R. & M. No. 3302

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

Some Mathematical Methods in Three-Dimensional Subsonic Flutter-Derivative Theory

By D. E. WILLIAMS

LONDON: HER MAJESTY'S STATIONERY OFFICE
1962
TWELVE SHILLINGS NET

Some Mathematical Methods in Three-Dimensional Subsonic Flutter-Derivative Theory

By D. E. WILLIAMS

COMMUNICATED BY THE DEPUTY CONTROLLER AIRCRAFT (RESEARCH AND DEVELOPMENT), MINISTRY OF AVIATION

*Reports and Memoranda No. 3302**
February, 1961

Summary. This report gives a survey of some of the methods currently used for evaluating the flutter derivatives for three-dimensional wings oscillating in subsonic flow.

Introduction. In this report are given some of the methods used in three-dimensional subsonic flutter-derivative theory. It is hoped that it will provide a basis for the reading of specific papers on the subject.

Two proofs are given of the basic integral equation: the first proof uses co-ordinates fixed in space, the second uses co-ordinates fixed in the wing. Analytical solutions of the integral equation can be found for very few planforms and in these particular cases the problem is more easily solved by starting from the differential equation. Nevertheless, for the sake of completeness, and to be consistent with the rest of the report a method is given for obtaining an analytical solution of the integral equation. The rest of the report deals with the numerical solution of the integral equation. Since Gaussian integration is fundamental to the method of solution it is explained in some detail. Two methods of chordwise integration and two methods of spanwise integration are applied to the solution of the integral equation. Computational details are not given since they vary from author to author and a reader interested in a particular variant of the general method of solution is referred to the relevant papers.

Several methods exist for evaluating incompressible derivatives but these have now been superseded by the methods used for compressible flow and so no account of them is given.

No attempt has been made to give an historical survey of the subject. The references give the places where results may conveniently be found; they are not necessarily original papers.

1. *Basic Equations.* In this Section we shall list the basic equations of hydrodynamics which we shall need. The equations are derived in Garrick[1] and Temple[2].

When the axes are fixed in space the perturbation velocity potential satisfies the differential equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \qquad (1)$$

where c is the velocity of sound in the medium.

* Previously issued as R.A.E. Tech. Note Math. 75—A.R.C. 23,168.

When the equation is referred to a set of axes $0x'y'z'$ moving with velocity V along the negative x-axis, then, since

$$x' = x + Vt\,, \qquad y' = y\,, \qquad z' = z\,, \qquad t = t$$

the equation becomes

$$(1-M^2)\frac{\partial^2\Phi}{\partial x'^2} + \frac{\partial^2\Phi}{\partial y'^2} + \frac{\partial^2\Phi}{\partial z'^2} - \frac{2M}{c}\frac{\partial^2\Phi}{\partial x'\partial t} - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad (2)$$

In fixed co-ordinates the pressure p, the acceleration potential ϕ and the velocity potential Φ are connected by the equations

$$-\frac{p}{\rho_\infty} = \phi = \frac{d\Phi}{dt} \qquad (3)$$

where ρ_∞ is the density of the undisturbed medium. In moving co-ordinates the equation is

$$-\frac{p}{\rho_\infty} = \phi = \frac{\partial\Phi}{\partial t} + V\frac{\partial\Phi}{\partial x'}.$$

The acceleration potential also satisfies equations (1) and (2).

2. *The Derivation of the Basic Integral Equation.* *Proof A* [*Richardson*[3]]. The source solution of equation (1) is

$$4\pi\phi = -\frac{H(t-r/c)}{r}$$

where H is the Heaviside step function, r is the distance between the observation point and the source, t is the time measured from the instant of the disturbance and c is the speed of sound in the medium. The observation point is affected at time t if the disturbance moving with velocity c can traverse the distance r in that time; the strength of the disturbance is then $-1/4\pi r$. Derivations of this formula are given in Morse and Feshbach[4].

Let $0\ x\ y\ z$ be co-ordinate axes fixed in space and let $0\ \xi\ \eta\ \zeta$ be axes fixed on a wing moving with the velocity V along the negative x-axis. The time is chosen so that the axes coincide at time $t = 0$. The co-ordinates of the point $(\xi, \eta, \zeta{:}t)$ referred to the fixed axes are $(\xi - Vt, \eta, \zeta{:}t)$.

A unit acceleration-potential source at the point $(\xi - Vt, \eta, 0)$ will induce an acceleration potential at the point (x, y, z) after some time t' and the magnitude of the potential will be $-1/4\pi r$ where

$$r^2 = (x-\xi+Vt)^2 + (y-\eta)^2 + z^2\,.$$

A doublet of strength $\Delta\phi(\xi, \eta{:}t)$ will induce a potential $\phi(x, y, z{:}t)$ of magnitude

$$-\Delta\phi(\xi, \eta{:}t)\frac{1}{4\pi}\frac{\partial}{\partial z}\left(\frac{1}{r}\right).$$

From equation (3) the velocity potential Φ and the acceleration potential ϕ are connected by the relation

$$\Phi(x, y, z{:}t) = \int_{-\infty}^{t} \phi(x, y, z{:}t)dt$$

and so the velocity potential at $(x, y, z{:}0)$ due to the moving doublet at (ξ, η) is

$$\Phi(x, y, z{:}0) = -\frac{1}{4\pi}\int_{-\infty}^{t_0} \Delta\phi(\xi, \eta{:}t)\frac{\partial}{\partial z}\left(\frac{1}{r}\right)dt$$

where t_0 is the last instant of time for which the moving doublet can affect the observation point. This time t_0 is easily obtained from the equation

$$r = ct,$$

i.e.,

$$[x - \xi + Vt_0]^2 + (y - \eta)^2 + z^2 = c^2 t_0^2 .$$

If we write $x - \xi = X$, $y - \eta = Y$, $z = Z$ this becomes

$$(X + Vt_0)^2 + Y^2 + Z^2 = c^2 t_0^2 .$$

If we put $V = Mc$ and rearrange we get

$$(1 - M^2)(ct_0)^2 - 2MX(ct_0) - (X^2 + Y^2 + Z^2) = 0$$

which gives

$$ct_0 = \frac{MX \pm \sqrt{\{M^2X^2 + (1 - M^2)(X^2 + Y^2 + Z^2)\}}}{(1 - M^2)}$$

$$= \frac{MX \pm R}{1 - M^2}$$

where $R^2 = X^2 + (1 - M^2)Y^2$. Since t_0 must be negative we take the negative sign and obtain

$$ct_0 = \frac{MX - R}{1 - M^2} .$$

If we write $X + Vt = \tau$ then

$$\tau_0 = X + Vt_0$$

$$= X + \frac{M(MX - R)}{1 - M^2}$$

$$= \frac{X - MR}{1 - M^2} .$$

The velocity potential at $(x, y, z, 0)$ due to the moving doublet at (ξ, η) then becomes

$$-\frac{1}{4\pi}\frac{1}{V}\int_{-\infty}^{\tau_0} \Delta\phi\left(\xi, \eta : \frac{\tau - X}{V}\right) \frac{\partial}{\partial z} \frac{1}{\sqrt{(\tau^2 + Y^2 + Z^2)}}\, d\tau .$$

The total velocity potential at $(x, y, z:0)$ is then given by the equation

$$4\pi\Phi(x, y, z:0) = -\frac{1}{V}\iint_S d\xi\, d\eta \int_{-\infty}^{\tau_0} \Delta\phi\left(\xi, \eta : \frac{\tau - X}{V}\right) \frac{\partial}{\partial z} \frac{1}{\sqrt{(\tau^2 + Y^2 + Z^2)}}\, d\tau$$

where the double integral is taken over the wing area.

Since the downwash $w(x, y)$ is given by

$$w(x, y) = \left[\frac{\partial\phi}{\partial z}\right]_{z=0}$$

we get the integral equation

$$4\pi w(x, y) = -\frac{1}{V}\iint_S d\xi\, d\eta \int_{-\infty}^{\tau_0} \Delta\phi\left(\xi, \eta : \frac{\tau - X}{V}\right) \left[\frac{\partial^2}{\partial z^2} \frac{1}{\sqrt{(\tau^2 + Y^2 + Z^2)}}\right]_{z=0} d\tau .$$

We now assume that the motion is periodic with frequency p and make all lengths non-dimensional with respect to some length s. We shall use the same symbols for the dimensional and non-dimensional lengths but this should cause no confusion. We shall also replace w and $\Delta\phi$ by non-dimensional equivalents.

We write

$$w(x, y{:}t) = VW(x, y) \exp(ipt)$$

and

$$\Delta\phi(x, y{:}t) = \Delta\phi(x, y) \exp(ipt)$$

$$= -\frac{\Delta p(x, y)}{\rho} \exp(ipt).$$

If we write $\Delta p = \rho V^2 \Gamma$ where Γ is a non-dimensional pressure, we get

$$\Delta\phi = -V^2\Gamma(x, y) \exp(ipt).$$

The integral equation then becomes

$$4\pi W(x, y) = \iint_S \Gamma(\xi, \eta) K[X, Y] \, d\xi \, d\eta$$

where

$$K[X, Y] = \lim_{z \to 0} \int_{-\infty}^{(X - MR)(1-M^2)} \frac{\partial^2}{\partial z^2} \left[\frac{\exp\{i\omega(\tau - X)\}}{\sqrt{(\tau^2 + Y^2 + Z^2)}} \right] d\tau$$

and $\omega = ps/V$.

When the differentiation is performed the kernel takes the form

$$K[X, Y] = \frac{M(MX+R)}{R(X^2+Y^2)} \exp\{iM\Omega(MX-R)\} + \exp(-i\omega x) \int_{(-X+MR)(1-M^2)}^{\infty} \frac{\exp(-i\omega\tau)}{(\tau^2+Y^2)^{3/2}} \, d\tau$$

where

$$\Omega = \omega/(1-M^2).$$

Proof B. The differential equation satisfied by the acceleration potential referred to co-ordinates fixed in the wing is

$$(1-M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{2M}{c}\frac{\partial^2\phi}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0.$$

If we now assume that $\phi = \phi(x, y, z) \exp(ipt)$ and make all lengths non-dimensional with respect to some length s then the equation becomes

$$(1-M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - 2i\omega M^2 \frac{\partial\phi}{\partial x} + M^2\omega^2\phi = 0$$

where $\omega = ps/V$ and the same symbols have been used for the dimensional and non-dimensional co-ordinates.

