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A Calculation Method for Three-Dimensional Turbulent Boundary Layers

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Summary.—A method for the calculation of three-dimensional turbulent boundary layers is given. It depends on the assumptions that the component of the velocity in the boundary layer parallel to the external streamlines follows a power law, and that the angle the direction of flow in the boundary layer makes with the external streamlines is small.

The external flow is expressed in streamline co-ordinates and the calculations follow a streamline in the external flow. The momentum thickness is determined by a quadrature and the angle the limiting streamlines make with the external streamlines is determined from a linear differential equation.

Only slight experimental checks are available; they suggest that the method is adequate, but more experimental results are needed in order to refine the method.

1. *Introduction*.—The calculation of two-dimensional turbulent boundary layers, whilst based on theoretical considerations, requires some appeal to experiment in order to fill in the details and to supply numerical constants. When we come to three-dimensional turbulent boundary layers, for which experimental evidence is scanty, it is necessary to rely to a certain extent on the two-dimensional methods, together with additional assumptions owing to the three-dimensional nature of the flow. Even then the analysis is complicated and so we make an additional simplifying assumption. This states that velocities normal to the external streamlines of the flow are everywhere small, together with their derivatives. This has been called the semi-independence principle by Eichelbrenner and Oudart[3], but it may be unwise to use such a name, since it may be confused with the 'independence principle' for yawed infinite wings, which is quite different, and which does not apply for turbulent flow in any case.

We assume, on the basis of some experiments by Wallace[4]. that the component of the velocity in the boundary layer parallel to the external streamlines follows a power law, and that the angle which the direction of flow makes with the external streamlines, which is to be small, is a quadratic function of distance from the surface.

The solution follows a straightforward procedure. Firstly, the momentum thickness in the direction of the external streamlines is determined by a quadrature very similar to that used in two dimensions, except that a quantity $\bar{p}$, determined by the three-dimensional character of the flow, is introduced. The direction of flow in the boundary layer (β) is then found by the solution of a linear differential equation. As regards the form parameter H, it is possible to go some way in the analysis for its determination, but more experiments are needed before this analysis can be completed.

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The two-dimensional criterion for separation may not be used, and must be replaced by making use of the fact that the separation line is the envelope of the limiting streamlines, which can be drawn when β is known everywhere.

It is difficult to give a physical picture of the quantity $\bar{\rho}$ mentioned above. It depends only on the external flow and on the shape of the body which are supposed known. Finding $\bar{\rho}$ and the streamline co-ordinates, though theoretically straightforward, will involve a large amount of computation. Once this is done the labour of the boundary-layer calculations themselves is not excessive.

It is difficult to verify this work by experiment. Attempts have been made here, which suggest that the method is reasonable for the determination of the momentum thickness and the direction of flow. More refined experiments are needed to verify the method.

2. *Momentum Equations*.—We take these from the work of Timman[1] and Zaat[2]. Referring to the external flow a streamline co-ordinate system is used, in which the line element ds is given by

$$ds^2 = \frac{1}{T}\,d\phi^2 + \frac{1}{\bar{\rho}T}\,d\psi^2 + d\zeta^2, \tag{1}$$

where T is the square of the external velocity, ϕ is the velocity potential and ψ is a stream function. ζ represents distance measured normal to the surface of the body. $\bar{\rho}$ is not completely determinate and in fact may be multiplied by any function of ψ. It satisfies the equation

$$\frac{1}{\bar{\rho}g}\,\frac{\partial(\bar{\rho}g)}{\partial\phi} = -\frac{2}{T}\left(\frac{\delta\bar{U}}{\delta x} + \frac{\delta\bar{V}}{\delta y}\right),$$

or

$$\frac{1}{\bar{\rho}g}\left[\bar{U}\,\frac{\delta}{\delta x} + \bar{V}\,\frac{\delta}{\delta y}\right](\bar{\rho}g) = -2\left[\frac{\delta\bar{U}}{\delta x} + \frac{\delta\bar{V}}{\delta y}\right],$$

where

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + z_x\,\frac{\partial}{\partial z}, \qquad \frac{\delta}{\delta y} = \frac{\partial}{\partial y} + z_y\,\frac{\partial}{\partial z}, \qquad g = 1 + z_x^{\,2} + z_y^{\,2},$$

suffixes denoting partial differentiation; x, y, z are Cartesian co-ordinates, and $z = z(x, y)$ is the equation of the body, with $\bar{U}$, $\bar{V}$, $\bar{W}$ component velocities along the axes, W the component of velocity normal to the surface. All velocities are taken to be just outside the boundary layer, that is, in the external flow. W is of course zero but not $\partial W/\partial\zeta$. These velocity components are shown in Fig. 1a.

In this co-ordinate system the momentum equations as shown in Appendix A of Ref. 2 are

$$\int_0^\infty \Bigg\{ U\,\frac{\partial(U-u)u}{\partial\phi} + \sqrt{\bar{\rho}}\,U\,\frac{\partial(U-u)v}{\partial\psi} + U(U-u)\,\frac{\partial U}{\partial\phi} - \sqrt{\bar{\rho}}\,vU\,\frac{\partial U}{\partial\psi} - \left(\frac{1}{2\bar{\rho}}\,\frac{\partial\bar{\rho}}{\partial\phi} + \frac{1}{U}\,\frac{\partial U}{\partial\phi}\right)U[(U-u)u + v^2] - \sqrt{\bar{\rho}}\,\frac{\partial U}{\partial\psi}\,[(U-u)v - uv]\Bigg\}\,d\zeta$$

$$= \nu\left(\frac{\partial u}{\partial\zeta}\right)_{\zeta=0} = \frac{\tau_{01}}{\rho}, \tag{2}$$

$$\int_0^\infty \left\{ \sqrt{\bar\rho} U \frac{\partial v^2}{\partial \psi} - U \frac{\partial (uv)}{\partial \phi} + \sqrt{\bar\rho} \frac{\partial U}{\partial \psi} (v^2 + U^2 - u^2) + 2uvU \left[\frac{1}{2\bar\rho} \frac{\partial \bar\rho}{\partial \phi} + \frac{1}{U} \frac{\partial U}{\partial \phi} \right] \right\} d\zeta$$

$$= \nu \left(\frac{\partial v}{\partial \zeta} \right)_{\zeta = 0} = \frac{\tau_{02}}{\bar\rho} . \tag{3}$$

In these equations u and v are velocities inside the boundary layer in the streamwise and normal directions, U is the external velocity in the streamwise direction, $\bar\rho$ is the density, ν the kinematic viscosity, and τ_{01} and τ_{02} the surface shear-stress components in the streamwise and normal directions. Velocity components are shown in Fig. 1b.

