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The Calculation of the Compressible Turbulent Boundary Layer in an Arbitrary Pressure Gradient—A Correlation of certain previous Methods

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COMMUNICATED BY THE DEPUTY CONTROLLER AIRCRAFT (RESEARCH AND DEVELOPMENT), MINISTRY OF SUPPLY

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Summary. The calculations of the compressible turbulent boundary-layer by Tucker (1951), Young (1953), Englert (1957), Mager (1957), Reshotko and Tucker (1957), Culick and Hill (1958), Michel (1959) and Spence (1959) have been examined, and extensions given simplifying the earlier two methods. It is shown that, for turbulent flow from the leading edge, the results of all these investigations may be expressed in the form

$$\theta = f(M) X R_X^{-b} \tag{1a}$$

where the symbols have the connotation shown in the Appendix; X is an equivalent flat plate length defined by

$$X = P^{-1} \int_0^x P\, dx \tag{1b}$$

P being a function of Mach number. The wall is assumed to be thermally insulating. For axi-symmetric flow P is replaced by Pr^{α}, where r is the radial distance from the axis and $\alpha = 1/(1 - b)$. Graphs of the functions $f(M)$ and P (*see* Figures 1 and 2) enable θ to be calculated according to any of the methods.

The average of the results may be represented to a close approximation by

$$\theta = 0 \cdot 022\, (1 + 0 \cdot 16 M^2)^{-0 \cdot 60} X R_X^{-1/6} \tag{2}$$

and

$$P = [M/(1 + M^2/5)]^4 \tag{3}$$

P being plotted in Figure 5. The Reynolds number R_X is based on the local free-stream conditions and the distance X, and may conveniently be obtained from Figure 6, or from the expression

$$R_X = UX/\nu = (a_0/\nu_0)\, XM(1 + M^2/5)^{-(3-\omega)} \tag{4}$$

where suffix 0 refers to stagnation.

Provisional working formulae are discussed in Section 12.0.

1.0. *Introduction.* Several methods are available for the calculation of the compressible turbulent boundary-layer. The earlier methods of Tucker (1951) and Young (1953) involve numerical integration, while the more recent methods of Englert (1957), Mager (1957), Reshotko and Tucker

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(1957), Culick and Hill (1958), Michel (1959) and Spence (1959) derive explicit formulae. It is found that the earlier methods may be developed analytically and that all the methods may then be correlated on the basis of Equation (1) of the Summary.

The methods of Tucker, Young and Michel start directly from the momentum equation for the compressible turbulent boundary-layer. Tucker and Young assume the profile shape to be independent of pressure gradient, a simplification which has been found useful in incompressible flow; Michel assumes a constant profile in any given flow but discriminates between favourable and adverse gradients. On the other hand, subsequent to a suggestion by Van Le (1953), the later methods use, with some empiricism, a form of the theoretical transformation of Stewartson in order to convert existing solutions of the boundary layer equation from incompressible to compressible flow. In all the methods except Mager's the variation of skin friction with Mach number is based indirectly on the limited amount of available experimental evidence. All the methods except those of Reshotko and Tucker and of Spence restrict consideration to an insulated wall, and the present paper is also thus restricted. Initial regions of laminar flow are omitted from the present paper for clarity, but may readily be treated by the standard techniques.

2.0. *Tucker's method.* Tucker shows that the momentum equation may be expressed

$$d\delta/dx + \phi\delta\, dM/dx = K\psi_{am}/X^{1/7} \tag{5}$$

where δ is the full thickness of the idealized boundary layer, K is a constant and ϕ and ψ_{am} are functions of Mach number defined in Reference 1. The quantity X is the equivalent length such that a boundary layer growing at a constant Mach number equal to the local Mach number, over a distance X, would attain the same thickness as the actual local boundary layer. Consequently $\delta \propto X^{6/7}$, i.e. $X^{1/7} \propto \delta^{1/6}$, so that, on temporarily eliminating X, the momentum equation becomes a linear differential equation in $\delta^{7/6}$ as dependent variable and either x, or M (by multiplying through by dx/dM), as independent variable. Its solution is readily obtained, and may be written in terms of M and X as

$$\theta = 0{\cdot}0153\,(1 + M^2/10)^{-5/7}\, X R_X^{-1/7} \tag{6}$$

and $$X = P^{-1}\int_0^x P\, dx$$

provided $$P = (dI/dM)^{7/6},$$

I being a function of Mach number defined and tabulated in Reference 1. Tabulations are given corresponding to power laws for the velocity profile, $u/U = (y/\delta)^{1/n}$, from $n = 5$ to $n = 11$. Variation of n has only a trivial effect on P and the value $n = 7$ has been used for Figure 2.

Tucker indicates experimental measurements in support of his results.

3.0. *Young's method.* Young integrates the momentum equation to obtain

$$\theta^{6/5} \exp \int_0^x F_2(x)dx = 0{\cdot}01054 R^{-1/5} \int_0^x G(x) \exp\left[\int_0^x F_2(x)dx\right] dx \tag{7}$$

where $$F_2(x) = 1{\cdot}2(H + 2 - M^2)\,(dU/dx)/U$$

and $$G(x) = (U/\nu)^{-1/5}h,$$

h being a function allowing for compressibility in a manner defined and discussed by Young. The value of h varies with Reynolds number, but only very slowly, and Young recommends that a single curve appropriate to the mean Reynolds number should be used throughout any given flow. Equation (7) may be put in the form

$$\theta = 0\cdot0225 h^{5/6} X R_X^{-1/6} \tag{8}$$

and

$$X = P^{-1} \int_0^x P\, dx$$

provided

$$P = M^{-1/5}(1 + M^2/5)^{19/45 h} \exp \int_1^M \frac{6}{5} \frac{(H + 2 - M^2)dM}{(1 + M^2/5)M}.$$

Values of H appropriate to the $\frac{1}{9}$th power law, as used by Young, have been adopted when calculating the function P for Figure 2.

At the higher Mach numbers Equation (8) is an extrapolation of Young's method as Young discusses his expression for h only up to $M = 5$. Values of θ/X given by Young's method at $R_X = 10^8$ become almost coincident with the mean of the other methods up to $M = 4$.

4.0. *Englert's method.* By using a form of Stewartson's transformation Englert adapts Truckenbrodt's[10] solution for the incompressible boundary layer. His formula for the momentum thickness can be rearranged as

$$\theta = 0\cdot0153\,(1 + M^2/5)^{-30/49} X R_X^{-1/7} \tag{9}$$

with

$$X = P^{-1} \int_0^x P\, dx$$

and

$$P = M^{10/3}(1 + M^2/5)^{-4\cdot0}.$$

Englert provides experimental measurements in support of his results.