If we now write

$$\phi = \exp(iM^2\Omega x)\phi^*(x, y, z)$$

where $\Omega = \omega/(1-M^2)$ the differential equation becomes

$$(1-M^2)\frac{\partial^2\phi^*}{\partial x^2} + \frac{\partial^2\phi^*}{\partial y^2} + \frac{\partial^2\phi^*}{\partial z^2} + \frac{M^2\omega^2}{1-M^2}\phi^* = 0.$$

We now write

$$x' = x, \quad y' = \sqrt{(1-M^2)}y, \quad z' = \sqrt{(1-M^2)}z$$

and the equation becomes

$$\frac{\partial^2\phi^*}{\partial x'^2} + \frac{\partial^2\phi^*}{\partial y'^2} + \frac{\partial^2\phi^*}{\partial z'^2} + k^2\phi^* = 0 \qquad (4)$$

where $k = M\Omega$.

We now have to solve this equation for ϕ^* when $(\partial\phi^*/\partial z')_{z'=0}$ is known over the transformed wing.

We shall here for the sake of convenience drop the asterisk and the dashes and replace them when they are needed.

To solve the equation we use Green's identity

$$\iiint_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,dv = \iint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] dS.$$

We take ϕ to be the solution we are seeking and take for ψ the 'elementary' solution of equation (4)

$$\psi = \frac{\exp(-ikr)}{r}$$

where

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2.$$

We apply this identity to the region bounded by a sphere of large radius R and the surface formed by the upper and lower surfaces of the wing.

Since

$$(\nabla^2+k^2)\left\{\frac{\exp(-ikr)}{r}\right\} = -4\pi\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta),$$

(Morse and Feshbach[4]) the left-hand side of the identity becomes

$$-4\pi\phi(x, y, z).$$

The integral over the sphere of radius R is equal to

$$\int_0^\pi d\theta \int_0^{2\pi} R^2 \sin\theta \left\{\phi\frac{\partial\psi}{\partial R} - \psi\frac{\partial\phi}{\partial R}\right\} d\varphi$$

where θ, φ are the Euler angles. The potential ϕ behaves like $\{\exp(-ikr)\}/r$ for large r and so its value on the sphere can be given by a series of the form

$$\phi = \frac{\exp(-ikR)}{R}\{a_0(\theta, \varphi) + a_1(\theta, \varphi)/R + 0(1/R^2)\}.$$

Since $\psi = \{\exp(-ikr)\}/r$ is approximately equal to $\{\exp(-ikR)\}/R$ for large R it can easily be seen that

$$\left\{\phi\frac{\partial\psi}{\partial R} - \psi\frac{\partial\phi}{\partial R}\right\} = 0(1/R^3).$$

The integral is then $0(1/R)$ and so tends to zero as $R \to \infty$.

If therefore we denote the values taken by ϕ on the upper and lower surfaces of the wing by ϕ_a and ϕ_b respectively we get

$$-4\pi\phi(x, y, z) = \iint_S \left[(\phi_a-\phi_b)\frac{\partial}{\partial\zeta}\left\{\frac{\exp(-ikr)}{r}\right\} - \left\{\frac{\exp(-ikr)}{r}\right\}\frac{\partial}{\partial\zeta}(\phi_a-\phi_b)\right] d\xi\, d\eta.$$

Since ϕ is an odd function of z we have

$$\frac{\partial}{\partial\zeta}(\phi_a-\phi_b) = 0$$

and so since $\frac{\partial}{\partial\zeta}\left\{\frac{\exp(-ikr)}{r}\right\} = \frac{\partial}{\partial z}\left\{\frac{\exp(-ikr)}{r}\right\}$ we have

$$-4\pi\phi(x, y, z) = \iint_S (\phi_a-\phi_b)\frac{\partial}{\partial z}\left\{\frac{\exp(-ikr)}{r}\right\} d\xi\, d\eta.$$

If we revert to the original notation we get

$$-4\pi\phi^*(x, y, z) = \iint_S (\phi_a^* - \phi_b^*) \frac{\partial}{\partial z}\left\{\frac{\exp(-ikr)}{r}\right\} d\xi\, d\eta$$

where

$$r^2 = (x-\xi)^2 + (1-M^2)[(y-\eta)^2 + z^2].$$

To obtain the velocity potential from the acceleration potential we use the relation

$$V \exp(i\omega x)\Phi(x, y, z) = \int_{-\infty}^{x} \exp(i\omega x)\phi(x', y, z)dx'$$

or the equivalent relation

$$V \exp(i\omega x)\Phi(x, y; z) = \int_{-\infty}^{x} \exp(i\Omega x')\phi^*(x', y, z)dx'.$$

We then have

$$-4\pi V \exp(i\omega x)\Phi(x, y, z) = \int_{-\infty}^{x} \exp(i\Omega x)dx \iint_S \Delta\phi^*(\xi, \eta) \frac{\partial}{\partial z}\left\{\frac{\exp(-ikr)}{r}\right\} d\xi\, d\eta$$

where we have written

$$\Delta\phi^* = \phi_a^* - \phi_b^*.$$

Since the downwash is given by

$$w(x, y) = \left[\frac{\partial\Phi}{\partial z}(x, y, z)\right]_{z=0}$$

the integral equation becomes, if we write $w(x, y) = VW(x, y)$ and $\Delta\phi^* = -V^2\Gamma^*$

$$4\pi \exp(i\omega x)W(x, y) = \int_{-\infty}^{x} \exp(i\Omega x)dx \iint_S \Gamma^*(\xi, \eta)\bar{K}[x-\xi, y-\eta]d\xi\, d\eta$$

where

$$\bar{K}[x-\xi, y-\eta] = \left[\frac{\partial^2}{\partial z^2}\left\{\frac{\exp(ikr)}{r}\right\}\right]_{z=0}.$$

If we replace Γ^* by $\{\exp(-iM^2\Omega\xi)\}\Gamma$ the integral equation becomes, after some rearrangement

$$4\pi W(x, y) = \iint_S \Gamma(\xi, \eta)K[x-\xi, y-\eta]d\xi\, d\eta$$

where

$$K = \exp(-i\omega x - iM^2\Omega\xi) \lim_{z\to 0} \frac{\partial^2}{\partial z^2}\int_{-\infty}^{x} \frac{\exp(i\Omega x' - ikr)}{r}dx'$$

$$= \exp\{-i\omega(x-\xi)\} \lim_{z\to 0} \frac{\partial^2}{\partial z^2}\int_{-\infty}^{x} \frac{\exp\{i\Omega(x'-\xi) - ikr\}}{r}dx'.$$

If in the integral we make the transformation

$$(1-M^2)\tau = (x'-\xi) - Mr$$

the integral takes the form given by Richardson[3].

3. *The Analytic Solution of the Integral Equation.* 3.1. Before we discuss the numerical methods of solving the integral equation we shall give a short discussion of an analytical method of solution. The method will however only give solutions when the wing planform is simple, for example, an elliptic wing or an infinite strip wing, but for these specific cases the problem is more easily solved by starting with the differential equation because the special functions needed are usually defined by means of differential equations and not by integral equations.

3.2. The integral equation which connects the downwash $VW(x, y)$ and the modified pressure $\rho V^2\Gamma(\xi, \eta)$ is of the form

$$\exp(i\omega x) W(x, y) = -\int_{-\infty}^{x} \exp(i\Omega x) dx \iint_S K[x, \xi : y, \eta] \Gamma(\xi, \eta) d\xi \, d\eta \tag{5}$$

where the double integral is taken over the wing surface S and the kernel* $K[x, \xi : y, \eta]$ is such that

$$K[x, \xi : y, \eta] = K[\xi, x : \eta, y] .$$

We can split up the integral equation into two simpler equations

$$\exp(i\omega x) W(x, y) = -\int_{-\infty}^{x} \exp(i\Omega x) G(x, y) dx \tag{6}$$

and

$$G(x, y) = \iint_S K[x, \xi : y, \eta] \Gamma(\xi, \eta) d\xi \, d\eta . \tag{7}$$

From equation (6) we get

$$\begin{aligned} \exp(i\Omega x) G(x, y) &= -\frac{d}{dx}[\exp(i\omega x) W(x, y)] \\ &= -\exp(i\omega x)\left[\frac{dW}{dx} + i\omega W\right] \\ &= -\exp(i\omega x) A(x, y) \end{aligned} \tag{8}$$

where $A(x, y)$ is the normal acceleration.

3.3. To solve equation (6) we need to know the eigenvalues and eigenfunctions of the homogeneous equation, i.e., numbers λ_n and functions $\phi_n(x, y)$ for which

$$\phi_n(x, y) = \lambda_n \iint_S K[x, \xi : y, \eta] \phi_n(\xi, \eta) d\xi \, d\eta .$$

Since the kernel is symmetric these eigenvalues and eigenfunctions will exist and the set of eigenfunctions will be complete. The functions are orthogonal, i.e.,

$$\iint_S \phi_m(x, y) \phi_n(x, y) dx \, dy = 0 \qquad m \neq n$$

and they can be chosen so that

$$\iint_S \phi_n^2(x, y) dx \, dy = 1 .$$

Let

$$G(x, y) = \sum_1^\infty G_n \phi_n(x, y)$$

where

$$G_n = \iint_S G(x, y) \phi_n(x, y) dx \, dy$$

and let

$$\Gamma(\xi, \eta) = \sum_1^\infty \Gamma_n \phi_n(\xi, \eta) .$$

---

* The K here is different from the K of Para. 3, it is in fact a constant multiple of $\bar{K}$. The precise form of K is not needed in this Section and so this should cause no confusion.

If we substitute these equations into equation (7) we get

$$\sum_1^\infty G_n\phi_n(x, y) = \iint_S K[x, \xi:y, \eta] \sum_1^\infty \Gamma_n\phi_n(\xi, \eta)d\xi\, d\eta$$

$$= \sum_1^\infty \frac{\Gamma_n\phi_n(x, y)}{\lambda_n},$$

i.e.,

$$\Gamma_n = \lambda_n G_n$$

and so

$$\Gamma = \sum_1^\infty \lambda_n G_n \phi_n(\xi, \eta).$$

We shall call this solution Γ_1.

To obtain this function Γ_1, we have used not the downwash $W(x, y)$ but the acceleration $A(x, y)$ and, as we shall see below, this series when substituted into equation (5) gives not the original downwash $W(x, y)$ but a downwash $W(x, y) + \alpha(y) \exp(-i\omega x)$ where α is a function of y. It is obvious from equation (6) that any downwash of the form $W(x, y) + \alpha(y) \exp(-i\omega x)$ will give the same value of $G(x, y)$. To remove this extra term we add to Γ_1 a singular solution Γ_0, a solution which gives G to be zero over the wing. Since the choice of this singular solution presents some difficulty we shall now discuss in general terms the choice of this singular function.

