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The Supersonic Flow Past a Circular Cone at Incidence

By B. A. Woods

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By B. A. WOODS

COMMUNICATED BY THE DEPUTY CONTROLLER AIRCRAFT (RESEARCH AND DEVELOPMENT), MINISTRY OF AVIATION

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Summary.

The problem of calculating the supersonic flow past a circular cone at small incidence α is treated by the method of inner and outer expansions, on the assumption that it can be expressed as a perturbation (in powers of α) of the corresponding axially symmetric flow. Stone's first-order solution, and the first-order vortical-layer solution are connected as the first-order terms in the outer and inner expansions for the flow. It is shown that the logarithmic infinities which occurred in Stone's second-order solution are removed from the final (composite) solution for second-order terms by application of the generalized matching principle, and the second-order terms in the expansions of the inner solution are obtained.

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* Replaces R.A.E. Tech. Note No. Aero. 2939—A.R.C. 25 876.

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1. *Introduction.*

In this report, the problem of calculating the supersonic flow past a circular cone at a small angle of incidence α to the undisturbed stream is considered. Published work on this problem is dominated by the two original papers by Stone[1, 2] which form the basis for the only available numerical data, the second and third volumes of the M.I.T. cone tables[3, 4]. A large proportion of papers on the topic which have appeared since has been devoted to elucidating, criticizing, or refining Stone's work. Below, Stone's theory and the main criticisms of it will be reviewed, and then the problem will be treated again, still using much of Stone's basic approach, but using also the ideas of the method of inner and outer expansions (also called the method of matched asymptotic expansions) due to Kaplun, Lagerstrom and Cole[5, 6, 7]. These meet many of the criticisms of Stone's theory, at least those concerning the first-order solution, and appear also to overcome a major difficulty in the second-order solution.

The review of Stone's theory, and of the criticism directed at it, is given in the next section. In Section 3 the first-order problem is treated. It is found that Stone's first-order solution, transformed to body coordinates, is the valid first-order 'outer' solution, while the 'inner' solution is identical to that obtained independently by Bulakh[8], Cheng[9] and Woods[10] for the flow close to the cone surface. In Section 4 it is shown that logarithmic infinities at the cone surface which occur in Stone's solution for the second-order terms for u and ρ are automatically cancelled from the final ('composite') expansions for these quantities by application of a generalized matching principle due to Kaplun[11]. Equations are derived for the finite ('non-cancelling') parts of the second-order terms in the outer expansions. Although these are not solved in this report, it seems justifiable to suggest that their solution would entail difficulties less severe than those reported by Kopal[4] in his account of the computation of Stone's second-order solution. Finally in this section a solution for the second-order terms in the inner expansions is given, assuming that the corresponding terms in the outer expansions have been obtained.

2. *An Account of Stone's Theory.*

The equations for the steady flow with conical similarity of a perfect inviscid compressible gas are as follow:

(i) equation for conservation of mass:

$$\nabla(\rho \mathbf{v}) = 2\rho u \sin\theta + \frac{\partial}{\partial\theta}(\rho v \sin\theta) + \frac{\partial}{\partial\varphi}(\rho w) = 0 \tag{1}$$

(ii) momentum equations:

$$v\frac{\partial u}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial u}{\partial\varphi} - v^2 - w^2 = 0 \tag{2a}$$

$$v\frac{\partial v}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial v}{\partial\varphi} + uv - w^2\cot\theta = -\frac{1}{\rho}\frac{\partial p}{\partial\theta} \tag{2b}$$

$$v\frac{\partial w}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial w}{\partial\varphi} + uw + vw\cot\theta = \frac{-1}{\rho\sin\theta}\frac{\partial p}{\partial\varphi} \tag{2c}$$

(iii) equation for convective conservation of entropy; Stone uses the equation:

$$\frac{\gamma p}{\rho} = \left(v\frac{\partial p}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial p}{\partial\varphi}\right)\Big/\left(v\frac{\partial\rho}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial\rho}{\partial\varphi}\right), \tag{3'}$$

which is equivalent to the more usual form;

$$v\frac{\partial S}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial S}{\partial\varphi} = 0, \tag{3}$$

together with the equation of state

$$(\text{const.})\exp\{S/C_v\} = p/\rho^\gamma. \tag{4}$$

In addition, Stone makes use of Bernoulli's equation, although it is not independent of those given above:

$$\frac{1}{2}(u^2+v^2+w^2) + \gamma p/(\gamma-1)\rho = \frac{1}{2}q_\infty{}^2 + \gamma p_\infty/(\gamma-1)\rho_\infty. \tag{5}$$

Here θ and φ are the angular coordinates in a spherical polar coordinate system (r, θ, φ) and u, v, w, are corresponding velocity components (*see* Fig. 1). p, ρ and S are pressure, density and specific entropy, C_v the specific heat at constant volume, and γ the ratio of specific heats of the gas.

In his first paper[1] Stone assumed that the solution of the equations (1), (2a) to (2c), and (3′) can be expressed in the form:

$$\left.\begin{aligned} u &= \bar{u} + \Sigma(x_n\cos n\varphi + X_n\sin n\varphi) \\ v &= \bar{v} + \Sigma(y_n\cos n\varphi + Y_n\sin n\varphi) \\ w &= \Sigma(z_n\sin n\varphi - Z_n\cos n\varphi) \\ p &= \bar{p} + \Sigma(\eta_n\cos n\varphi + H_n\sin n\varphi) \\ \rho &= \bar{\rho} + \Sigma(\xi_n\cos n\varphi + \Xi_n\sin n\varphi). \end{aligned}\right\} \tag{6}$$

Here the barred quantities $\bar{u}$, $\bar{v}$, etc. are the values of u, v, etc. for the axially symmetric case, and are known from the solution due to Taylor and Maccoll[12,13], and the perturbation quantities x_n, y_n, etc. are functions of θ only, and are assumed to be small. The requirement of symmetry about the plane defined by the cone axis and the incident stream direction (which is also the plane

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$\varphi = 0, \pi$) leads immediately to the deduction that *all* the perturbation terms denoted by capital letters vanish, and although Stone does not make this assumption, it will be made hereafter in this report. Similarly to (6), the equation for the shock wave is given as

$$\theta = \bar{\theta}_s + \sum \epsilon_n \cos n\varphi \tag{7}$$

where the ϵ_n are small constants.

When the expansions of equation (6) are inserted into the equation of motion (1), (2a) to (2c), (3′) and (5), and terms of first order of smallness equated, a system of ordinary differential equations for the perturbation coefficients results:

$$\frac{dy_n}{d\theta} + \left\{\cot\theta + \frac{d}{d\theta}(\ln\bar{\rho})\right\} y_n + 2x_n + \frac{nz_n}{\sin\theta} + \bar{v}\frac{d}{d\theta}\left(\frac{\xi_n}{\rho}\right) = 0 \tag{8}$$

$$\frac{dx_n}{d\theta} - y_n = 0 \tag{9a}$$

$$\bar{v}\frac{dy_n}{d\theta} + \left(\frac{d^2\bar{u}}{d\theta} + \bar{u}\right) y_n + \frac{d\bar{u}}{d\theta}x_n + \frac{1}{\rho}\frac{d\eta_n}{d\theta} - \frac{\xi_n}{\rho^2}\frac{d\bar{p}}{d\theta} = 0 \tag{9b}$$

$$\bar{v}\frac{dz_n}{d\theta} + (\bar{u} + \bar{v}\cot\theta)z_n - n\frac{\eta_n}{\bar{\rho}\sin\theta} = 0 \tag{9c}$$

$$\frac{d}{d\theta}(\eta_n/\bar{p} - \gamma\xi_n/\bar{\rho}) = 0$$

or

$$\eta_n/\bar{p} - \gamma\xi_n/\bar{\rho} = d_n \tag{10}$$

where d_n is a constant to be determined,

$$\bar{v}y_n + \bar{u}x_n + \eta_n/\bar{\rho} + d_n\bar{p}/(\gamma-1)\bar{\rho} = 0. \tag{11}$$

Stone reduces these to a second-order linear equation, with variable coefficients, for x_n; in his notation, this is

$$\frac{d^2x_n}{d\theta^2} + B(\theta)\frac{dx_n}{d\theta} + (C - n^2D)x_n = \frac{n^2d_n}{\gamma - 1}DT(\theta) \tag{12}$$

where B, C, D are known regular functions of θ (being combinations of the Taylor-Maccoll variables $\bar{u}$, $\bar{v}$ etc., none of which vanishes within the range of integration), and $T(\theta)$ is a function got by a preliminary integration, which can be represented, close to the undisplaced cone surface θ_c, as

$$T(\theta) = \bar{p}(\theta_c)/\bar{\rho}(\theta_c)\bar{u}(\theta_c) + O\{(\theta - \theta_c)^{1/2}\}. \tag{13}$$

