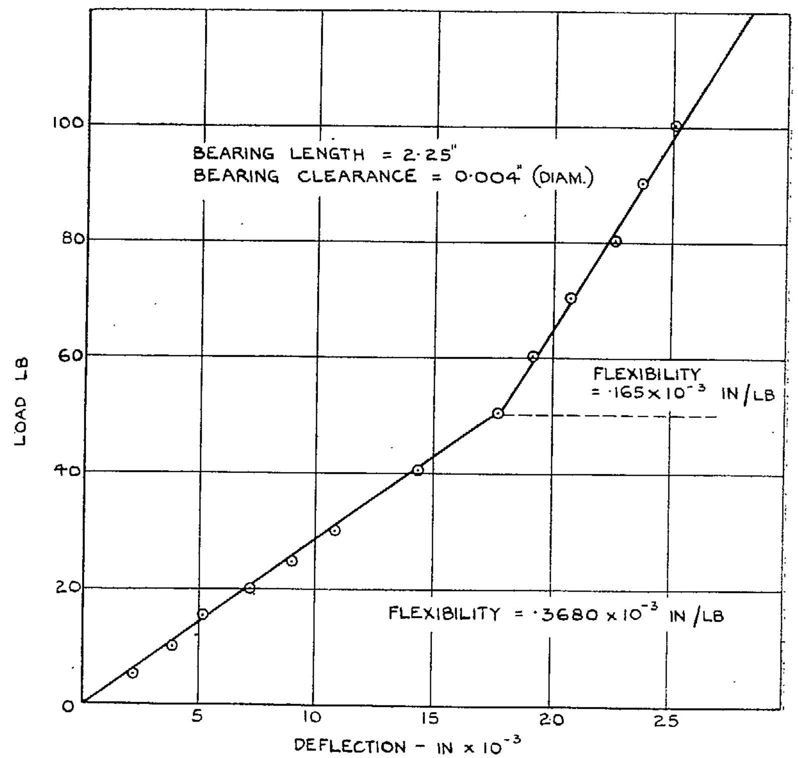



FIG. 2. Static load–deflection curve for a shaft supported in plain bearings.

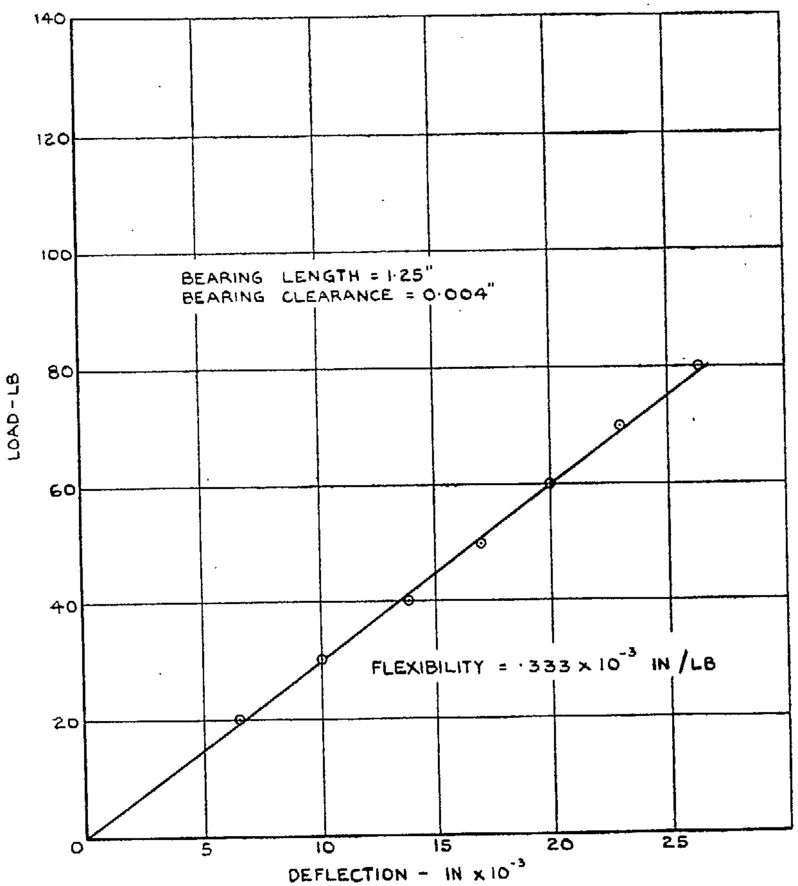



FIG. 3. Static load–deflection curve for a shaft supported in plain bearings.

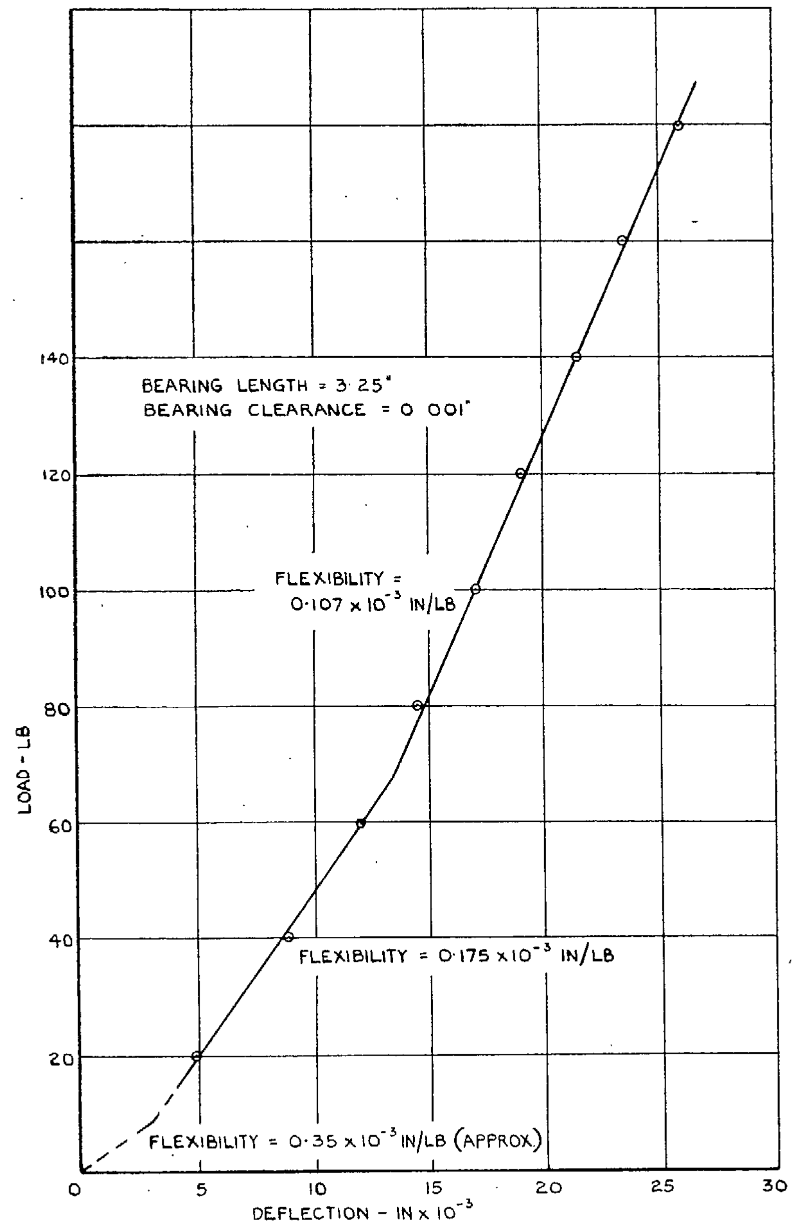



FIG. 4. Static load–deflection curve for a shaft supported in plain bearings.

