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The Calculation of Generalised Forces on Oscillating Wings in Supersonic Flow by Lifting Surface Theory

By G. Z. HARRIS, Ph.D.
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Summary.
A numerical method of solving the integral equation connecting the lift and downwash on a wing oscillating harmonically in a supersonic flow of any Mach number is described. The integral equation is replaced by a matrix equation connecting the values of the lift and downwash at sets of points on the wing, and the generalised aerodynamic forces acting on the wing are found simply as a matrix product. Comparisons are made between aerodynamic derivatives calculated by this method, and theoretical and experimental derivatives from other sources.

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1. *Introduction.*

For many years the methods available for the prediction of the aerodynamic forces on oscillating wings in linearised supersonic flow were all subject to certain restrictions. For example, methods existed for delta wings and for planforms whose edges were supersonic; in other cases results could readily be found only under some limitation in frequency or for rigid-body oscillations. Recently, with large digital computers available, attention has turned towards methods of general application which can be used with no restriction on either Mach number, frequency, nature of planform or shape of the mode in which the wing distorts. These methods usually consider the known distribution of downwash on the planform and attempt to find the lift by numerical solution of an integral equation involving the downwash. Lifting surface theory, of the type suggested originally by Multhopp and extended to general unsteady flow by Richardson, is the basis of the approach considered in this report.

Multhopp's lifting surface theory[1] was originally devised for steady subsonic flow; the first extension to unsteady flow was that of the Multhopp-Garner theory[2], valid for low frequencies in subsonic flow. Richardson[3] proposed an extension of the Multhopp theory to cover unsteady flow at any frequency parameter and also for any Mach number. The actual applications of the Multhopp-Richardson method to subsonic, transonic and supersonic flow have been carried out separately, and the present report deals with a modification of the method applied to supersonic flow. This modified method has been used in programmes written for the Mercury digital computer, and these are described in a separate report[4] which gives details of the actual numerical work.

The method employs the integral equation which expresses the downwash in terms of the lift. This is too intractable for analytical solution and numerical methods have to be used. Two sets of points are chosen on the wing; at one set the lift (which is not known) is evaluated and at the other set the downwash values (which are known) are taken. The integral equation can then be replaced by a matrix equation connecting the lift values at the points of the one set with the downwash values at the points of the other set. From this matrix equation the lift values, and thence the generalised aerodynamic forces, can be found.

2. *The Integral Equation.*

Take Cartesian co-ordinates x, y and z fixed relative to the wing so that the x-axis lies in the wing mean plane and is in the direction of the airflow of velocity V, the z-axis is downwards and the y-axis is directed to starboard (*see* Fig. 1). x, y and z are referred to a reference length $\bar{c}$, the wing mean chord. Other reference lengths could be chosen, but the mean chord has advantages when derivatives are the end product of the calculation. Then, if the wing oscillates with angular frequency ω, the associated frequency parameter being $\nu = \omega\bar{c}/V$, the integral equation connecting downwash $V e^{i\nu t}\, w(x,y)$ and lift $\rho V^2 e^{i\nu t}$ $l(x,y)$ is

$$w(x',y') = -\frac{1}{4\pi}\iint_S \frac{K(x'-x,\, y'-y)}{(y'-y)^2}\, l(x,y)\, dx\, dy, \tag{1}$$

where (*see*, for example, Watkins and Berman[5] or the slightly different form given by Richardson[3])

$$\left.\begin{aligned} K(X,Y) &= \frac{2X}{R}\exp\left\{\frac{-i\nu M^2 X}{\beta^2}\right\}\cos\left\{\frac{MR\nu}{\beta^2}\right\} + i\nu|Y|e^{-i\nu X}\int_{(X-MR)/(\beta^2|Y|)}^{(X+MR)/(\beta^2|Y|)} \frac{\tau}{\sqrt{(1+\tau^2)}}\, e^{-i\nu\tau|Y|}d\tau \\ &\qquad \text{if } X>\beta|Y| \text{ and } Y \neq 0, \\ K(X,Y) &= 2\,e^{-i\nu X} \qquad \text{if } X>\beta|Y| \text{ and } Y = 0, \\ K(X,Y) &= 0 \qquad \text{if } X\leqslant \beta|Y|, \end{aligned}\right\} \tag{2}$$

where $R = \sqrt{(X^2-\beta^2Y^2)}$ and $\beta^2 = M^2-1$. The integration in (1) is taken over the whole wing; t is a reduced time.

In view of the form of the kernel $K(X,Y)$ in (2), only that part of the integral taken over the area of the wing cut off by the forward Mach cone through $(x',y',0)$ need be considered. The area is shaded for the wing of Fig. 1.

The singularities of the function $K(X,Y)$ are considered in Ref. 5; near the forward Mach lines through $(x',y',0)$ it becomes large like a multiple of $1/R$.