The boundary conditions at the shock wave are set by applying shock relations at the surface given by (7), and then, by using a Taylor theorem expansion, are applied at the undisturbed position of the shock, $\theta = \bar{\theta}_s$. Stone formulated the problem in wind coordinates (the line $\theta = 0$ corresponds to the direction of the undisturbed flow), and the values of x_n, y_n, z_n, etc. at $\theta = \bar{\theta}_s$ are all given as multiples of the corresponding ϵ_n; in particular

$$\left.\begin{aligned} x_n(\bar{\theta}_s) &= K_1\epsilon_n \\ y_n(\bar{\theta}_s) = \left.\frac{dx_n}{d\theta}\right|_{\theta=\bar{\theta}_s} &= K_2\epsilon_n, \qquad \text{and} \\ d_n &= K_3\epsilon_n \end{aligned}\right\} \tag{14}$$

where the K's are constants. It is clear that the differential equation (12) and the boundary conditions (14) are sufficient to determine the function $x_n(\theta)/d_n = f_n(\theta)$ say, and this is computed in the range

$\bar{\theta}_s \geqslant \theta \geqslant \theta_c$ by numerical integration. The value of d_n is then determined by the boundary condition at the cone surface. In wind coördinates this surface is given to the required order of accuracy by

$$\theta = \theta_c + \alpha \cos\varphi, \tag{15}$$

and here, $v = 0$. However, this boundary condition is actually set at $\theta = \theta_c$. By Taylor's theorem, at the cone surface

$$\begin{aligned} \bar{v} &= \bar{v}(\theta_c) + \left.\frac{d\bar{v}}{d\theta}\right|_{\theta=\theta_c} \alpha \cos\varphi \\ &= -2\bar{u}(\theta_c)\alpha\cos\varphi, \qquad \text{since } \bar{v}(\theta_c) = 0, \end{aligned}$$

and hence at $\theta = \theta_c$

$$\left.\begin{aligned} y_1 &= 2\alpha\bar{u}(\theta_c) \\ y_n &= 0 \qquad (n \neq 1). \end{aligned}\right\} \tag{16}$$

From equation (9), this gives

$$\left.\begin{aligned} \left.\frac{du_n}{d\theta}\right|_{\theta=\theta_c} = d_n \left.\frac{df_n}{d\theta}\right|_{\theta=\theta_c} &= 2\alpha\bar{u}(\theta_c) \quad \text{for} \quad n = 1 \\ &= 0 \quad \text{for} \quad n \neq 1. \end{aligned}\right\} \tag{17}$$

The possibility that $df_n/d\theta = 0$ at $\theta = \theta_c$ in some exceptional cases is discarded and since $df_n(\theta)/d\theta$ is already known from the integration of (12), d_n is determined by (17) as a multiple of α. The first-order solution is thus obtained, and can be expressed as:

$$\left.\begin{aligned} u &= \bar{u} + \alpha x \cos\varphi \\ v &= \bar{v} + \alpha y' \cos\varphi \\ w &= \alpha z \sin\varphi \\ p &= \bar{p} + \alpha\eta \cos\varphi \\ \rho &= \bar{\rho} + \alpha\xi \cos\varphi \\ \theta_s &= \bar{\theta}_s + \alpha\epsilon \cos\varphi. \end{aligned}\right\} \tag{18}$$

In the second of his papers[2] Stone obtained solutions for the second-order perturbation terms, i.e. those of order α^2. He began by observing that the boundary conditions for the problem could be satisfied by, and in general required, the following form of solution

$$\left.\begin{aligned} u &= \bar{u} + \alpha x \cos\varphi + \alpha^2(u_0 + u_2 \cos 2\varphi) \\ v &= \bar{v} + \alpha y \cos\varphi + \alpha^2(v_0 + v_2 \cos 2\varphi) \\ w &= \alpha z \sin\varphi + \alpha^2 w_2 \sin 2\varphi \\ p &= \bar{p} + \alpha\eta \cos\varphi + \alpha^2(p_0 + p_2 \cos 2\varphi) \\ \rho &= \bar{\rho} + \alpha\xi \cos\varphi + \alpha^2(\rho_0 + \rho_2 \cos 2\varphi) \\ \theta_{\text{shock}} &= \bar{\theta}_s + \alpha\epsilon \cos\varphi + \alpha^2(\beta_0 + \beta_1 \cos 2\varphi) \end{aligned}\right\} \tag{19}$$

where the second-order terms u_0, u_2, v_0, v_2, etc., are, like the x, y, z, etc., functions of θ only. When the expansions (19) are inserted in the equations of motion, and terms of second order of smallness equated, a system of ordinary differential equations, analogous to (8) to (11), results. The reduction and solution of these will not be described in this report; in the words of Kopal, who directed the numerical computations of the second-order solution 'when the second-order terms are taken into account, the theory becomes . . . so complex as to defy effectively any attempt at presenting it in a concise form and still retaining intelligibility[4]'. The problem is examined in some detail in Section 4.1 below, when it is shown that the straightforward linearization process used by Stone leads to equations which predict values of u and ρ of $O\{\ln(\theta-\theta_c)\}$; i.e. which become infinite at the cone surface. This was recognised by Stone who remarked that, since $\bar{v}$ vanishes at θ_c, his procedure is invalid near the surface of the cone. However, he assumed that the expansions (19) are valid everywhere in the flow field except for a region of very small angular extent close to the cone surface. He observed that this assumption is plausible because the divergent terms in the solution are multiplied by the quantities $(x+z\sin\theta)$ and d, which are very small at moderate supersonic speeds, and is supported by the fact that 'reasonable' values of ρ_i and u_i at θ_c are obtained in (4) by extrapolation of the solutions got by numerical integration of $\theta > \theta_c$.

Criticisms of Stone's solutions have been made on two separate grounds. The more serious was raised initially by Ferri[14], who pointed out that Stone's first-order solution {equation (18)} predicts a sinusoidal variation of entropy at the cone surface. He argued that, because this surface must be wetted by a sheet of streamlines originating on the windward generator of the conical shock, it will be at constant entropy (except for the leeward generator where the entropy is many-valued). Ferri introduced the concept of the 'vortical layer', a thin layer close to the cone surface through which the entropy changes from the sinusoidally varying value given by Stone's solution to the constant value at the cone surface, and adduced physical arguments to show that this layer is of $O(\alpha^2)$ in thickness. The flow picture proposed by Ferri is given in Fig. 2a. This is a spherical projection of the flow through the origin on to a surface r = constant; 'streamlines' represent sheets of streamlines originating from rays through the origin on the shock wave. At first these 'streamlines' lie close to lines of constant φ; then as they approach the cone surface, and the inward velocity (v) becomes small, they are swept around the cone and all tend towards the leeward generator on the cone, which has in this projection the appearance of a sink. Since entropy is different on each streamline, it is many valued at this generator. The structure of Ferri's vortical layer was later investigated by Cheng[9], Woods[10] and Bulakh[8], who independently gave a solution for the flow close to the cone surface which both agreed with Stone's solution away from the cone, and conformed with the physical requirement that the entropy be constant on the cone surface. The basis for this solution, as obtained by the authors mentioned above, is a form of hybrid linearization, in which terms of the first order in α are equated with terms of first order in $(\theta-\theta_c)$ for small $(\theta-\theta_c)$. In the next section the same result is obtained as the solution for the first order in the inner expansion for the problem, and the apparent inconsistency of the earlier solution disappears. In his paper, Woods went further in criticizing Stone's solution, and asserted that, because of the properties of the vortical layer, in particular the fact that in it the derivative $\partial u/\partial\theta$ is of order $\alpha^2/(\theta-\theta_c)^{1-C\alpha}$, the method used by Stone to set the boundary condition at the cone surface is invalid. This assertion is mistaken, as will be clear from the discussion in 3.4 below.

Bulakh[8] considered both the first-order vortical-layer solution, and the logarithmic singularities in Stone's second-order solution and, in fact, identified the latter with what are in effect second-order terms in outer expansions of the former. He concluded that these singularities are due to truncation (at order 2, in this case) of the expansions for the physical quantities in power series in α, and that such expansions are therefore invalid close to the cone surface. As has already been remarked, this difficulty is overcome in the method of inner and outer expansion, by applying a matching principle which is described below in Section 3.1.3.