3.4. *The Singular Solution.* Since the set of functions $\{\phi_n(x, y)\}$ is complete the equation

$$\sum_1^\infty \phi_n(x, y)\phi_n(\xi, \eta) = \delta(x-\xi)\delta(y-\eta) \qquad (9)$$

is satisfied. By using this result we see that the series

$$\Gamma = \sum_1^\infty \lambda_n\phi_n(\xi, \eta)\phi_n(x', y') \qquad (10)$$

is a singular solution for

$$\iint_S \Gamma(\xi, \eta)K[x, \xi:y, \eta]d\xi\, d\eta = \sum_1^\infty \lambda_n\phi_n(x', y') \iint_S \phi_n(\xi, \eta)K[x, \xi:y, \eta]d\xi\, d\eta$$

$$= \sum_1^\infty \phi_n(x', y')\phi_n(x, y)$$

$$= \delta(x-x')\delta(y-y').$$

It is also easy to see that

$$\Gamma = \sum_1^\infty \lambda_n\phi_n(\xi, \eta)\frac{\partial}{\partial x'}\phi_n(x', y') \qquad (11)$$

or any other function obtained by differentiating (10) with respect to the variables x' and y' is a singular solution. It is difficult to give precise mathematical reasons why one singular solution should be chosen rather than another but we shall try to justify the choice of (11) on physical grounds by considering the case of a control surface in two-dimensional steady flow.

If the leading edge of the control surface is at $x = 0$ and the wing profile is such that

$$Z = H(x)$$

where $H(x)$ is the Heaviside step function, then the downwash is

$$W(x) = \delta(x)$$

and the acceleration is

$$A(x) = \delta'(x).$$

The leading edge of a wing behaves locally in the same way as the leading edge of a control surface with aerodynamic balance and so the acceleration must have a $\delta'(x)$ singularity at the leading edge of the wing.

The singular solution

$$\Gamma_0 = \sum_1^\infty \lambda_n \phi_n(\xi, \eta) \left[\frac{\partial}{\partial x'} \phi_n(x', y')\right]_{x'=x_{\mathrm{LE}}(y')}$$

will give zero acceleration over the wing and the correct singularity at one point of the leading edge.

The singular solution

$$\int \alpha(y') \left\{ \sum_1^\infty \lambda_n \phi_n(\xi, \eta) \left[\frac{\partial}{\partial x'} \phi_n(x', y')\right]_{x'=x_{\mathrm{LE}}(y')} \right\} dy'$$

where the integration is taken from wing tip to wing tip and $\alpha(y')$ is some function still to be determined, will therefore give zero acceleration over the wing and the correct singularity at all points of the leading edge.

3.5. *The Determination of* $\alpha(y)$. The complete solution is therefore

$$\Gamma(\xi, \eta) = \int \Gamma_0(\xi, \eta{:}y')\alpha(y')dy' + \Gamma_1(\xi, \eta).$$

Let

$$\bar{G}(x, y) = \iint_S K[x, \xi{:}y, \eta] \left\{ \int \Gamma_0(\xi, \eta{:}y')\alpha(y')dy' + \Gamma_1(\xi, \eta)\right\} d\xi\, d\eta$$
$$= \int \alpha(y')\bar{G}_0(x, y{:}y')dy' + \bar{G}_1(x, y).$$

When the point (x, y) is on the wing

$$\bar{G}(x, y) = G(x, y).$$

The downwash $\bar{W}(x, y)$ induced by this pressure is then given by the equation

$$\exp(i\omega x)\bar{W}(x, y) = -\int_{-\infty}^{x} \exp(i\Omega x)\bar{G}(x, y)dx$$
$$= -\int_{-\infty}^{x_{\mathrm{LE}}(y)} \exp(i\Omega x)\bar{G}(x, y)dx - \int_{x_{\mathrm{LE}}(y)}^{x} \exp(i\Omega x)\bar{G}(x, y)dx.$$

When the point (x, y) is on the wing by using (6) we see that the equation can be written in the form

$$\exp(i\omega x)\bar{W}(x, y) = A(x, y) + \int \alpha(y')B(x, y{:}y')dy' +$$
$$+ \exp(i\omega x)W(x, y) - [\exp(i\omega x)W(x, y)]_{x=x_{\mathrm{LE}}(y)}$$

where

$$A(x, y) = -\int_{-\infty}^{x_{\mathrm{LE}}(y)} \exp(i\Omega x)\bar{G}_1(x, y)dx$$

and

$$B(x, y{:}y') = -\int_{-\infty}^{x_{\mathrm{LE}}(y)} \exp(i\Omega x)\bar{G}_0(x, y{:}y')dx.$$

Therefore $W = \bar{W}$ if

$$A(x, y) + \int \alpha(y')B(x, y:y')dy' = [\exp(i\omega x)W(x, y)]_{x=x_{\mathrm{LE}}(y)}$$

which is an integral equation for the function $\alpha(y')$.

A detailed application of the method to an infinite-strip wing is given by Williams[5]. The differential equation for the two-dimensional case was used by Timman and van de Vooren[6]. Küssner[7] has in addition given results for elliptic wings but his singular solution is believed to be incorrect (van de Vooren[8]).

4. *Methods of Integration and the Choice of Downwash Points.* In solving the integral equation Gaussian integration will be used. An account of this method will first be given: it follows very closely the account given by Mineur[9].

4.1. We wish to find an integration formula for the integral

$$\int_a^b w(x)f(x)dx \tag{12}$$

in the form

$$\int_a^b w(x)f(x)dx = \sum_1^n H_\alpha f(x_\alpha) \tag{13}$$

where $w(x)$ is a weight function, so that the formula is exact for a polynomial $f(x)$ of as high a degree as possible. Since we have $2n$ unknowns $x_1, \ldots, x_n, H_1, \ldots, H_n$ the highest order is $(2n-1)$.

We can calculate these constants by putting $f(x) = x^r$, $r = 0, \ldots, (2n-1)$ and solving the equations which result. There is however a simple method.

Let $\{G_r(x)\}$ be a set of polynomial functions which are orthogonal with respect to the weight function $w(x)$, i.e.,

$$\int_a^b w(x)G_r(x)G_s(x)dx = 0 \qquad r \neq s. \tag{14}$$

We assume that the formula (13) is exact for a polynomial of order $(2n-1)$. Then if $\Phi(x)$ is a polynomial of order $(n-1)$, $G_n(x)\Phi(x)$ is a polynomial of order $(2n-1)$ and we have

$$\int_a^b w(x)G_n(x)\Phi(x)dx = \sum_1^n H_\alpha G_n(x_\alpha)\Phi(x_\alpha).$$

But $\Phi(x)$ can be expressed as a linear function of the polynomials $G_0(x), \ldots, G_{n-1}(x)$ and so by the orthogonality conditions (14) the integral is zero and we have

$$\sum_1^n H_\alpha G_n(x_\alpha)\Phi(x_\alpha) = 0.$$

The polynomial $\Phi(x)$ has been chosen arbitrarily and so we must have

$$G_n(x_\alpha) = 0 \qquad \alpha = 1, \ldots, n.$$

The points to be chosen are then the zeros of $G_n(x)$ and we have

$$G_n(x) = \lambda \prod_{\alpha=1}^n (x - x_\alpha)$$

where λ is some constant.

We now want to calculate H_α. The function

$$\frac{G_n(x)}{x - x_\alpha}$$

is a polynomial of order $(n-1)$ with zeros $x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_n$ and so, using (13) we get

$$\int_a^b w(x)\, \frac{G_n(x)}{x - x_\alpha}\, dx = H_\alpha G_n'(x_\alpha),$$

i.e.,

$$H_\alpha = \frac{1}{G'(x_\alpha)} \int_a^b w(x)\, \frac{G_n(x)}{x - x_\alpha}\, dx\,.$$

4.2. From the zeros of $G_n(x)$ we can form Lagrangian interpolation polynomials $\{g_r^*(x)\}$ of order $(n-1)$ with the property

$$g_r^*(x_s) = \delta_{rs}.$$

These polynomials are also orthogonal with respect to $w(x)$, for if $r \neq s$ we can write

$$g_r^*(x) g_s^*(x) = G_n(x) P_{n-2}(x)$$

where P_{n-2} is a polynomial of order $(n-2)$, and we have

$$\int_a^b w(x) g_r^*(x) g_s^*(x) dx = \int_a^b w(x) G_n(x) P_{n-2}(x) dx$$

$$= 0\,.$$

It is convenient to construct the functions

$$g_r(x) = \frac{w(x)}{w(x_r)}\, g_r^*(x)\,.$$

These obviously have the property that

$$g_r(x_s) = \delta_{rs}.$$

The orthogonality property then becomes

$$\int_a^b g_r(x) g_s^*(x) dx = 0 \qquad r \neq s.$$

We shall use such functions later as interpolation functions.

4.3. *Particular Weight Functions.* We shall always transform our co-ordinates so that the range of integration is $(-1, 1)$. We shall consider three kinds of weight functions.

(A)

$$w(x) = \frac{1}{\sqrt{(1-x^2)}}.$$

The orthogonal functions now satisfy the equation

$$\int_{-1}^{+1} \frac{G_m(x) G_n(x)}{\sqrt{(1-x^2)}}\, dx = 0 \qquad m \neq n\,.$$

The functions are therefore the Chebyshev polynomials $T_n(x)$. If we write $x = \cos\theta$, the integral becomes

$$\int_0^\pi G_m(\theta)G_n(\theta)d\theta = 0$$

and so we choose $G_n(\theta)$ to be a multiple of $\cos n\theta$, since this function, unlike $\sin n\theta$ is a polynomial in $\cos\theta$.

We then have

$$T_n(x) = \frac{1}{2^{n-1}}\cos n(\cos^{-1}x)$$

where the factor $1/2^{n-1}$ has been chosen to make the coefficient of x^n unity.