It is more convenient in discussing (1) to use wing co-ordinates ξ and η defined by

$$\left.\begin{aligned} x-\bar{x}(y) &= \tfrac{1}{2}\xi\, c(\eta) \\ \text{and} \qquad y &= s\eta, \end{aligned}\right\} \tag{3}$$

$x = \bar{x}(y)$ being the equation of the wing mid-chord line, $\bar{c}c(\eta)$ being the local chord and $s\bar{c}$ the semispan. ξ' and η' are defined similarly. The integral equation (1) then becomes

$$w(\xi',\eta') = -\frac{1}{8\pi s}\int_{\eta_1(\xi',\eta')}^{\eta_2(\xi',\eta')} \frac{c(\eta)\, d\eta}{(\eta'-\eta)^2}\int_{-1}^{\xi_M(\xi',\eta',\eta)} K(x'-x,y'-y)\, l(\xi,\eta)\, d\xi \tag{4}$$

where

$$\xi_M(\xi',\eta',\eta) = \min\{\xi_1(\xi',\eta',\eta), 1\},$$

and $\xi = \xi_1(\xi',\eta',\eta)$ is the equation of the reversed Mach lines through (x',y'); for planforms having no swept-forward part of the leading edge these cut the leading edge or tips at two points where $\eta = \eta_1(\xi',\eta')$ and $\eta = \eta_2(\xi',\eta')$ (*see* Fig. 2). More complicated planforms, for which part of the leading edge is swept forward and where the reversed Mach lines through a point may cut the leading edge at four or more points, may be dealt with using a form of (4) containing additional limits in η. They are not considered specifically in what follows.

3. *The Approximation to the Lift and Downwash Distributions.*

3.1. *The Approximate Lift.*

The lift distribution has singular behaviour near the edges of the wing in certain circumstances. Near a subsonic leading-edge it becomes infinite like $1/\sqrt{(1+\xi)}$, and near a subsonic trailing edge it contains terms involving $\sqrt{(1-\xi)}$. There is no singularity at supersonic edges.

Thus, when considering the lift along a chordwise section of the wing a function $f(\xi)$ is used which takes these singularities into account; this function is defined by

$$f(\xi) = \left\{ \begin{array}{ll} \sqrt{\left(\dfrac{1-\xi}{1+\xi}\right)} & \text{(subsonic leading edge, subsonic trailing edge)} \\ 1/\sqrt{(1+\xi)} & \text{(subsonic leading edge, supersonic trailing edge)} \\ \sqrt{(1-\xi)} & \text{(supersonic leading edge, subsonic trailing edge)} \\ 1 & \text{(supersonic leading and trailing edges)} \end{array} \right\} \quad (5)$$

Similarly, in the spanwise variation of lift, the function $\sqrt{(1-\eta^2)}$ is used to account for the singular behaviour at the tips $\eta = \pm 1$. These functions do not account for singularities other than those at leading and trailing edges.

For numerical solution of the integral equation (4), the values of the lift at a set of points on the wing are taken as unknowns and the lift at points between is interpolated, taking into account the singularities in lift mentioned above. The position of these lift points is considered first.

3.2. *Choice of Lift Points.*

Suppose the chordwise lift distribution $l(\xi)$ at any spanwise station is approximated at m points $\xi_1, \ldots, \xi_m$ by a function $\bar{l}(\xi) f(\xi)$ in such a way that $\bar{l}(\xi)$ is a polynomial of degree $(m-1)$, and

$$\bar{l}(\xi_\alpha) f(\xi_\alpha) = l(\xi_\alpha). \qquad (\alpha = 1, \ldots, m) \quad (6)$$

Consider the effect of adding t $(t \leq m)$ points $\xi_{m+1}, \ldots, \xi_{m+t}$ to the set $\xi_1, \ldots, \xi_m$ and approximating to $l(\xi)$ at the $(m+t)$ points $\xi_1, \ldots, \xi_{m+t}$ in such a way that

$$\bar{l}_t(\xi_\alpha) f(\xi_\alpha) = l(\xi_\alpha), \qquad (\alpha = 1, \ldots, m+t), \quad (7)$$

where $\bar{l}_t(\xi)$ is a polynomial of degree $(m+t-1)$. If

$$\phi(\xi) = (\xi-\xi_1) \ldots (\xi-\xi_m)$$

then

$$\bar{l}_t(\xi) - \bar{l}(\xi) = \phi(\xi)N(\xi)$$

for some polynomial $N(\xi)$ of degree $(t-1)$, since $\bar{l}_t(\xi)$ and $\bar{l}(\xi)$ agree at the points $\xi_1, \ldots\ldots, \xi_m$. So

$$\int_{-1}^{1} \xi^p \{\bar{l}_t(\xi) - \bar{l}(\xi)\} f(\xi) d\xi = \int_{-1}^{1} \xi^p N(\xi)\phi(\xi) f(\xi) d\xi, \; (p = 0, \ldots, (m-t)). \tag{8}$$