Equations (2) and (3) apply for a laminar boundary layer. Just as in two dimensions, we may use them for turbulent flow provided that u and v are now to be taken as mean velocities and certain other terms involving the perturbations of the velocities from their mean values are ignored.

If u', v' and w' are the perturbations of u, v and w from their mean values, then the mean values of $\rho u'^2$, $\rho v'^2$ and $\rho w'^2$ come into the equations of motion as 'Reynolds stresses' which behave like pressure. See, for instance, Schlichting[9]. These are small compared with the local pressures. They occur in the momentum equations integrated across the boundary layer, and we have ignored them.

The mean values of $\rho u'w'$ and $\rho v'w'$ behave like shear-stress components, and their integrated values can be incorporated in τ_{01} and τ_{02} respectively. The mean value of $\rho u'v'$ is also a shear-stress component and we ignore it because its integrated value is small compared with τ_{01} and τ_{02}. In fact, there seems to be no reason why u' and v' should be correlated as there is for instance in the case of u' and w' (*See* Ref. 9).

Hence the effect of the fluctuating components of velocity is to bring in additional components of shear stress but otherwise it leaves equations (2) and (3) unchanged, except that empirical values for τ_{01} and τ_{02} must be given.

Equations (2) and (3) still require further simplification before they can be used, and we make the additional assumption of 'semi-independence' made by Eichelbrenner and Oudart[3] and by Zaat[2]. The assumption is that the velocity v is small compared to u. We again point out that this has nothing to do with the 'independence principle' for infinite swept wings.

In many problems this assumption is a reasonable one to make. If it is true then equation (3) shows that the term

$$\sqrt{\bar\rho} \frac{\partial U}{\partial \psi} (U^2 - u^2)$$

must also be small. Now if dn denotes the element of length normal to the streamlines equation (1) shows that

$$\sqrt{\bar\rho} U \frac{\partial U}{\partial \psi} = \frac{\partial U}{\partial n} ,$$

and hence our assumption implies that $\partial U/\partial n$ is small.

Making allowance for the fact that the operator $\sqrt{\bar\rho}\,(\partial/\partial\psi)$ operating on any velocity is in general of the same order of smallness as v, we see that ignoring terms in v in equation (2) means in general ignoring terms of order v^2. Again, keeping in equation (3) only terms in v, together with the term

$$\sqrt{\bar{p}}\,\frac{\partial U}{\partial \psi}\,(U^2 - u^2)$$

means in fact that we are ignoring terms in v^3 in that equation.

The assumption implies that the direction of flow inside the boundary layer does not differ much from that outside. If the angle between the two directions is ε (*See* Fig. 1b), then we may suppose that ε is small.

This may not be true near the surface, where u is small. If β is the value of ε at the surface, we have

$$\tan\beta = \frac{\left(\dfrac{\partial v}{\partial \zeta}\right)_{\zeta=0}}{\left(\dfrac{\partial u}{\partial \zeta}\right)_{\zeta=0}}\,,$$

and this may be supposed to be small in general, except where $(\partial u/\partial \zeta)_{\zeta=0}$ is small. We may not therefore push the solution too far towards the point where the skin friction in the direction of the external flow would vanish. Separation seems normally to take place before this stage is reached. One may say that our assumption implies that ε is small provided that the solution is not pushed to a region too near to separation.

If the resultant velocity in the boundary layer is u_m we shall have approximately

$$u = u_m \cos\varepsilon \simeq u_m, \qquad v = u_m \sin\varepsilon \simeq \varepsilon u_m, \tag{4}$$

since ε is small. ε will vary as we pass through the boundary layer normal to the surface. Thus it will be zero at the edge of the layer and will take a limiting value β at the surface of the body. β is in fact the angle between the limiting streamline and the external streamline.

If the resultant shear stress at the surface is τ_0, which is in the same direction as the limiting streamline, we shall have

$$\tau_{01} = \tau_0 \cos\beta \simeq \tau_0, \qquad \tau_{02} = \tau_0 \sin\beta \simeq \beta\tau_0\,. \tag{5}$$

3. *Assumed Velocity Profiles.*—We shall suppose that u_m follows a power law as is done in two dimensions. If u_m/U is plotted against ζ/δ where δ is the boundary-layer thickness, as is done by Wallace[4], we obtain curves very much like those known in two-dimensional flow. If H is defined by

$$H = \frac{\displaystyle\int_0^\infty \left(1 - \frac{u_m}{U}\right) d\zeta}{\displaystyle\int_0^\infty \frac{u_m}{U}\left(1 - \frac{u_m}{U}\right) d\zeta} = \frac{\delta^*}{\theta}\,, \tag{6}$$

we find that the power law

$$\frac{u_m}{U} = \left(\frac{\zeta}{\delta}\right)^{(H-1)/2} \tag{7}$$

fits the profiles of Ref. 4 quite well. In order to avoid doubts as to the definition of δ we write equation

(7) in the form

$$\frac{u_m}{U} = \left(\frac{\gamma\zeta}{\theta}\right)^{(H-1)/2},$$

where

$$\gamma = \frac{(H-1)}{H(H+1)}.$$

Fig. 2 shows that the agreement is very good in the outer part of the boundary layer. The inner part of the boundary layer would be better fitted by a logarithmic law, and of course nearer still to the surface there is the laminar sub-layer. The discrepancies in the inner regions have little effect on δ^*, θ and H.

So far all is much the same as in two dimensions, provided that the u of the two-dimensional profiles is replaced by u_m, the resultant velocity in the boundary layer. This leads us to believe that we may make the same assumptions as in two dimensions for the skin friction τ_{01}. A full account of the two-dimensional problem is given in Chapter II of Ref. 5.