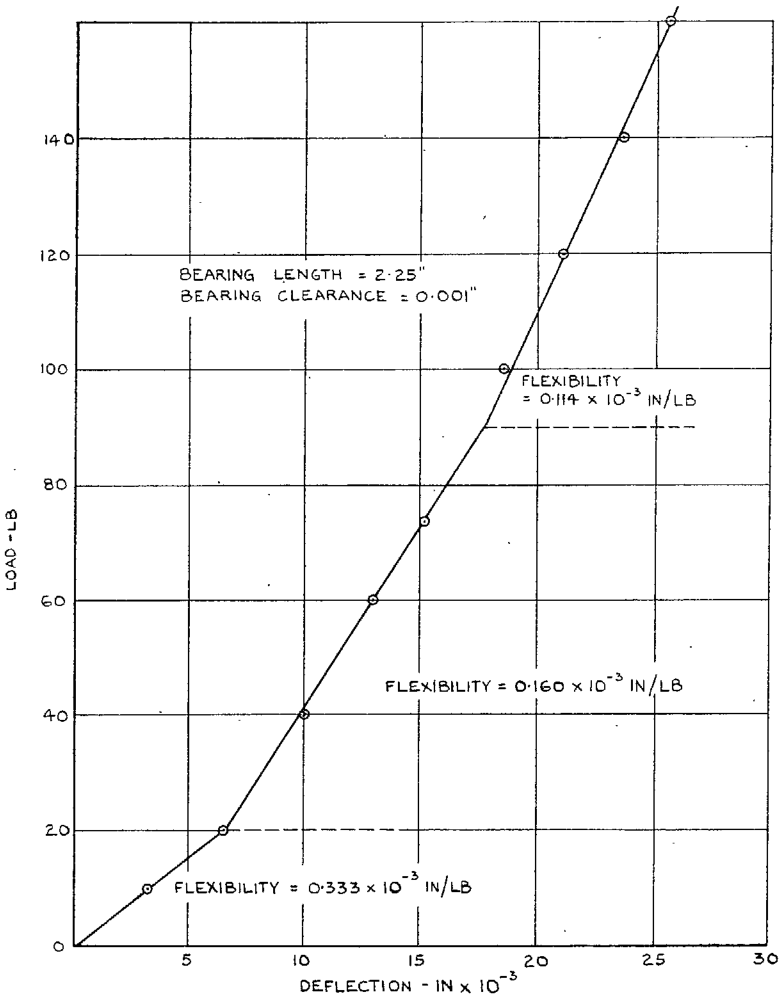



Fig. 5. Static load–deflection curve for a shaft supported in plain bearings.

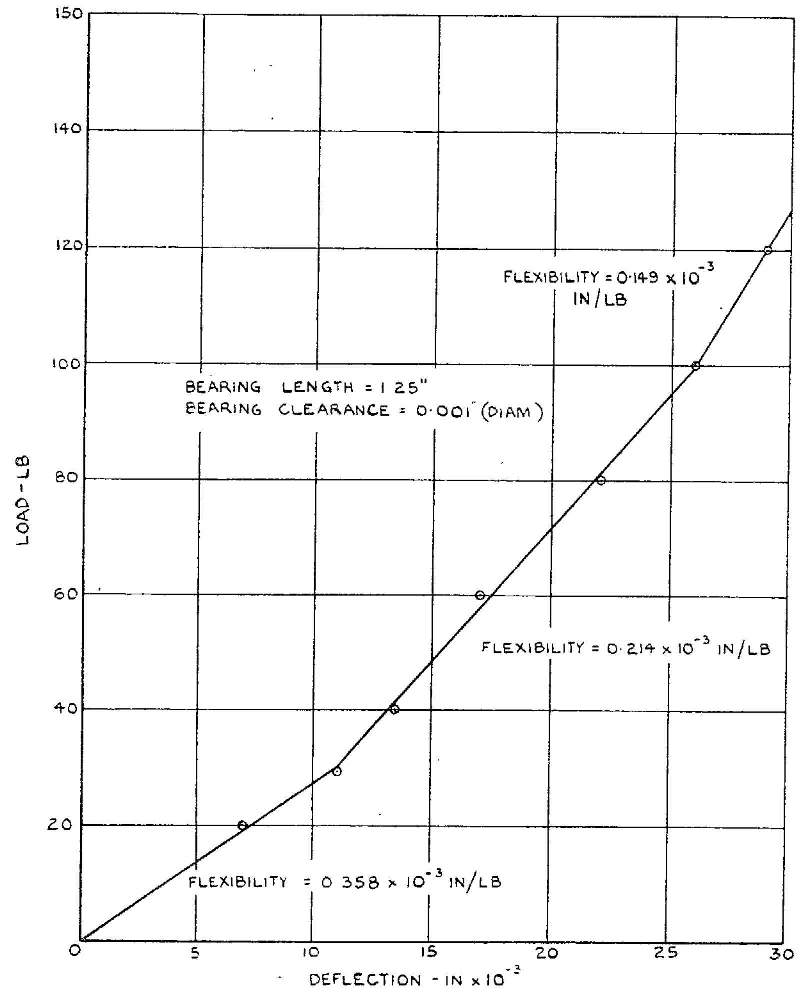



Fig. 6. Static load–deflection curve for a shaft supported in plain bearings.

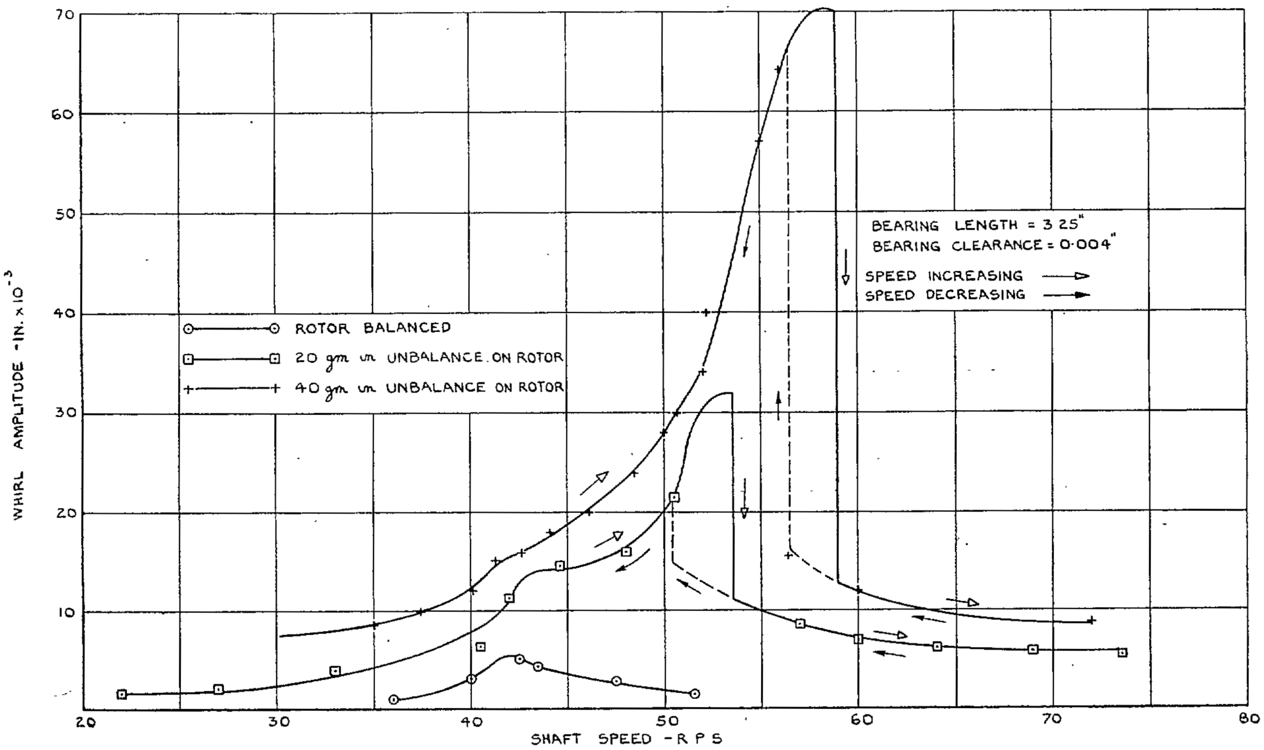



FIG. 7. The effect of rotor unbalance on the critical whirling speed.