We then have

$$\int_{-1}^{+1}\frac{T_n^2(x)}{\sqrt{(1-x^2)}}\,dx = \frac{1}{2^{n-2}}\int_0^\pi \cos^2 n\theta\,d\theta = \frac{\pi}{2^{2n-1}}.$$

The first few polynomials are

$$T_0(x) = 2$$
$$T_1(x) = x$$
$$T_2(x) = \tfrac{1}{2}(2x^2-1).$$

The integration points are the roots of $T_n(x)$, i.e., the points for which $\cos n\theta = 0$. These are the points

$$n\theta = \frac{\pi}{2} + r\pi$$

$$\theta = \frac{(2r+1)}{n}\frac{\pi}{2}.$$

Since

$$\frac{dT_n(x)}{dx} = -\frac{1}{\sin\theta}\frac{dT_n(\theta)}{d\theta} = \frac{n}{2^n}\frac{\sin n\theta}{\sin\theta}.$$

we have

$$H_\alpha = \frac{\sin\theta_\alpha}{n\sin n\theta_\alpha}\int_0^\pi\frac{\cos n\theta}{\cos\theta-\cos\theta_\alpha}$$

$$= \frac{\pi}{n}.$$

The integration formula then becomes

$$\int_{-1}^{+1}\frac{f(x)}{\sqrt{(1-x^2)}}\,dx = \frac{\pi}{n}\sum_{\alpha=0}^{n-1} f(x_\alpha).$$

(B)

$$w(x) = \sqrt{(1-x^2)}.$$

The orthogonal functions $\{G_n(x)\}$ satisfy the equation

$$\int_{-1}^{+1}\sqrt{(1-x^2)}\,G_m(x)G_n(x)dx = 0 \qquad m \neq n$$

and so are the Chebyshev polynomials $U_n(x)$. If we write $x = \cos\theta$ the integral becomes

$$\int_0^\pi \sin^2\theta G_n(\theta)G_m(\theta)d\theta = 0$$

and so $U_n(\theta)$ is taken to be a multiple of $\sin(n+1)\theta/\sin\theta$. We take

$$U_n(\theta) = \frac{1}{2^n}\frac{\sin(n+1)\theta}{\sin\theta},$$

the coefficient of x^n will then be unity. The first few polynomials are

$$U_0(x) = 1$$

$$U_1(x) = x$$

$$U_2(x) = x^2 - \tfrac{1}{4}.$$

The integration points are the roots of $U_n(x)$ and so are the points

$$\theta_\alpha = \frac{\alpha\pi}{n+1}.$$

It can easily be shown that

$$H_\alpha = \frac{\pi}{n+1}(1-x_\alpha^2).$$

The integration formula then becomes

$$\int_{-1}^{+1}\sqrt{(1-x^2)}f(x)dx = \sum_1^n B_\alpha f(x_\alpha)$$

where

$$B_\alpha = \frac{\pi}{(n+1)}(1-x_\alpha^2).$$

We shall call the interpolation function associated with this weight function $\{g_r(x)\}$.

(C)

$$w(x) = \sqrt{\left(\frac{1-x}{1+x}\right)}.$$

The polynomials satisfy the equation

$$\int_{-1}^{+1}\sqrt{\left(\frac{1-x}{1+x}\right)}\,G_m(x)G_n(x)dx = 0.$$

If we write $x = -\cos\theta$ this becomes

$$\int_0^\pi(1+\cos\theta)G_m(\theta)G_n(\theta)\,d\theta = 0$$

or

$$\int_0^\pi\left[\cos\frac{\theta}{2}G_m(\theta)\right]\left[\cos\frac{\theta}{2}G_n(\theta)\right]d\theta = 0.$$

If we take

$$\cos\frac{\theta}{2}G_n(\theta) = \cos(2n+1)\frac{\theta}{2}$$

the integral becomes

$$\int_0^\pi\cos(2n+1)\frac{\theta}{2}\cos(2m+1)\frac{\theta}{2} = 4\int_0^\pi\cos(2m+1)\theta\cos(2n+1)\theta\,d\theta$$

$$= 0.$$

Therefore

$$G_n(\theta) = \frac{\cos(2n+1)(\theta/2)}{\cos(\theta/2)}$$

or, since $x = -\cos\theta$

$$G_n(x) = \frac{\cos(2n+1)\left[\cos^{-1}\sqrt{\left(\frac{1-x}{2}\right)}\right]}{\cos\left[\cos^{-1}\sqrt{\left(\frac{1-x}{2}\right)}\right]}$$

$$= 2^{2n}\frac{T_{2n+1}\left\{\sqrt{\left(\frac{1-x}{2}\right)}\right\}}{\left\{\sqrt{\left(\frac{1-x}{2}\right)}\right\}}.$$

The integration points are the points for which

$$(2n+1)\frac{\theta}{2} = \frac{\pi}{2} + \alpha\pi,$$

i.e.,

$$\theta = \frac{(2\alpha+1)}{(2n+1)}\pi.$$

The coefficient H_α is given by

$$H_\alpha = \frac{1}{G_n'(x_\alpha)}\int_{-1}^{+1}\sqrt{\left(\frac{1-x}{1+x}\right)}\frac{G_n(x)}{x-x_\alpha}dx$$

$$= -\frac{1}{G_n'(x_\alpha)}\int_0^\pi \frac{(1+\cos\theta)}{\cos\theta-\cos\theta_\alpha}\frac{\cos(2n+1)(\theta/2)}{\cos(\theta/2)}d\theta$$

$$= -\frac{1}{G_n'(x_\alpha)}\int_0^\pi \frac{\cos(n+1)\theta+\cos n\theta}{\cos\theta-\cos\theta_\alpha}$$

$$= -\frac{1}{G_n'(\theta_\alpha)}\pi\frac{\sin(n+1)\theta_\alpha+\sin n\theta_\alpha}{\sin\theta_\alpha}$$

$$= -\frac{1}{G_n'(\theta_\alpha)}\pi\frac{\sin(2n+1)(\theta_\alpha/2)}{\sin(\theta_\alpha/2)}.$$

Now

$$\left[\frac{dG_n(x)}{dx}\right]_{x=x_\alpha} = -\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\frac{\cos(2n+1)(\theta/2)}{\cos(\theta/2)}\right\}\right]_{\theta=\theta_\alpha}$$

$$= \frac{(2n+1)}{2}\frac{\sin(2n+1)(\theta_\alpha/2)}{\sin\theta_\alpha\cos(\theta_\alpha/2)}.$$

Therefore

$$H_\alpha = \frac{2\pi}{2n+1}2\cos^2(\theta_\alpha/2) = \frac{2\pi}{2n+1}(1-x_\alpha).$$

The integration formula then becomes

$$\int_{-1}^{+1}\sqrt{\left(\frac{1-x}{1+x}\right)}f(x)dx = \sum_{\alpha=0}^{n-1} C_\alpha f(x_\alpha)$$

where $C_\alpha = \frac{2\pi}{2n+1}(1-x_\alpha)$ and $x_\alpha = -\cos\left(\frac{2\alpha+1}{2n+1}\right)\pi$.

We shall call the interpolation functions associated with this weight function, $\{f_r(x)\}$.

4.4. *The Choice of the Downwash Points.* The integral equation which connects the chordwise pressure $\Gamma(y)$ and the downwash $W(x)$ in two-dimensional steady incompressible flow is

$$2\pi W(x) = \int_{-1}^{+1} \frac{\Gamma(y)}{x-y}\, dy.$$

The solution of this integral equation is

$$\frac{\pi}{2}\sqrt{\left(\frac{1-y}{1+y}\right)} \int_{-1}^{+1} \sqrt{\left(\frac{1+x}{1-x}\right)} \frac{W(x)}{x-y}\, dx.$$

We want to approximate to $W(x)$ by a polynomial $\overline{W}(x)$ of order $(n-1)$ which gives the correct values at n points $x_1, \ldots, x_n$. We want to choose these n points so that the corresponding approximate pressure $\overline{\Gamma}(y)$ will be as good an approximation to $\Gamma(y)$ as possible.

The error $\{\Gamma(y) - \overline{\Gamma}(y)\}$ is given by the equation

$$\frac{\pi}{2}\{\Gamma(y) - \overline{\Gamma}(y)\} = \sqrt{\left(\frac{1-y}{1+y}\right)} \int_{-1}^{+1} \sqrt{\left(\frac{1+x}{1-x}\right)} \frac{[W(x) - \overline{W}(x)]}{x-y}\, dx.$$

We shall try to choose the points so that

$$\int_{-1}^{+1} [\Gamma(y) - \overline{\Gamma}(y)]dy = 0,$$

i.e.,

$$\int_{-1}^{+1} \sqrt{\left(\frac{1+x}{1-x}\right)} [W(x) - \overline{W}(x)]dx \int_{-1}^{+1} \sqrt{\left(\frac{1-y}{1+y}\right)} \frac{dy}{x-y} = 0$$

or since the value of the inner integral is π,

$$\int_{-1}^{+1} \sqrt{\left(\frac{1+x}{1-x}\right)} [W(x) - \overline{W}(x)]dx = 0.$$

Now if $W(x)$ is a polynomial of order $(2n-1)$ we can write

$$W(x) = \overline{W}(x) + \prod_{\alpha=1}^{n} (x-x_\alpha)P_{n-1}(x)$$

where $P_{n-1}(x)$ is some polynomial of order $(n-1)$. The condition then becomes

$$\int_{-1}^{+1} \sqrt{\left(\frac{1+x}{1-x}\right)} \prod_{\alpha=1}^{n} (x-x_\alpha)P_{n-1}(x)dx = 0.$$

By writing $x = -x$ we see from (C) that the points $(-x_\alpha)$ are the integration points for the weight function $\sqrt{\{(1-x)/(1+x)\}}$.

The condition can be written

$$\int_{-1}^{+1} \sqrt{\left(\frac{1+x}{1-x}\right)} \prod_{\alpha=1}^{n} (x-x_\alpha)x^r dx = 0 \qquad r = 0, \ldots, n-1.$$

Since the value of the integral $\int_{-1}^{+1} \sqrt{\left(\frac{1+y}{1-y}\right)} \frac{y^s}{x-y}$ is a polynomial of degree s in x these same conditions would be obtained if we wished to approximate to a downwash polynomial of order $(2n-r)$ in terms of its values at n points subject to the conditions that

$$\int_{-1}^{+1} [\Gamma(y) - \overline{\Gamma}(y)]y^s dy = 0 \qquad s = 0, \ldots, r-1.$$

In particular if we approximate to a polynomial of order n by the polynomial of order $(n-1)$ we have

$$\int_{-1}^{+1} [\Gamma(y) - \bar{\Gamma}(y)]y^s dy = 0 \qquad s = 0, \ldots, n-1.$$

We find the best spanwise collocation points in a similar manner. The integral equation which connects the two-dimensional steady incompressible spanwise downwash with the pressure is

$$2\pi W(x) = \int_{-1}^{+1} \frac{\Gamma(y)}{(x-y)^2} dy.$$

The solution of the integral equation is

$$2\pi\Gamma(y) = \sqrt{(1-y^2)} \int_{-1}^{+1} \frac{\sqrt{(1-x^2)}}{(x-y)^2} W(x)dx.$$

As before we choose the approximate $\bar{W}$ so that for the corresponding $\bar{\Gamma}$ we have

$$\int_{-1}^{+1} \{\Gamma(y) - \bar{\Gamma}(y)\} dy = 0,$$

i.e.,

$$\int_{-1}^{+1} \sqrt{(1-x^2)}\{W(x) - \bar{W}(x)\} dx \int_{-1}^{+1} \frac{\sqrt{(1-y^2)}}{(x-y)^2} dy = 0.$$

The value of the inner integral is $-\pi$ and so we must have

$$\int_{-1}^{+1} \sqrt{(1-x^2)}\{W(x) - \bar{W}(x)\} dx = 0.$$

If $W(x)$ is a polynomial of order $(2n-1)$ and $\bar{W}$ is a polynomial which gives the correct values at n points $x_1, \ldots, x_n$, then as before we must have

$$\int_{-1}^{+1} \sqrt{(1-x^2)} \prod_{\alpha=1}^{n} (x-x_\alpha)P_{n-1}(x)dx = 0$$

for any polynomial of order $(n-1)$. By referring to (B) we see that these points are the roots of $U_n(x)$.