The most often quoted value for τ_{01} is the Ludwieg and Tillman formula

$$\frac{\tau_{01}}{\rho U^2} = \alpha(H)\left(\frac{U\theta}{\nu}\right)^{-n}, \qquad (8)$$

where

$$\alpha(H) = 0{\cdot}123 \times 10^{-0{\cdot}678H}, \qquad n = 0{\cdot}268.$$

However, we shall here suppose that in equation (2) we may write

$$\frac{\tau_{01}}{\rho U^2} = 0{\cdot}00885\left(\frac{U\theta}{\nu}\right)^{-1/5}, \qquad (9)$$

which is appropriate for a flat plate. This form is due to Young[10].

Finally we must assume a form for the distribution of ε. It must be such that $\varepsilon = \beta$ when $\zeta = 0$ and $\varepsilon = 0$ when $\zeta = \delta$. A simple relation which fits the measurements of Wallace[4] fairly well is

$$\varepsilon = \beta\left(1 - \frac{\zeta}{\delta}\right)^2. \qquad (10)$$

We show in Fig. 3 a comparison between Wallace's experimental values of ε/β and the curve $(1 - \zeta/\delta)^2$. The spread is almost within the error merely of reading off values from Wallace's curves. The relation (10) was also given by Mager[6], from experiments on flow in ducts.

4. *The Streamline Equation.*—Making the assumption of small v, we may write equation (2) in the form

$$U\frac{\partial}{\partial\phi}\left(U^2\delta_{11}\right) + U^2\delta_1\frac{\partial U}{\partial\phi} - \left(\frac{1}{2\bar{\rho}}\frac{\partial\bar{\rho}}{\partial\phi} + \frac{1}{U}\frac{\partial U}{\partial\phi}\right)U^3\delta_{11} = \frac{\tau_{01}}{\rho} \qquad (11)$$

$$U^2\delta_{11} = \int_0^\delta u\,(U-u)\,d\zeta, \qquad U\delta_1 = \int_0^\delta (U-u)\,d\zeta\,. \qquad (12)$$

Since $u \simeq u_m$ we have

$$\delta_{11} \simeq \theta, \qquad \delta_1 \simeq \delta^*. \tag{13}$$

and so equation (11) may be written

$$U \frac{\partial \theta}{\partial \phi} + (H+1)\theta \frac{\partial U}{\partial \phi} - \frac{1}{2\bar{\rho}} \frac{\partial \bar{\rho}}{\partial \phi} U\theta = \frac{\tau_{01}}{\rho U^2}. \tag{14}$$

Excellent results[5] can be obtained by giving τ_{01} the value in equation (9) and assuming that H is constant and equal to 1·5, in the two-dimensional equation corresponding to (14). We shall do this here.

If we write

$$\Theta = \theta \left(\frac{U\theta}{\nu}\right)^{1/5}$$

we may verify that equation (14) may be written

$$\frac{\partial}{\partial \phi}\left\{\frac{\Theta U^{14/5}}{\bar{\rho}^{3/5}}\right\} = 0{\cdot}0106 \frac{U^{9/5}}{\bar{\rho}^{3/5}} \tag{15}$$

It can be shown[2] that if our problem reduces to a two-dimensional one, then $\bar{\rho} = A/U^2$, $d\phi = U\,dx$, and equation (15) then reduces to

$$\frac{\partial}{\partial x}\left(\theta U^4\right) = 0{\cdot}0106 U^4,$$

which is Spence's form for the two-dimensional solution[5].

Thus we see that equation (15) for the flow along the streamlines is a generalised form of the two-dimensional equation, modified to take into account the effect of the three-dimensional nature of the flow.

In the case of a body of revolution Zaat[2] shows that $\bar{\rho} = 1/r^2U^2$, where r is the radius of a section. Thus for a body of revolution equation (15) becomes

$$\frac{\partial}{\partial s}\left(\Theta U^4 r^{6/5}\right) = 0{\cdot}0106 U^4 r^{6/5},$$

where s is the arc measured along a section through the axis.

5. *The Cross-Flow Equation.*—From equations (4), (7) and (10) we have

$$v = \beta\left(1 - \frac{\zeta}{\delta}\right)^2 \left(\frac{\zeta}{\delta}\right)^{(H-1)/2},$$

and we define δ_{21} by the equation

$$U^2\delta_{21} = -\int_0^{\delta} uv\, d\zeta. \tag{16}$$

Hence we have, using equation (7),

$$\delta_{21} = -2\beta\delta/H(H+1)(H+2),$$

whilst

$$\theta = \delta_{11} = \delta(H-1)/H(H+1).$$

Hence

$$\delta_{21}/\theta = -\beta/f, \quad \text{where} \quad f = \tfrac{1}{2}(H-1)(H+2). \tag{17}$$

Equation (3), on making the assumption of small v, may be written

$$U\frac{\partial}{\partial\phi}\left(U^2\delta_{21}\right) + \sqrt{\bar{\rho}}U^2\left(\delta_1 + \delta_{11}\right)\frac{\partial U}{\partial\psi} - 2U^3\delta_{21}\left(\frac{1}{2\bar{\rho}}\frac{\partial\bar{\rho}}{\partial\phi} + \frac{1}{U}\frac{\partial U}{\partial\phi}\right) = \alpha\beta U^2\left(\frac{U\theta}{\nu}\right)^{-n},$$

using equations (5), (9), (12) and (16).

Writing $R = U\theta/\nu$ and using equation (17) we have

$$-U^3\frac{\partial}{\partial\phi}\left(\frac{\beta\theta}{f}\right) + \sqrt{\bar{\rho}}U^2(1+H)\theta\frac{\partial U}{\partial\psi} + \frac{U^3\theta\beta}{f\bar{\rho}}\frac{\partial\bar{\rho}}{\partial\phi} = \alpha\beta U^2R^{-n},$$

that is,

$$\frac{\partial}{\partial\phi}\left(\frac{\beta\theta}{\bar{\rho}f}\right) = \frac{(1+H)\theta}{U\sqrt{\bar{\rho}}}\frac{\partial U}{\partial\psi} - \frac{\alpha\beta R^{-n}}{U\bar{\rho}}. \tag{18}$$

If we do not assume, as we did before, that H is constant we may as well use the more accurate equation (8) for the skin friction, instead of assuming α constant and n equal to 1/5.