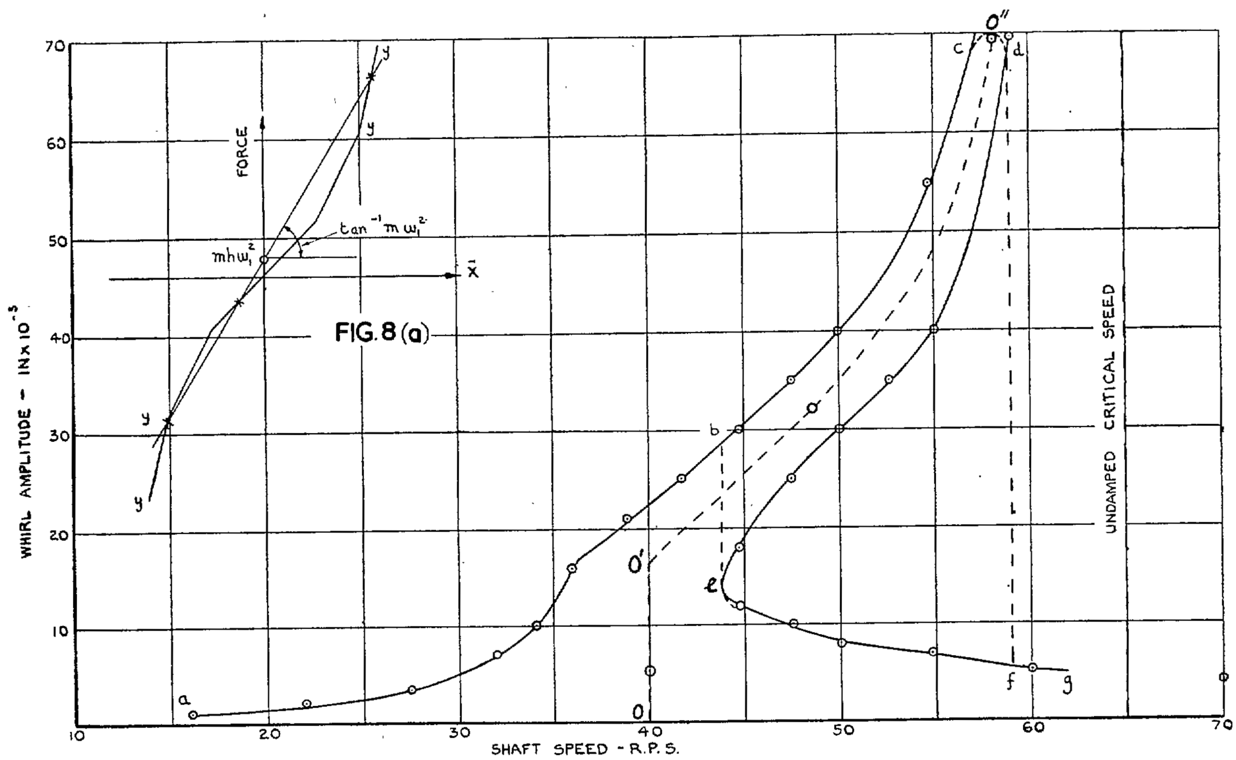



FIG. 8. Theoretical amplitude–frequency curve for a system with non-linear stiffness,

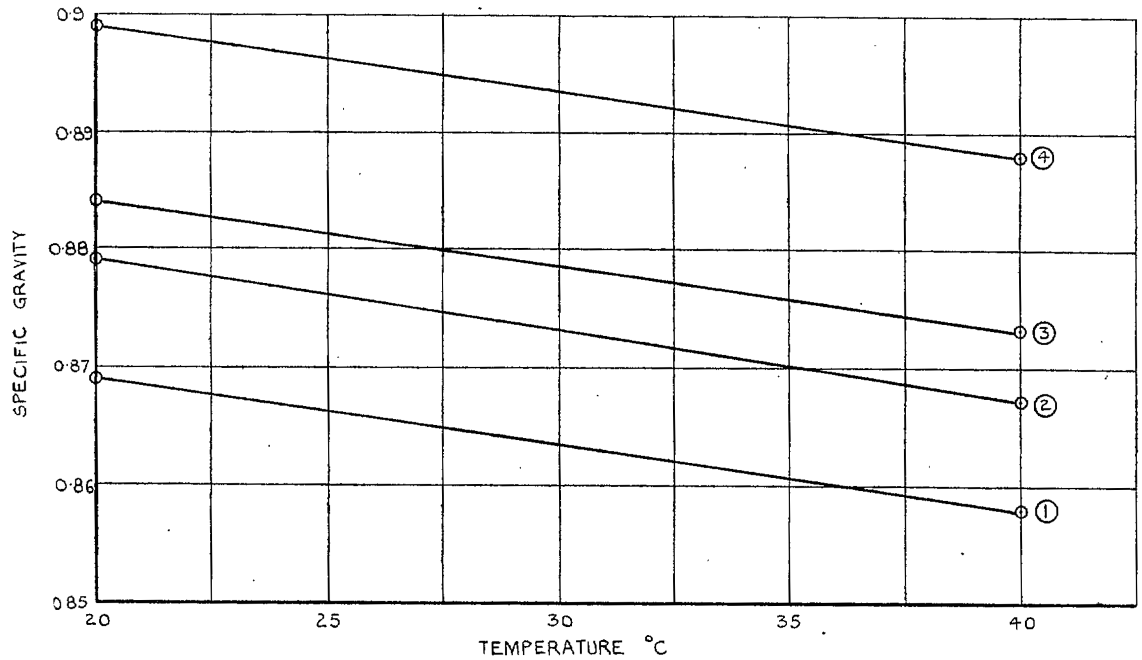



FIG. 9. Specific gravity–temperature chart for the four grades of oil used for lubrication.