The left-hand side of (8) is the difference in p^{th} moment between the two approximate lift distributions $\bar{l}(\xi)f(\xi)$ and $\bar{l}_t(\xi)f(\xi)$. If $\phi(\xi)$ is now chosen in such a way that

$$\int_{-1}^{1} \xi^q \phi(\xi) f(\xi) = 0, \qquad (q = 0, \ldots, m-1),$$

then the right-hand side of (8) will be zero whatever the function $N(\xi)$, that is whatever the choice of additional points $\xi_{m+1}, \ldots\ldots, \xi_{m+t}$. These relations imply that $\phi(\xi)$ is the m^{th} degree polynomial of the set orthogonal with respect to the weight function $f(\xi)$ over the interval $(-1,1)$ (*see*, for example Hildebrand[6], page 271) and the points $\xi_1, \ldots\ldots, \xi_m$ are its zeros. Equation (8) then gives

$$\int_{-1}^{1} \xi^p \{\bar{l}_t(\xi) - l(\xi)\} f(\xi)\, d\xi = 0, \qquad (p = 0, \ldots, (m-t)),$$

so the approximate lift distribution $\bar{l}(\xi)f(\xi)$ has

the same lift as any approximation obtained by approximating at any m additional points,

the same lift and moment as any approximation obtained by approximating at any $(m-1)$ additional points,

and so on, up to

the same lift, moment, ..., $(m-1)^{th}$ moment as any approximation obtained by approximating at any one additional point.

In this sense, the approximation found by fitting values to $l(\xi)$ at the zeros of the appropriate orthogonal polynomial is the best possible.

With this choice of $\phi(\xi)$, define interpolation functions

$$h_\alpha(\xi) = \frac{\phi(\xi)f(\xi)}{(\xi - \xi_\alpha)\phi'(\xi_\alpha)f(\xi_\alpha)}, \qquad (\alpha = 1, \ldots, m), \tag{9}$$

which will have the property

$$h_\alpha(\xi_\gamma) = \begin{cases} 0 \text{ if } \alpha \neq \gamma \\ 1 \text{ if } \alpha = \gamma. \end{cases}$$

The functions $h_\alpha(\xi)$ have the property[3] that

$$\int_{-1}^{1} h_\alpha(\xi)\, p(\xi)\, d\xi = 2H_\alpha\, p(\xi_\alpha), \qquad (10)$$

where

$$H_\alpha = \tfrac{1}{2}\int_{-1}^{1} h_\alpha(\xi)\, d\xi, \qquad (11)$$

for polynomials $p(\xi)$ of degree m or less in ξ. The lift distribution $l(\xi)$ is then approximated by

$$\sum_{\alpha=1}^{m} l(\xi_\alpha)\, h_\alpha(\xi).$$

An exactly similar argument may be used to provide a choice of spanwise position for the points at which the lift is to be evaluated. Across any section for which ξ = constant the lift $l(\eta)$ may be approximated by a function $\lambda(\eta)\sqrt{(1-\eta^2)}$; [$\lambda(\eta)$ being an $(n-1)^{th}$ degree polynomial] which agrees with the true lift at n points $\eta_1, \ldots, \eta_n$ so that

$$l(\eta_\beta) = \lambda(\eta_\beta)\sqrt{(1-\eta_\beta^2)}, \qquad (\beta = 1, \ldots, n).$$

If the points $\eta_1, \ldots, \eta_n$ are chosen to be the zeros of the n^{th} degree polynomial

$$\psi(\eta) = (\eta-\eta_1) \ldots (\eta-\eta_n)$$

of the set orthogonal with respect to the weight function $\sqrt{(1-\eta^2)}$ over the interval $(-1,1)$, then the approximation to the lift will be the best possible in the sense just described for the chordwise lift distribution, with the proviso that here we are considering the spanwise moments of lift. Interpolation functions

$$g_\beta(\eta) = \frac{\psi(\eta)\sqrt{(1-\eta^2)}}{(\eta-\eta_\beta)\,\psi'(\eta_\beta)\sqrt{(1-\eta_\beta^2)}}, \qquad (\beta = 1, \ldots, n), \qquad (12)$$

with the property

$$g_\beta(\eta_\gamma) = \begin{cases} 0 \text{ if } \beta \neq \gamma \\ 1 \text{ if } \beta = \gamma \end{cases}$$

can also be defined, so that the spanwise lift distribution can be approximated by

$$\sum_{\beta=1}^{n} l(\eta_\beta)\, g_\beta(\eta).$$

3.3. *Choice of Downwash Points.*

These are also chosen on the basis of two dimensional theory.

In two-dimensional steady flow, the downwash and lift for various leading and trailing-edge conditions are related by an integral equation of the form

$$w(\xi) = \int_{-1}^{1} K_0(\xi-\xi')\, l(\xi')\, d\xi'. \tag{13}$$

The exact form of the integral and of the kernel K_0 has been set out in Ref. 3. $K_0(\xi-\xi')$ has a singularity at $\xi-\xi' = 0$ and can be shown to have the property that for any positive integer p

$$\int_{-1}^{1} \xi'^{p} f(\xi')\, K_0(\xi-\xi')\, d\xi' = R_p(\xi), \tag{14}$$

where $R_p(\xi)$ is a polynomial in ξ of degree p. From (13) and (14) it follows that

$$\int_{-1}^{1} w(-\xi)\, \xi^p f(\xi)\, d\xi = \int_{-1}^{1} R_p(-\xi)\, l(\xi)\, d\xi. \tag{15}$$

If $l(\xi)$ is of the form $f(\xi) \times$ (a polynomial of degree m or less in ξ) then it follows from (13) and (14) that $w(\xi)$ is a polynomial in ξ of the same degree.