Finally, then, we may write equation (18), which is a linear equation for β, in the form

$$\frac{\partial}{\partial\phi}(C\beta) = A - B\beta,$$

where

$$C = \frac{\theta}{\bar{\rho}f}, \qquad A = \frac{(1+H)\theta}{U\sqrt{\bar{\rho}}}\frac{\partial U}{\partial\psi}, \qquad B = \frac{\alpha R^{-n}}{U\bar{\rho}}, \qquad R = \frac{U\theta}{\nu}.$$

In the equation for β we take $n = 0{\cdot}268$ and

$$\alpha = 0{\cdot}123 \times 10^{-0{\cdot}678H}.$$

Alternatively we may simplify equation (18) considerably by assuming, as we did in obtaining equation (15), that H is constant and equal to $1{\cdot}5$, $n = 1/5$, and $\alpha = 0{\cdot}00885$. Putting in these values and making use of equation (14), we may reduce equation (18) to the form

$$\frac{d\eta}{d\phi} + \frac{0{\cdot}0166}{U\Theta}\eta = \frac{2{\cdot}187}{U^{3\cdot5}}\frac{\partial U}{\partial\psi}, \tag{19}$$

where

$$\eta = \frac{\beta}{\sqrt{\bar{\rho}}U^{2\cdot5}}.$$

It may well be that equation (19), with different numerical constants, to be determined by experiment, may suffice for the determination of β. In the example given below, where H changes from 1·5 to 1·67, the values of β, as found by this equation, agree fairly well with experiment. Strictly, however, equation (18) requires knowledge of H before it can be solved, and we proceed to consider the determination of H.

6. *The Determination of H.*—We attack this problem by a method analogous to that used in two dimensions. In streamline co-ordinates the u equation of motion is given by Zaat[2] as

$$U\left(u\frac{\partial u}{\partial \phi} - U\frac{\partial U}{\partial \phi}\right) + vU\sqrt{\bar{\rho}}\frac{\partial u}{\partial \psi} + w\frac{\partial u}{\partial \zeta} - uv\sqrt{\bar{\rho}}\frac{\partial U}{\partial \psi} + v^2\frac{\partial U}{\partial \phi} + \frac{v^2 U}{2\bar{\rho}}\frac{\partial \bar{\rho}}{\partial \phi} = \nu\frac{\partial^2 u}{\partial \zeta^2}.$$

Making the assumption that terms in v are small, this may be written

$$U\left(u\frac{\partial u}{\partial \phi} - U\frac{\partial U}{\partial \phi}\right) + w\frac{\partial u}{\partial \zeta} = \nu\frac{\partial^2 u}{\partial \zeta^2}. \qquad (20)$$

The equation of continuity is

$$\frac{\partial w}{\partial \zeta} = -U\frac{\partial u}{\partial \phi} + u\frac{\partial U}{\partial \phi} + \frac{uU}{2\bar{\rho}}\frac{\partial \bar{\rho}}{\partial \phi} - U\sqrt{\bar{\rho}}\frac{\partial v}{\partial \psi} + v\sqrt{\bar{\rho}}\frac{\partial U}{\partial \psi},$$

or again if v is small

$$\frac{\partial w}{\partial \zeta} = -U\frac{\partial u}{\partial \phi} + u\frac{\partial U}{\partial \psi} + \frac{uU}{2\bar{\rho}}\frac{\partial \bar{\rho}}{\partial \phi}. \qquad (21)$$

We note also that, integrating through the boundary layer,

$$\int_0^\infty uw\frac{\partial u}{\partial \zeta}\,d\zeta = \Big[u^2 w\Big]_0^\infty - \int_0^\infty \left(u^2\frac{\partial w}{\partial \zeta} + uw\frac{\partial u}{\partial \zeta}\right)d\zeta,$$

and so

$$\int_0^\infty uw\frac{\partial u}{\partial \zeta}\,d\zeta = -\frac{1}{2}\int_0^\infty u^2\frac{\partial w}{\partial \zeta}\,d\zeta = \frac{1}{2}\int_0^\infty (U^2 - u^2)\frac{\partial w}{\partial \zeta}\,d\zeta, \qquad (22)$$

making use of the fact that w vanishes at both limits.

We now multiply equation (20) by u and integrate with respect to ζ, making use of equations (21) and (22), and obtain after some reduction

$$\frac{1}{2}\int_0^\infty U^2\frac{\partial}{\partial \phi}\left\{\frac{u(u^2 - U^2)}{U}\right\}d\zeta - \frac{U}{4\bar{\rho}}\frac{\partial \bar{\rho}}{\partial \phi}\int_0^\infty u(u^2 - U^2)\,d\zeta = -\nu\int_0^\infty\left(\frac{\partial u}{\partial \zeta}\right)^2 d\zeta.$$

We have here integrated the right-hand side by parts.

If we define δ_3 by

$$\delta_3 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)d\zeta,$$

the equation becomes

$$U^2 \frac{\partial}{\partial \phi}\left(U^2 \delta_3\right) - \frac{U^4}{2\bar{\rho}} \frac{\partial \bar{\rho}}{\partial \phi} \delta_3 = \frac{2U}{\rho} \int_0^\infty \tau \frac{\partial}{\partial \zeta}\left(\frac{u}{U}\right) d\zeta$$

where

$$\tau = \mu \frac{\partial u}{\partial \zeta},$$

or

$$\frac{\partial \delta_3}{\partial \phi} + \frac{2\delta_3}{U} \frac{\partial U}{\partial \phi} - \frac{1}{2\bar{\rho}} \frac{\partial \bar{\rho}}{\partial \phi} \delta_3 = \frac{2\tau_s}{\rho U^3}, \tag{23}$$

where

$$\tau_s = \int_0^\infty \tau \frac{\partial}{\partial \zeta}\left(\frac{u}{U}\right) d\zeta .$$

If we assume the power law equation (7) it can be shown that

$$\frac{\delta_3}{\theta} = \frac{\delta_3}{\delta_{11}} = \frac{4H}{3H-1} . \tag{24}$$

For our purpose here it will be sufficient to assume that

$$\delta_3 = \theta f(H),$$

which has been verified experimentally in two dimensions.