The other ground for criticism of Stone's theory, which is applicable to the solutions for both first- and second-order terms, is the inconvenience caused expressing the physical variables in wind coordinates. To determine, say, the pressure distribution on the cone surface proper, the solution has to be transformed into body coordinates. Rules for this have been given by Young and Siska[15] and Roberts and Riley[16]. These are quite straightforward, and, in the case of the first-order solution, valid; but in the case of the second-order solution, some of the rules become, like the solution itself, singular at the cone surface. For example, in the expression given by Roberts and Riley for deriving the second-order terms for w, the circumferential velocity, on the cone surface from the tabulated solution in wind coordinates, the term $dz/d\theta$ occurs, and indeed these authors give a finite expression for $dz/d\theta$ at $\theta = \theta_c$ obtained by 'using l'Hospital's rule'. In fact, it is clear from Stone's first paper {*see*, equation (36)[1], and the text immediately following it}, that $dz/d\theta$ is of $O\,[(\theta-\theta_c)^{-1/2}]$ as $\theta \to \theta_c$, and so the rule is theoretically inapplicable there. It is for this reason that the second-order terms of the outer expansions must be calculated in terms of body coordinates *ab initio*.

3. *First-Order Solution*†.

3.1. *Expansions Leading to Linearized Equations.*

We begin by giving a brief account of the method of inner and outer expansions as it applies to the problem in hand. The method was devised initially to deal with problems in viscous flow at low Reynolds number, in which the solution to the full Navier-Stokes equation could not be obtained and for which different approximate linearized forms of the full equations were appropriate in different regions of the flow field. This feature (the necessity for different forms of the linearized equations in different regions) also appears in the present problem.

3.1.1. *Outer expansions.*—In the region away from the cone surface, it is assumed that the flow quantities may be expressed as expansions of the form (taking u as example)

$$u = \mathrm{O}\exp(u) = \bar{u}(\theta) + \alpha\hat{u}_1(\theta, \varphi) + \alpha^2\hat{u}_2(\theta, \varphi) + \dots \tag{20}$$

These expansions (and the equations which govern them) are got by repeated applications of the *outer limit*

$$\mathrm{O}\lim(f) = \lim_{\alpha\to 0}(f); \qquad \theta, \varphi, \qquad \text{fixed}, \qquad \theta \neq \theta_c, \tag{21}$$

and are called the *outer expansions*. The natural coordinates θ and φ are the *outer variables.*

3.1.2. *Inner expansions.*—Near the cone surface, as θ approaches θ_c the linearized equations for successive terms in the outer expansions become invalid. To overcome this new *inner variables* ζ, φ' are chosen such that the equations for successive terms of the *inner expansions* of the flow variables, e.g.

$$u = \mathrm{I}\exp(u) = \bar{u}(\theta\{\zeta\}) + \alpha u_1{}^*(\zeta, \varphi') + \alpha^2 u_2{}^*(\zeta, \varphi') + \dots \tag{22}$$

---

† In the following analysis the natural coordinates θ, φ will always (unless otherwise specified) refer to a body coordinate frame, in which the origin coincides with the cone vortex, the line $\theta = 0$ with the cone axis, and the plane $\varphi = 0, \pi$ is defined by the axis and the undisturbed wind direction, with the half-plane $\varphi = 0$ extending to the windward side of the cone. This body coordinate system is shown in Fig. 1.

are valid. The terms of the inner expansions (and the equations for them) are got by repeated application of the *inner limit*,

$$\text{I lim}\,(f) = \lim_{\alpha\to 0}\,(f), \qquad \zeta, \varphi' \quad \text{fixed}, \qquad \zeta \neq 0. \tag{23}$$

3.1.3. *Matching conditions.*—In general, the outer solution, being valid in a certain region, can be expected to satisfy only those boundary conditions set in that region, and the same is true of the inner solution. In the present application, the shock relations furnish boundary conditions in the region of validity of the outer expansion, while the flow tangency condition $v = 0$ at the cone surface is within that of the inner expansion. In neither case are there enough boundary conditions to determine solutions of the inner or outer equations fully, and the lacking boundary conditions are derived from the requirement that the inner and outer expansions should 'match' in some sense where the regions of validity overlap. Lagerstrom[11] distinguishes between two matching principles; a restricted matching principle, which has been applied to boundary-layer-type problems, which stems from the proposition that if first the I lim and then the O lim are applied to some physical quantity, the result should be the same as that got by reversing the order of the limits, and a generalized matching principle, which asserts that the inner expansion of the outer expansion of a physical quantity should agree within the order considered with the outer expansion of the inner expansion. If we use the following notation for outer and inner expansions of up to some given order n for some physical quantity f

$$\left.\begin{aligned} \sum_{i=0}^{n} \alpha^i \hat{f}_i(\theta, \varphi) &= \text{O exp}^{(n)}(f) = \hat{f}^{(n)}, \\ \text{and} \qquad \sum_{i=0}^{n} \alpha^i f_i^*(\zeta, \varphi') &= \text{I exp}^{(n)}(f) = f^{*(n)}, \end{aligned}\right\} \tag{24}$$

then the generalized matching principle can be stated as

$$\text{I exp}^{(n)}(\hat{f}^{(n)}) = \text{O exp}^{(n)}(f^{*(n)}). \tag{25}$$

The generalized matching principle has been found to be appropriate for the problem at hand.

3.1.4. *Composite expansions.*—When the outer and inner expansions are known for some quantity up to order n, then an expansion for that quantity up to order n which is valid throughout the flow field is given by

$$f^{(n)} = \hat{f}^{(n)} + f^{*(n)} - \text{O exp}^{(n)}(f^{*(n)}). \tag{26}$$

Of course, from the generalized matching principle (25), the last term of (26) could equally have been $\text{I exp}^{(n)}(\hat{f}^{(n)})$.

3.2. *First-Order Outer Equations and Solution.*

From the foregoing it is clear that Stone's linearization, applied to the first-order terms, is equivalent to an application of the outer limit to the basic equation. Anticipating the result obtained later that the correct boundary condition at $\theta = \theta_c$ is $v = 0$, i.e. unchanged by the matching condition, we assert that his first-order solution is thus a valid representation of the first terms of the outer expansions. However, in order to formulate the inner equations, it is necessary to work in body coordinates, so that Stone's solution must be transformed to these, according to the rules given by Roberts and Riley[16].

In effect, then, the first-order outer equations are (8), (9a) to (9c) and (10) in conjunction with the expansions (6). The first terms in the outer expansions are given by (18), in which we understand that the transformation to body coordinates has been carried out, i.e.

$$\left.\begin{aligned} \hat{u}_1(\theta, \varphi) &= x(\theta) \cos \varphi \\ \hat{v}_1(\theta, \varphi) &= y(\theta) \cos \varphi \\ \hat{w}_1(\theta, \varphi) &= z(\theta) \sin \varphi \\ \hat{p}_1(\theta, \varphi) &= \eta(\theta) \cos \varphi \\ \hat{\rho}_1(\theta, \varphi) &= \xi(\theta) \cos \varphi . \end{aligned}\right\} \tag{27}$$

3.3. *First-Order Inner Equations and Solution.*

We have first to choose appropriate inner variables. When the equations of motion are linearized by applying the outer-limit process, i.e. in terms of the natural coordinates, the following estimates for the θ- and φ-components of velocity are implied:

$$v(\theta, \varphi) = O(1)$$

$$w(\theta, \varphi) = O(\alpha).$$

Consequently, in deriving the equations for the first-order terms in the outer expansions, the operator $v\, \partial/\partial\theta$ applied to such terms is retained (as $\bar{v}\, \partial/\partial\theta$), but the operator $(w/\sin\theta)\partial/\partial\varphi$ is discarded as being an order of magnitude less in effect.

However, the flow tangency condition at the cone wall requires that v vanish as $\theta \to \theta_c$; from the Appendix we have that $\bar{v}(\theta) \sim 2u_c(\theta-\theta_c) = -2u_c\delta$, say, and from Stone's first-order solution, it can be shown that $y(\theta) \sim A(\theta-\theta_c) + B(\theta-\theta_c)^{3/2} = A\delta + B\delta^{3/2}$. Inner variables are therefore defined for the region close to the cone surface in terms of which both operators are equivalent in magnitude. The circumferential coordinate remains unchanged

$$\varphi' = \varphi \tag{28}$$

(and the prime will be omitted hereafter), and the coordinate normal to the cone surface, ζ, defined by requiring that

$$v\frac{\partial}{\partial\theta} \sim -2u_c(\theta-\theta_c)\frac{\partial}{\partial\theta} = \alpha u_c \frac{\partial}{\partial\zeta} \tag{29}$$

whence

$$(\theta-\theta_c) = \delta = \exp(-2\zeta/\alpha),$$

or

$$\begin{aligned} \zeta &= -\frac{1}{2}\alpha \log(\theta-\theta_c) \\ &= -\frac{1}{2}\alpha \log(\delta). \end{aligned} \tag{30}$$

The variable ζ ranges from $\zeta = \infty$ (at the cone surface, $\theta = \theta_c$) to $\zeta = 0$. The latter value is got by applying the outer limit ($\alpha \to 0$, θ fixed $\neq \theta_c$) to it, and is the value of ζ in the matching region.