Now suppose that, as in Section 3.2, we have approximations $\bar{l}(\xi) f(\xi)$ and $l_1(\xi) f(\xi)$ to the lift distribution which agree with $l(\xi)$ as shown in equations (6) and (7); note that $l_1(\xi) f(\xi)$ approximates to $l(\xi)$ at only one point more than does $\bar{l}(\xi) f(\xi)$. Then, by virtue of the remarks following (15), the corresponding downwash distributions $\bar{w}(\xi)$ and $\bar{w}_1(\xi)$ are polynomials of degree $(m-1)$ and m respectively. Further,

$$\int_{-1}^{1} \{l_1(\xi)-\bar{l}(\xi)\}\, R_p(-\xi) f(\xi)\, d\xi = \int_{-1}^{1} \{\bar{w}_1(-\xi)-\bar{w}(-\xi)\}\, \xi^p f(\xi)\, d\xi, \tag{16}$$

from (15). The two approximate lift distributions have the same lift, moment, ..., $(m-1)^{th}$ moment and so

$$\int_{-1}^{1} \{\bar{w}_1(-\xi)-\bar{w}(-\xi)\}\, \xi^p f(\xi)\, d\xi = 0, \qquad (p = 0,\ldots,m-1), \tag{17}$$

since the left-hand side of (16) vanishes for these values of p. Since $\{\bar{w}_1(-\xi)-\bar{w}(-\xi)\}$ is a polynomial of degree m, it follows from (17) that

$$\bar{w}_1(-\xi)-\bar{w}(-\xi) = \text{constant} \times \phi(\xi).$$

The two downwash distribution $\bar{w}_1(\xi)$ and $\bar{w}(\xi)$ will thus have the same values at the points $-\xi_1,\ldots, -\xi_m$ which are the negative of the zeros of the orthogonal polynomial $\phi(\xi)$. This suggests that these points will give a suitable set for evaluation of the downwash; they are denoted by $\bar{\xi}_r$ $(r = 1,\ldots,m)$, and are taken as the downwash points in our calculation.

It should be noted that, from (17) the two approximate downwash distributions will also agree in their first m moments with respect to the weight function $f(\xi)$.

A similar argument applied to the integral equation

$$w(\eta) = \frac{1}{2\pi}\int_{-1}^{1}\frac{l(\eta')}{(\eta-\eta')^2}\,d\eta'$$

which connects lift and downwash in slender wing theory gives a choice for the spanwise downwash points. These are in fact $\eta_1, \ldots, \eta_n$, and are the same as the lift points.

4. *The Numerical Solution of the Integral Equation.*

4.1. *Approximation of the Lift on the Wing.*

The solution of (4) may now be considered. With a set of points for evaluating the lift now chosen according to Section 3.2, an approximation $\bar{l}(\xi,\eta)$ may be taken to $l(\xi,\eta)$, where

$$c(\eta)\bar{l}(\xi,\eta) = \sum_{\alpha=1}^{m} P_\alpha(\eta)\frac{h_\alpha(\xi)}{H_\alpha}. \tag{18}$$

Here

$$P_\alpha(\eta) = \sum_{\beta=1}^{n} P_{\alpha\beta}\frac{g_\beta(\eta)}{G_\beta}, \tag{19}$$

$$P_{\alpha\beta} = H_\alpha\, G_\beta\, c(\eta_\beta)\, l(\xi_\alpha,\eta_\beta) \tag{20}$$

and

$$G_\beta = \tfrac{1}{2}\int_{-1}^{1} g_\beta(\eta)\,d\eta. \tag{21}$$

The $P_{\alpha\beta}$ are used in place of the actual lift values $l(\xi_\alpha,\eta_\beta)$ at the lift points (ξ_α,η_β) since, for reasons which will appear later, they are fundamental in the discussion of the generalised forces.