Substituting in equation (23) and making use of equation (14) we have

$$\theta \frac{dH}{d\phi} = \frac{f}{f'}(H-1)\frac{\theta}{U}\frac{\partial U}{\partial \phi} - \frac{f}{f'}\frac{\tau_{01}}{\rho U^3} + \frac{2}{f'}\frac{\tau_s}{\rho U^3} . \tag{25}$$

It will be noticed that $\bar{\rho}$ has disappeared, except in so far as it is implicitly involved in the determination of θ from equation (15). If $f(H)$ is given by equation (24), then equation (25) is the same as equation II (64) in Ref. 5, provided that the equation is supposed taken along a streamline, so that $d\phi = U\,dx$ by equation (1). The rest of the arguments of Ref. 5 may therefore be continued, much as though this were a two-dimensional problem. Using the expression (8) for τ_{01}, and the expression

$$\frac{\tau_s}{\rho U^2} = \left(\frac{U\theta}{\nu}\right)^{-n} F_1(H) - \theta \frac{dU}{d\phi} F_2(H),$$

suggested in Ref. 5, for τ_s, where $F_1(H)$ and $F_2(H)$ are functions of H, we may reduce equation (25) to the form

$$\Theta \frac{dH}{d\phi} = \bar{\phi}(H)\,\Gamma - \frac{1}{U}\bar{\psi}(H), \tag{26}$$

where

$$\Theta = \theta\left(\frac{U\theta}{\nu}\right)^{-n}, \qquad \Gamma = -\frac{1}{U}\frac{dU}{d\phi} .$$

It remains to determine $\bar{\phi}$ and $\bar{\psi}$. We may obtain $\bar{\psi}$ in a similar manner to Spence,[5] by consideration of the flat-plate problem to which equation (26) reduces when $\Gamma = 0$. Spence finds that, with $n = 1/5$

$$\bar{\psi}(H) = 0{\cdot}00307(H - 1)^2. \tag{27}$$

We may adopt this value here, but the determination of $\bar{\phi}$ must await further experiment. We may only remark here that the value

$$\bar{\phi}(H) = 9{\cdot}524(H - 1{\cdot}21)(H - 1), \tag{28}$$

used in Ref. 5 in the two-dimensional case, is not suitable for the problem of Ref. 4, as may be seen in Table 3 below.

7. *Separation.*—In two dimensions separation takes place when H is in the range $2{\cdot}0 < H < 2{\cdot}8$. One is tempted to try to use the same criterion in three dimensions. If this were true any work which was undertaken for the express purpose of determining the separation line would be greatly simplified, because it would not be necessary to solve equation (19). Consideration of the laminar case leads to the conclusion that it would be incorrect to use the criterion given above. Two-dimensional laminar separation takes place when the surface shear stress vanishes, and writers have similarly been tempted to say that three-dimensional laminar separation takes place when the shear-stress component along the streamlines vanishes. Certain laminar calculations on delta wings have shown this to be highly inaccurate and we must use the criterion[8] that the separation line is tangential to the limiting streamlines. This means that we must determine β by equation (19) before we can find the separation line.

8. *Examples.*—Two attempts were made to compare with experiment, though the experimental results are not nearly detailed enough for accurate computation. However, the curves of Wallace[4] at station 40·41 and angle of incidence 8 deg gave the following values:

TABLE 1

x/c	U^2	β (deg)	H	θ/c	$\bar{p}$	Angle of flow (deg)
0·3	1·69	7	1·50	0·00104	0·704	$11\frac{1}{2}$
0·5	1·44	9	1·57	0·00176	0·855	9
0·7	1·21	12	1·64	0·00288	1·064	7
0·9	1·04	17	1·67	0·00449	1·299	$4\frac{1}{2}$

The values of $\bar{\rho}$ are very much of an estimate, on the assumption that locally, for the purpose of calculating $\bar{\rho}$ only, the wing may be considered as an infinite yawed wing. It was in fact tapered. In order to obtain any results from these it was also necessary to assume that the external streamlines were straight. They were not so, as shown by the last column, termed 'angle of flow', which gives the angle the external streamlines makes with the chord, measured towards the centre-section (*See* Fig. 4). Naturally the results are crude with so few points, but on integrating the equations (15) and (19) the following results were found:

TABLE 2

x/c	θ/c (calc)	θ/c (expt)	β (calc) (deg)	β (expt) (deg)
0·3	0·00105	0·00105	7	7
0·5	0·00183	0·00180	10	9
0·7	0·00293	0·00288	$14\frac{1}{2}$	12
0·9	0·00425	0·00449	19	17

The calculated values of H from equation (26) are shown in Table 3. It has already been pointed out that the value for $\bar{\phi}$ may not be suitable for this problem, and must await further experiments.

TABLE 3

x/c	H (calc)	H (expt)
0·3	1·50	1·50
0·5	1·61	1·57
0·7	1·94	1·64
0·9	3·01	1·67

The second attempt at comparison was on a wing tested by Brebner (unpublished). It was a swept wing of aspect ratio 3·4, angle of sweep 55 deg, thickness/chord ratio $4\frac{1}{2}$ per cent, at a lift coefficient of 0·2, the measurements being made half-way out along the span. Figs. 5 and 6 show that the agreement is quite good. This time there is much better agreement for H than for the first example.

9. *Solution Procedure*.—It is first necessary to know the external flow and to transform into streamline co-ordinates. If Cartesian co-ordinates were in use originally, with velocity components $\bar{U}$, $\bar{V}$

and $\bar{W}$, the projection of a streamline on the xy plane is obtained from

$$\frac{dy}{dx} = \frac{\bar{V}}{\bar{U}},$$

and the corresponding z co-ordinates follow from the equation of the surface $z = z(x, y)$.

Integrals with respect to ϕ are evaluated on the understanding that ψ is kept constant; hence $d\phi$ may be replaced by $(T/\bar{U})\, dx$ or $(T/\bar{V})\, dy$ where $T = \bar{U}^2 + \bar{V}^2 = U^2$.

Derivatives with respect to ϕ and ψ are given by

$$T\frac{\partial}{\partial\phi} = \bar{U}\frac{\delta}{\delta x} + \bar{V}\frac{\delta}{\delta y}, \qquad T\sqrt{(\bar{\rho}g)}\frac{\partial}{\partial\psi} = \frac{\delta\phi}{\delta y}\frac{\delta}{\delta x} - \frac{\delta\phi}{\delta x}\frac{\delta}{\delta y},$$

where

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + z_x\frac{\partial}{\partial z}, \qquad \frac{\delta}{\delta y} = \frac{\partial}{\partial y} + z_y\frac{\partial}{\partial z},$$

$$\frac{\delta\phi}{\delta x} = \bar{U}(1 + z_x^{\,2}) + \bar{V}z_x z_y, \qquad \frac{\delta\phi}{\delta y} = \bar{V}(1 + z_y^{\,2}) + \bar{U}z_x z_y.$$

$\bar{\rho}$ is found from formulae given in Section 2.