The equations for u_1^*, v_1^*, w_1^*, p_1^* and $\bar{\rho}_1^*$ can now be derived from (1), (2a), (2b), (2c) and (3) {in conjunction with (4)}. The derivation of the continuity equation will be sketched in some detail. It is convenient to work in terms of the two small quantities α and δ $\{= (\theta - \theta_c)\}$. From Appendix, the non-yaw quantities and their derivatives are known in terms of δ for small δ;

$$\bar{u} = u_c(1 - \delta^2 + \ldots)$$

$$\bar{v} = -2u_c\delta(1 - \frac{1}{2}\cot\theta_c\delta + \ldots)$$

$$\bar{\rho} = \rho_c(1 - M_c^2\delta^2 + \ldots).$$

Then $\bar{u} + \alpha u_1^*$, $\bar{v} + \alpha v_1^*$, αw_1^* and $\bar{\rho} + \alpha\rho_1^*$ are substituted for u, v, w and ρ in (1), and $(-\alpha/2\delta)\partial/\partial\zeta$ for $\partial/\partial\theta$ when that operator is applied to the first-order quantities ($\partial\bar{v}/\partial\theta$, and $\partial\bar{\rho}/\partial\theta$ are known directly in terms of δ). In the equation then obtained by subtracting the axially symmetric continuity equation and dividing the remainder by α, terms of orders (α) and (δ) are discarded, and terms of orders unity, (α/δ) and (α^2/δ) retained, to give

$$2\rho_c u_1^* \sin\theta_c - \left(\frac{\alpha^2}{\delta}\right)\frac{v_1^*}{2}\frac{\partial\rho_1^*}{\partial\zeta}\sin\theta_c - \left(\frac{\alpha}{\delta}\right)\frac{\rho_c}{2}\sin\theta_c\left(1 + \alpha\frac{\rho_1^*}{\rho_c}\right)\frac{\partial v_1^*}{\partial\zeta} +$$

$$+ \rho_c v_1^* \cos\theta_c + \rho_c\frac{\partial w_1^*}{\partial\varphi} = 0.$$

Further, because of the boundary condition $v = 0$ at the cone surface, it is assumed that $v_1^* = O(\delta)$; the second and fourth terms in the above equations are then discarded, and we have

$$\frac{\partial v_1^*}{\partial\zeta} = \frac{2\delta}{\alpha}\left(\frac{\partial w_1}{\partial\varphi} + 2u_1^*\sin\theta_c\right)\left(1 - \alpha\frac{\rho_1^*}{\rho_c}\right)\operatorname{cosec}\theta_c$$

$$= O\left(\frac{\delta}{\alpha}\right). \qquad (31)$$

The two small quantities α and δ are connected by the relation (30); in the inner limit ($\alpha \to 0$, ζ fixed, $\zeta \neq 0$) δ is exponentially small—smaller than any power of α—and we have finally

$$\frac{\partial v_1^*}{\partial\zeta} = 0.$$

This equation can be integrated immediately to give

$$v_1^*(\zeta, \varphi) = F_1(\varphi)$$

or, more precisely, by virtue of (31),

$$v_1^*(\zeta, \varphi) = F_1(\varphi) + O(\delta)$$

and from the boundary conditions $v_1^*(\infty, \varphi) = 0$, we have $F_1(\varphi) \equiv 0$, whence

$$v_1^*(\zeta, \varphi) = 0. \qquad (32)$$

{The assumption that $v_1^* = O(\delta)$ is confirmed *a posteriori.*}

The remainder of the inner equations are derived similarly. They are given below, and in brackets at the right-hand side of each, the orders of magnitude of the terms neglected are indicated

$$u_c \frac{\partial u_1^*}{\partial \zeta} + w_1^* \frac{1}{\sin \theta_c} \frac{\partial u_1^*}{\partial \varphi} - w_1^{*2} = 0 \qquad [O(\alpha)] \tag{33}$$

$$\frac{\partial p_1^*}{\partial \zeta} = 0 \qquad [O(\delta/\alpha)] \tag{34}$$

$$u_c w_1^* + \frac{1}{\rho_c \sin \theta_c} \frac{\partial p_1^*}{\partial \varphi} = 0 \qquad [O(\alpha)] \tag{35}$$

$$u_c \frac{\partial}{\partial \zeta} (p_1^*/p_c - \gamma \rho_1^*/\rho_c) + \frac{w_1^*}{\sin \theta_c} \frac{\partial}{\partial \varphi} (p_1^*/p_c - \gamma \rho_1^*/\rho_c) = 0 \qquad [O(\alpha)]. \tag{36}$$

The boundary conditions for u_1^*, w_1^*, p_1^* and ρ_1^* are given by the matching principle and to first order are simply that

$$(u_1^*, w_1^*, \ldots \text{etc.})|_{\zeta=0} = (\hat{u}, \hat{w}, \ldots \text{etc.})|_{\theta=\theta_c}. \tag{37}$$

From (34) we have immediately

$$\begin{aligned} p_1^*(\zeta, \varphi) &= F_2(\varphi) \\ &= \eta_c \cos \varphi \end{aligned} \tag{38}$$

(from the matching principle) and this determines w_1^* from (35), as

$$w_1^*(\zeta, \varphi) = \frac{\eta_c}{\rho_c u_c \sin \theta_c} \sin \varphi. \tag{39}$$

Equation (39) does not conflict with the matching requirement for w_1^*, viz. $w_1^* = z_c \sin \varphi$; it follows from (9c) that for $\theta = \theta_c$, $z_c = \eta_c/\rho_c u_c \sin \theta_c$.

With w_1^* known, (33) and (36) can both be integrated, to yield

$$\left.\begin{aligned} u_1^* &= F_3(\mu) - \sin \theta_c z_c \cos \varphi, \\ \text{and} \quad p_1^* p_c - \gamma \rho_1^*/\rho_c &= F_4(\mu) \end{aligned}\right\} \tag{40}$$

respectively, where

$$\mu = \exp\left(\zeta \frac{z_c}{u_c \sin \theta_c}\right) \cot \varphi/2. \tag{41}$$

The matching conditions enable F_3 and F_4 to be fixed; we have from (40) and (27), using (37),

$$F_3(\cot \varphi/2) - \sin \theta_c z_c \cos \varphi = x_c \cos \varphi$$

or

$$F_3(\mu) = (x_c + z_c \sin \theta_c)\Phi(\mu)$$

where

$$\Phi(\mu) = \frac{\mu^2 - 1}{\mu^2 + 1} \tag{42}$$

and

$$\begin{aligned} F_4(\mu) &= [\eta_c/p_c - \gamma \xi_c/\rho_c]\Phi(\mu) \\ &= \Phi(\mu) d, \end{aligned}$$

say, where we have followed Stone in denoting the first-order entropy perturbation coefficient by d.

Thus finally,

$$u_1^*(\zeta, \varphi) = (x_c + z_c \sin \theta_c)\Phi(\mu) - \sin \theta_c z_c \cos \varphi \tag{43}$$

$$\rho_1^*(\zeta, \varphi) = \eta_c \cos \varphi / a_c^2 - \frac{\rho_c d}{\gamma} \Phi(\mu). \tag{44}$$

To complete the solution for first-order terms in the inner expansions, we anticipate the need in considering the second-order terms for a first-order correction to the operator $v\partial/\partial\theta$, by calculating $V_1^* = v_1^*/\bar{v}$. Observing that in the inner limit the term $\partial v_1/\partial\theta$ can be represented as $-2u_c V_1^*$, we get from the continuity equation (1),

$$2\rho_c u_1^* - 2\rho_c u_c V_1^* + \frac{\rho_c}{\sin \theta_c} \frac{\partial w_1^*}{\partial \varphi} = 0 \qquad [O(\alpha)]$$

or

$$V_1^* = \frac{u_1^*}{u_c} + \frac{z_c}{2u_c \sin \theta_c} \cos \varphi. \tag{45}$$

In deriving the equations for second-order terms in the inner expansions, we shall replace the operator $v\, \partial/\partial\theta$ by

$$\bar{v}(1+\alpha V_1^*) \frac{\partial}{\partial \theta} = (1+\alpha V_1^*)\alpha u_c \frac{\partial}{\partial \zeta}.$$

3.4. *Remarks on Boundary Conditions and Order of Solution.*

In solving for the first-order terms in the outer and inner expansions, the logical sequence would be: (i) to derive the appropriate equations, i.e. the systems (8) to (11) and (31) to (35), (ii) to solve (31) for v_1^*, since this is the only equation which with its boundary condition $v_1^*(\infty, \varphi) = 0$ is autonomous, i.e. does not rely on a matching condition, (iii) now knowing from the matching principle that $y(\theta_c) = v^*(0, \varphi) = 0$, to solve the remainder of the inner equations in sequence, appealing to the matching conditions as boundary conditions.