5.0. *Mager's method.* As an example of his method of applying Stewartson's transformation Mager adapts Maskell's[11] solution for the incompressible boundary layer. The resulting expression for the momentum thickness can be written

$$\theta = 0\cdot0258(1 + M^2/5)^{-0\cdot645\omega} X R_X^{-0\cdot1775} \tag{10}$$

with

$$X = P^{-1} \int_0^x P\, dx$$

and

$$P = M^{4\cdot2}\,(1 + M^2/5)^{-(3+\omega)}$$

Mager's results suggest that the skin friction in compressible flow is quite sensitive to the value of ω, the exponent in the viscosity-temperature relationship. The value $\omega \doteq 0\cdot76$, as used by Mager, has been adopted for Equation (10) in Figures 1 and 2. It may be observed in passing that Mager's method appears to be less empirical than are some of the others reviewed here.

6.0. *The method of Reshotko and Tucker.* For general values of the wall temperature, Reshotko and Tucker integrate the transformed momentum equation using the technique developed by Maskell[11] for incompressible flow. They obtain

$$\theta_{tr}^{1\cdot2155} = 0\cdot01173(\nu_0/a_0)^{0\cdot2155} M_e^{-B-0\cdot2155} \int_0^x M_e^B \left(\frac{t_e}{t_{ref}}\right)^{0\cdot732} \left(\frac{t_e}{t_0}\right)^{3\cdot268} dx \tag{11}$$

θ_{tr} being the transformed momentum thickness. The other quantities are defined in Reference 5, the suffix e referring to conditions in the local free-stream. For consistency with the derivation of Equation (11) the conversion of this equation to the form of Equation (1) must be made assuming the viscosity to be proportional to temperature and the temperature of the insulated wall to be equal to the stagnation temperature. The result may then be expressed in the form

$$\theta = 0\cdot0258(1 + 0\cdot144M^2)^{-0\cdot602} (1 + M^2/5)^{-0\cdot043} XR_X{}^{-0\cdot177} \tag{12}$$

where $X = P^{-1} \int_0^x P\,dx$

and $P = M^{4\cdot2}(1 + 0\cdot144M^2)^{-0\cdot732} (1 + M^2/5)^{-3\cdot268}$.

7.0. *The method of Culick and Hill.* Culick and Hill apply a modified form of Stewartson's transformation to Truckenbrodt's solution for incompressible flow. When the parameter N in their Reynolds number exponent is put equal to 4—a value which applies to most of Figure 1—and when $\mu \propto T^{0\cdot84}$, their result becomes

$$\theta = 0\cdot0355(1 + M^2/5)^{-0\cdot432} XR_X{}^{-1/5} \tag{13}$$

where $X = P^{-1} \int_0^x P\,dx$

and $P = M^{3\cdot5} (1 + M^2/5)^{-3\cdot75}$

8.0. *Michel's method.* Michel's integration of the momentum equation yields

$$\theta = \left(\frac{L}{\phi} \frac{M^{0\cdot268}}{(1 + M^2/5)^{0\cdot566}}\right)^{0\cdot789} XR_X{}^{-0\cdot211} \tag{14}$$

where $X = P^{-1} \int_0^x P\,dx$

and $P = L$

L and ϕ being functions of Mach number defined and tabulated by Michel and taking slightly different values in accelerating and retarded flows. For a flat plate Michel suggests a simpler formula

$$\theta = 0\cdot0221(1 + 0\cdot14M^2)^{-0\cdot685} xR_x{}^{-1/6} \tag{15}$$

which illustrates his basic assumption for C_f/C_{fi}.

9.0. *The method of Spence.* For general values of the wall temperature Spence uses a modification of Stewartson's transformation to reduce the momentum equation to incompressible form. Integrations are carried out for an insulated and for a constant temperature wall. For the insulated wall his result is equivalent to

$$\theta = 0\cdot0226(1 + 0\cdot128M^2)^{-0\cdot685} XR_X{}^{-1/6} \tag{16}$$

where $X = P^{-1} \int_0^x P\,dx$

and $P = M^4(1 + M^2/5)^{-3\cdot343} (1 + 0\cdot128M^2)^{-0\cdot822}$

or, for his second set of empirical constants,

$$\theta = 0\cdot0258(1 + 0\cdot128M^2)^{-0\cdot665} XR_X{}^{-0\cdot177} \tag{17}$$

where $X = P^{-1} \int_0^x P\,dx$

and $P = M^{4\cdot2} (1 + M^2/5)^{-3\cdot368} (1 + 0\cdot128M^2)^{-0\cdot808}$.

10.0. *Axi-symmetric flow.* The six methods which consider axi-symmetric flow assume that the local skin friction coefficient for a given boundary-layer thickness is identical with that for two-dimensional flow. Reference to the original papers then shows that the function P should be replaced by Pr^{α}, where $\alpha = 1/(1 - b)$ and where $(-b)$ is the exponent of Reynolds number in the formula for boundary-layer thickness.

11.0. *Comparison between the methods.* Values of the non-dimensional momentum thickness θ/X are shown for the various methods in Figure 1, while the pressure gradient function P is plotted in Figure 2. A boundary-layer thickness if calculated by all the eight methods could be expected to show a variation about a mean of the order of $\pm$ 10 to 20 per cent, depending upon the Mach number and the steepness of the pressure gradients.

The methods all concur in the conclusion that the momentum thickness may be expressed in the form $\theta = f(M)XR_X^{-b}$ where $X = P^{-1}\int_0^x P\,dx$. Now for flow on a flat plate the Mach number, and hence the function P, would be constant, so that $X = x$ and $\theta = f(M)xR_x^{-b}$. In effect, therefore, all the methods concur in saying that the momentum thickness in a flow with pressure gradient may be obtained from the expression for a flat plate, provided the actual distance x is replaced by an equivalent distance X, X being calculated from the integral form $P^{-1}\int_0^x P\,dx$. Consequently the methods can only differ from one another in the expression for the flow on a flat plate and in the value of the function P. With the exception of Mager all the methods use empirical expressions for the flat-plate flow, such expressions being in effect data on which the method is based rather than fundamental to the method itself. The only part of the result which can differ owing to differences in the analytical treatments is therefore the function P.

The function P is of the form $M^B(T_1/T_0)^C$ or $M^B(T_1/T_0)^C\,(T_1/T_{ref})^D$ in each of the five methods using a transformation. The Mach number exponent B varies from $3\frac{1}{3}$ in Englert's method to $4\cdot2$ in the methods using Maskell's form of analysis; consequently the ratio $P/P_{1\cdot0}$ varies by a factor of $7\cdot4$ at $M = 10$ due to the variations in B. The remaining terms cause a maximum variation between the methods of $2\frac{1}{2}$ to 1 at $M = 10$. Now the various values for the exponent B have been carried over from the solutions for the incompressible boundary layer. Moreover the variation in B accounts for most of the variation in P, the remaining variations being relatively small. Until, therefore, the variations have been resolved for the incompressible flow there would seem little point in trying to assign an order of merit to the present solutions for compressible flow. The most expedient working formula for P would appear to be the simplest which represents a reasonable average of all the methods, and it is on this basis that $[M/(1 + M^2/5)]^4$ has been suggested in Equation (3).