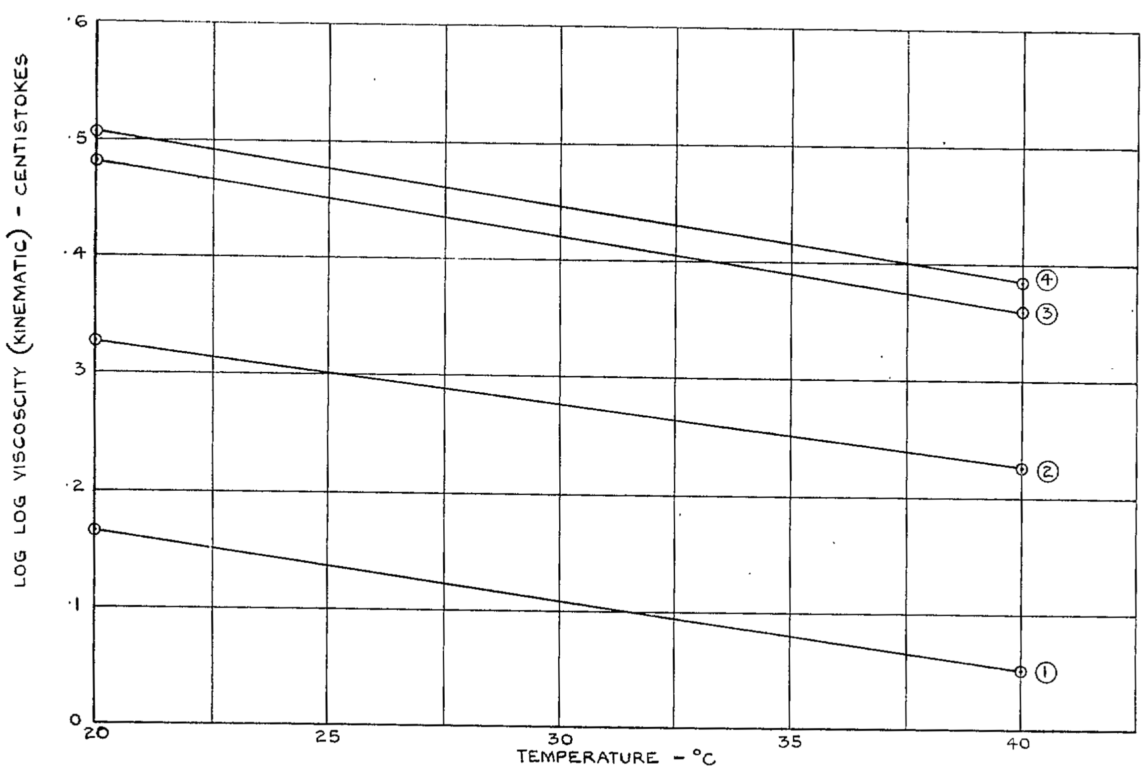



FIG. 10. Log log viscosity–temperature chart for the four oils used for lubrication.

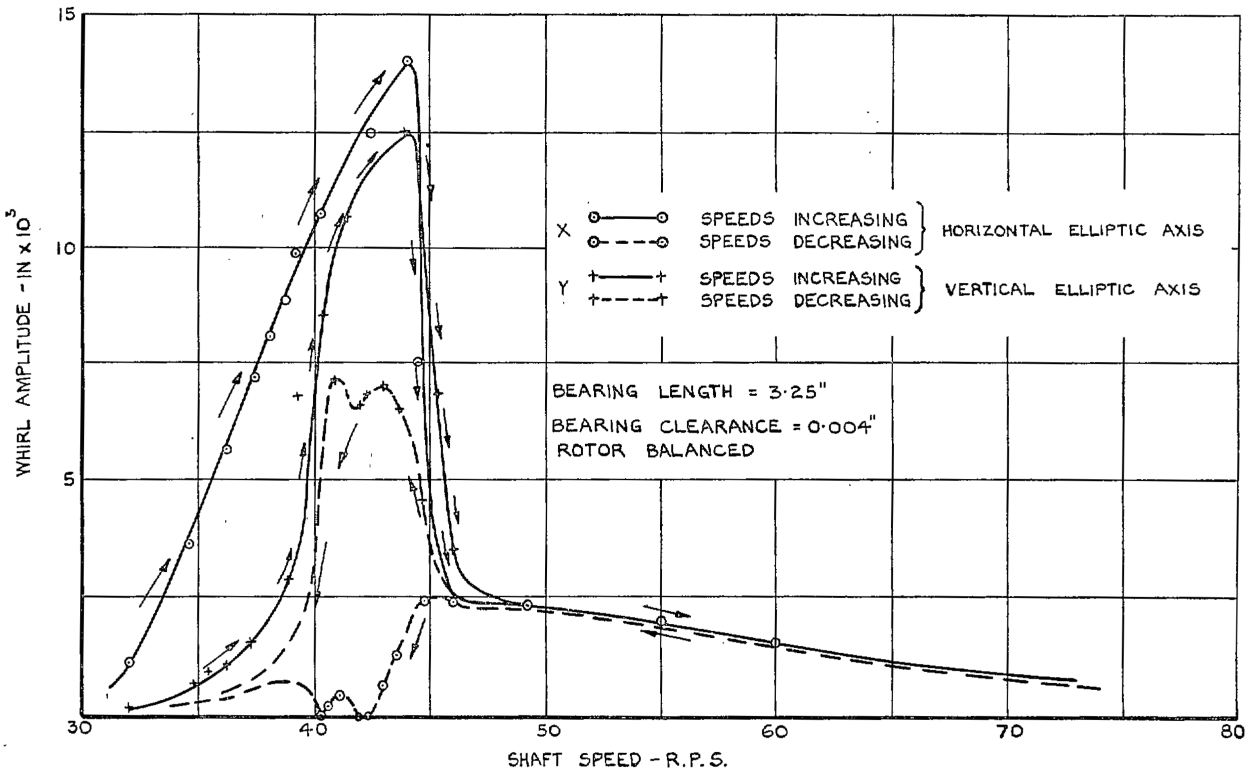



FIG. 11. The whirling characteristics of a shaft running in dry bearings.

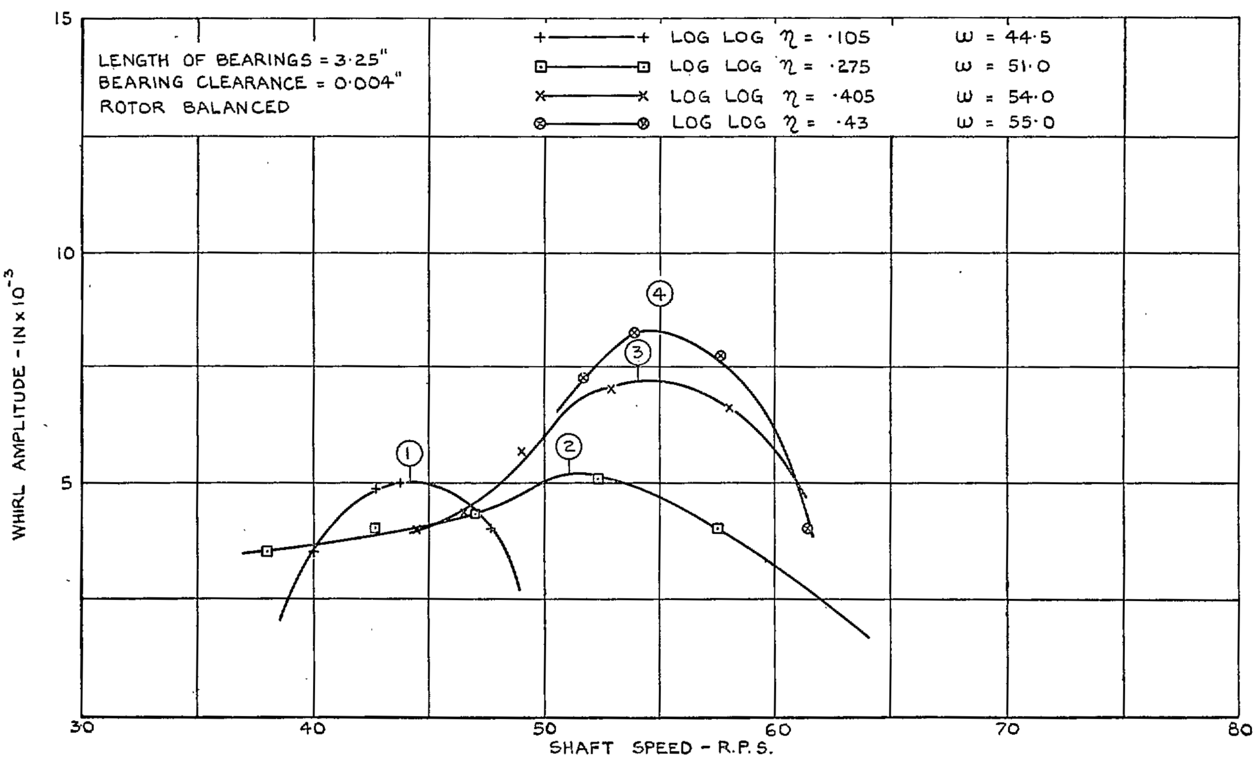



FIG. 12. The effect of lubrication on the critical whirling speed.