Equation (18) and the integral equation (4) give

$$w(\xi',\eta') \doteqdot -\frac{1}{8\pi s}\sum_{\alpha=1}^{m}\int_{\eta_1(\xi',\eta')}^{\eta_2(\xi',\eta')}\frac{P_\alpha(\eta)\,i_\alpha(\xi',\eta',\eta)}{(\eta-\eta')^2}\,d\eta \tag{22}$$

where

$$i_\alpha(\xi',\eta',\eta) = \frac{1}{H_\alpha}\int_{-1}^{\xi_M(\xi',\eta',\eta)} h_\alpha(\xi)\,K(x'-x,y'-y)\,d\xi. \tag{23}$$

4.2. *The Logarithmic Singularity.*

By developing K into a series and integrating term by term, it may be shown that $i_\alpha(\xi',\eta',\eta)$ has a singularity of the type $(\eta-\eta')^2 \log|\eta-\eta'|$ when $\eta-\eta' = 0$; in fact

$$i_\alpha(\xi',\eta',\eta) = L_\alpha(\xi',\eta')\,(\eta-\eta')^2 \log|\eta-\eta'| + i^*(\xi',\eta',\eta) \qquad (24)$$

where

$$L_\alpha(\xi',\eta') = \frac{s^2}{H_\alpha\{c(\eta')\}^2}\left[\begin{array}{l} 2\,i(M^2+1)\nu\, c(\eta')\, h_\alpha(\xi') + 4(M^2-1)h_\alpha'(\xi') \\ + \nu^2\{c(\eta')\}^2 \displaystyle\int_{-1}^{\xi'} e^{-\frac{1}{2}i\nu c(\eta')(\xi'-\xi)} h_\alpha(\xi)\, d\xi \end{array}\right] \qquad (25)$$

and $i_\alpha^*(\xi',\eta',\eta)$ contains only logarithmic terms such as $(\eta-\eta')^4 \log|\eta-\eta'|$, $(\eta-\eta')^6 \log|\eta-\eta'|$ etc. The derivation of the term $L_\alpha(\xi',\eta')$ is due to Minhinnick and is given in Appendix A.

The logarithmic term in (24) clearly needs special attention, since $i_\alpha(\xi',\eta',\eta)$ occurs in the integrands on the right-hand side of (22). This term can be dealt with by using the method of Mangler and Spencer[7]. Thus, when later dealing with the numerical solution of (22) we will write

$$P_\alpha(\eta) i_\alpha(\xi',\eta',\eta) = P_\alpha(\eta') \sqrt{\left(\frac{1-\eta^2}{1-\eta'^2}\right)}\, L_\alpha(\xi',\eta')\,(\eta-\eta')^2 \log|\eta-\eta'| +$$

$$+ \left\{ P_\alpha(\eta) i_\alpha(\xi',\eta',\eta) - P_\alpha(\eta') \sqrt{\left(\frac{1-\eta^2}{1-\eta'^2}\right)}\, L_\alpha(\xi',\eta')\,(\eta-\eta')^2 \log|\eta-\eta'| \right\}. \qquad (26)$$

The term within the curly brackets on the right of (26) has no singularity of the type $(\eta-\eta')^2 \log|\eta-\eta'|$.

4.3. *The Chordwise Integration.*

The numerical solution of the integral equation may now be considered, and we first take the chordwise integral of (23).

If the upper limit of the integral in (23) is unity, i.e. if the chordwise integral is taken between leading and trailing edges, the only singularity in the integrand arises from $h_\alpha(\xi)$. If, however, the upper limit is $\xi_1(\xi',\eta',\eta)$ (corresponding to the Mach line) the singularities in the integrand arise from both $h_\alpha(\xi)$ and K. A co-ordinate ζ is introduced so that $\zeta = -1$ at the wing leading edge and $\zeta = 1$ at the trailing edge or Mach line, whichever is further upstream at the chordwise section being considered. In fact

$$\zeta(\xi,\eta,\xi',\eta') = \frac{1-\xi_M(\xi',\eta',\eta)+2\xi}{1+\xi_M(\xi',\eta',\eta)}. \qquad (27)$$

A function $k(\zeta)$ containing the singular terms of $h_\alpha K$ in (23) may then be defined by

$$k(\zeta) = \left\{ \begin{array}{ll} \sqrt{\left(\dfrac{1-\zeta}{1+\zeta}\right)} & \text{(Limits of (23) at a subsonic leading edge and a subsonic trailing edge)} \\ 1/\sqrt{(1-\zeta^2)} & \text{(Limits at subsonic leading edge and the Mach line)} \\ 1/\sqrt{(1-\zeta)} & \text{(Limits at supersonic leading edge and Mach line)} \\ \sqrt{(1-\zeta)} & \text{(Limits at supersonic leading edge and subsonic trailing edge)} \\ 1 & \text{(Limits at supersonic leading edge and supersonic trailing edge)} \\ 1/\sqrt{(1+\zeta)} & \text{(Limits at subsonic leading edge and supersonic trailing edge)} \end{array} \right\} \quad (28)$$

The way in which these singularities may be achieved for different planforms is shown in Fig. 3.