Amongst those methods using a transformation the Prandtl number, σ, and the exponent, ω, in the viscosity temperature relationship are both given realistic values only by Culick and Hill and by Spence. Of these two methods that of Culick and Hill appears to use the more accurate form for the shape parameter H, at least on the experimental evidence available. Spence's method—along with Reshotko and Tucker—has the advantage of including the effects of heat transfer.

As an aside it is interesting to notice that the methods not using the transformation and existing solutions for incompressible flow come closer to the mean of all the methods for the function P.

The values given by the various empirical expressions adopted for the flow on a flat plate are, in effect, shown in Figures (1a) and (1b). Englert's values are probably rather low as he was concerned more with the principle of the method rather than with the accuracy of his empirical formula for skin friction. Otherwise there is probably insufficient experimental data to judge between the various curves. All the methods except that of Mager use the concept of a mean, or reference, temperature. On this concept the expression for the skin-friction coefficient is assumed to be the same function of the dynamic head and Reynolds number as in incompressible flow provided the density and viscosity are evaluated at the mean temperature in the boundary layer, or at the reference temperature. The mean or reference temperature is chosen to give values for the skin friction in agreement with experiment. The extrapolation of Young's expression to the higher Mach numbers gives higher values of skin friction, at the lower Reynolds numbers, than are given by the other methods—partly because Young has used a logarithmic expression for the skin friction in incompressible flow, whereas the other methods use power laws, and partly because Reynolds numbers evaluated at Young's particular definition of reference temperature are very low. Despite the numerical majority against Young's use of the log law it could be argued that in this respect Young's is the more logical approach. The power laws are each essentially restricted in their range of Reynolds numbers; consequently they may not be able to represent the range of reference temperature Reynolds number that corresponds, at constant R_X, to a range of Mach number from, say, 0 to 10.

Examination of the results from any of the methods shows that where dP/dM is strongly negative, *i.e.* above a Mach number of about $2 \cdot 5$, favourable pressure gradients tend to cause an increase in boundary layer thickness and adverse gradients a decrease, so that there is a reversal of behaviour from incompressible flow. However, when considering, say, the flow in a duct the quantity of interest is the ratio of the area occupied by the boundary layer to that of the duct. Calculations show that this ratio varies in the same direction as for incompressible flow with, in fact, a greater sensitivity than incompressible flow to changes of main-stream velocity—as might be expected from the reduced density in the supersonic boundary-layer.

12.0. *Provisional working formulae.* Equations (2) and (3) of the Summary are close to an average of the methods. In finding the average, Young's result has been included for the function P, but in the interests of simplicity, not for θ/X; in the former Young's value for $R = 10^7$ has been selected as representative of his method. For a given value of X values of θ resulting from Equation (2) are within about 1 per cent of the average from the seven methods at all Mach numbers up to 10. The function P of Equation (3) is within about 3 per cent of the average from the eight methods at supersonic speeds up to $M = 5$, but at higher Mach numbers the scatter excludes an accurate assessment. The function P is shown plotted in Figure 2, but the results from Equation (2) have been omitted from Figure 1 for clarity.

The value of $C_f/C_{f,i}$, i.e., the ratio of the skin friction in compressible flow to that in incompressible flow at the same Reynolds number R_X, for flow on a flat plate, is given by Equation (2) to be $(1 + 0 \cdot 16M^2)^{-0 \cdot 60}$. In comparison with the experimental results—for example, as analysed by Chapman and Kester (1954)—this quantity is about 0 to 10 per cent low up to a Mach number of 3, and 0 to 15 per cent low between Mach numbers of 3 and 5. In comparison with the experimental mean line proposed by Chapman and Kester it is less than 1 per cent low at $M = 2$, 5 per cent at $M = 3$ and 11 per cent at $M = 5$. The broader limits have been suggested, however, in order to take some account of the experimental scatter and of the difference, mentioned by Chapman and Kester, between wind-tunnel tests and flight.

Equations (2) to (4), with (1b), could be taken as provisional working formulae for the compressible turbulent boundary-layer.

When θ is known the displacement thickness δ^* and the full idealized boundary-layer thickness δ may be determined from the ratios δ^*/θ and θ/δ. In mild pressure gradients there is little error in assuming these ratios to be the same as for flow on a flat plate. Now as mentioned, for example, by Cope (1951) the velocity profile on a flat plate appears to be almost the same in compressible as in incompressible flow; consequently the use of the ratios given by, say, the $\frac{1}{7}$th power law, see Figures 3 and 4, enable δ^* and δ to be calculated. When the displacement thickness δ^* is required in a strong pressure gradient, the ratio δ^*/θ should be found taking into account the change in the shape of the velocity profile of the boundary layer resulting from the pressure gradient, as is possible using Englert's, Mager's or Reshotko and Tucker's method.

It will be found that the use of the $\frac{1}{7}$th power law in conjunction with Equation (2) gives values of the thickness δ which become 10 per cent lower than for incompressible flow at $M = 4$, and 10 per cent higher at $M = 10$. This is in broad agreement with experimental observations, which have suggested that δ is almost independent of Mach number[14, 15], and there is probably insufficient evidence to determine the behaviour to closer than 10 per cent. Consequently an alternative approach to the production of a set of provisional working formulae is to assume zero variation of δ with Mach number, so that an accepted formula for incompressible flow may be used—for example, as given in Reference 16—and to derive θ and δ^* from δ using algebraic expressions for the ratios θ/δ and δ^*/δ arranged to fit the values given by the $\frac{1}{7}$th power law. The following set of formulae may then be obtained.

For free-stream Reynolds numbers R_X of the order of 10^6,

$$\left.\begin{aligned} \delta &= 0\cdot 37 X R_X^{-1/5} \\ \theta &= 0\cdot 036(1 + M^2/10)^{-0\cdot 70} X R_X^{-1/5} \\ \delta^* &= 0\cdot 046(1 + 0\cdot 8M^2)^{0\cdot 44} X R_X^{-1/5} \end{aligned}\right\} \qquad (18a$$

For free-stream Reynolds numbers of the order of 10^7,

$$\left.\begin{aligned} \delta &= 0\cdot 23 X R_X^{-1/6} \\ \theta &= 0\cdot 022(1 + M^2/10)^{-0\cdot 70} X R_X^{-1/6} \\ \delta^* &= 0\cdot 028(1 + 0\cdot 8M^2)^{0\cdot 44} X R_X^{-1/6} \end{aligned}\right\} \qquad 18b)$$

where

$$X = P^{-1} \int_0^x P\, dx \qquad (1b)\text{bis}$$

$$P = [M/(1 + M^2/5)]^4 \qquad (3)\text{bis}$$

and where R_X may be obtained from Figure 6 (compiled with the aid of Reference 17) or from the expression

$$R_X = UX/\nu = (a_0/\nu_0)\, XM(1 + M^2/5)^{-(3-\omega)} \qquad (4)\text{bis}$$

suffix 0 referring to stagnation. The quantity ω is the exponent in the viscosity-temperature relation $\mu \propto T^\omega$, a reasonable value for ω being $0\cdot 75$, except at very high temperatures, where $\omega = \frac{1}{2}$.