The spanwise integration points coincide with the downwash points given in (B).

5. *The Numerical Solution of the Integral Equation.* We are now in a position to solve the integral equation

$$4\pi W(x', y') = \iint_S \Gamma(x, y) \frac{K[x-x', y-y']}{(y-y')^2} dx\, dy$$

numerically.

We shall transfer the co-ordinates to new co-ordinates ξ, η defined by the equations

$$y = s\eta$$

$$x = x_{C_L} + \tfrac{1}{2}sc(y)\xi$$

where s is the semi-span, x_{C_L} is the co-ordinate of the centreline at spanwise position y, and $sc(y)$ is the chord at this section.

We shall denote the downwash points by the suffices r, s and the integration points by the suffices α, β. The y' co-ordinate will only depend on s and will be written y_s, the x' co-ordinate will depend on r and s and will be written x_{rs}.

The integral equation then becomes

$$8\pi W(x_{rs}, y_s) = \int_{-1}^{+1} \frac{c(\eta)d\eta}{(\eta-\eta_s)^2} \int_{-1}^{+1} \Gamma(\xi, \eta) K[x_{rs} - x, y_s - y]\, d\xi .$$

5.1. *Chordwise Integration.* We shall first consider the chordwise integral

$$\int_{-1}^{+1} \Gamma(\xi, \eta) K[x_{rs} - x, y_s - y]\, d\xi .$$

If we write

$$\Gamma(\xi, \eta) = \sqrt{\left(\frac{1-\xi}{1+\xi}\right)}\, \Gamma^*(\xi, \eta)$$

the integral becomes

$$\int_{-1}^{+1} \sqrt{\left(\frac{1-\xi}{1+\xi}\right)}\, \Gamma^*(\xi, \eta) K[x_{rs} - x, y_s - y]\, d\xi$$

and using integration formula (C) its value is

$$\sum_1^m C_\alpha \Gamma^*(\xi_\alpha, \eta) K[x_{rs} - x_\alpha, y_s - y]$$

or

$$\sum_1^m C_\alpha{}' \Gamma(\xi_\alpha, \eta) K[x_{rs} - x_\alpha, y_s - y]$$

where

$$C_\alpha{}' = \sqrt{\left(\frac{1+\xi_\alpha}{1-\xi_\alpha}\right)}\, H_\alpha = \frac{2\pi}{2n+1} \sqrt{(1-\xi_\alpha{}^2)} .$$

We can also use an alternative method due to Multhopp[10].

The function

$$\Gamma_n(\xi) = 2^n \sqrt{\left(\frac{1-\xi}{1+\xi}\right)} \frac{T_{2n+1}\left[\sqrt{\left(\frac{1-\xi}{2}\right)}\right]}{\left[\sqrt{\left(\frac{1-\xi}{2}\right)}\right]}$$

has the property, that

$$\int_{-1}^{+1} \Gamma_n(\xi) \xi^r d\xi = 0$$

for $r = 0, \ldots, n-1$. We may consider Γ_n, $n \neq 0$, to be a pressure function which gives the lift and the first $(n-1)$ moments to be zero. We can expand the chordwise pressure in the form

$$\Gamma_0 = \sum_1^\infty a_n \Gamma_n$$

and so if we only require the lift and the first $(n-1)$ moments we need only calculate $a_0, \ldots, a_n$.

If as in (C) we write $x = -\cos\theta$ the first two of these pressure functions are

$$\Gamma_0 = \cot\frac{\theta}{2}$$

$$\Gamma_1 = \cot\frac{\theta}{2}\frac{\cos(3\theta/2)}{\cos(\theta/2)}$$

$$= \cot\frac{\theta}{2} - 2\sin\theta.$$

We therefore take for $\Gamma(\xi, \eta)$ a finite series

$$\Gamma(\xi, \eta) = \sum_1^m a_n(\eta)\Gamma_n(\xi)$$

and we have to evaluate the integrals

$$G_n(\xi', \xi, \eta) = \int_{-1}^{+1} \Gamma_n(\xi)K[\xi, \xi', \eta, \eta']\, d\xi.$$

These integrals can, if wished, be evaluated by using the Gaussian integration formula (C).

5.2. *Spanwise Integration.* With the first method of chordwise integration we are left with the spanwise integral

$$\int_{-1}^{+1} \frac{\Gamma(\xi_\alpha, \eta)K[\xi', \xi_\alpha : \eta, \eta_s]}{(\eta-\eta_s)^2}\, d\eta$$

and with the second method we are left with

$$\int_{-1}^{+1} \frac{G_n(\xi', \eta, \eta_s)a_n(\eta)}{(\eta-\eta_s)^2}.$$

All these integrals are of the form

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}\, G(\eta, \eta_s)}{(\eta-\eta_s)^2}\, d\eta.$$

The spanwise integration is more difficult than the chordwise integration because the integral to be evaluated is a principal-value integral. We shall now discuss some possible methods.

5.2.1. *Multhopp's method.* The method of spanwise integration usually used is due to Multhopp. The function $G(\eta, \eta_s)$ is expanded in terms of the interpolation functions $\{g_r^*(\eta)\}$ given in Section 4.3. The series

$$\sum_1^N G(\eta_\beta, \eta_s)g_\beta^*(\eta)$$

is then an approximation to $G(\eta, \eta_s)$ which in addition to having the correct values at the points $\eta_1, \ldots \eta_N$ has the property that the equation

$$\int_{-1}^{+1} \sqrt{(1-\eta^2)}\, G(\eta, \eta_s)d\eta = \sum_1^N G(\eta_\beta, \eta_s) \int_{-1}^{+1} \sqrt{(1-\eta^2)}g_\beta^*(\eta)d\eta$$

holds if $G(\eta, \eta_s)$ is a polynomial in η of order less than $(2N-1)$.

If we use this approximation we get

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}\, G(\eta, \eta_s)}{(\eta-\eta_s)^2}\, d\eta = \sum_1^N G(\eta_\beta, \eta_s) \int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)} g_\beta^*(\eta)}{(\eta-\eta_s)^2}\, d\eta$$

$$= \sum_1^N G_{s\beta} J_{s\beta}$$

where

$$G_{s\beta} = G(\eta_\beta, \eta_s) \text{ and } J_{s\beta} = \int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)} g_\beta^*(\eta)}{(\eta-\eta_s)^2}\, d\eta .$$

It is more usual when approximating to a function by means of a polynomial to use as interpolation points the roots of the Chebyshev polynomial $T_{n+1}(x)$. The analysis would be very similar but the integrals $J_{s\beta}$ would be a little more difficult to evaluate. We shall now give an alternative method of deriving the integration formula.

Since $U_n(\theta)$ is a multiple of $\sin (n+1)\theta/\sin \theta$ it can easily be shown by integrating Glauerts integral

$$\int_0^\pi \frac{\cos n\theta}{\cos \theta - \cos \varphi}\, d\theta = \pi \frac{\sin n\varphi}{\sin \varphi}$$

by parts, that

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}\, U_n(\eta)}{(\eta-\eta')^2}\, d\eta = -(n+1)\pi U_n(\eta')$$

and using this result we get the formula

$$\frac{1}{(\eta-\eta')^2} = -\pi \sum_0^\infty \frac{(n+1)U_n(\eta)U_n(\eta')}{\lambda_n} = K(\eta, \eta')$$

where $\lambda_n = \int_{-1}^{+1} \sqrt{(1-\eta^2)}\, U_n^2(\eta) d\eta$. The series is divergent but it has a first Cesaro sum. We shall define an N'th convergent K_N to the function $K(\eta, \eta')$ by taking the first N terms of this series.

We then get the following approximation to the value of the spanwise integral

$$\int_{-1}^{+1} \sqrt{(1-\eta^2)}\, G(\eta, \eta_s) K_N(\eta, \eta_s) d\eta$$

$$= \int_{-1}^{+1} \sqrt{(1-\eta^2)}\, G(\eta, \eta_s) \left\{ -\pi \sum_0^\infty (n+1) \frac{U_n(\eta)U_n(\eta_s)}{\lambda_n} \right\} d\eta .$$

We can now evaluate the integral numerically by using the Gaussian integration formula (B). The integral is then approximately equal to

$$\sum_{\beta=1}^N B_\beta G(\eta_\beta, \eta_s) \left\{ -\pi \sum_{n=0}^N (n+1) \frac{U_n(\eta_s)U_n(\eta_\beta)}{\lambda_n} \right\} .$$

We shall show that this reduces to

$$\sum_{\beta=1}^N J_{s\beta} G_{s\beta}$$

as in Multhopp's method.

We have

$$J_{s\beta} = \int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)} g_\beta^*(\eta) d\eta}{(\eta-\eta_s)^2} .$$

We first want to expand $g_\beta^*(\eta)$ in a series of the form

$$g_\beta^*(\eta) = \sum_{r=0}^{N-1} \alpha_r U_r(\eta).$$

The coefficients α_r are given by

$$\lambda_r \alpha_r = \int_{-1}^{+1} \sqrt{(1-\eta^2)} g_\beta^*(\eta) U_r(\eta) d\eta.$$

Now $g_\beta^*(\eta)\ U_r(\eta)$ is a polynomial of order $(2N-2)$ at most and so can be evaluated exactly by the Gaussian formula (B). We then have

$$\lambda_r \alpha_r = \sum_{\mu=1}^{N} B_\mu g_\beta^*(\eta_\mu) U_r(\eta_\mu) = B_\beta U_r(\eta_\beta).$$

Therefore

$$\begin{aligned} J_{s\beta} &= \int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}}{(\eta-\eta_s)^2} g_\beta^*(\eta) d\eta \\ &= -\pi \sum_{n=1}^{\infty} \frac{(n+1)U_n(\eta_s)}{\lambda_n} \sum_{r=1}^{N-1} \frac{B_\beta U_r(\eta_\beta)}{\lambda_r} \int_{-1}^{+1} \sqrt{(1-\eta^2)} U_r(\eta) U_n(\eta) d\eta \\ &= -\pi B_\beta \sum_{n=1}^{N-1} \frac{U_n(\eta_s) U_n(\eta_\beta)}{\lambda_n}. \end{aligned}$$

Since $U_N(\eta_s) = U_N(\eta_\beta) = 0$

$$J_{s\beta} = -\pi B_\beta \sum_{\mu=0}^{N} \frac{U_n(\eta_s) U_n(\eta_\beta)}{\lambda_n}.$$

The Multhopp method is then equivalent to using a degenerate kernel to approximate to the kernel $1/(\eta-\eta_s)^2$ and evaluating the integrals by Gaussian integration. This was recognised by K. Jaeckel (Collatz)[11].