Thus the boundary condition at the cone surface for the outer equations is precisely that which Stone used, so that the criticism made by Woods[10] on this count is mistaken. However, this boundary condition does not follow immediately from the boundary condition $v = 0$ at the cone surface, which properly applies to the inner equations, but rather from a matching principle.

4. *Second-Order Solution.*

4.1. *Second-Order Outer Equations and the Matching Principles.*

We shall follow Stone in assuming the form of the second-order terms to be as given in equation (19), and shall use his notation. A second application of the outer limit to (1), (2a), (2c), (3′) and (5) then gives precisely those equations derived by him in (2):

$$\frac{dv_i}{d\theta} + v_i \left(\cot \theta + \frac{d}{d\theta}(\log \bar{\rho})\right) + 2u_i + 2w_2 \operatorname{cosec} \theta + + \bar{v} \frac{d}{d\theta}\left(\frac{\rho_i}{\rho}\right) + \frac{1}{2}\left(y - \frac{\bar{v}\xi}{\rho}\right) \frac{d}{d\theta}\left(\frac{\xi}{\rho}\right) \mp \frac{\xi z}{2\rho \sin \theta} = 0 \tag{46}$$

$$\frac{du_i}{d\theta} - v_i = \pm \frac{zt \operatorname{cosec} \theta}{2\bar{v}} \tag{47}$$

$$\bar{v}\frac{dw_2}{d\theta} + (\bar{u}+\bar{v}\cot\theta)w_2 - \frac{2p_2}{\bar{\rho}\sin\theta} +$$

$$+\frac{1}{2}\left\{y\frac{dz}{d\theta} + z(x+y\cot\theta+z\,\text{cosec}\,\theta) + \frac{\xi\eta}{\bar{\rho}^2\sin\theta}\right\} = 0 \quad (48)$$

$$\frac{d}{d\theta}\left(\frac{p_i}{\bar{p}}\right) - \gamma\frac{d}{d\theta}\left(\frac{\rho_i}{\bar{\rho}}\right) = \pm\frac{z\,\text{cosec}\,\theta d}{2\bar{v}} - \frac{1}{2\bar{\rho}\bar{p}}\left(\frac{d\eta}{d\theta}\xi - \gamma\eta\frac{d\xi}{d\theta}\right) \quad (49)$$

$$\bar{u}u_i + \bar{v}v_i + \frac{1}{4}(x^2+y^2\pm z^2) + \frac{\gamma}{\gamma-1}\frac{\bar{p}}{\bar{\rho}}\left\{\frac{p_i}{\bar{p}} - \frac{\rho_i}{\bar{\rho}} + \frac{1}{2}\frac{\xi}{\rho}\left(\frac{\xi}{\rho} - \frac{\eta}{p}\right)\right\} = 0 \quad (50)$$

Here the suffix i takes the values 0 and 2, the symbol $\pm$ means $(-1)^{i/2}$, and $t = x + z\,\text{cosec}\,\theta$.

It is not proposed to reduce and solve the above equations in the present report, and the expressions for the boundary conditions at the shock wave will not be given. Stone gives the appropriate expressions in [2], for the problem set in the wind coordinate system; in the body coordinate system they would be somewhat different. As regards the boundary condition at $\theta = \theta_c$, we anticipate the result obtained for the inner solution that $v_2^*(\zeta, \varphi') = 0$, so that $v_i(\theta_c) = 0$; again, the vortical layer has no displacement effect.

By inspection, knowing that $\bar{v} = O(\theta-\theta_c)$, we deduce from (47) and (49) that u_i and $(p_i/\bar{p}-\gamma\rho_i/\bar{\rho})$ must both contain terms of $O[\log(\theta-\theta_c)]$, which become infinite at the cone surface. We shall show that these terms are eliminated by formal application of the matching principle, and that the resultant composite expansions (to order 2) will give finite values for all the physical quantities throughout. We put

$$u_i = u_i' \mp \frac{z_c t_c\,\text{cosec}\,\theta_c}{4u_c}\log(\theta-\theta_c) \quad (51)$$

$$\frac{\rho_i}{\bar{\rho}} = \left(\frac{\rho_i}{\bar{\rho}}\right)' \pm \frac{z_c d\,\text{cosec}\,\theta_c}{4\gamma u_c}\log(\theta-\theta_c) \quad (52)$$

and

$$v_i = v_i' \pm \frac{z_c t_c\,\text{cosec}\,\theta_c}{2u_c}(\theta-\theta_c)\log(\theta-\theta_c), \quad (53)$$

and convert (49) to (53) thereby into equations for u_i', v_i', w_2, p_i, and $(\rho_i/\bar{\rho})'$. These are:

$$\frac{dv_i'}{d\theta} + v_i'\left(\cot\theta + \frac{d}{d\theta}(\log\bar{\rho})\right) + 2u_i' + 2w_2\,\text{cosec}\,\theta +$$

$$+\bar{v}\frac{d}{d\theta}\left(\frac{\rho_i}{\bar{\rho}}\right)' + \frac{1}{2}\left(y - \frac{\bar{v}\xi}{\bar{\rho}}\right)\frac{d}{d\theta}(\xi/\bar{\rho}) \mp \frac{\xi z}{2\rho\sin\theta} \pm \frac{z_c t_c\,\text{cosec}\,\theta_c}{2u_c} +$$

$$\pm\frac{z_c t_c\,\text{cosec}\,\theta_c}{2u_c}(\theta-\theta_c)\log(\theta-\theta_c)\left(\cot\theta + \frac{d}{d\theta}(\log\bar{\rho})\right) \pm \frac{\bar{v}}{(\theta-\theta_c)}\frac{z_c d\,\text{cosec}\,\theta_c}{4\gamma u_c} = 0 \quad (54)$$

$$\frac{du_i}{d\theta} - v_i' = \pm\left\{\frac{zt\,\text{cosec}\,\theta}{2\bar{v}} + \frac{z_c t_c\,\text{cosec}\,\theta_c}{4u_c(\theta-\theta_c)}\right\} \mp \frac{z_c t_c\,\text{cosec}\,\theta_c}{2u_c}(\theta-\theta_c)\log(\theta-\theta_c) \quad (55)$$

$$\frac{d}{d\theta}\left(\frac{p_i}{\bar{p}}\right) - \gamma\frac{d}{d\theta}\left(\frac{\rho_i}{\bar{\rho}}\right)' = \pm\left\{\frac{zd\,\text{cosec}\,\theta}{2\bar{v}} + \frac{z_c d\,\text{cosec}\,\theta_c}{4u_c(\theta-\theta_c)}\right\} - \frac{1}{2\bar{p}\bar{\rho}}\left(\frac{d\eta}{d\theta}\xi - \gamma\frac{d\xi}{d\theta}\eta\right) \quad (56)$$

and

$$\bar{u}u_i' + \bar{v}v_i + \frac{1}{4}(x^2+y^2\pm z^2) + \frac{\gamma}{\gamma-1}\frac{\bar{p}}{\bar{\rho}}\left\{\frac{p_i}{\bar{p}} - \left(\frac{\rho_i}{\bar{\rho}}\right)' + \frac{1}{2}\frac{\xi}{\bar{\rho}}\left(\frac{\xi}{\bar{\rho}} - \frac{\eta}{\bar{p}}\right)\right\} \pm$$

$$\pm\bar{v}\frac{z_c t_c\,\text{cosec}\,\theta_c}{2u_c}(\theta-\theta_c)\log(\theta-\theta_c) \mp \frac{z_c\,\text{cosec}\,\theta_c}{4u_c}\log(\theta-\theta_c)\left\{\bar{u}t_c + \frac{\bar{p}}{(\gamma-1)\bar{\rho}}d\right\} = 0. \quad (57)$$

Equation (48) remains the same. With the aid of what is known already from Stone's investigations it can be shown that no infinities arise in the solution of equations (54) to (57) and (48). In (54), $\bar{v}/(\theta-\theta_c) \to -2u_c$, so the last term is finite: in (54) and (56) the first terms on the R.H.S. are both of $O(\theta-\theta_c)^{-1/2}$ as $\theta \to \theta_c$ so that u_i' and $(\rho_i/\bar{\rho})'$ both have the form {const. + $O(\theta-\theta_c)^{1/2}$} as $\theta \to \theta_c$. In (57), we recall from equation (37), Ref. 1*, that $t_c = -\{d/(\gamma-1)\}(p_c/u_c\rho_c)$, so that the last term is of $O\{(\theta-\theta_c)^2 \log(\theta-\theta_c)\}$.