With the change of variables from ξ to ζ equation (23) becomes

$$i_\alpha(\xi',\eta',\eta) = \frac{1}{2H_\alpha}\left(1+\xi_M(\xi',\eta',\eta)\right)\int_{-1}^{1} h_\alpha(\xi)\, K(x'-x,y'-y)\, d\zeta$$

and a Gaussian integration formula which takes the appropriate singularity $k(\zeta)$ of $h_\alpha(\xi)\, K$ into account may be used to approximate to $i_\alpha(\xi',\eta',\eta)$. In fact

$$i_\alpha(\xi',\eta',\eta) \doteqdot \frac{1}{2H_\alpha}\left\{1+\xi_M(\xi',\eta',\eta)\right\}\sum_{\lambda=1}^{p}\frac{W_\lambda}{k(\zeta_\lambda)}\left[h_\alpha(\xi)K(x'-x,y'-y)\right]_{\zeta=\zeta_\lambda} \quad (29)$$

for a p-point integration formula, where W_λ and ζ_λ are the Gaussian weights and zeros in the integration formula appropriate to the singularity $k(\zeta)$. Also, since a p-point Gaussian integration formula is exact for polynomials of degree $(2p-1)$ or less (*see*, for example, Hildebrand[6], page 319) (29) would be exact were $Kh_\alpha(\xi)$ of the form $k(\zeta)$ (polynomial in ζ of degree $(2p-1)$). This provides an estimate of how good a chordwise representation of K is provided by the number of integration points taken.

4.4. *The Spanwise Integration.*

The integration of (22) can now be performed approximately. (26) and (22) give

$$w(\xi',\eta') \doteqdot -\frac{1}{8\pi s}\sum_{\alpha=1}^{m}\left\{\int_{\eta_1(\xi',\eta')}^{\eta_2(\xi',\eta')}\frac{u_\alpha(\xi',\eta',\eta)}{(\eta-\eta')^2}\,d\eta + L_\alpha(\xi',\eta')\frac{P_\alpha(\eta')}{\sqrt{(1-\eta'^2)}}\int_{\eta_1}^{\eta_2}\sqrt{(1-\eta^2)}\log|\eta-\eta'|\,d\eta\right\} \quad (30)$$

where

$$u_\alpha(\xi',\eta',\eta) = P_\alpha(\eta)i_\alpha(\xi',\eta',\eta) - P_\alpha(\eta')\sqrt{\left(\frac{1-\eta^2}{1-\eta'^2}\right)}L_\alpha(\xi',\eta')\,(\eta-\eta')^2\log|\eta-\eta'|. \quad (31)$$

The first integral within the brackets on the right of (30) is a principal value integral to which we wish to approximate; the numerator has no $(\eta-\eta')^2 \log|\eta-\eta'|$ term so we may hope to approximate to it reasonably by a polynomial fitted at q points in the interval (η_1,η_2). If a variable

$$\phi = \frac{2\eta-\eta_1-\eta_2}{\eta_2-\eta_1} \tag{32}$$

is defined so that $\phi = (-1,1)$ corresponds to $\eta = (\eta_1,\eta_2)$, a suitable set of points at which to fit a polynomial to $u_\alpha(\xi',\eta',\eta)$ is the set of zeros of the q^{th} Chebyshev polynomial. This is the set of points

$$\phi_\gamma = \cos\pi\left(\frac{2q-2\gamma+1}{2q}\right), \qquad (\gamma = 1,\ldots,q). \tag{33}$$

A set of interpolation polynomials based on these zeros may also be defined such that $p_\gamma(\phi)$, a polynomial in ϕ of degree $(q-1)$, satisfies

$$p_\gamma(\phi_\delta) = \begin{cases} 0 \text{ if } \gamma \neq \delta \\ 1 \text{ if } \gamma = \delta\,. \end{cases} \tag{34}$$

$u_\alpha(\xi',\eta',\eta)$ may then be approximated by the expression

$$u_\alpha(\xi',\eta',\eta) \doteqdot \sum_{\gamma=1}^{q} u_\alpha\left(\xi',\eta',\bar\eta_\gamma(\xi',\eta')\right) p_\gamma(\phi),$$

where $\bar\eta_\gamma(\xi',\eta')$ corresponds to ϕ_γ in (32). It follows that

$$\int_{\eta_1}^{\eta_2} \frac{u_\alpha(\xi',\eta',\eta)}{(\eta-\eta')^2}\,d\eta = \frac{2}{\eta_2-\eta_1}\int_{-1}^{1}\frac{u_\alpha(\xi',\eta',\eta)}{(\phi-\phi')^2}\,d\phi,$$

which is an exact equation, and hence that

$$\int_{\eta_1}^{\eta_2} \frac{u_\alpha(\xi',\eta',\eta)}{(\eta-\eta')^2}\,d\eta \doteqdot \frac{2}{\eta_2-\eta_1}\sum_{\gamma=1}^{q} u_\alpha(\xi',\eta',\bar\eta_\gamma(\xi',\eta'))\int_{-1}^{1}\frac{p_\gamma(\phi)}{(\phi-\phi')^2}\,d\phi, \tag{35}$$

which is approximate. The integrals $\int_{-1}^{1}\frac{p_\gamma(\phi)}{(\phi-\phi')^2}\,d\phi$ can be evaluated exactly, as shown in Appendix B.

This provides a method for evaluating the first integral within the brackets on the right-hand side of (30); the remaining integral

$$\int_{\eta_1}^{\eta_2}(1-\eta^2)^{\frac{1}{2}}\log|\eta-\eta'|\,d\eta$$

can be found numerically for any particular value of η'.