For axi-symmetric flow

$$\left.\begin{aligned} \alpha &= 5/4 \text{ when } R_X \sim 10^6 \\ \alpha &= 6/5 \text{ when } R_X \sim 10^7 \end{aligned}\right\}. \qquad (19)$$

The equations for δ and δ^* would be applicable only in mild or zero pressure gradients, while that for θ could be used generally.

Values of θ/X from Equation (18) are shown in Figure 1, while the ratios θ/δ, etc. are compared with values from the power laws in Figures 3 and 4. Clearly the expression for δ^* would not be satisfactory above $M = 10$.

The value of $C_f/C_{f,i}$ implicit in Equation (18) is $(1 + M^2/10)^{-0\cdot70}$, which is about 0 to 5 per cent higher than the band of experimental results up to $M = 3$, and within about $\pm$ 10 per cent of the experimental values between $M = 3$ and $M = 5$. In comparison with the experimental mean line proposed by Chapman and Kester the value of $C_f/C_{f,i}$ implicit in Equation (18) is correct almost to within 5 per cent up to $M = 5$.

When the boundary-layer thickness is required at only a single point the equations quoted, *i.e.* Equations (2) or (18), with (1b) and (3), are in a suitable form for the calculation. However, when values are required at several positions along a surface the calculation would be simplified by rearrangement. Equation (2) with (1b), (3) and (4) can be rearranged as

$$\theta = gQ^{5/6}\,L(a_0L/\nu_0)^{-1/6} \tag{20}$$

where

$$Q = \int_0^x Pd(x/L), \tag{21}$$

where L is any reference length, and P and g are functions which may be plotted on a large scale, for direct computational reference, and are shown in Figure 5. Actually P is the function of Equation (3), while

$$g = 0\cdot022\,(1 + 0\cdot16M^2)^{-0\cdot60}\,(1 + M^2/5)^{3\cdot71}\,M^{-3\cdot5} \tag{22}$$

The quantity a_0L/ν_0 may for convenience be derived from Figure 7. Equation (18) similarly yields:—
for free-stream Reynolds numbers R_X of the order of 10^6,

$$\left.\begin{aligned}\delta &= g_1Q^{4/5}L(a_0L/\nu_0)^{-1/5}\\ \theta &= g_2Q^{4/5}L(a_0L/\nu_0)^{-1/5}\\ \delta^* &= g_3Q^{4/5}L(a_0L/\nu_0)^{-1/5}\end{aligned}\right\} \tag{23a}$$

for free-stream Reynolds number of the order of 10^7

$$\left.\begin{aligned}\delta &= g_4Q^{5/6}L(a_0L/\nu_0)^{-1/6}\\ \theta &= g_5Q^{5/6}L(a_0L/\nu_0)^{-1/6}\\ \delta^* &= g_6Q^{5/6}L(a_0L/\nu_0)^{-1/6}\end{aligned}\right\} \tag{23b}$$

where the g functions are shown in Figures 8 and 9,

where

$$Q = \int_0^x Pd(x/L) \tag{21bis}$$

$$P = \{M/(1 + M^2/5)\}^4 \tag{3bis}$$

P being shown in Figure 5, while L is any reference length and the quantity a_0L/ν_0 may be derived from Figure 7.

The recommendations for provisional working formulae are therefore as follows. If θ is the only thickness required, Equations (2) or (20) may be used, P being taken from Equation (3). If δ^* or δ are required, and if the pressure gradients are not very steep, Equations (18) or (23) are rather more convenient, P again being taken from Equation (3). These equations are, of course, also suitable for calculating θ. In strong pressure gradients θ should still be given adequately by the above equations but δ^* should be calculated using the ratio δ^*/θ as obtained taking into account the change in shape of the velocity profile of the boundary layer resulting from the pressure gradient, as is possible using Englert's, Mager's or Reshotko and Tucker's method. Equations (2) and (18) are suitable for calculating values at a single point; Equations (20) and (23) are more suitable when values are required at several positions along a surface.

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LIST OF SYMBOLS

x	Distance from the leading edge
X	An equivalent distance defined by Equation (1b) and discussed in Section 2.0
L	A reference length (also, a specialised function in Michel's method only)
y	Distance from the surface
r	Radial distance from the axis
U	Velocity in the mainstream
u	Velocity in the boundary layer
M	Mach number in the mainstream
ρ_1	Density in the mainstream
ρ	Density in the boundary layer
δ	Thickness of the idealized boundary layer: $u/U = (y/\delta)^{1/n}$
θ	Momentum thickness of the boundary layer, i.e. $\theta = \int_0^\infty (\rho u/\rho_1 U)(1 - u/U)dy$
δ^*	Displacement thickness of the boundary layer, i.e. $\delta^* = \int_0^\infty (1 - \rho u/\rho_1 U)dy$
T_1	Static temperature in the mainstream
T_0	Stagnation temperature
ν	Kinematic viscosity in the mainstream
ν_0	Kinematic viscosity at stagnation
ω	Exponent in the viscosity-temperature relation $\mu \propto T^\omega$; a reasonable value for ω is $0{\cdot}75$, except at very high temperatures, where $\omega = \frac{1}{2}$
a_0	Velocity of sound
p_0	Stagnation pressure
R_X	Reynolds number UX/ν
R'_X	Value of R_X when $X = 1$ ft and $p_0 = 1$ atmosphere (data taken from Reference 17)
R_0'	Value of a_0L/ν_0 when $L = 1$ ft and $p_0 = 1$ atmosphere (data taken from Reference 17)
f, P	Functions of Mach number defined by Equations (1a) and (1b)
$-b$	Reynolds number exponent in Equation (1a)
α	$1/(1 - b)$
g, Q	Functions of Mach number defined by Equations (20) to (22) etc.