5.2.2. There is however the difficulty that $G(\eta, \eta_s)$ is really of the form

$$G_1(\eta, \eta_s) + (\eta-\eta_s)^2 \log |\eta - \eta_s| G_2(\eta, \eta_s)$$

where G_1 and G_2 are well behaved functions. The logarithmic term is only significant near $\eta = \eta_s$ and so we shall take the function to be of the form

$$G_1(\eta, \eta_s) + \lambda(\eta_s)(\eta-\eta_s)^2 \log |\eta - \eta_s|.$$

The function $G_1(\eta, \eta_s)$ has no singularities and so we can approximate to it by means of the functions $\{g_\beta^*(\eta)\}$. We have

$$\begin{aligned} G_1(\eta, \eta_s) &= \sum_{\beta=1}^{n} G_1(\eta_\beta, \eta_s) g_\beta^*(\eta) \\ &= \sum_{\beta=1}^{n} \{G(\eta_\beta, \eta_s) - \lambda(\eta_s)(\eta_\beta-\eta_s)^2 \log |\eta_\beta - \eta_s|\} g_\beta^*(\eta). \end{aligned}$$

Therefore

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}G(\eta, \eta_s)}{(\eta-\eta_s)^2}\, d\eta = \int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}G_1(\eta, \eta_s)}{(\eta-\eta_s)^2}\, d\eta + \lambda(\eta_s) \int_{-1}^{+1} \sqrt{(1-\eta^2)} \log |\eta - \eta_s|\, d\eta$$

$$= \sum_{\beta=1}^{n} \{G(\eta_\beta, \eta_s) - \lambda(\eta_s)(\eta_\beta - \eta_s)^2 \log |\eta_\beta - \eta_s|\} \int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)} g_\beta^*(\eta)}{(\eta-\eta_s)^2}\, d\eta +$$

$$+ \lambda(\eta_s) \int_{-1}^{+1} \sqrt{(1-\eta^2)} \log |\eta - \eta_s|\, d\eta \,.$$

Since

$$\int_{-1}^{+1} \sqrt{(1-\eta^2)} \log |\eta - \eta_s|\, d\eta = \pi\{\tfrac{1}{2}\eta_s^2 - \tfrac{1}{4} - \tfrac{1}{2}\log 2\}$$

we get

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}G(\eta, \eta_s)}{(\eta-\eta_s)^2}\, d\eta = \sum_{\beta=1}^{n} G(\eta_\beta, \eta_s) J_{s\beta} +$$

$$+ \lambda(\eta_s) \Big[\pi\{\tfrac{1}{2}\eta_s^2 - \tfrac{1}{4} - \tfrac{1}{2}\log 2\} -$$

$$- \sum_{\beta=1}^{n} (\eta_\beta - \eta_s)^2 \log |\eta_\beta - \eta_s| J_{s\beta}\Big] \,.$$

This modification is due to Mangler and Spencer[12]. The evaluation of $\lambda(\eta_s)$ is difficult, possibly the best approximation is due to I. T. Minhinnick: it is derived in Appendix I.

5.3. An alternative method is due to Hsu[13]. We can remove the singularity from the integrand by writing it in the form

$$\frac{G(\eta_s, \eta)}{(\eta-\eta_s)^2} = \frac{G(\eta_s, \eta) - G(\eta_s, \eta_s) - (\eta - \eta_s)\left(\dfrac{\partial G}{\partial \eta}\right)_{\eta=\eta_s}}{(\eta-\eta_s)^2} +$$

$$+ \frac{G(\eta_s, \eta_s)}{(\eta-\eta_s)^2} + \frac{1}{(\eta-\eta_s)^2}\left[\frac{\partial G}{\partial \eta}\right]_{\eta=\eta_s} .$$

Since

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}}{(\eta-\eta_s)^2}\, d\eta = -\pi\eta_s$$

and

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}}{(\eta-\eta_s)^2}\, d\eta = -\pi$$

we get

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}G(\eta_s, \eta)}{(\eta-\eta_s)^2}\, d\eta = -\pi G(\eta_s, \eta_s) - \pi\eta_s \left(\frac{\partial G}{\partial \eta}\right)_{\eta=\eta_s} +$$

$$+ \int_{-1}^{+1} \sqrt{(1-\eta^2)} \left\{\frac{G(\eta_s, \eta) - G(\eta_s, \eta_s) - (\eta-\eta_s)\left(\dfrac{\partial G}{\partial \eta}\right)_{\eta=\eta_s}}{(\eta-\eta_s)^2}\right\} d\eta \,. \quad (15)$$

The integrand of the integral on the right-hand side has no singularity at $\eta = \eta_s$ we can apply integration formula (B) to the integral. We get

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}G(\eta_s, \eta)}{(\eta-\eta_s)^2}\, d\eta = -\pi G(\eta_s, \eta_s) - \pi\eta_s \left(\frac{\partial G}{\partial \eta}\right)_{\eta=\eta_s} +$$

$$+ \frac{\pi}{N+1}\sum_{\beta=1}^{N}\left\{\frac{G(\eta_s, \eta_\beta)}{(\eta_\beta-\eta_s)^2} - \frac{G(\eta_s, \eta_s)}{(\eta_\beta-\eta_s)^2} - \frac{1}{(\eta_\beta-\eta_s)^2}\left(\frac{\partial G}{\partial \eta}\right)_{\eta=\eta_s}\right\}$$

$$= \frac{\pi}{N+1}\sum_{\beta=1}^{N}(1-\eta_\beta{}^2)\frac{G_{s\beta}}{(\eta_\beta-\eta_s)^2} -$$

$$- \pi G_{ss}\left\{1 + \frac{1}{N+1}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)}{(\eta_s-\eta_\beta)^2}\right\} -$$

$$- \pi\left(\frac{\partial G}{\partial \eta}\right)_{\eta=\eta_s}\left\{\eta_s + \frac{1}{N+1}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)}{\eta_\beta-\eta_s}\right\}. \qquad (16)$$

The formula as it stands is of little use because it involves the derivative of $G(\eta_s, \eta)$. If however the points η_s could be chosen to be the roots of the equation

$$\eta_s + \frac{1}{N+1}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)}{(\eta_\beta-\eta_s)} = 0$$

then this difficulty would be removed. These points can easily be calculated and the equation (16) becomes

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}\, G(\eta_s, \eta)}{(\eta-\eta_s)^2}\, d\eta = \frac{\pi}{N+1}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)G_{s\beta}}{(\eta_s-\eta_\beta)^2} - \pi\lambda_s G_{ss}$$

where

$$\lambda_s = 1 + \frac{1}{N+1}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)}{(\eta_\beta-\eta_s)^2}.$$

The points η_s are not related to the downwash points obtained in Section 4.4. If however we write the integral in equation (15) as

$$\int_{-1}^{+1}\frac{(1-\eta^2)}{\sqrt{(1-\eta^2)}}\frac{\left[G(\eta_s, \eta) - G(\eta_s, \eta_s) - (\eta-\eta_s)\left(\frac{\partial G}{\partial \eta}\right)_{\eta=\eta_s}\right]}{(\eta-\eta_s)^2}\, d\eta$$

and use integration formula (A) for the weight function $1/\sqrt{(1-\eta^2)}$ we obtain equation (16) with the factor $1/(N+1)$ replaced by $1/N$. The derivative $(\partial G/\partial \eta)_{\eta=\eta_s}$ will be removed if we choose the downwash points η_s to satisfy the equation

$$\eta_s + \frac{1}{N}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)}{(\eta_\beta-\eta_s)^2} = 0.$$

It is shown in Appendix II that these points are the roots of $U_{N-1}(x)$. With this choice of downwash points the expression

$$1 + \frac{1}{N}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)}{(\eta_s-\eta_\beta)^2}$$

is equal to N. The value of the integral is then

$$\int_{-1}^{+1} \frac{\sqrt{(1-\eta^2)}\, G(\eta, \eta_s)}{(\eta-\eta_s)^2}\, d\eta = \frac{\pi}{N}\sum_{\beta=1}^{N}\frac{(1-\eta_\beta{}^2)G_{s\beta}}{(\eta_s-\eta_\beta)^2} - N\pi G_{ss}.$$

With this method we have one more integration point than downwash point, and so with the chordwise integration we must choose more downwash points than integration points. If we choose N spanwise integration points and M chordwise integration points then the number of chordwise points S must be chosen so that

$$MN = (N-1)S$$

i.e., M must be a multiple of $(N-1)$.

5.4. We have now two possible methods of chordwise integration and two methods of spanwise integration. Acum[14, 15, 16] and Garner[17] have used Multhopp pressure functions for the chordwise integration and Multhopp's method for spanwise integration, Richardson[3] has used Gaussian integration for chordwise integration and Multhopp's method for spanwise integration, and Hsu[13] has used Gaussian integration for chordwise integration and method of his own for spanwise integration. These methods are given in detail in the references quoted.

We shall give details of a method similar to that used by Richardson. If we neglect the logarithmic correction the integral equation is easily solved. The integral equation is

$$8\pi W_{rs} = \int_{-1}^{+1} \frac{c(\eta)d\eta}{(\eta-\eta_s)^2} \int_{-1}^{+1} \Gamma(\xi, \eta) K[x_{rs} - x, y_s - y] d\xi .$$

If we apply integration formula (C) to the inner integral

$$I = \int_{-1}^{+1} \Gamma(\xi, \eta) K[x_{rs} - x, y_s - y] d\xi$$

we get

$$I = \sum_{\alpha=1}^{m} C_\alpha' \Gamma(\xi_\alpha, \eta) K[x_{rs} - x, y_s - y] .$$

If we now integrate spanwise we get

$$8\pi W_{rs} = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} C_\alpha' \Gamma(\xi_\alpha, \eta_\beta) c(\eta_\beta) K[x_{rs} - x_{\alpha\beta}, y_s - y_\beta] J_{s\beta}$$

$$= \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} C_\alpha' c_\beta K_{rs, \alpha\beta} J_{s\beta} \Gamma_{\alpha\beta}$$

where

$$c_\beta = c(\eta_\beta), \; \Gamma_{\alpha\beta} = \Gamma(\xi_\alpha, \eta_\beta)$$

and

$$K_{rs,\alpha\beta} = K[x_{rs} - x_{\alpha\beta}, y_s - y_\beta] .$$

These equations can easily be solved for $\Gamma_{\alpha\beta}$.