In places where the flow has become turbulent the surface is usually not very highly curved. In such places we may suppose that z_x and z_y are small, if the surface approximates to $z = 0$. We then have

$$\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial\phi} = -\frac{2}{U^2}\left(\frac{\partial\bar{U}}{\partial x} + \frac{\partial\bar{V}}{\partial y}\right),$$

and equation (15) may be written

$$\frac{\partial}{\partial\phi}\left(\Theta U^{14/5}\right) + \frac{1\cdot2}{U^2}\left(\frac{\partial\bar{U}}{\partial x} + \frac{\partial\bar{V}}{\partial y}\right)\Theta U^{14/5} = 0\cdot0106U^{9/5},$$

which is a linear equation for $\Theta U^{14/5}$, replacing the simple quadrature of equation (15) but avoiding the determination of $\bar{\rho}$.

Again, it may be shown that along a streamline $\psi =$ constant

$$\frac{\partial}{\partial\phi} = \frac{1}{U}\frac{\partial}{\partial s},$$

and so we have

$$\frac{\partial}{\partial s}\left(\Theta U^{14/5}\right) + \frac{1\cdot2}{U}\left(\frac{\partial\bar{U}}{\partial x} + \frac{\partial\bar{V}}{\partial y}\right)\Theta U^{14/5} = 0\cdot0106U^{14/5}. \qquad (29)$$

Again, along a streamline we have

$$ds = \frac{1}{U}\,d\phi = \frac{1}{U}(\bar{U}\,dx + \bar{V}\,dy) = \frac{U}{\bar{U}}\,dx,$$

since $dy = (\bar{V}/\bar{U})\,dx$ on a streamline; and so equation (29) may be written

$$\frac{\partial}{\partial x}\left(\Theta U^{14/5}\right) + \frac{1 \cdot 2}{\bar{U}}\left(\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y}\right)\Theta U^{14/5} = \frac{0 \cdot 0106 U^{19/5}}{\bar{U}} .$$

We fix our attention on a particular streamline and all that follows is understood to apply to that particular streamline only.

Initially the flow is laminar and calculations are made by any method suitable for laminar flow, the obvious choice being the method of Zaat[2]. This is continued to the point of transition, if this is known. Herein lies a difficulty, in that it is not certain what the criterion for transition is in three-dimensional flow. For the purpose of experiment transition is often deliberately caused by wires or other artificial means. In this case the position is known. If it is not known, Spence[7] recommends that at high Reynolds numbers the best assumption to make is that transition begins at the velocity maximum.

Starting from the transition point and following a streamline Θ is found from equations (15) or (29). This is expected to give Θ with sufficient accuracy. The initial value of Θ is taken from the laminar calculations.

Next β is determined from equation (19), the starting value again being known from the laminar calculations. The equation is linear and may be solved by a step-by-step method.

Once Θ and β are known much of the physical information about the turbulent boundary layer that may be required is known.

10. *Conclusions.*—The method given here is computationally feasible provided the streamline co-ordinates are obtained, or, more specifically, provided that $\bar{\rho}$ is known everywhere.

Very little experimental check is available. What there is can only give very crude checks and suggests that equation (15) may be adequate for the determination of θ. Equation (19) for β seems to give reasonable results, but later work may show that the constants should be modified. Equation (26) for H, with $\bar{\psi}$ given by equation (27), may be the right form when a suitable expression is found for $\bar{\phi}$. The lack of information here is not so serious, as knowledge of H is not required for the determination of all physical properties of the boundary layer; it is not required, for instance, for separation in the three-dimensional case.

LIST OF SYMBOLS

c	Local chord
x, y, z	Cartesian co-ordinates
s	Arc length
$\bar{U}, \bar{V}, \bar{W}$	External velocity components parallel to axes x, y, z
U	External velocity component parallel to surface
W	External velocity component perpendicular to surface
T	Square of resultant external velocity
u, v, w	Boundary-layer velocity components parallel to external streamlines, perpendicular to external streamlines and normal to surface
u_m	Resultant velocity in the boundary layer parallel to surface
f	$\frac{1}{2}(H-1)(H+2)$
g	$1 + z_x^2 + z_y^2$
R	$U\theta/\nu$
ζ	Distance measured normal to surface
$\alpha(H)$	$0{\cdot}123 \times 10^{-0{\cdot}678H}$
β	Angle between limiting streamlines and external streamlines
γ	$(H-1)/H(H+1)$
δ	Boundary-layer thickness
δ^*	$= \int_0^\infty \left(1 - \frac{u_m}{U}\right) d\zeta$
δ_1	$= \int_0^\delta \left(1 - \frac{u}{U}\right) d\zeta$
δ_{11}	$= \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) d\zeta$
δ_{21}	$= -\int_0^\delta uv\, d\zeta$
δ_3	$= \int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) d\zeta$
ε	Angle between flow direction in the boundary layer and external streamlines
θ	$= \int_0^\infty \frac{u_m}{U}\left(1 - \frac{u_m}{U}\right) d\zeta$

Θ	$=$	$\theta\left(\frac{U\theta}{\nu}\right)^{1/5}$
H	$=$	δ^*/δ
μ		Coefficient of viscosity
ν		Kinematic viscosity
ρ		Density
$\bar{\rho}$		Integrating factor
τ	$=$	$\mu\frac{\partial u}{\partial \zeta}$
τ_0		Resultant skin friction
τ_{01}, τ_{02}		Components of the skin friction along and perpendicular to the external streamlines
τ_s		$\int_0^\infty \tau \frac{\partial}{\partial \zeta}\left(\frac{u}{U}\right) d\zeta$
ϕ		Velocity potential
$\bar{\phi}$		Defined by equation (28)
ψ		Stream function
$\bar{\psi}$		Defined by equation (27)

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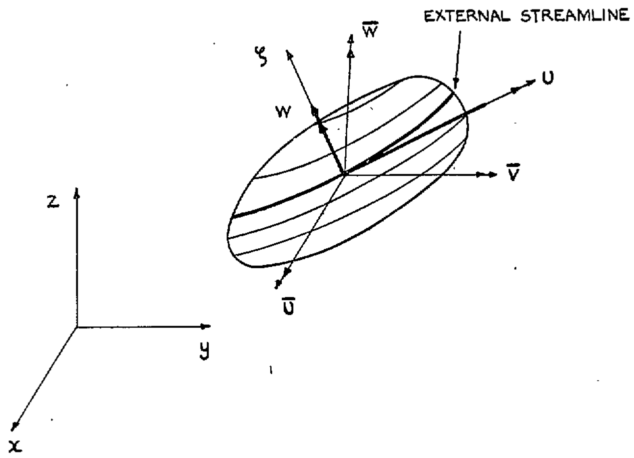

FIG. 1a. External velocity components.