4.5. *Solution of the Integral Equation.*

A set of mn lift points (ξ_α,η_β) at which the lift distribution is evaluated has been chosen over the wing by the method of Section 3.2. We may also choose a set of mn points $(\bar{\xi}_r,\eta_s)$ at which the downwash is evaluated by the method described in Section 3.3. If m chordwise and n spanwise stations are taken we get, over the whole wing, sets of lift and downwash points as shown in the example of Fig. 4; in view of the remarks of Section 3.3, the spanwise positions of the lift and downwash points are the same.

To obtain the downwash at any particular point $(\bar{\xi}_r,\eta_s)$ combine the equations (19), (22), (26), (29), (30), (31) and (35) and substitute $(\bar{\xi}_r,\eta_s)$ for (ξ',η') to get

$$-8\pi s\, w(\bar{\xi}_r,\eta_s) = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} P_{\alpha\beta}\, C_{\alpha\beta}(\bar{\xi}_r,\eta_s) \tag{36}$$

where

$$\begin{aligned} C_{\alpha\beta}(\bar{\xi}_r,\eta_s) &= \frac{1}{G_\beta(\eta_2(\bar{\xi}_r,\eta_s)-\eta_1(\bar{\xi}_r,\eta_s))} \sum_{\gamma=1}^{q}\sum_{\lambda=1}^{p} \frac{g_\beta(\eta_{rs,\gamma}) W_\lambda \{1+\xi_M(\bar{\xi}_r,\eta_s,\eta_{rs,\gamma})\} h_\alpha(\xi_{rs,\lambda\gamma})}{H_\alpha\, k(\zeta_\lambda)} \times \\ &\times K(x_{rs}-x_{rs,\lambda\gamma}, y_s - y_{rs,\gamma}) \int_{-1}^{1} \frac{p_\gamma(\phi)}{(\phi-\phi_{rs})^2}\, d\phi + \\ &+ \delta_{\beta s} \frac{L_\alpha(\bar{\xi}_r,\eta_s)}{G_s\sqrt{(1-\eta_s^2)}} \left\{ \int_{\eta_1(\bar{\xi}_r,\eta_s)}^{\eta_2(\bar{\xi}_r,\eta_s)} \sqrt{(1-\eta^2)} \log|\eta-\eta_s|\, d\eta \,- \right. \\ &- \left. \frac{2}{\eta_2(\bar{\xi}_r,\eta_s)-\eta_1(\bar{\xi}_r,\eta_s)} \sum_{\gamma=1}^{q} \sqrt{(1-\eta_{rs,\gamma}^2)} \left(\eta-\eta_{rs,\gamma}\right)^2 \log|\eta-\eta_{rs,\gamma}| \int_{-1}^{1} \frac{p_\gamma(\phi) d\phi}{(\phi-\phi_{rs})^2} \right\}. \end{aligned} \tag{37}$$

The H_α is included in the double summation since the lift singularity $k(\zeta)$ may have different forms at different spanwise positions. A simplification occurs during the substitution, as (19) gives

$$P_\alpha(\eta_s) = P_{\alpha s}/G_s,$$

since the spanwise positions of the lift and downwash points are the same.

Taking (36) for each of the mn downwash points we have a set of mn equations for the mn unknowns $P_{\alpha\beta}$.

5. *Determination of Generalised Forces.*

Suppose that the wing deforms harmonically in j modes $Z_1(x,y), \ldots\ldots, Z_j(x,y)$ in such a way that the displacement of a point on the wing is $cZ(x,y)$, where

$$Z(x,y) = \sum_{u=1}^{j} q_u\, Z_u(x,y)$$

and q_u is a generalised co-ordinate. In a small displacement $\delta q_1, \ldots, \delta q_j$ the virtual work done by the airforces is

$$-\rho V^2 \bar{c}^3 e^{ivt} \sum_{u=1}^{j} \delta q_u \iint_S l(x,y)\, Z_u(x,y)\, dx\, dy.$$

The negative sign occurs since lift and displacement are measured in opposite directions. Comparing with the expression

$$\rho V^2 \bar{c}^3 e^{ivt} \sum_{u=1}^{j} Q_u \,\delta q_u$$

for the virtual work, where $\rho V^2 \bar{c}^3 e^{ivt} Q_u$ is the generalised aerodynamic force in the u^{th} mode, we get

$$Q_u = -\iint_S l(x,y)\, Z_u(x,y)\, dx\, dy. \tag{38}$$

Now, if $q_u = \bar{q}_u e^{ivt}$ and we write

$$l(x,y) = \sum_{v=1}^{j} \bar{q}_v\, l_v(x,y)$$

then (38) gives

$$Q_u = \sum_{v=1}^{j} Q_{uv}\, \bar{q}_v \tag{39}$$

where

$$Q_{uv} = -\iint_S l_v(x,y)\, Z_u(x,y)\, dx\, dy. \tag{40}$$

The expressions (18) and (19) may be used to provide an approximation $\bar{l}_v(x,y)$ to $l_v(x,y)$. Substituting these in (40) and changing the integration to one in ξ and η we have as an approximation to Q_{uv}