Thus, a reduction of the system (54) to (57), (48) analogous to Stone's reduction of the system (46) to (50) should yield two ordinary differential equations† which with the appropriate boundary conditions could be integrated numerically to give, ultimately, values of $u_i'(\theta)$, $v_i'(\theta)$, $p_i(\theta)$, $(\rho_i/\rho)'$ and $w_2(\theta)$ which should be finite within the range $\theta_s \geqslant \theta \geqslant \theta_c$. In the following discussion, we shall assume that this is the case.

The outer expansion for u, say, can now be expressed as:

$$\hat{u}^{(2)} = \bar{u}(\theta) + \alpha\hat{u}_1(\theta, \varphi) + \alpha^2\hat{u}_2(\theta, \varphi) + \ldots \tag{58}$$

where

$$\left.\begin{aligned} \hat{u}_1(\theta, \varphi) &= x(\theta)\cos\varphi \\ \hat{u}_2(\theta, \varphi) &= \left\{u_0'(\theta) - \frac{z_c t_c \operatorname{cosec}\theta_c}{4u_c}\log(\theta-\theta_c)\right\} + \\ &\quad + \left\{u_2'(\theta) + \frac{z_c t_c}{4u_c}\operatorname{cosec}\theta_c \log(\theta-\theta_c)\right\}\cos 2\varphi \end{aligned}\right\} \tag{59}$$

while the inner expansion can be expressed as

$$u^{*(2)} = u_c + \alpha u_1^*(\zeta, \varphi) + \alpha^2 u_2^*(\zeta, \varphi). \tag{60}$$

where

$$\left.\begin{aligned} u_1^* &= t_c\frac{\mu^2-1}{\mu^2+1} - \sin\theta_c z(\theta_c)\cos\varphi, \\ \mu &= \exp\left(\zeta\frac{z_c}{\bar{u}_c\sin\theta_c}\right)\cot\frac{\varphi}{2} \end{aligned}\right\} \tag{61}$$

and u_2^* is not involved in the present discussions. The generalized matching principle requires that

$$\text{I}\exp^{(2)}(\hat{u}^{(2)}) = \text{O}\exp^{(2)}(u^{*(2)}).$$

Now, in outer variables (from (29))

$$\mu^2 = (\theta-\theta_c)^{-\alpha z_c/u_c \sin\theta_c}\cot^2\varphi/2$$

and the first 2 terms of the outer expansion of $(\mu_2{}^2-1)/(\mu^2+1)$ are:

$$\cos\varphi - \alpha\frac{z_c \operatorname{cosec}\theta_c}{4u_c}\log(\theta-\theta_c)(1-\cos 2\varphi). \tag{62}$$

Thus,

$$\begin{aligned} \text{O}\exp^{(2)}(u^{*(2)}) &= \bar{u}_c + \alpha x_c\cos\varphi + \\ &\quad + \alpha^2\left[u_2^*(0, \varphi) - \frac{z_c \operatorname{cosec}\theta_c}{4u_c}\log(\theta-\theta_c)(1-\cos 2\varphi)\right]. \end{aligned} \tag{63}$$

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* The value of $t(\theta)$ is unchanged by a transformation from wind to body coordinates.

† cf. equation (28), Ref. 2.

Similarly, from (23) and (29) it can be deduced that if $\hat{f}(\theta)$ can be expressed as

$$\hat{f}(\theta) = \hat{f}_c + g(\theta),$$

where f_c is a constant, and $g(\theta)$ vanishes as $\theta \to \theta_c$ either like $(\theta - \theta_c)^n$ or $(\theta - \theta_c)^n \log(\theta - \theta_c)$, where $n > 0$, then the inner expansion of $\hat{f}(\theta)$ of whatever order is given simply by $\hat{f}_c$, while

$$I \lim \{\log(\theta - \theta_c)\} = -\frac{2\zeta}{\alpha}$$
$$= \log(\theta - \theta_c)$$

in terms of the outer variables.

Thus the generalised matching principle applied to u gives

$$u_c + \alpha x_c \cos\varphi + \alpha^2 \left[\{u_0'(\theta_c) + u_2'(\theta_c)\cos 2\varphi\} - \frac{z_c t_c \operatorname{cosec}\theta_c}{4u_c}\log(\theta - \theta_c)(1 - \cos 2\varphi)\right]$$
$$= u_c + \alpha x_c \cos\varphi + \alpha^2\left[u_2^*(0, \varphi) - \frac{z_c t_c \operatorname{cosec}\theta_c}{4u_c}\log(\theta - \theta)(1 - \cos 2\varphi)\right]. \quad (64)$$

The terms in $\log(\theta - \theta_c)$ thus match automatically and in determing the second term in the inner expansion for u, we need only set

$$u_2^*(0, \varphi) = u_0'(\theta_c) + u_2'(\theta_c)\cos 2\varphi. \quad (65)$$

Further, when the composite expansion for u is made up from the inner and outer expansions, the divergent terms in $\log(\theta - \theta_c)$ cancel, and the resultant expression, which represents the physical quantity, will be finite throughout the flow field.

It can readily be shown, by a similar use of the matching principle to the above, that the composite expansion for ρ is also finite, and that in determining $\rho_2^*(\zeta, \varphi)$, we again need only set

$$\rho_2^*(0, \varphi) = \rho_c\left(\frac{\rho_0}{\bar{\rho}}\right)' + \rho_c\left(\frac{\rho_2}{\bar{\rho}}\right)\cos 2\varphi. \quad (66)$$

Finally, in a more straightforward way, it can be shown that the matching principle gives

$$w_2^*(0, \varphi) = w_2(\theta_c)\sin 2\varphi$$
$$p_2^*(0, \varphi) = p_0(\theta_c) + p_2(\theta_c)\cos 2\varphi.$$

4.2. *Second-Order Inner Equations and Solution.*

A second application of the inner limit to equations (1), (2a), (2c) and (3) yields:

$$\frac{\partial v_2^*}{\partial \zeta} = 0 \quad (67)$$

$$u_c\frac{\partial u_0^*}{\partial \zeta} + w_1^* \operatorname{cosec}\theta_c \frac{\partial u_2^*}{\partial \varphi} = -u_c V_1^* \frac{\partial u_1^*}{\partial \zeta} - w_2^* \operatorname{cosec}\theta_c \frac{\partial u_1^*}{\partial \varphi} + 2w_1^* w_2^* \quad (68)$$

$$\frac{\partial p_2^*}{\partial \zeta} = 0 \quad (69)$$

$$\operatorname{cosec}\theta_c w_1^* \frac{\partial w_1^*}{\partial \varphi} + w_1^* u_1^* + u_c w_2^* = -\frac{1}{\rho_c \sin\theta_c}\frac{\partial p_2^*}{\partial \varphi} + \frac{\rho_1^*}{\rho_c^2 \sin\theta_c}\frac{\partial p_1^*}{\partial \varphi} \quad (70)$$

$$u_c\frac{\partial}{\partial \zeta}\left(\frac{s_2^*}{C_v}\right) + w_1^* \operatorname{cosec}\theta_c \frac{\partial}{\partial \varphi}\left(\frac{s_2^*}{C_v}\right) = -u_c V_1^* \frac{\partial}{\partial \zeta}\left(\frac{s_1^*}{C_v}\right) - w_2^* \operatorname{cosec}\theta^2 \frac{\partial}{\partial \varphi}\left(\frac{s_1^*}{C_v}\right). \quad (71)$$

Immediately from (67) and the boundary condition $v_2^*(\infty, \varphi') = 0$, we have

$$v_2^*(\zeta, \varphi) = 0, \tag{72}$$

and from (69) and the matching principle

$$p_2^*(\zeta, \varphi) = p_0(\theta_c) + p_2(\theta_c)\cos 2\varphi. \tag{73}$$

Then (70), together with the solutions for the first-order terms, gives

$$w_2^* = W_1 \sin 2\varphi + W_2 \Phi(\mu) \sin\varphi \tag{74}$$

where

$$W_1 = \frac{2p_2'(\theta_c)\,\mathrm{cosec}\,\theta_c}{\rho_c u_c} - \frac{1}{2}\frac{[\eta_c]^2}{\gamma u_c p_c \rho_c} - \frac{[z_c]^2}{u_c}(\mathrm{cosec}\,\theta_c - \sin\theta_c)$$

$$W_2 = \frac{\eta_c d}{\gamma u_c \rho_c \sin\theta_c} - \frac{t_c z_c}{u_c}.$$

With $w_2^*(\zeta, \varphi)$ known {and we again observe that its determination from (70) also satisfies the matching condition $w_2^*(0, \varphi) = w_2(\theta_c)\sin 2\varphi$, as can be confirmed from equation (48)}, it is now possible to determine $u_2^*(\zeta, \varphi)$ and (s_2^*/C_v) from equations (68) and (71), and the appropriate matching conditions. $u_2^*(\zeta, \varphi)$ is given by