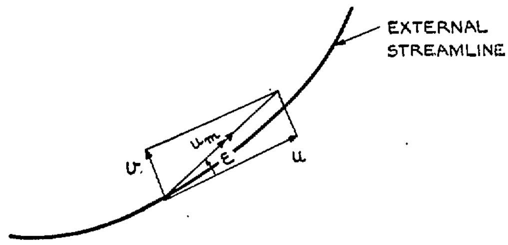

FIG. 1b. Boundary layer velocity components.

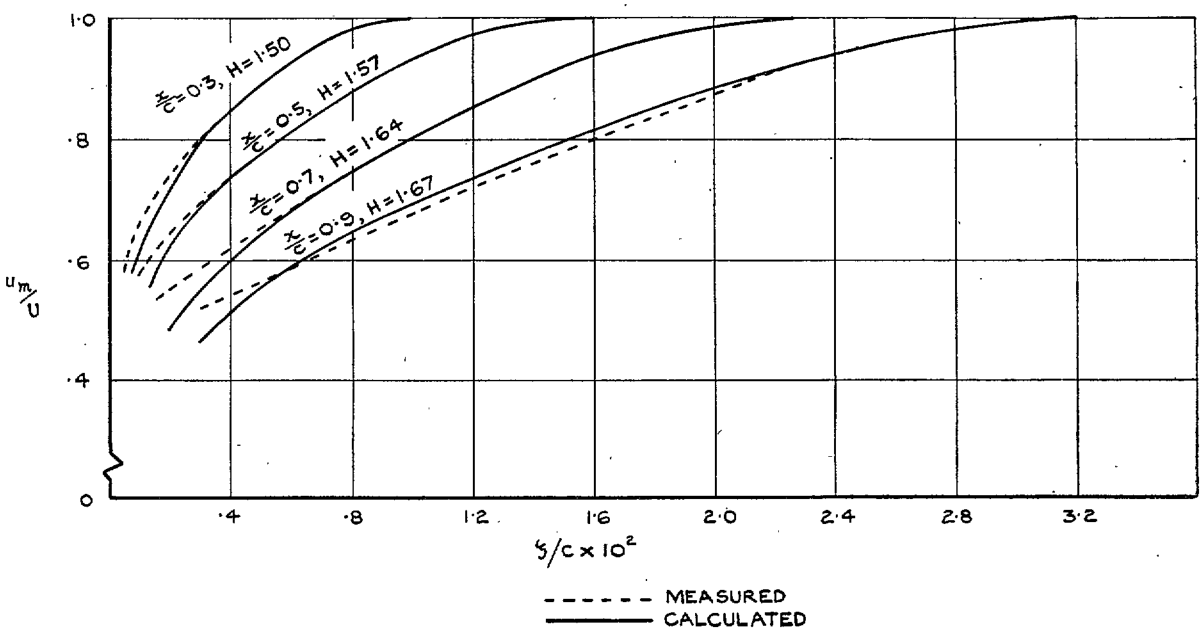

FIG. 2. Velocity profiles from Ref. 4. Station 40·41, $\alpha = 8$ deg.

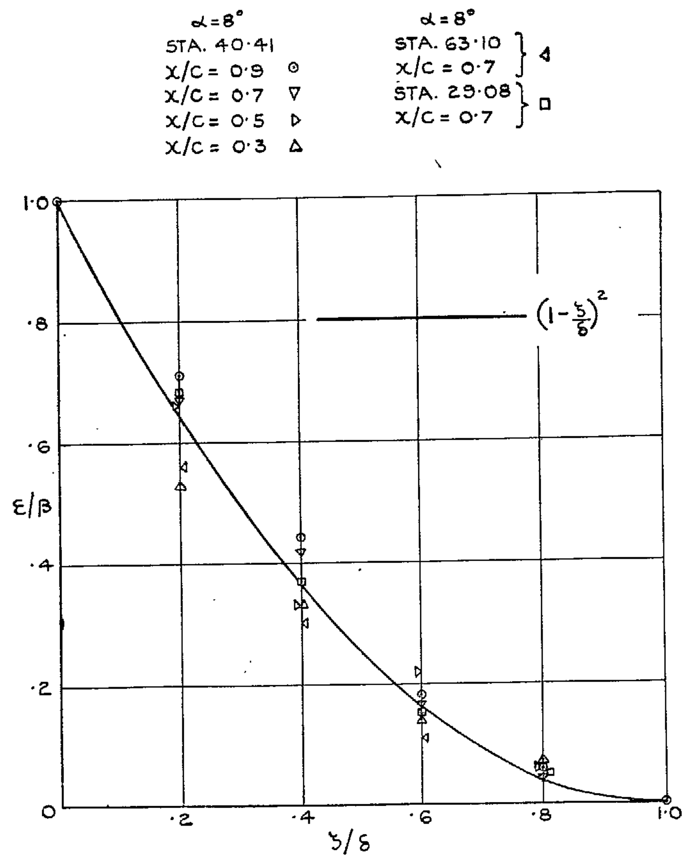

Fig. 3. Values of ε/β from Ref. 4.