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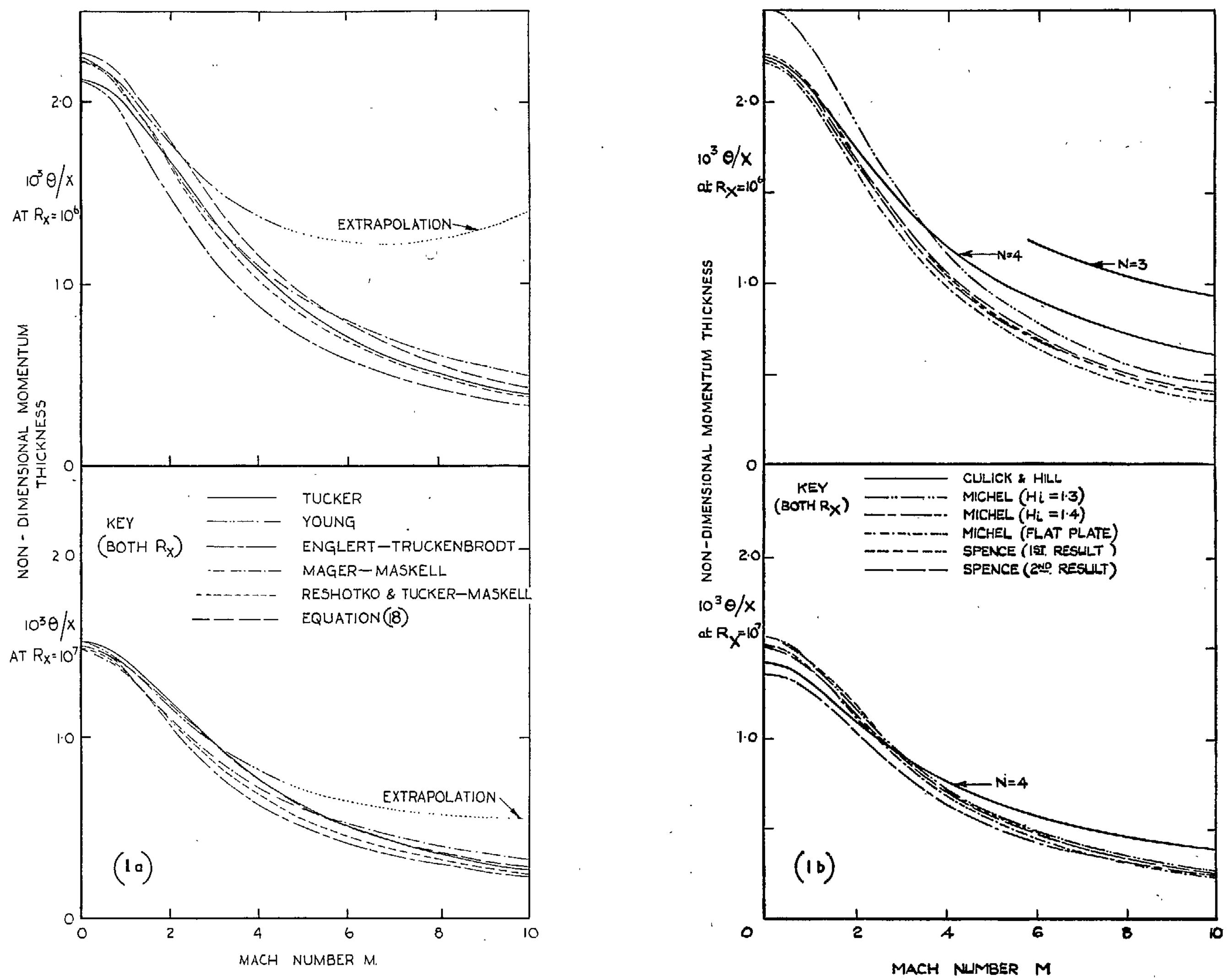

FIG. 1. The momentum thickness θ at Reynolds numbers R_X of 10^6 and 10^7.

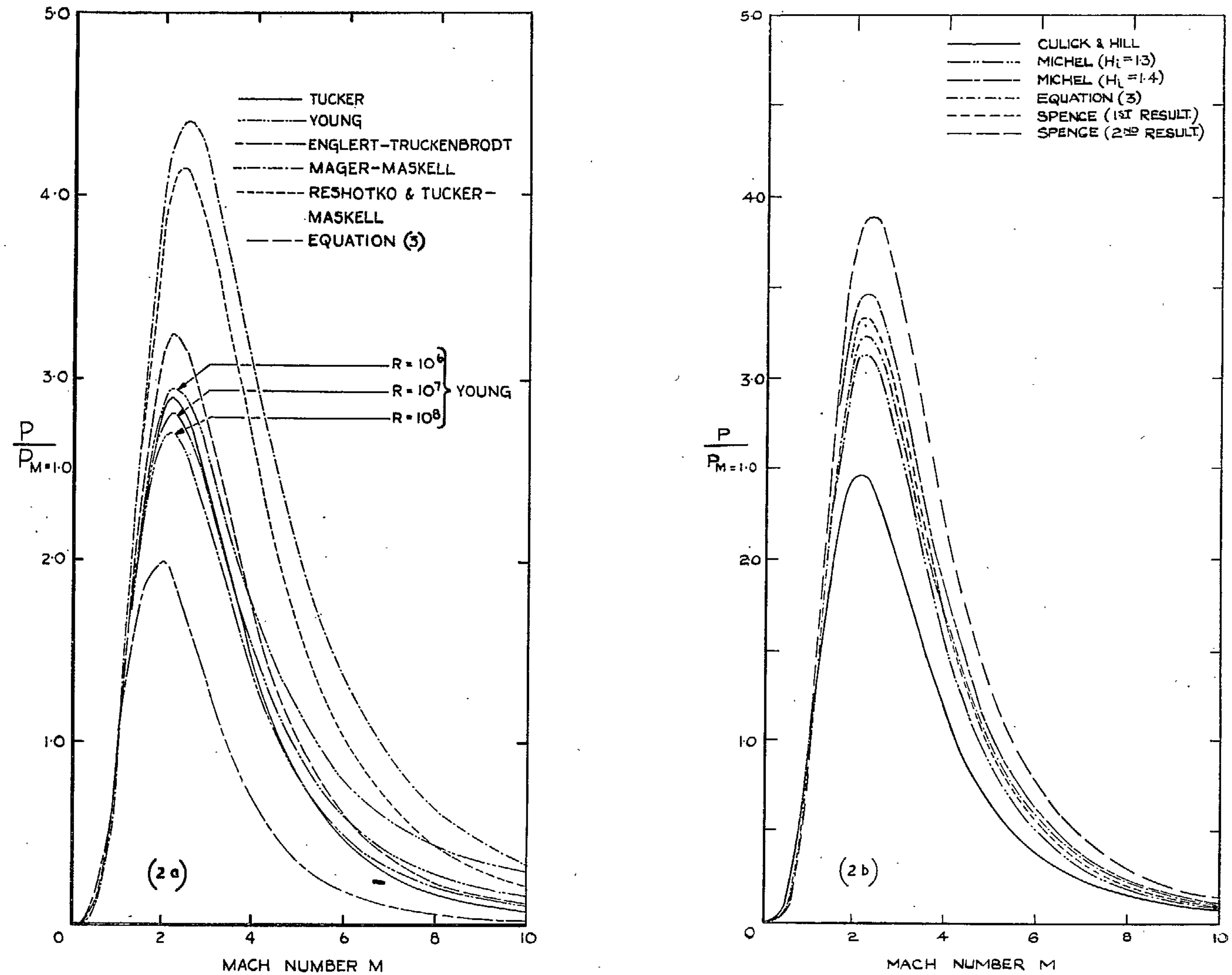

FIG. 2. The pressure gradient function P.