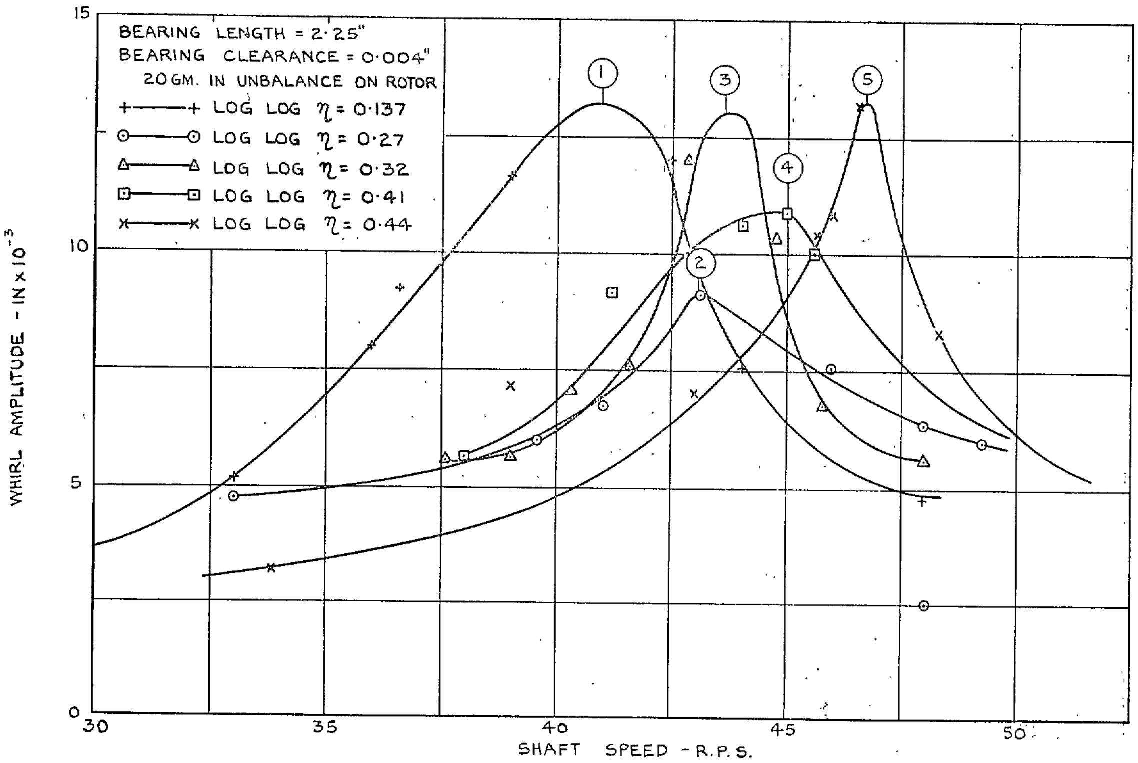



FIG. 13. The effect of lubrication on the critical whirling speed.

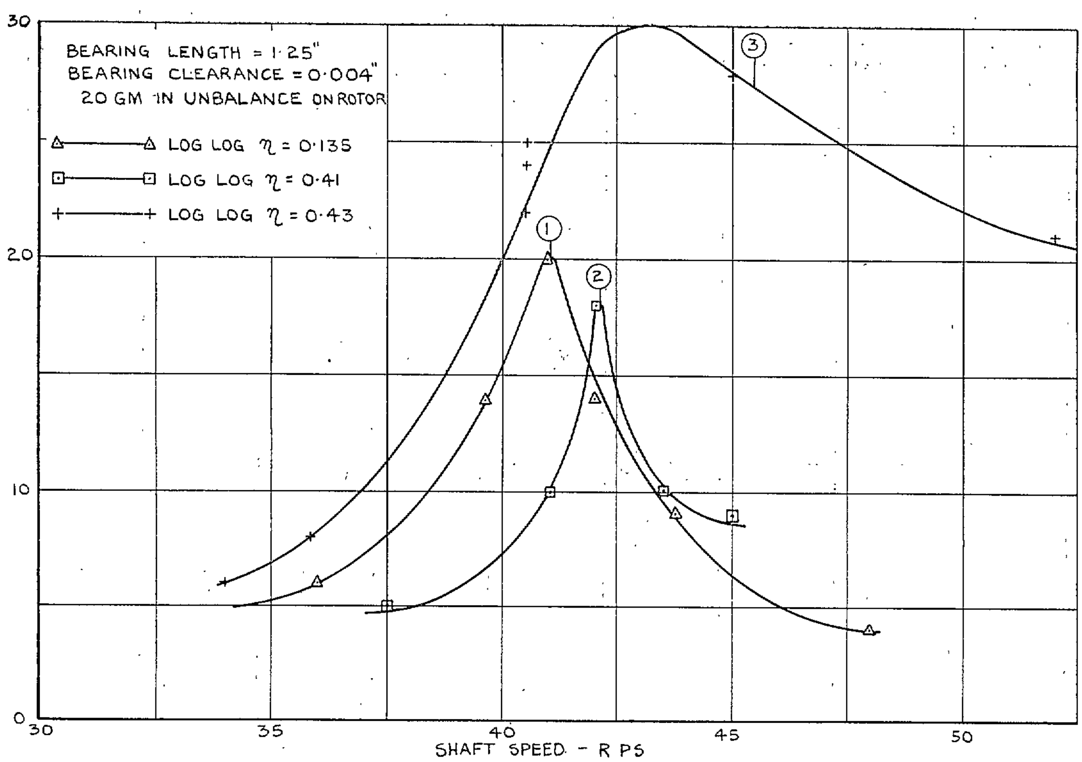



FIG. 14. The effect of lubrication on the critical whirling speed.

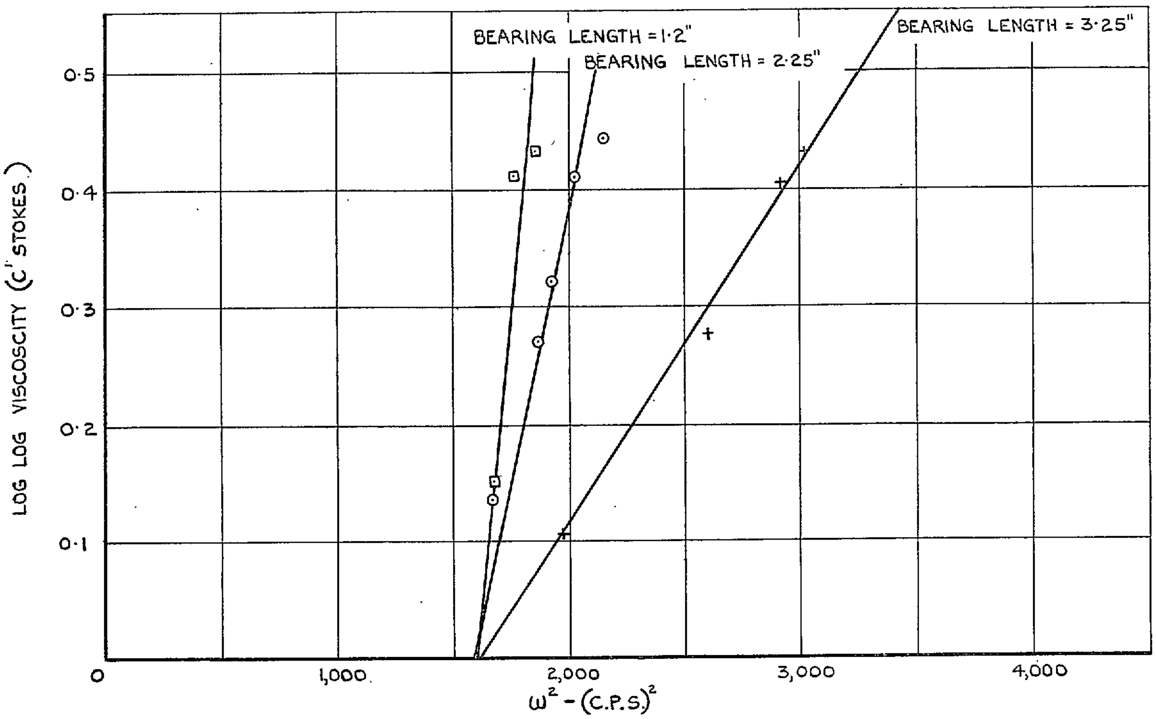



FIG. 15. The effect of lubrication on the critical whirling speed (Bearing clearance 0·004 in.).