$$\begin{aligned} u_2^*(\zeta, \varphi) = {} & u_0'(\theta_c) + u_2'(\theta_c)\{2\Phi^2(\mu) - 1\} + K_1 \mu \Phi(\mu)\frac{d\Phi(\mu)}{d\mu}\log\left(\frac{\cot\varphi/2}{\mu}\right) + \\ & + K_2 \mu \frac{d\Phi(\mu)}{d\mu}\log\left(\frac{\cot^2\varphi/2 + 1}{\mu^2+1}\,\frac{\mu}{\cot\varphi/2}\right) + \\ & + K_3\{\cos 2\varphi - 2\Phi^2(\mu) + 1\} + K_4 \Phi(\mu)\{\cos\varphi - \Phi(\mu)\} \end{aligned} \tag{75}$$

where μ and $\Phi(\mu)$ are defined above, and

$$K_1 = -t_c\left[\frac{W_2}{z_c} - \frac{t_c}{u_c}\right]$$

$$K_2 = t_c\left[\frac{2W_1}{z_c} + \sin\theta_c \frac{z_c}{u_c} - \frac{z_c}{2u_c \sin\theta_c}\right]$$

$$K_3 = -\frac{1}{2} W_1 \sin\theta_c$$

$$K_4 = -W_2 \sin\theta_c$$

and

$$\begin{aligned} \frac{s_2^*}{C_v} = {} & \left[\frac{p_0(\theta_c)}{p_c} - \gamma\frac{\rho_0'(\theta_c)}{\rho_c} - \frac{1}{4}\left\{\left(\frac{\eta_c}{p_c}\right)^2 - \gamma\left(\frac{\xi_c}{\rho_c}\right)^2\right\}\right] + \\ & + \left[\frac{p_2(\theta_c)}{p_c} - \gamma\frac{\rho_2'(\theta_c)}{\rho_c} - \frac{1}{4}\left\{\left(\frac{\eta_c}{p_c}\right)^2 - \gamma\left(\frac{\xi_c}{\rho_c}\right)^2\right\}\right][2\Phi^2(\mu) - 1] + \\ & + K_5 \mu \Phi(\mu)\frac{d\Phi(\mu)}{d\mu}\log\left(\frac{\cot\varphi/2}{\mu}\right) + K_6 \mu \frac{d\Phi(\mu)}{d\mu}\log\left(\frac{\cot^2\varphi/2+1}{\mu^2+1}\,\frac{\mu}{\cot\varphi/2}\right) \end{aligned} \tag{76}$$

where

$$K_5 = -d\left(\frac{W_2}{z_c} - \frac{t_c}{U_c}\right)$$

$$K_6 = -d\left(\frac{2W_1}{z_c} + \sin\theta_c\frac{z_c}{u_c} - \frac{z_c}{2u_c\sin\theta_c}\right).$$

Finally, using (4), we have for $\rho_2{}^*(\zeta, \varphi)$:

$$\frac{\rho_2{}^*(\zeta, \varphi)}{\rho_c} = \frac{1}{\gamma}\left[\frac{p_2{}^*}{p_c} - \frac{s_2{}^*}{C_v}\right] - \frac{1}{2}\left[\frac{1}{\gamma}\frac{\eta_c{}^2}{p_c{}^2}\cos^2\varphi - \left(\frac{\rho_1{}^*}{\rho_c}\right)^2\right]. \tag{77}$$

5. *Discussion.*

We begin by writing down (terms of outer variables) the composite solutions for the flow quantities, made up from the various solutions obtained or sketched out in this report:

$$\begin{aligned} u^{(2)} = {} & \bar{u} + \alpha x \cos\varphi + \alpha^2(u_0{}' + u_2{}' \cos 2\varphi) + \alpha[t_c\Phi(\mu) - \sin\theta_c z_c \cos\varphi] + \\ & + \alpha^2\left[u_2{}'(\theta_c)\{2\Phi^2(\mu) - 1\} + K_1\mu\Phi(\mu)\frac{d\Phi(\mu)}{d\mu}\log\left(\frac{\cot\varphi/2}{\mu}\right) + \right. \\ & + K_2\mu\frac{d\Phi(\mu)}{d\mu}\log\left(\frac{\cot^2\varphi/2 + 1}{\mu^2 + 1}\,\frac{\mu}{\cot\varphi/2}\right) + K_3\{\cos 2\varphi - 2\Phi^2(\mu) + 1\} + \\ & \left. + K_4\Phi(\mu)\{\cos\varphi - \Phi(\mu)\}\right] - \alpha x_c \cos\varphi - \alpha^2 u_2{}'(\theta_c)\cos 2\varphi \end{aligned} \tag{78}$$

$$v^{(2)} = \bar{v} + \alpha y \cos\varphi + \alpha^2(v_0{}' + v_2{}' \cos 2\varphi) \tag{79}$$

$$w^{(2)} = \alpha z \sin\varphi + \alpha^2\{w_2 \sin 2\varphi + W_2\Phi(\mu)\sin\varphi - W_2\cos\varphi\sin\varphi\} \tag{80}$$

$$p^{(2)} = \bar{p} + \alpha\eta\cos\varphi + \alpha^2(p_0 + p_2\cos 2\varphi) \tag{81}$$

$$\begin{aligned} \rho^{(2)} = {} & \bar{\rho} + \alpha\xi\cos\varphi + \alpha^2\left\{\bar{\rho}\left(\frac{\rho_0}{\bar{\rho}}\right)' + \bar{\rho}\left(\frac{\rho_2}{\bar{\rho}}\right)'\cos 2\varphi\right\} + \alpha\left\{\rho_c\frac{\eta_c}{\gamma p_c}\cos\varphi - \frac{\rho_c d}{\gamma}\Phi(\mu)\right\} + \\ & + \alpha^2\left[\frac{\rho_c}{\gamma}\left\{\frac{p_0(\theta_c)}{p_c} + \frac{p_2(\theta_c)}{p_c}\cos 2\varphi - \frac{s_2{}^*}{C_v}\right\} + \frac{\rho_c}{2}\left\{\frac{1}{\gamma}\frac{\eta_c{}^2}{p_c{}^2}\cos^2\varphi - \left(\frac{\eta_c}{\gamma p_c}\cos\varphi - \frac{d}{\gamma}\Phi(\mu)\right)^2\right\}\right] - \\ & - \alpha\frac{\rho_c}{\gamma}\left\{\frac{\eta_c}{p_c} - d\right\}\cos\varphi - \alpha^2[\rho_0{}'(\theta_c) + \rho_2{}'(\theta_c)\cos 2\varphi] \end{aligned} \tag{82}$$

where $s_2{}^*/C_v$ is given by equation (79), and μ and $\Phi(\mu)$ are defined in (42), (42) above.

The expansions are written down in the order O exp + I exp − O exp (I exp) {or I exp (O exp) as the case may be}, and identical terms have been cancelled. We observe that in equation (79), we have written the second-order term for v as being given by $v_0{}'(\theta) + v_2{}'(\theta)\cos 2\varphi$, i.e. by the solution of equations (54) to (57). This choice arises from application of the generalized matching principle to the expression $v/\bar{v}(\theta)$, from whence it is seen that the logarithmic terms cancel out in the composite expansion, as do those in the expressions for $\hat{u}_2(\theta, \varphi)$ and $\hat{p}_2(\theta, \varphi)$.

The solution represented by the expressions (78) to (82) is clearly non-analytic. All the terms in the inner solutions (save $w_1{}^*$, $p_1{}^*$ and $p_2{}^*$) are functions of the variable μ, so that they have infinite first derivatives with respect to θ at $\theta = \theta_c$, on the cone surface. The outer solutions, expanded in series close to $\theta = \theta_c$, all contain in addition to regular series of integral powers of $(\theta - \theta_c)$ series of the form $\sum_{j=N}^{\infty} C_j(\theta - \theta_c)^{j+1/2}$; in the first-order terms, $N = 0$ for $z(\theta)$, 1 for $y(\theta)$ and 2 for $x(\theta)$, $\eta(\theta)$ and $\xi(\theta)$. In the second-order terms (which have not been thoroughly investigated) it is evident by inspection of equations (54) to (57) that at least $u_n{}'(\theta)$ and $\rho_n{}'(\theta)$ contain such expansions, with $N = 0$, and also that terms like $(\theta - \theta_c)\log(\theta - \theta_c)$ occur.