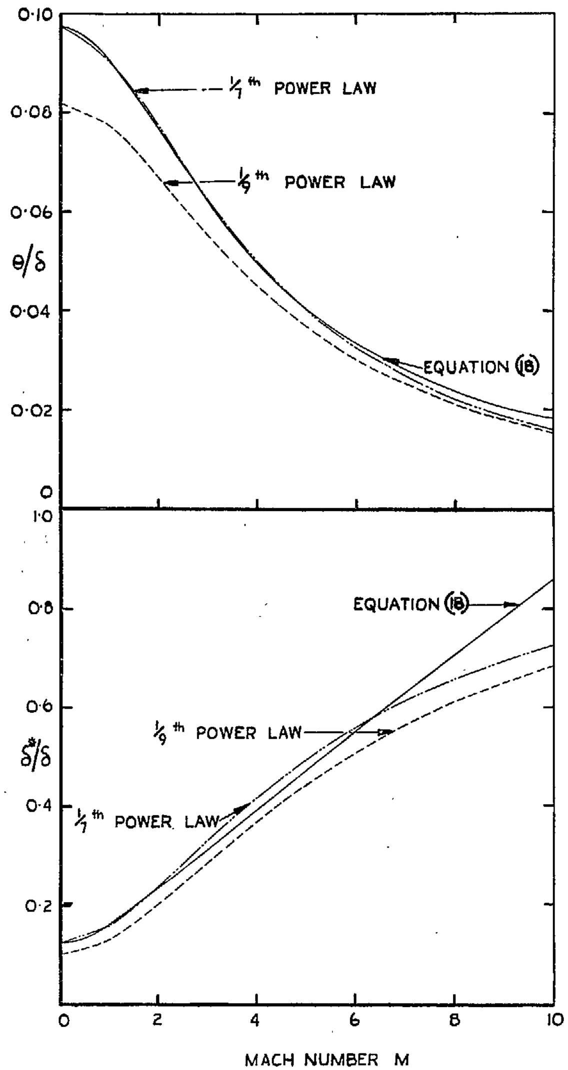

FIG. 3. Profile shape parameters θ/δ and δ^*/δ.

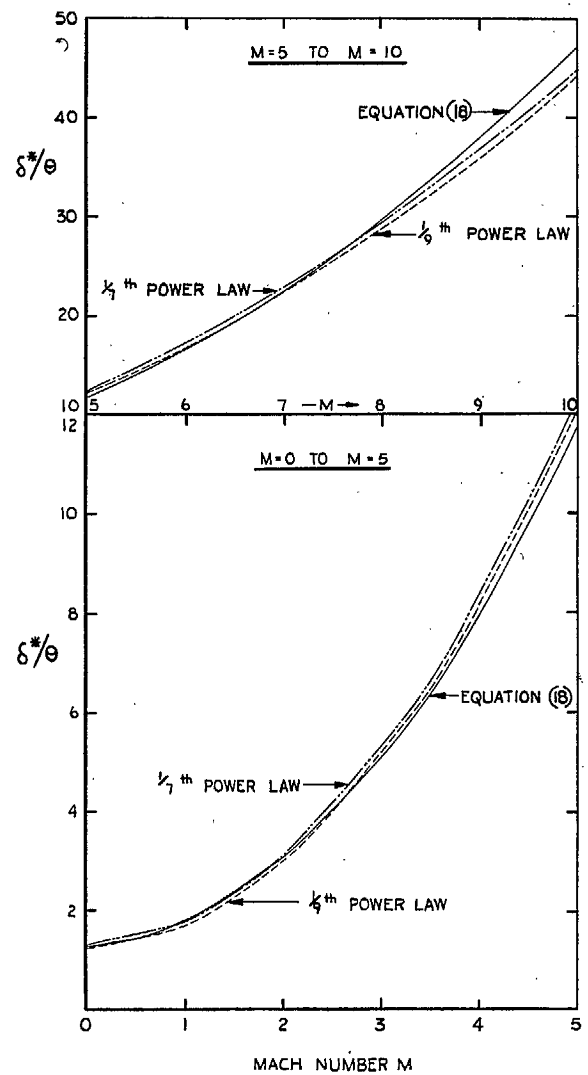

FIG. 4. Profile shape parameter δ^*/θ.