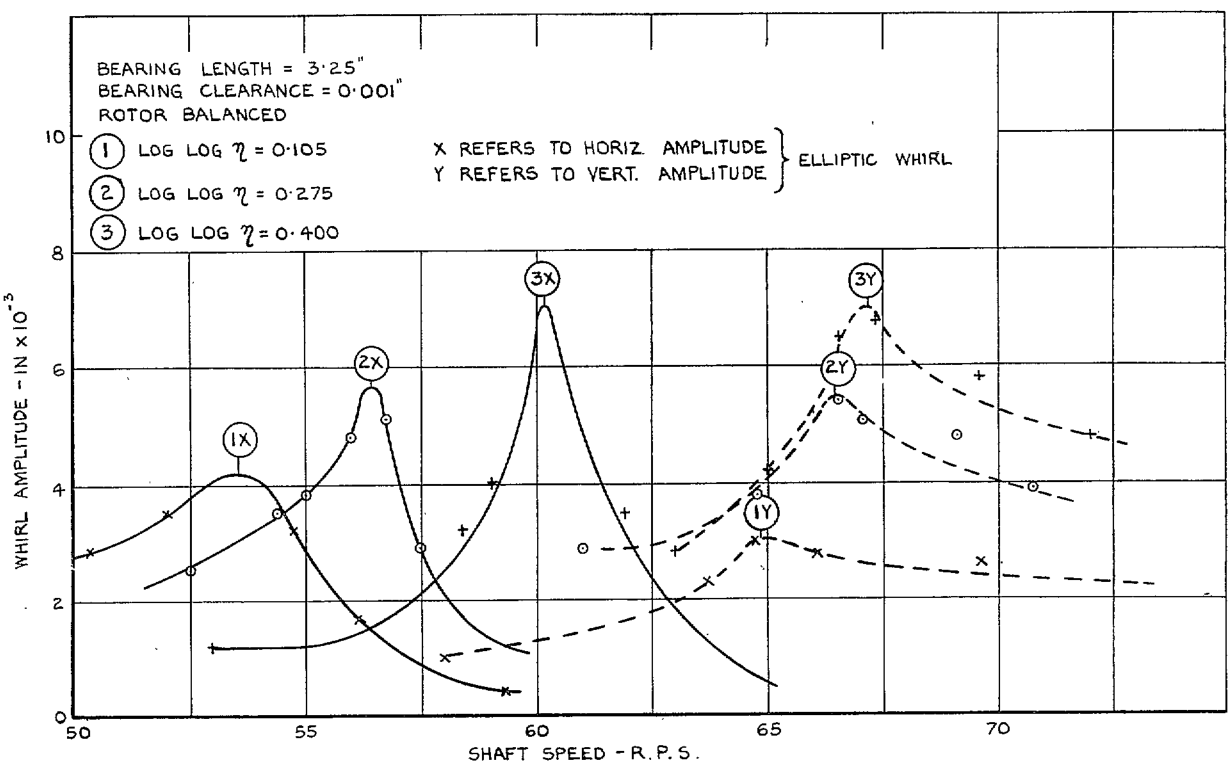



FIG. 16. The effect of lubrication on the critical whirling speed.



B

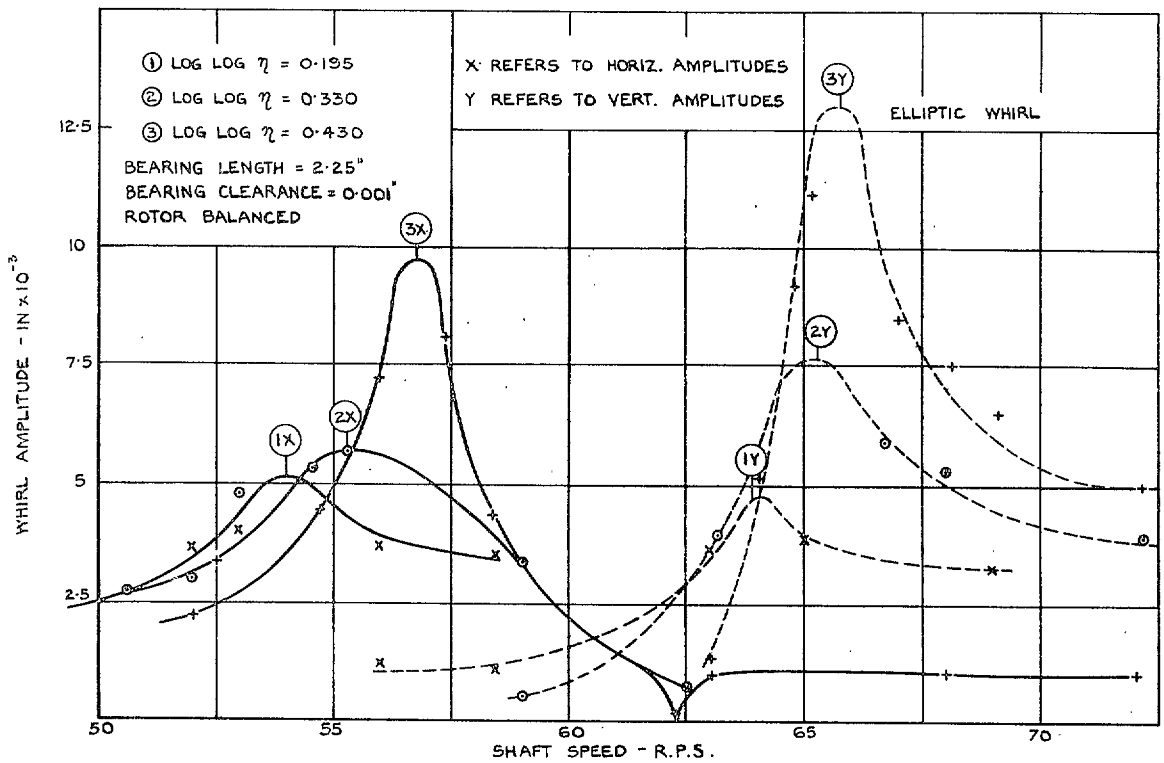



FIG. 17. The effect of lubrication on the critical whirling speed.

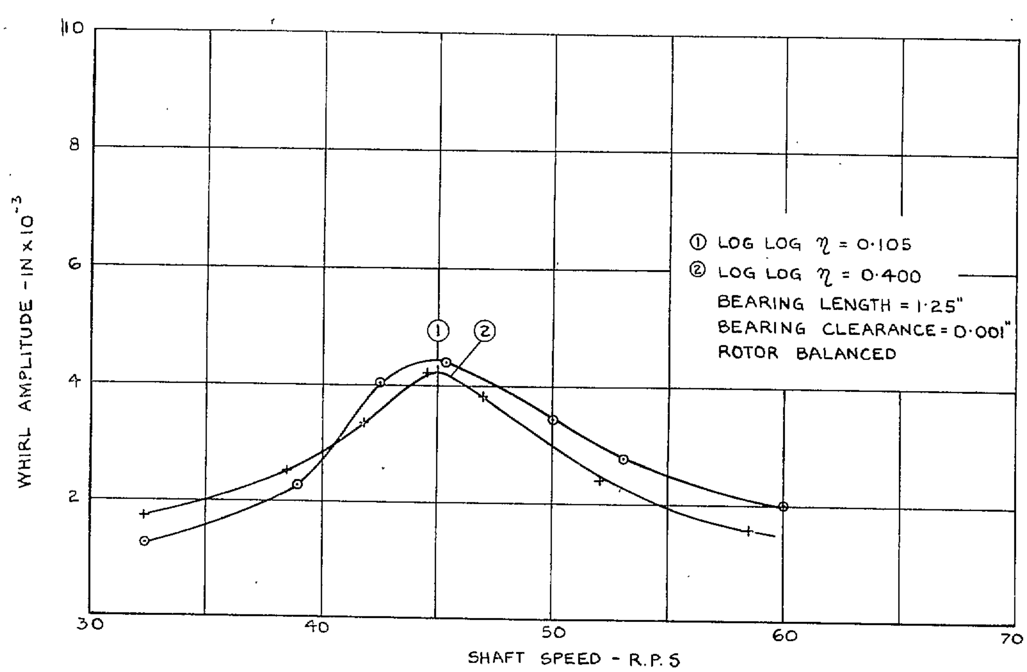



FIG. 18. The effect of lubrication on the critical whirling speed.