If we are to include the logarithmic correction the integral of

$$c(\eta) K[x_{rs} - x, y_s - y] \Gamma(\xi_{\alpha}, \eta)$$

becomes

$$\sum_{1}^{n} c_\beta K_{rs,\alpha\beta} J_{s\beta} \Gamma_{\alpha\beta} + \lambda_{rs} \mu_s$$

where

$$\mu_s = \pi [\tfrac{1}{2}\eta_s^2 - \tfrac{1}{4} - \tfrac{1}{2}\log 2] - \sum_{1}^{n} (\eta_\beta - \eta_s)^2 \log |\eta_\beta - \eta_s| J_{s\beta} .$$

It is shown in Appendix II that

$$\lambda_{r,s} = \left(\frac{2}{c_s}\right)^2 \left[-(1-M^2)\frac{\partial\Gamma(\xi_r,\eta_s)}{\partial\xi} + (1+M^2)i\kappa\Gamma(\xi_r,\eta_s) + \kappa^2\int_{-1}^{\xi_r}\Gamma(\xi,\eta_s)\exp(-i\kappa(\xi_1-\xi)\}d\xi\right]$$

where

$$\kappa = (\omega c_s/2).$$

We then have for $\lambda_{r,s}$

$$\lambda_{r,s} = \left(\frac{2}{c_s}\right)^2 \sum_{\alpha=1}^{m} \Gamma_{\alpha\beta}\delta_{s\beta}\left\{ -(1-M^2)f_\alpha'(\xi) + (1+M^2)i\kappa f_\alpha(\xi_r) + \right.$$

$$\left. + \kappa^2\int_{-1}^{\xi_r} f_\alpha(\xi)\exp\{-i\kappa(\xi_r-\xi)\}d\xi\right\}$$

$$= \sum_{\alpha=1}^{m} \delta_{s\beta}L_{rs,\alpha}\Gamma_{\alpha\beta}$$

where

$$L_{rs,\alpha} = \left(\frac{2}{c_s}\right)^2 \left[-(1-M^2)f_\alpha'(\xi_r) + (1+M^2)i\kappa f_\alpha(\xi_r) + \kappa^2\int_{-1}^{\xi_r} f_\alpha(\xi)\exp\{-i\kappa(\xi_r-\xi)\}d\xi\right].$$

The complete equation is then

$$8\pi W_{rs} = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} C_\alpha' C_\beta J_{s\beta}K_{rs,\alpha\beta}\Gamma_{\alpha\beta} + \mu_s \sum_{\alpha=1}^{m} \delta_{s\beta}L_{rs,\alpha}\Gamma_{\alpha\beta}.$$

This set of equations can also be put in matrix form and solved for $\Gamma_{\alpha\beta}$, details are given by Williams and Birchall[18].

These interpolation functions have already been used implicitly; the original equations for I can be obtained by expanding Γ in terms of these functions.

Conclusions. The methods used for incompressible flow and some of the methods for compressible flow were devised so as to reduce the amount of computation at the expense of complicating the method. With automatic digital computers, however, it is not so necessary to restrict the amount of computation but it is an advantage to have a simple method. The methods given here are suitable for digital computers and would be of little use without them. The few published results available show that they are capable of giving the derivatives needed for flutter calculations.

Acknowledgement. My thanks are due to Dr. G. Z. Harris for his valuable criticism and advice.

REFERENCES

No.	Author	Title, etc.
1	I. E. Garrick	Non-steady wing characteristics. *High speed aerodynamics and jet propulsion.* Vol. 7. Section F. Oxford. 1947.
2	G. Temple	Unsteady motion. *Modern developments in fluid dynamics high speed flow.* Ch. 9. Oxford. 1956.
3	J. R. Richardson	A method for calculating the lifting forces on wings (unsteady subsonic and supersonic lifting surface theory). Fairey Aviation Company Ltd. Tech. Report 165. April, 1955.
4	P. M. Morse and H. Feshbach ..	*Methods of theoretical physics.* Vol. I. McGraw Hill. 1953.
5	D. E. Williams	On the integral equations of two-dimensional subsonic flutter derivative theory. A.R.C. R. & M. 3057. June, 1955.
6	R. Timman and A. I. van de Vooren	Theory of the oscillating wing with aerodynamically balanced control surface in two-dimensional subsonic compressible flow. N.L.L. Report F.54. June, 1949.
7	H. G. Küssner	A general method for solving problems of the unsteady lifting surface theory in the subsonic range. *J. Ae. Sci.* Vol. 2. January, 1954.
8	A. I. van de Vooren	Unsteady airfoil theory. *Advances in applied mechanics.* Vol. V. Academic Press. 1958.
9	H. Mineur	*Techniques de Calcul Numérique.* Librairie Polytechnique Ch. Beranger, Paris et Liège. 1952.
10	H. Multhopp	Methods for calculating the lift distribution of wings (subsonic lifting-surface theory). A.R.C. R. & M. 2884. January, 1950.
11	L. Collatz	*Numerische Behandlungen von Differentialgleichungen.* Springer-Verlag. 1951.
12	K. W. Mangler and B. R. F. Spencer	Some remarks on Multhopp's subsonic lifting surface theory. A.R.C. R. & M. 2926. August, 1952.
13	Pao-Tan Hsu	Some recent developments in the flutter analysis of low aspect ratio wings. Proceedings of the National Specialists Meeting on Dynamics and Aeroelasticity, Fort Worth, Texas, 6th–7th November, 1958. (Institute of the Aeronautical Sciences.)
14	W. E. A. Acum	Calculation of force on wings oscillating with finite frequency in a high-subsonic stream. A.R.C. 19,229. April, 1957.

REFERENCES—*continued*

No.	*Author*	*Title, etc.*
15	W. E. A. Acum	Forces on a rectangular wing oscillating in a stream of high subsonic Mach number. A.R.C. 18,630. August, 1956.
16	W. E. A. Acum	Theory of lifting surfaces oscillating at general frequencies in a stream of high subsonic Mach number. A.R.C. 17,824. August, 1955.
17	H. C. Garner	Multhopp's subsonic lifting-surface theory of wings in slow pitching oscillation. A.R.C. R. & M. 2885. July, 1952.
18	D. E. Williams and P. C. Birchall ..	The determination, by Richardson's method, of the subsonic flutter derivatives for a rectangular wing of aspect ratio two. Unpublished M.o.A. Report.
19	C. E. Watkins, H. L. Runyon and D. S. Woolston.	On the kernel function of the integral equation relating the lift and downwash distributions of oscillating wings in subsonic flow. N.A.C.A. Report 1234. 1955.

APPENDIX I

The Evaluation of the Coefficient of $y^2 \log |y|$

When x and y are small the kernel $K(x, y)$ can be expanded in the series

$$y^2K(x, y) = \exp(-i\omega x)\left[1 + \frac{x}{\sqrt{(x^2+\beta^2y^2)}} - \frac{i\omega y^2}{\sqrt{(x^2+\beta^2y^2)}} + \frac{\omega^2y^2}{2}\left\{A + \frac{x}{\beta^2\sqrt{(x^2+\beta^2y^2)}}\right\} + \right.$$
$$\left. + \frac{\omega^2y^2}{2}\log\left\{\omega\frac{x+\sqrt{(x^2+\beta^2y^2)}}{2(1-M)}\right\}\ldots\right]$$

or alternatively

$$y^2K(x, y) = \left[1 + \frac{x}{\sqrt{(x^2+\beta^2y^2)}} - i\omega x - \frac{i\omega(x^2+y^2)}{\sqrt{(x^2+\beta^2y^2)}} + \frac{\omega^2y^2}{2}\left\{A - \frac{(1-2M^2)x}{\beta^2\sqrt{(x^2+\beta^2y^2)}}\right\} - \right.$$
$$\left. - \frac{\omega^2}{2}\left\{x^2 + \frac{x^3}{\sqrt{(x^2+\beta^2y^2)}}\right\} + \frac{\omega^2y^2}{2}\exp(-i\omega x)\log\left\{\omega\frac{\sqrt{(x^2+\beta^2y^2)}-x}{2(1-M)}\right\}\right]$$
$$= y^2K_1(x, y) + \tfrac{1}{2}\omega^2y^2\exp(-i\omega x)\log\left\{\omega\frac{\sqrt{(x^2+\beta^2y^2)}-x}{2(1-M)}\right\}$$

where

$$A = \gamma - \tfrac{1}{2} - \frac{M}{\beta^2} + \frac{i\pi}{2}$$

and γ is Euler's constant (Watkins *et al*[19]).

We want to find the coefficient of $y^2 \log y$ in the expansion of the integral

$$F(x', y) = \int_{x_{\rm LE}}^{x_{\rm TE}} f(x)y^2K(x'-x, y)dx$$
$$= \int_{x_{\rm LE}}^{x_{\rm TE}} f(x)y^2K_1(x'-x, y)dx +$$
$$+ \frac{\omega^2y^2}{2}\int_{x_{\rm LE}}^{x_{\rm TE}}\exp\{-i\omega(x'-x)\}f(x)\log\frac{\omega\sqrt{\{(x'-x)^2+\beta^2y^2\}}-(x'-x)}{2(1-M)}\,dx. \qquad (17)$$

If we write $x' - x = X$ then the first integral becomes

$$-y^2\int_{X_1}^{X_2} f(x'-X)K_1(X, y)dX$$

where

$$X_2 = x' - x_{\rm TE}, \qquad X_1 = x' - x_{\rm LE}.$$

Now we can write

$$f(x'-X) = f(x') - Xf'(x') + \frac{X^2}{2!}f''(X')\ldots$$

and if we substitute this into the integral we get, with an obvious notation,

$$F_1(x', y) = -f(x')\int_{X_1}^{X_2} K_1(X, y)dX + f'(x')\int_{X_1}^{X_2} XK_1(X, y)dX -$$
$$-\frac{f''(x')}{2!}\int_{X_1}^{X_2} X^2K_1(X, y)dX.$$