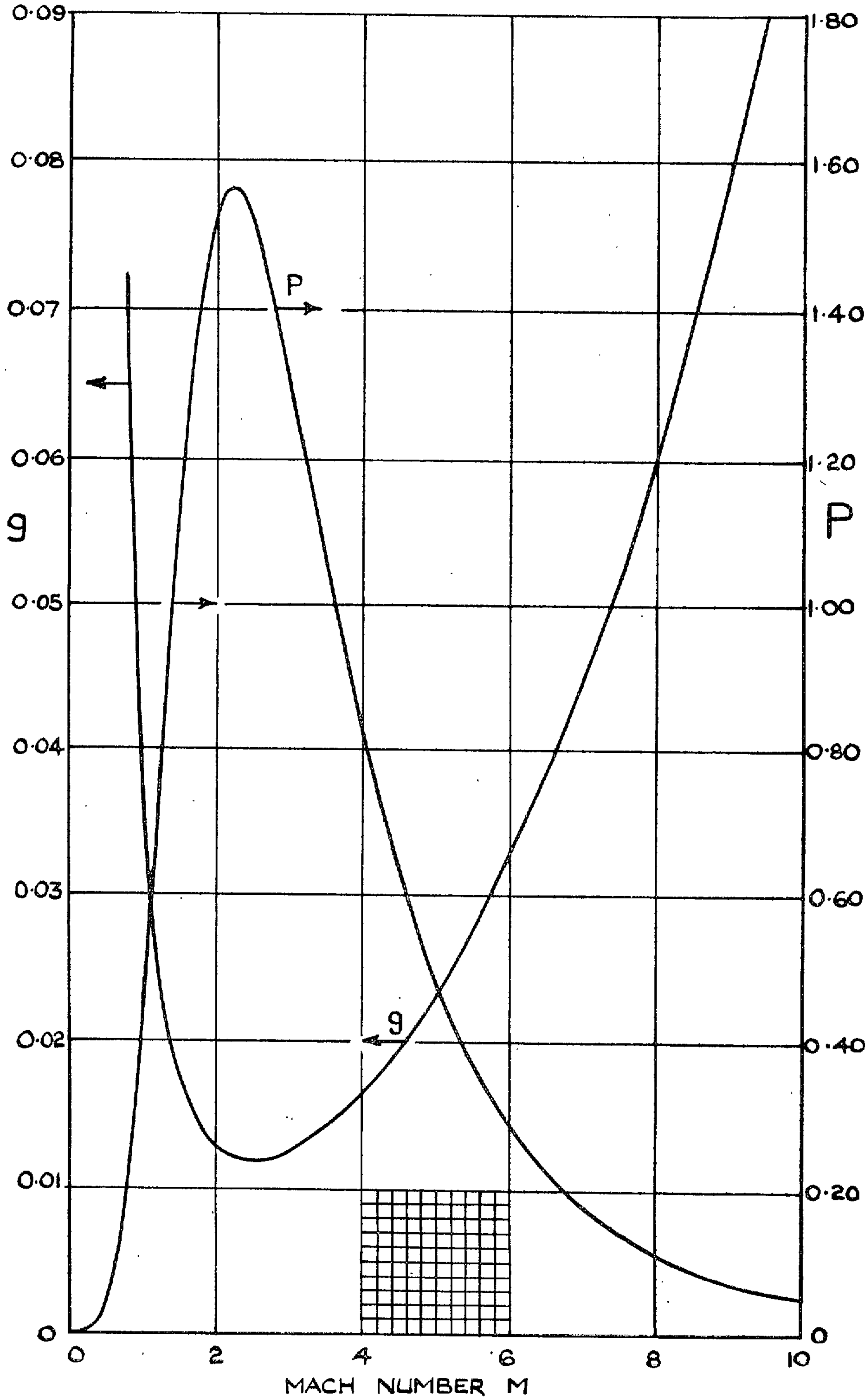

FIG. 5. The functions g and P.

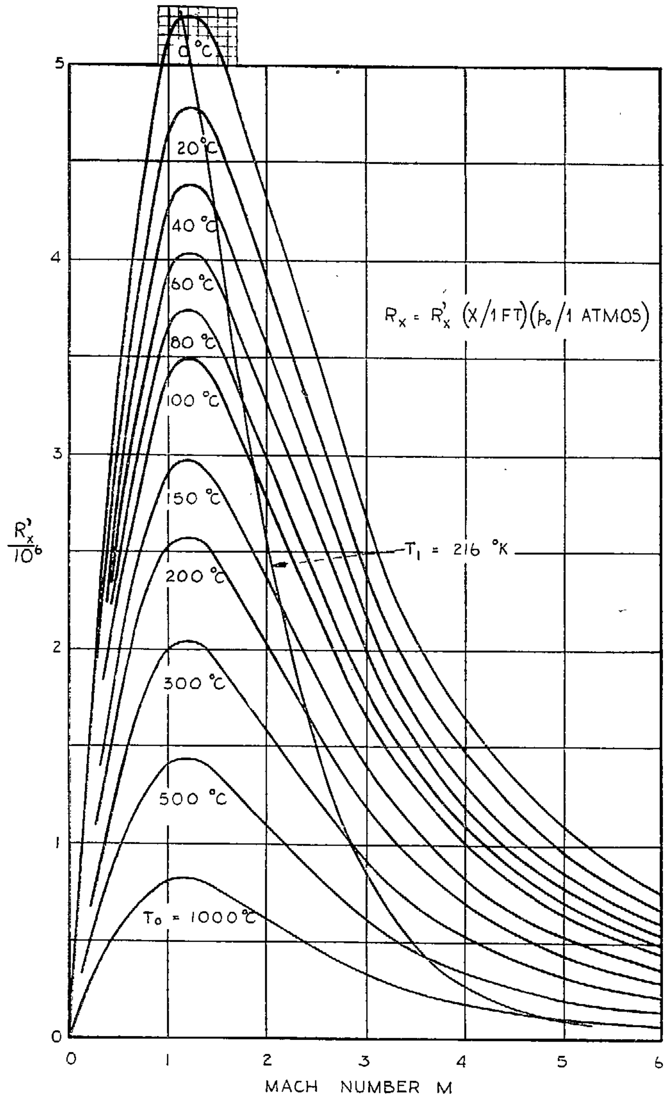

Fig. 6. The Reynolds number function R'_X.

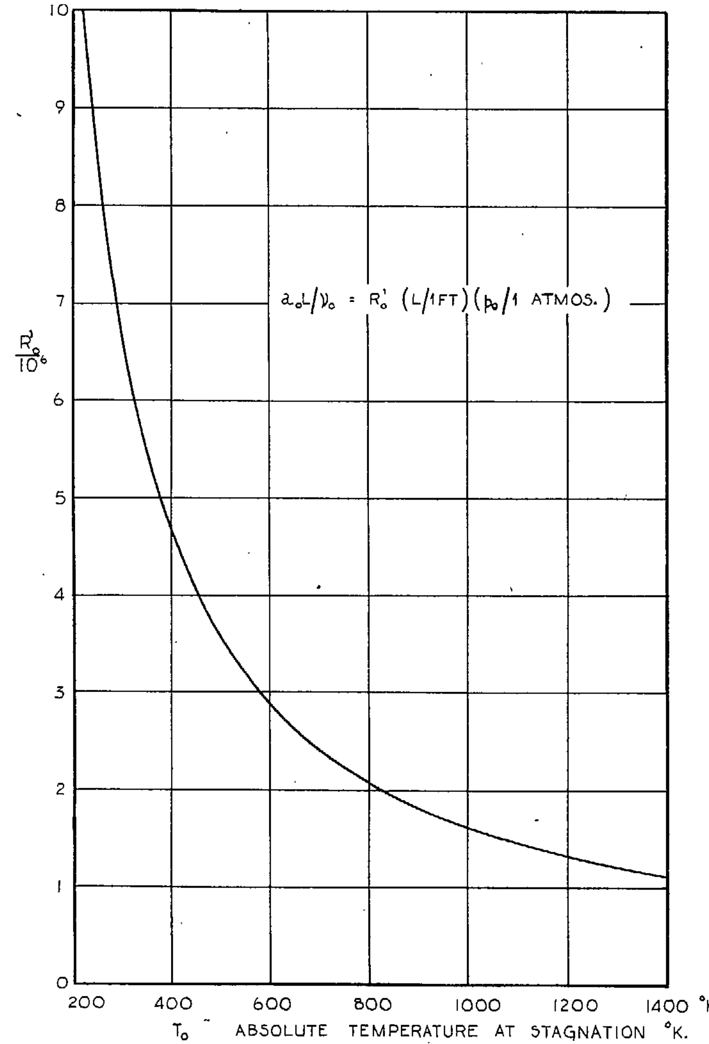

Fig. 7. The Reynolds number function R'_0.