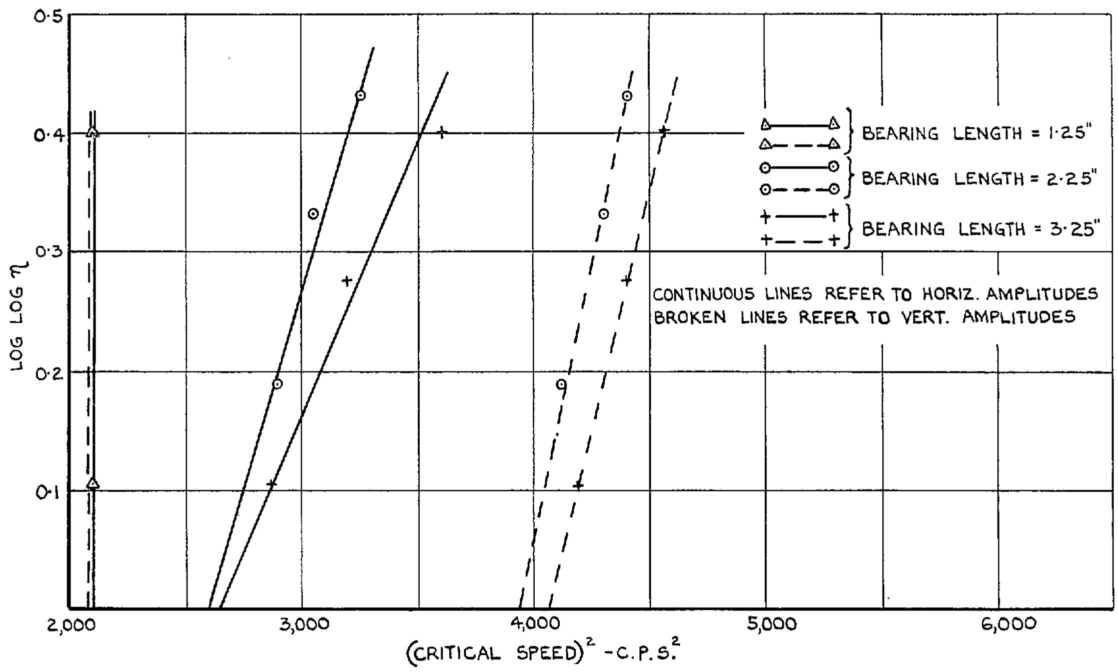



FIG. 19. The effect of lubrication on the critical whirling speed (Bearing clearance = 0·001 in.).

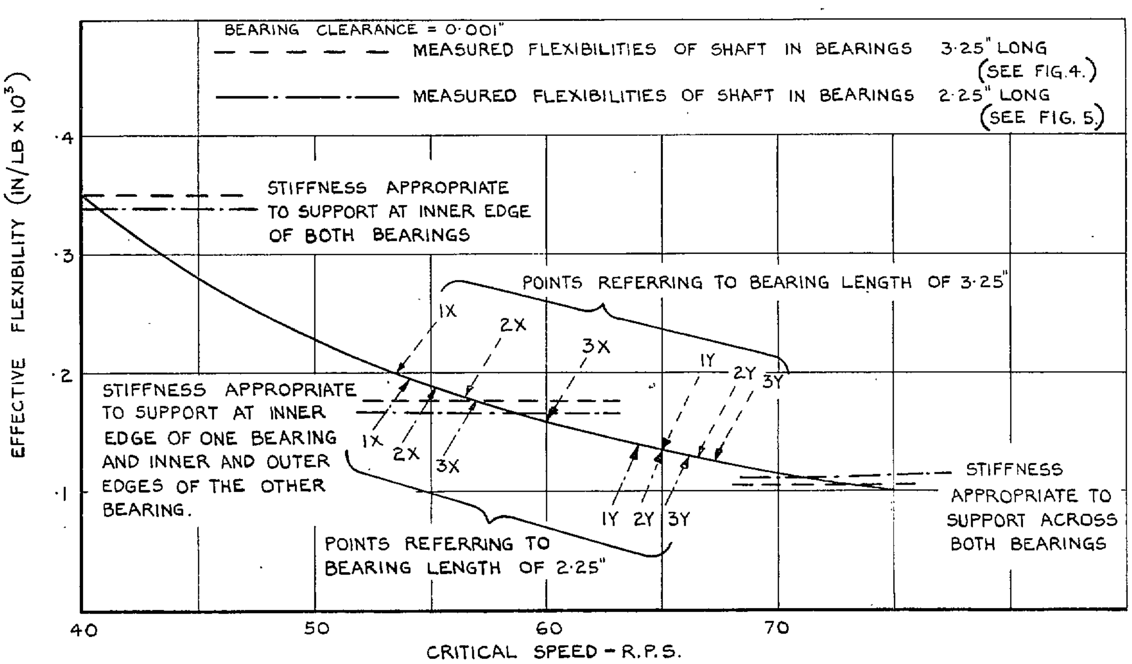



FIG. 20. The effective flexibilities of the shaft showing critical whirling speeds obtained from Figs. 15 and 16.