The only terms in K_1 which can give rise to terms in $\log y$ are those which give rise to integrals of the form

$$I_n = \int_{X_1}^{X_2}\frac{x^n}{\sqrt{(x^2+\beta^2y^2)}}\,dx.$$

Now

$$I_0 = \Big[\log\{x + \sqrt{(x^2+\beta^2y^2)}\}\Big]_{X_1}^{X_2} = \Lambda(X_1, X_2)$$

$$I_1 = \Big[\sqrt{(x^2+\beta^2y^2)}\Big]_{X_1}^{X_2}$$

and

$$\frac{n+2}{n+1} I_{n+2} + \beta^2y^2 I_n = \left[\frac{x^{n+1}}{n+1}\sqrt{(x^2+\beta^2y^2)}\right]_{X_1}^{X_2}$$

and so if we only retain the logarithmic terms and ignore the algebraic terms we have

$$\begin{aligned} I_0 &= \Lambda(X_1, X_2) \\ I_1 &= 0 \\ I_2 &= -\tfrac{1}{2}\beta^2y^2\Lambda(X_1, X_2) \\ I_3 &= 0 \\ I_4 &= \tfrac{3}{8}\beta^4y^4\Lambda(X_1, X_2). \end{aligned}$$

Therefore if we single out the terms which can contribute to the term in $y^2 \log y$ we have

$$\begin{aligned} F_1(x', y) = f(x')\omega(y^2 - \tfrac{1}{2}\beta^2y^2)\Lambda(X_1, X_2) + \\ + f'(x')(-\tfrac{1}{2}\beta^2y^2)\Lambda(X_1, X_2). \end{aligned}$$

Now for small y, $X_1 = x' - x_{\mathrm{LE}}$ is positive and $X_2 = x' - x_{\mathrm{TE}}$ is negative. The term $\log[X_1 + \sqrt{(X_1^2+\beta^2y^2)}]$ has no singularity at $y = 0$. The term $\log[X_2 + \sqrt{(X_2^2+\beta^2y^2)}]$ can be written as

$$-\log|[-X_2 + \sqrt{(X_2^2+\beta^2y^2)}]| + \log\beta^2y^2.$$

The first term has no singularity at $y = 0$ and so then we have

$$\Lambda(X_1, X_2) = 2\log|y|.$$

Therefore

$$F_1(x', y) = y^2\log|y|\{+\omega(1+M^2)f(x') - (1-M^2)f'(x')\} + \ldots$$

Similarly in the integrand of the second integral of (17), $\log|[X - \sqrt{(X^2+\beta^2y^2)}]|$ can only have a logarithmic singularity at $y = 0$ if X is positive and the value of the singularity is then $\log\beta^2y^2$. The logarithmic term in the second integral is then

$$-\omega^2y^2\log|y| \int_X^{x'} \exp(-i\omega X)f(x'-X)dX.$$

The coefficient of the term in $y^2 \log |y|$ is then

$$+\omega(1+M^2)f(x') - (1-M^2)f'(x') + \omega^2\int_{x_{\mathrm{LE}}}^{x'} \exp\{-i\omega(x'-x)\}f(x)dx.$$

When referred to the ξ co-ordinate this result becomes

$$\begin{aligned} \int_{-1}^{+1} f(\xi)y^2K(\xi'-\xi, y)d\xi = \ldots \left(\frac{2}{c}\right)^2 \Bigg[-(1-M^2)\left(\frac{\partial f}{\partial\xi}\right)_{\xi=\xi'} + \\ + ik(1+M^2)f(\xi') + \\ + k^2\int_{-1}^{\xi'} \exp\{-i\omega(\xi'-\xi)\}f(\xi)d\xi\Bigg] y^2\log|y| \end{aligned}$$

where $k = \omega c/2$, and c is the chord at the integration section.

APPENDIX II

Some Relations between the Zeros of Chebyshev Polynomials

Let $y_1, \ldots y_n$ be the zeros of $T_n(y)$, then

$$T_n(y) = \lambda(y-y_1)\ldots(y-y_n). \tag{18}$$

By logarithmic differentiation we get

$$\sum_{\alpha=1}^{n} \frac{1}{y-y_\alpha} = \frac{T_n'(y)}{T_n(y)} \tag{19}$$

and differentiating again

$$\sum_{\alpha=1}^{n} \frac{1}{(y-y_\alpha)^2} = \left[\frac{T_n'(y)}{T_n(y)}\right]^2 - \frac{T_n''(y)}{T_n(y)}. \tag{20}$$

If we choose y to be a zero of $T_n'(y)$, i.e., a root of $U_{n-1}(y)$ equation (19) becomes

$$\sum_{\alpha=1}^{n} \frac{1}{y-y_\alpha} = 0 \tag{21}$$

and equation (20) becomes

$$\sum_{\alpha=1}^{n} \frac{1}{(y-y_\alpha)^2} = -\frac{T_n''(y)}{T_n(y)}. \tag{22}$$

Since $T_n(y)$ satisfies the differential equation

$$(1-y^2)T_n''(y) - yT_n'(y) + n^2T_n(y) = 0 \tag{23}$$

equation (22) reduces to

$$\sum_{\alpha=1}^{n} \frac{1}{(y-y_\alpha)^2} = \frac{n^2}{1-y^2}.$$

We need the sums of the series

$$\sum_{\alpha=1}^{n} \frac{(1-y_\alpha)^2}{(y-y_\alpha)^2}, \qquad \sum_{\alpha=1}^{n} \frac{(1-y_\alpha^2)}{y-y_\alpha}.$$

If we write $(1-y_\alpha^2)$ as

$$(1-y)^2 + 2y(y-y_\alpha) - (y-y_\alpha)^2$$

the first series becomes

$$(1-y^2)\sum_{\alpha=1}^{n} \frac{1}{(y-y_\alpha)^2} + 2y\sum_{\alpha=1}^{n} \frac{1}{y-y_\alpha} - n$$

and so its sum is $n^2 - n = n(n-1)$,

i.e.,

$$\sum_{\alpha=1}^{n} \frac{(1-y_\alpha)^2}{(y-y_\alpha)^2} = n(n-1). \tag{24}$$

The second series becomes

$$(1-y^2)\sum_{\alpha=1}^{n} \frac{1}{y-y_\alpha} + 2ny - \sum_{\alpha=1}^{n}(y-y_\alpha).$$

Since $\sum_{\alpha=1}^{n} y_\alpha = 0$ we have therefore

$$\sum_{\alpha=1}^{n} \frac{(1-y_\alpha^2)}{y-y_\alpha} = ny$$

or

$$y + \frac{1}{n}\sum_{\alpha=1}^{n} \frac{(1-y_\alpha^2)}{y_\alpha - y} = 0.$$

(85744) Wt. 64/1857 K5 11/62 Hw.

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75*s.* (post 2*s.* 9*d.*)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47*s.* 6*d.* (post 2*s.* 3*d.*)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80*s.* (post 2*s.* 6*d.*)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90*s.* (post 2*s.* 9*d.*)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84*s.* (post 3*s.*)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84*s.* (post 3*s.*)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130*s.* (post 3*s.* 6*d.*)
Vol. II. Aircraft, Airscrews, Controls. 130*s.* (post 3*s.* 6*d.*)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130*s.* (post 3*s.* 3*d.*)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130*s.* (post 3*s.* 3*d.*)

1946 Vol. I. Accidents, Aerodynamics, Aerofoils and Hydrofoils. 168*s.* (post 3*s.* 9*d.*)
Vol. II. Airscrews, Cabin Cooling, Chemical Hazards, Controls, Flames, Flutter, Helicopters, Instruments and Instrumentation, Interference, Jets, Miscellaneous, Parachutes. 168*s.* (post 3*s.* 3*d.*)
Vol. III. Performance, Propulsion, Seaplanes, Stability, Structures, Wind Tunnels. 168*s.* (post 3*s.* 6*d.*)

1947 Vol. I. Aerodynamics, Aerofoils, Aircraft. 168*s.* (post 3*s.* 9*d.*)
Vol. II. Airscrews and Rotors, Controls, Flutter, Materials, Miscellaneous, Parachutes, Propulsion, Seaplanes, Stability, Structures, Take-off and Landing. 168*s.* (post 3*s.* 9*d.*)

1948 Vol. I. Aerodynamics, Aerofoils, Aircraft, Airscrews, Controls, Flutter and Vibration, Helicopters, Instruments, Propulsion, Seaplane, Stability, Structures, Wind Tunnels. 130*s.* (post 3*s.* 3*d.*)
Vol. II. Aerodynamics, Aerofoils, Aircraft, Airscrews, Controls, Flutter and Vibration, Helicopters, Instruments, Propulsion, Seaplane, Stability, Structures, Wind Tunnels. 110*s.* (post 3*s.* 3*d.*)

Special Volumes

Vol. I. Aero and Hydrodynamics, Aerofoils, Controls, Flutter, Kites, Parachutes, Performance, Propulsion, Stability. 126*s.* (post 3*s.*)
Vol. II. Aero and Hydrodynamics, Aerofoils, Airscrews, Controls, Flutter, Materials, Miscellaneous, Parachutes, Propulsion, Stability, Structures. 147*s.* (post 3*s.*)
Vol. III. Aero and Hydrodynamics, Aerofoils, Airscrews, Controls, Flutter, Kites, Miscellaneous, Parachutes, Propulsion, Seaplanes, Stability, Structures, Test Equipment. 189*s.* (post 3*s.* 9*d.*)

Reviews of the Aeronautical Research Council

1939–48 3*s.* (post 6*d.*) 1949–54 5*s.* (post 5*d.*)

Index to all Reports and Memoranda published in the Annual Technical Reports

1909–1947 R. & M. 2600 (out of print)

Indexes to the Reports and Memoranda of the Aeronautical Research Council

Between Nos. 2351–2449	R. & M. No. 2450	2*s.* (post 3*d.*)
Between Nos. 2451–2549	R. & M. No. 2550	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2551–2649	R. & M. No. 2650	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2651–2749	R. & M. No. 2750	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2751–2849	R. & M. No. 2850	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2851–2949	R. & M. No. 2950	3*s.* (post 3*d.*)
Between Nos. 2951–3049	R. & M. No. 3050	3*s.* 6*d.* (post 3*d.*)
Between Nos. 3051–3149	R. & M. No. 3150	3*s.* 6*d.* (post 3*d.*)

HER MAJESTY'S STATIONERY OFFICE

from the addresses overleaf

Printed and published by
HER MAJESTY'S STATIONERY OFFICE

To be purchased from
York House, Kingsway, London W.C.2
423 Oxford Street, London W.1
13A Castle Street, Edinburgh 2
109 St. Mary Street, Cardiff
39 King Street, Manchester 2
50 Fairfax Street, Bristol 1
35 Smallbrook, Ringway, Birmingham 5
80 Chichester Street, Belfast 1
or through any bookseller

Printed in England

R. & M. No. 3302

S.O. Code No. 23-3302