The non-analytic nature of the inner (vortical-layer) solution seems to be characteristic of perturbation solutions for non-axially-symmetric conical flows. A discussion of these has been given by Cheng[17]. The 'half-power' series which cause the outer solution to be non-analytic as well are analogous to those which have been found in another perturbation problem based on the Taylor-Maccoll solution, namely the calculation of the flow past a pointed body of revolution at zero incidence, treated by Shen and Lin[18]. This latter case has been rather fully discussed by Van Dyke[19]. Whether or not these solutions are non-analytic because of some inherent property of the full equations and boundary conditions, or because of linearization procedures, cannot be discussed on the basis of the work reported here. We shall merely remark that because these solutions are non-analytic, particular care should be taken in their computation. In this connexion, some methods described by Kaplan[20] may be of value.

The solution corresponds qualitatively to Ferri's description of the vortical layer. From (40), (41) and (42), we see that close to the cone, entropy is given to first order in α by $\Phi(\mu)d$; $\Phi(\mu)$ tends to $\cos\varphi$ at any finite distance from the surface, but at the surface, where $\mu\{= (\theta - \theta_c)^{-\alpha z_c/2u_c \sin\theta_c} \cot\varphi/2\}$ is in general infinite; $\Phi(\mu) = 1$, and the entropy is constant, equal to that along the sheet of streamlines originating from the most windward generator of the shock wave. At $\varphi = \pi$, however, $\cot\varphi/2$ is zero, and μ is indeterminate; physically, all the streamlines tend towards the leeward generator of the cone, where the entropy is many-valued.

From (76), it is clear that this picture is little altered by including second-order effects. Again, the second-order entropy term tends to its outer value at finite distance from the cone and is constant on the surface, equal to its value on the sheet of streamlines originating from the most windward shock generator.

For moderately high values of incidence, alternative flow pictures have been proposed. That shown in Fig. 2b corresponds to a change in sign of the circumferential velocity w at S. If the plausible assumption that $\{u + (1/\sin\theta)\partial w/\partial\varphi\} \neq 0$ is made it may be deduced from equation (2c) that this would occur if the pressure passed through a minimum at S, i.e. at some value of φ less than π, and this behaviour has been observed experimentally. The entropy on the cone surface is then piecewise-constant: on BS its value is that gained in passage through the shock at A, and on CS in passage through the shock at D. At S the entropy is many-valued. The streamline pattern in Fig. 2c was suggested by Ferri[21]. In this case, the singular ray represented by the point S has been lifted off the cone surface.

Neither of these flow patterns could be derived by analysis of the sort used in this report, in which, in the passage to the limit $\alpha \to 0$, the hierarchy $\bar{v} \gg \alpha v_1 \gg \alpha^2 v_2$, $\alpha w_1 \gg \alpha^2 w_2$, etc., must be maintained. For the flow represented by Fig. 2b to occur, it would be necessary for $\alpha^2 w_2$ to exceed αw_1 in magnitude for a finite range of φ, while for that in 2c, $\alpha^2 v_2$ would have to exceed $v_0 + \alpha v_1$ similarly.

Conclusions.

The method of inner and outer expansions has been applied to the problem of calculating the supersonic flow about a circular cone at small incidence. Stone's first-order solution for this problem, and the vortical-layer solution obtained separately by Cheng[9], Bulakh[8], and Woods [10], are more satisfactorily connected than hitherto as the first terms respectively of inner and outer expansions for the problem. It is found that the terms in Stone's second-order solution which are theoretically infinite at the cone surface are in fact removed from the final expressions for physical quantities by the generalized matching process. In practice this means that the computation of the second-order solution, which in the M.I.T. cone tables was restricted to low Mach numbers by the divergent terms, can now be extended to hypersonic speeds, using equations which have been derived above.

The second-order terms in the inner expansion have been obtained in closed form. These, however, contain expressions which depend on the values at $\theta = \theta_c$ of the corresponding terms in the outer expansions, which have as yet not been computed.

SYMBOLS

C_v		Specific heat at constant volume
d		First-order entropy perturbation coefficient {equation (10)}
p		Pressure
r, θ, φ		Spherical polar coordinates
u, v, w		Corresponding velocity components
S		Specific entropy
t	=	$x + z \sin \theta$
x		First-order perturbation coefficient for u (equation (27))
y		First-order perturbation coefficient for v (equation (27))
z		First-order perturbation coefficient for w (equation (27))
α		Angle of incidence
β		Second-order perturbation coefficient for shock shape {equation (19)}
γ		Adiabatic index
ϵ		First-order perturbation coefficient for shock shape {equation (7)}
ζ, ϕ'		Inner coordinates {equation (29)}
η		First-order pressure perturbation coefficient {equation (27)}
ξ		First-order density perturbation coefficient {equation (27)}
ρ		Density
$(\bar{\ })$		Quantities in Taylor-Maccoll solution
$(\hat{\ })$		Quantities in outer expansions
$(\ )^*$		Quantities in inner expansions
$(\ )_c$		Quantities at cone surface
$(\ )_s$		Quantities at shock wave
$(\ )_{1,2}$		(When associated with circumflex and asterisk) first- and second-order terms in expansions in powers of α {in general: $(\ )_n$}
$(\ )_{0,2}$		(When *not* associated with circumflex and asterisk) in coefficients of $\cos(0 \times \varphi)$ and $\cos 2\varphi$ in Stone's second-order solution {in general: $(\ )_i$}

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APPENDIX

Expansions of the Taylor-Maccoll Solution in Powers of $(\theta - \theta_c)$

$$\bar{u}(\theta) = u_c - u_c(\theta - \theta_c)^2 + u_c \cot \theta_c \frac{(\theta - \theta_c)^3}{3} + \ldots$$

$$\bar{v}(\theta) = -2u_c(\theta - \theta_c) + u_c \cot \theta_c(\theta - \theta_c)^2 - u_c \left\{\cot^2 \theta_c + \frac{4}{3} M_c^2\right\} (\theta - \theta_c)^3 + \ldots$$

$$\bar{p}(\theta) = p_c - \rho_c u_c^2(\theta - \theta_c)^2 + \frac{5}{3} \rho_c u_c^2 \cot \theta_c(\theta - \theta_c)^3 + \ldots$$

$$\bar{\rho}_0(\theta) = \rho_c - \rho_c M_c^2(\theta - \theta_c)^2 + \frac{5}{3} \rho_c M_c^2 \cot \theta_c(\theta - \theta_c)^3 + \ldots$$

Here the subscript c refers to conditions at

$$\theta = \theta_c,$$

and

$$M_c^2 = \frac{\rho_c u_c^2}{\gamma p_c}.$$

(91679) Wt. 67/7689 K.5 7/65 Hw.

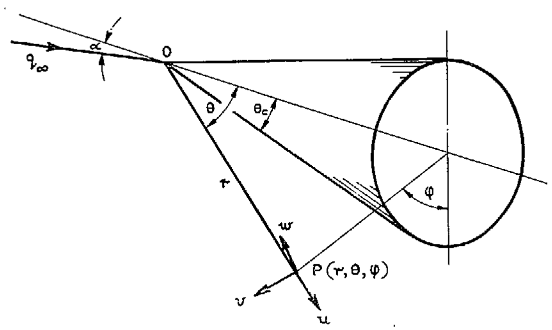

FIG. 1. Body coordinate system.

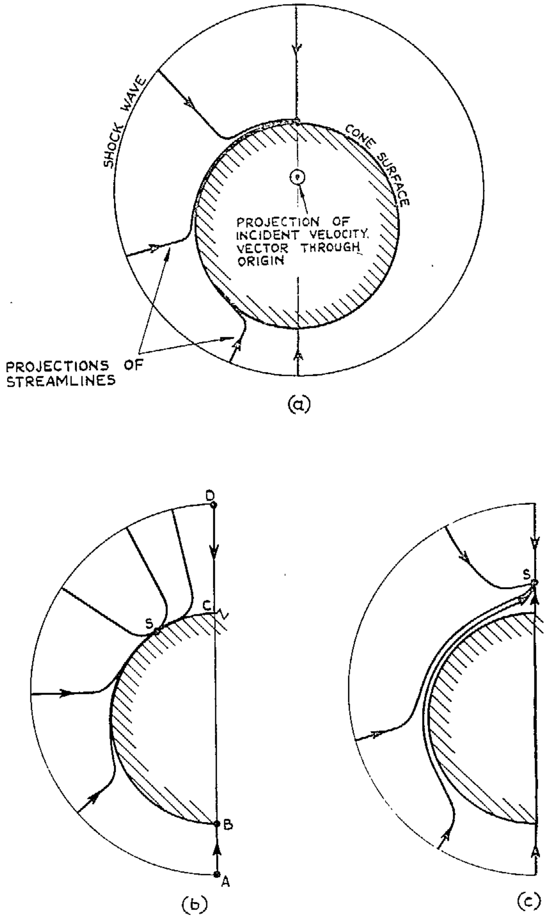

FIG. 2. Flow patterns.

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