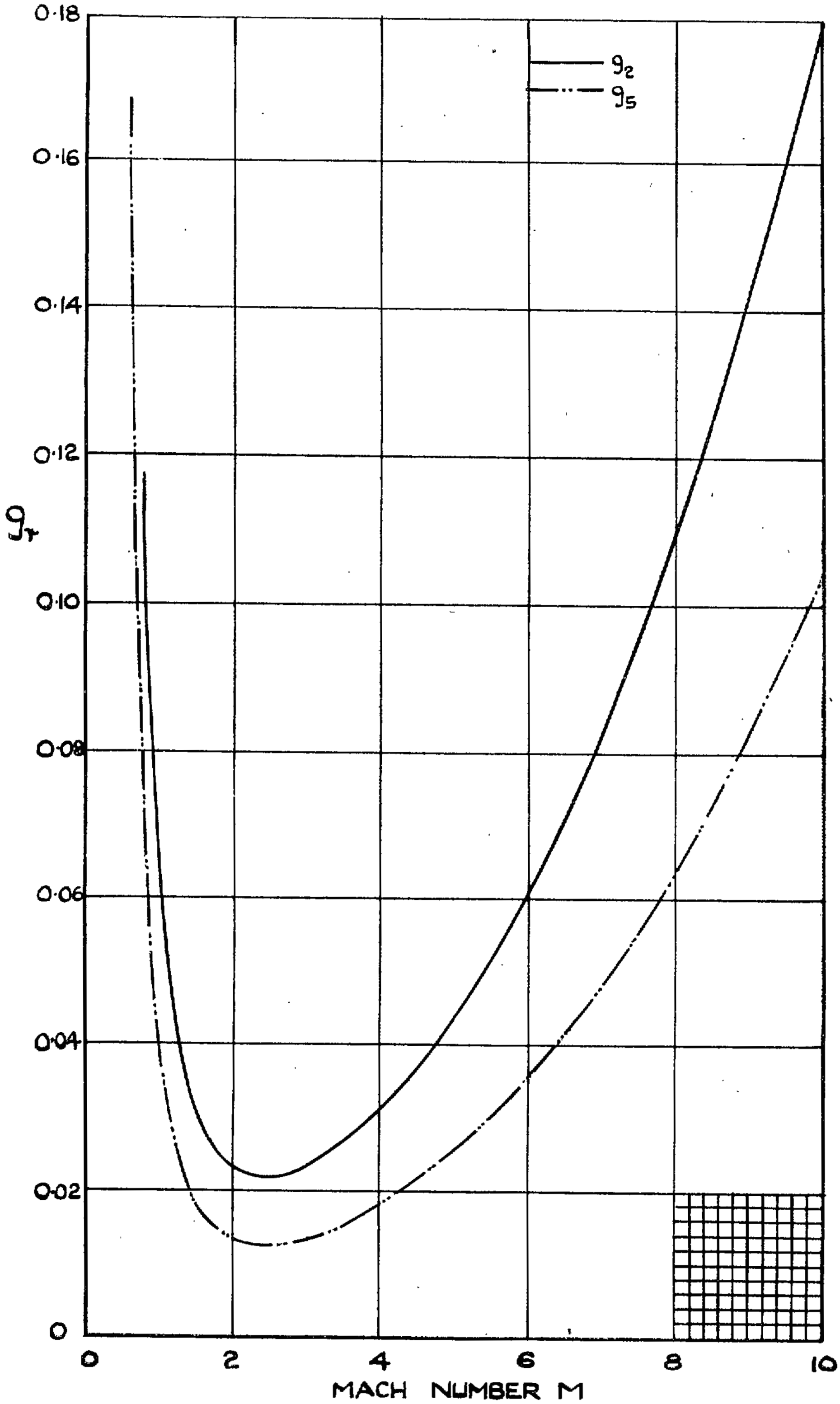

FIG. 8. The functions g_2 and g_5.

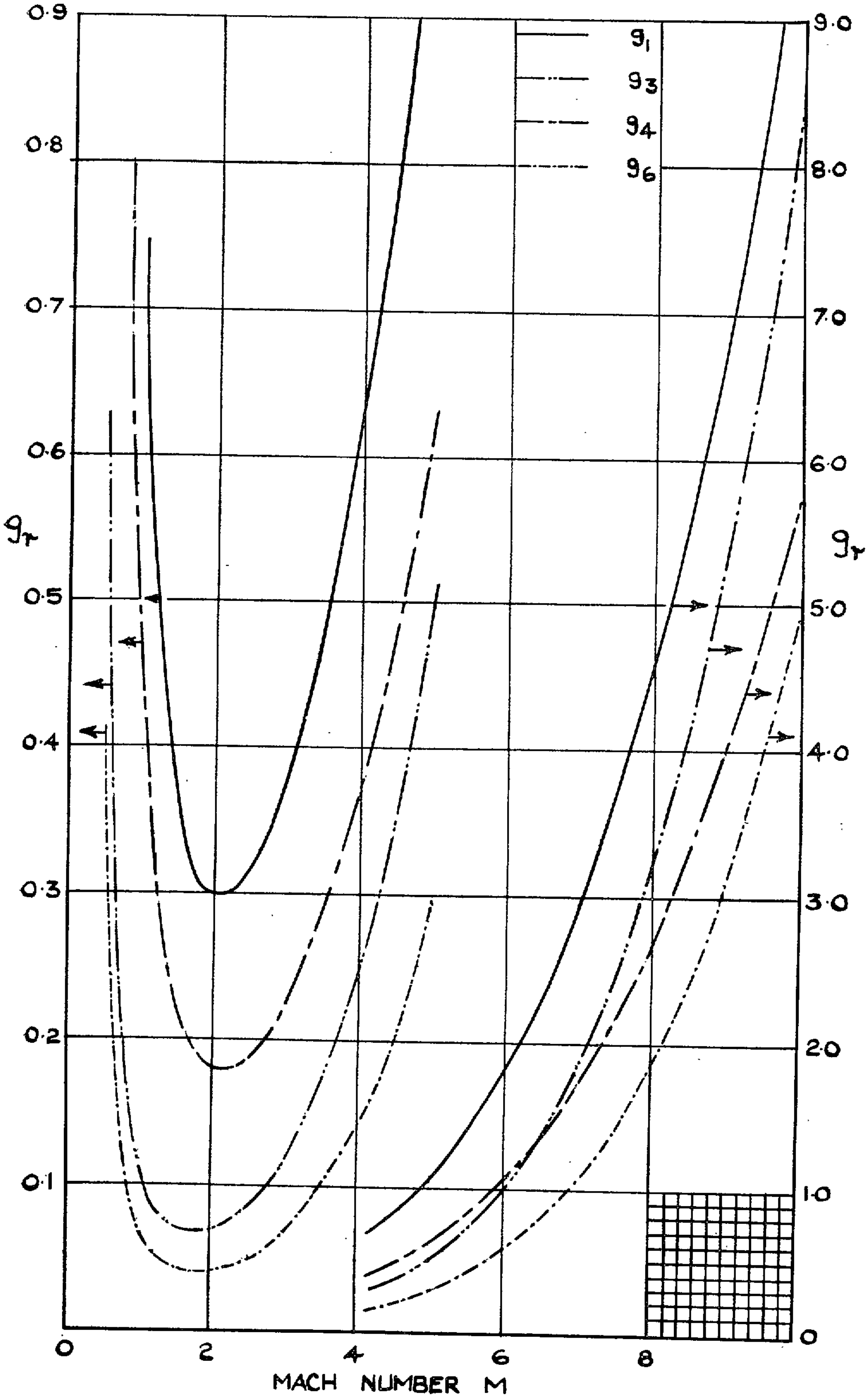

FIG. 9. The functions g_1, g_3, g_4 and g_6.

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