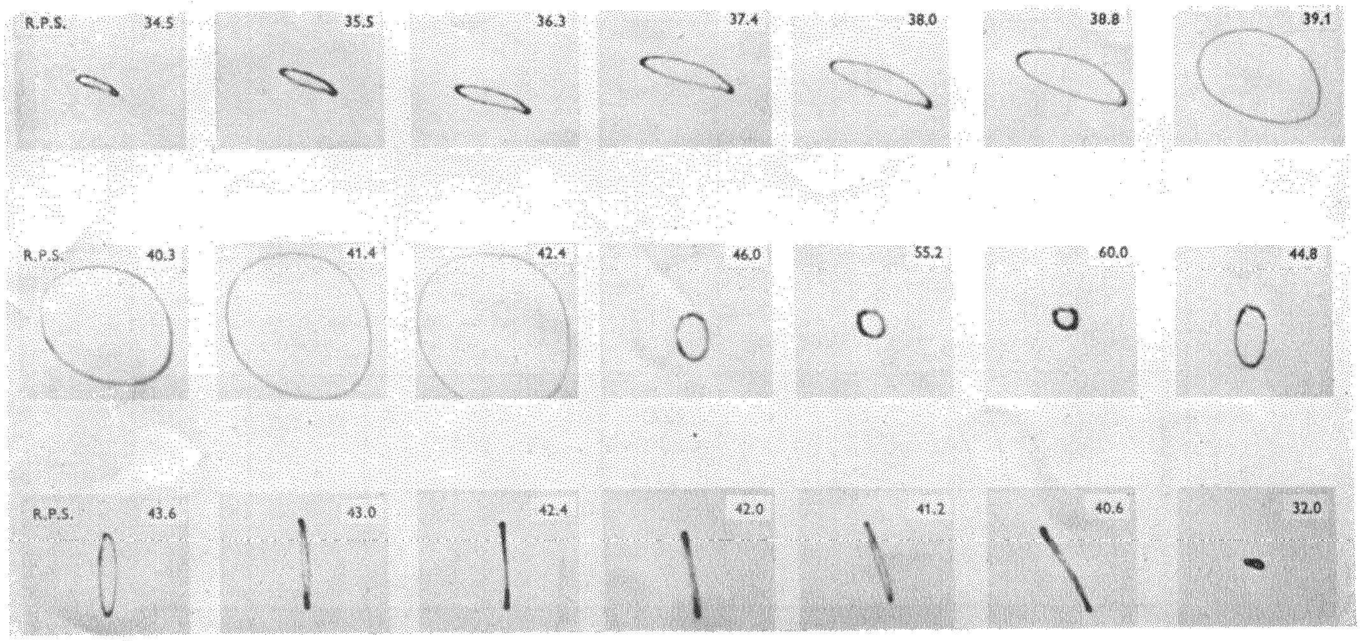



FIG. 21. Records of whirl amplitudes and paths of a shaft–rotor system rotating in dry bearings.

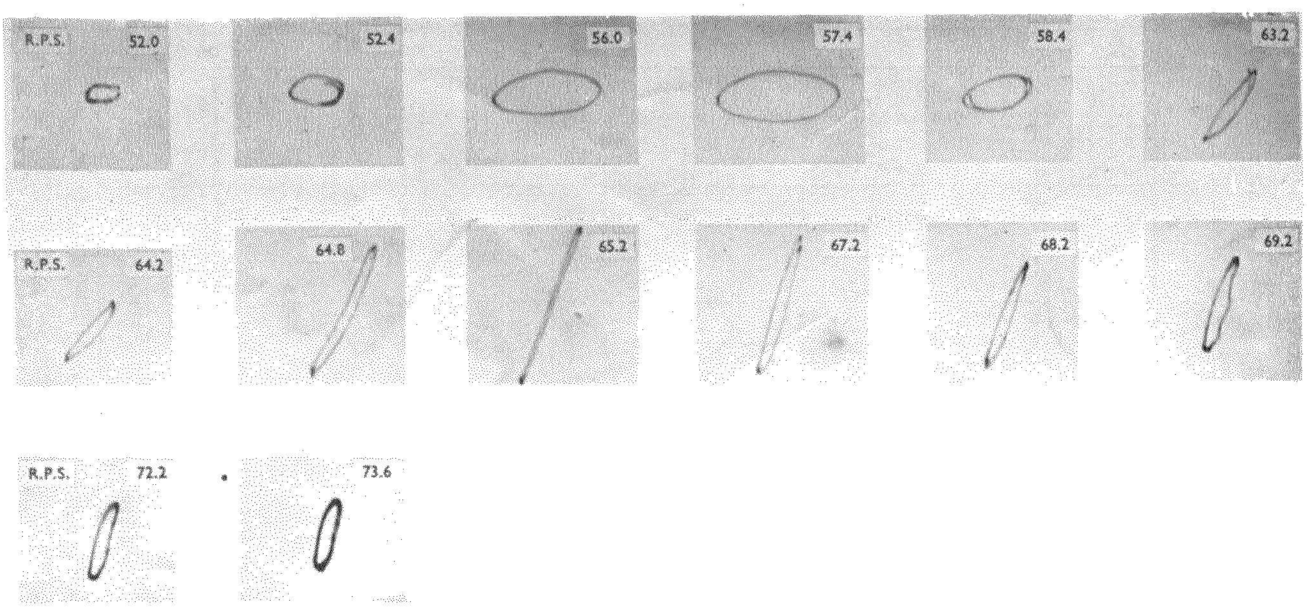



FIG. 22. Records of whirl amplitudes and paths of a shaft rotating in plain bearings 2·25 in. long and 0·001 in. clearance.

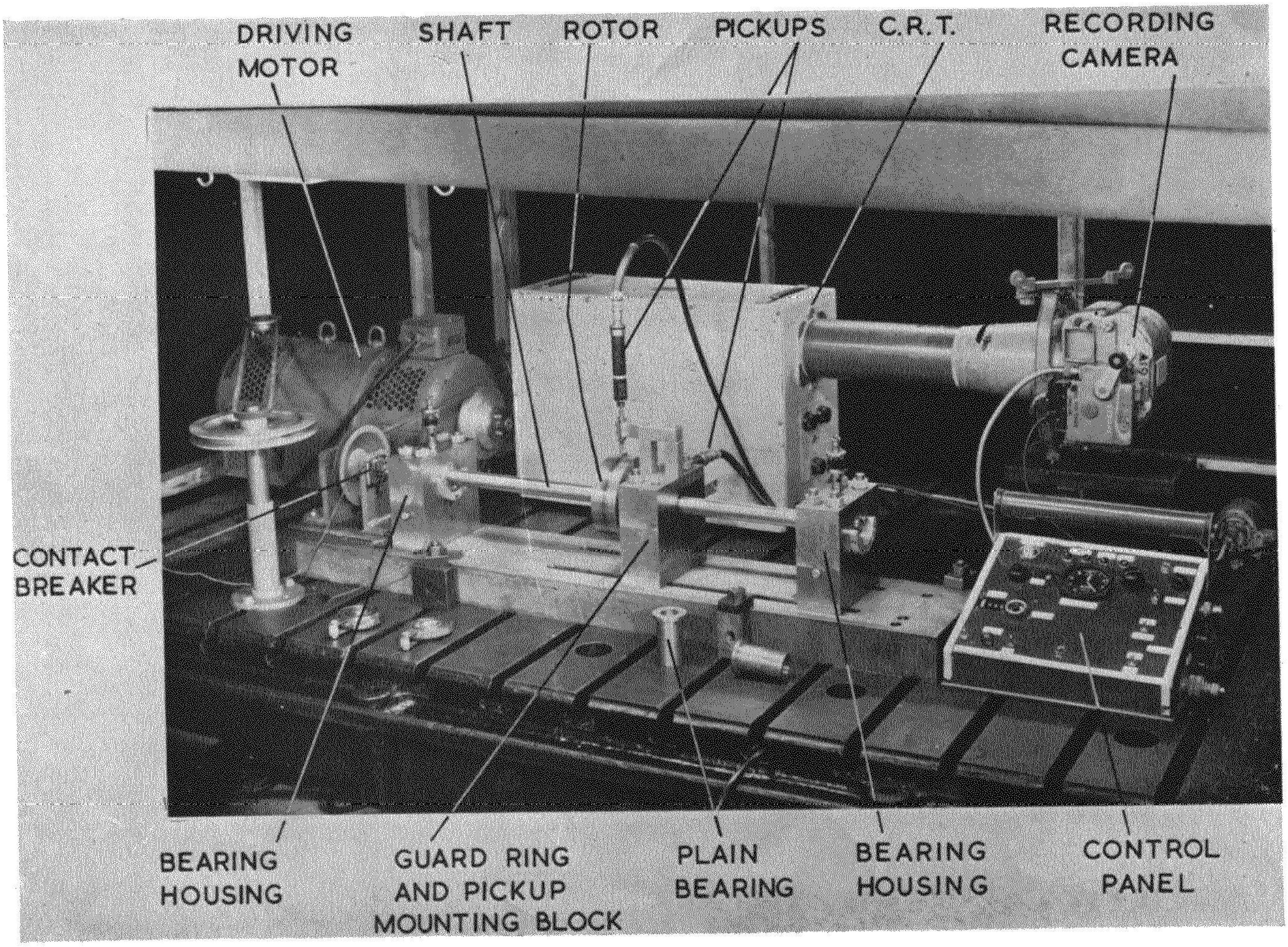



FIG. 23. Details of the experimental rig.



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