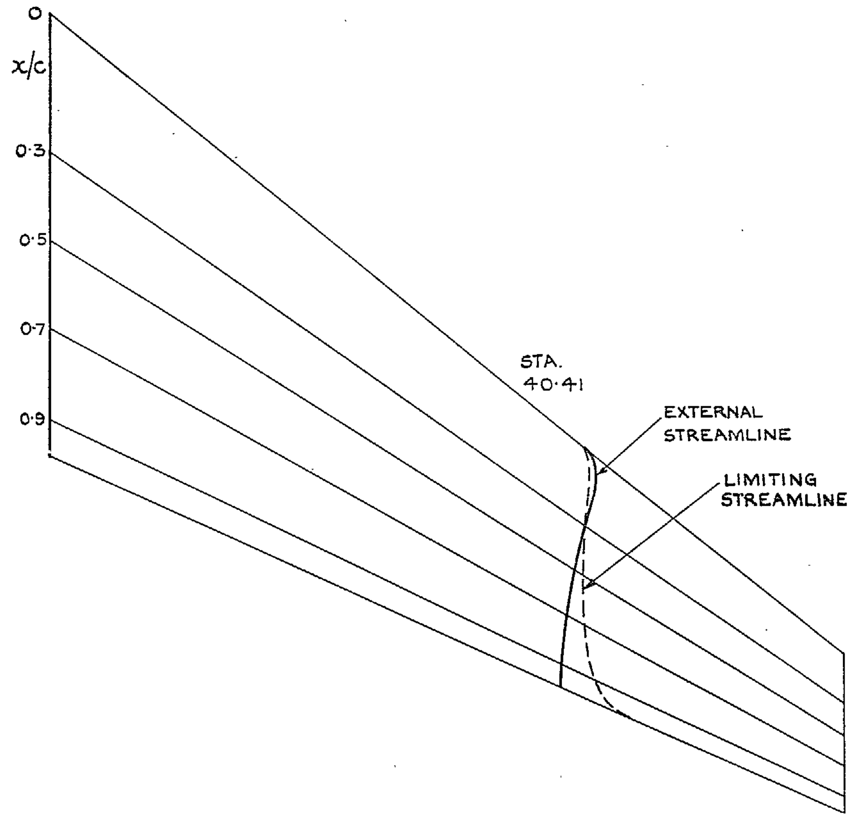

Fig. 4. Streamlines at station 40·41, Ref. 4.

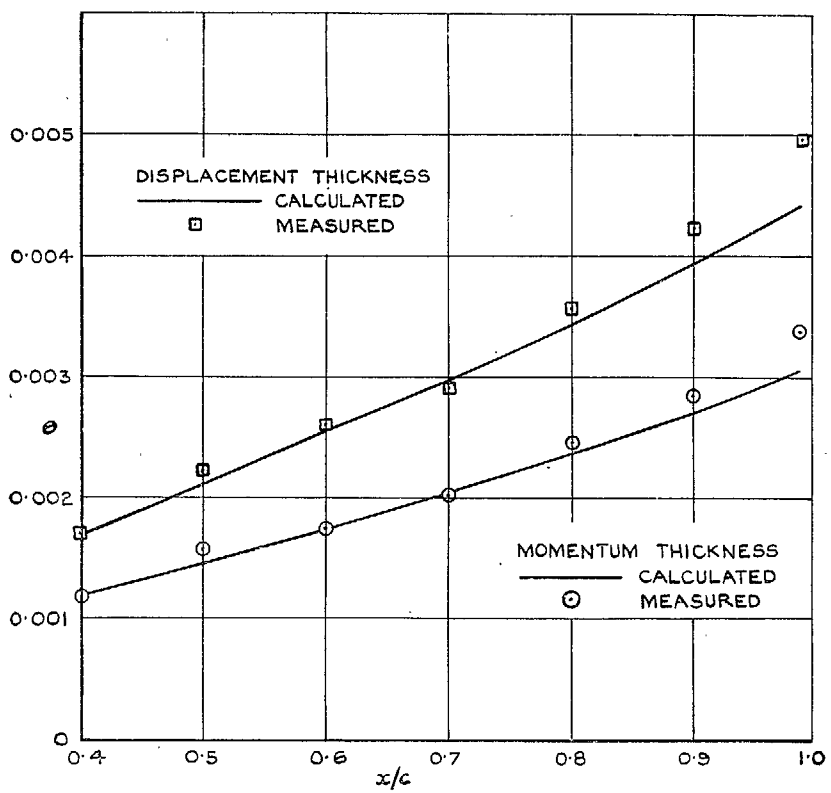

FIG. 5. Momentum and displacement thicknesses for swept wing RAE 101, thickness/chord ratio 4·5 per cent, angle of sweep 55 deg (Brebner).

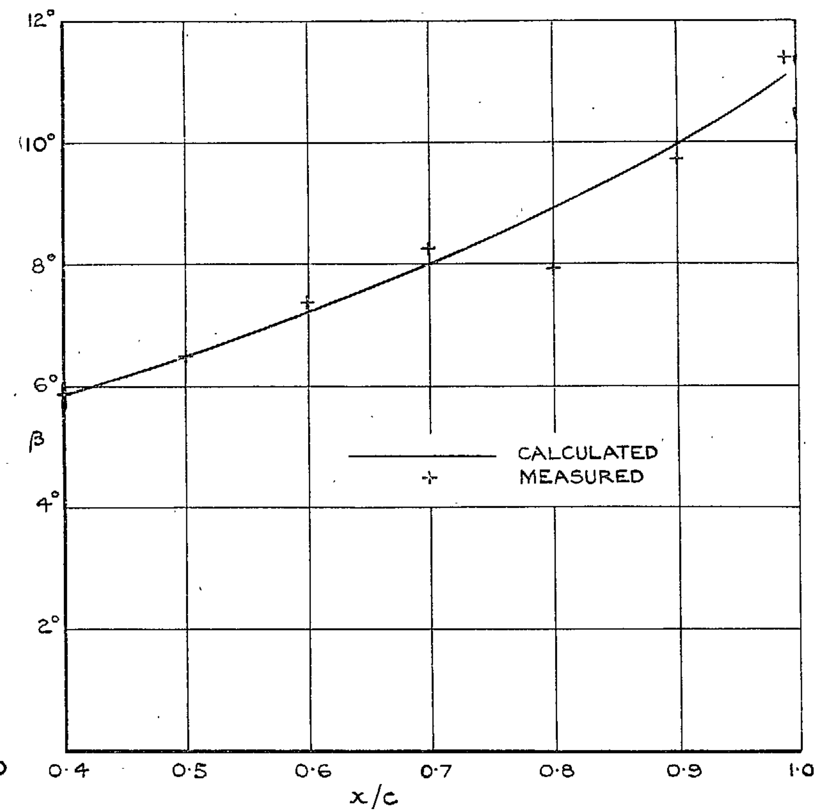

FIG. 6. Values of β, the angle between streamlines and limiting streamlines, for wing of FIG. 5.

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