$$Q_{uv} \doteqdot -\tfrac{1}{2}s \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} \frac{P_{\alpha\beta,v}}{G_\beta} \int_{-1}^{1} \frac{g_\beta(\eta)}{H_\alpha}\, d\eta \int_{-1}^{1} h_\alpha(\xi)\, Z_u(\xi,\eta)\, d\xi, \tag{41}$$

where the $P_{\alpha\beta,v}$ are the values of $P_{\alpha\beta}$ occurring in the equations (18) and (19) for $\bar{l}_v(x,y)$. Now (*see*, e.g. Richardson[3]) for polynomials $q(\eta)$ of degree n or less in η

$$\int_{-1}^{1} g_\beta(\eta) q(\eta)\, d\eta = 2 G_\beta q(\eta_\beta).$$

So, in view of this together with (10), we may make in (41) the further approximation of assuming $Z_u(\xi,\eta)$ a double polynomial of degree m in ξ and n in η; the double integral on the right-hand side of (41) will be approximated by

$$4H_\alpha\, G_\beta\, Z_u\,(\xi_\alpha,\eta_\beta)$$

and we obtain

$$Q_{uv} \doteqdot -2s\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} P_{\alpha\beta,v} Z_u(\xi_\alpha,\eta_\beta). \tag{42}$$

The remark made in section 4.1 about the $P_{\alpha\beta}$ being used instead of the lifts at the points (ξ_α,η_β) is now clarified; in (42) it is the $P_{\alpha\beta,v}$ which multiply the displacements $Z_u(\xi_\alpha,\eta_\beta)$ in this expression.

6. *Matrix Formulation of the Problem.*

6.1. *Solution of the Integral Equation.*

Generally, the lift and downwash distributions will have no particular symmetry. However, an asymmetric lift and downwash distribution can be split into a symmetric part, consisting of a downwash $\frac{1}{2}\{w(x,y)+w(x,-y)\}$ with a corresponding lift $\frac{1}{2}\{l(x,y)+l(x,-y)\}$, and an anti-symmetric part, consisting of a downwash $\frac{1}{2}\{w(x,y)-w(x,-y)\}$ and lift $\frac{1}{2}\{l(x,y)-l(x,-y)\}$. In these symmetric and anti-symmetric distributions the lift and downwash on the starboard half of the wing only need be considered.

Equation (36) gives the downwash at any of the mn downwash points on the wing in terms of the mn unknowns $P_{\alpha\beta}$; attention will thus be restricted to the downwash points on the starboard half-wing.

It is convenient at this stage to confine discussion to the case in which the number of spanwise lift and downwash stations is odd; that is there is a station at the wing centre-line. Discussion for even n would, basically, merely mean omitting the terms relevant to this centre-line section in what follows.

There are $\frac{1}{2}m(n+1)$ lift and downwash points on the starboard half-wing; the $\frac{1}{2}m(n+1)$ equations which follow from (36) may be written

$$-8\pi s\, W = CP \tag{43}$$

where W is a column matrix of $\frac{1}{2}m(n+1)$ elements and P a column matrix of mn elements defined by

$$\left.\begin{aligned} W' &= \left(w\left(\xi_1,\eta_{\frac{1}{2}(n+1)}\right)\ldots\ldots\ldots w\left(\xi_m,\eta_{\frac{1}{2}(n+1)}\right)\ldots\ldots\ldots w\left(\xi_1,\eta_n\right)\ldots\ldots w\left(\xi_m,\eta_n\right)\right) \\ &\text{and} \\ P' &= \left(P_{11}\ldots\ldots P_{m1}\ldots\ldots P_{1,\frac{1}{2}(n+1)}\ldots\ldots P_{m,\frac{1}{2}(n+1)}\ldots\ldots P_{1n}\ldots\ldots P_{mn}\right) \end{aligned}\right\} \tag{44}$$

respectively. In (43), C is a $\frac{1}{2}m(n+1)$ by mn matrix having $C_{\alpha\beta}(\xi_r,\eta_s)$ as the element in its $\{\frac{1}{2}m(2s-n-1)+r\}^{th}$ row and $\{m(\beta-1)+\alpha\}^{th}$ column. C can be regarded as made up of submatrices, namely

$$C = \begin{pmatrix} C_{0-} & C_{00} & C_{0+} \\ C_{+-} & C_{+0} & C_{++} \end{pmatrix} \tag{45}$$

where the two suffices relate to the position of the lift and downwash points respectively, and $+$, 0 and $-$ refer to points on the starboard side, centre section and port side of the wing. C_{0+} and C_{0-} are m by $\frac{1}{2}m(n-1)$ matrices, C_{++} and C_{+-} are $\frac{1}{2}m(n-1)$ by $\frac{1}{2}m(n-1)$, C_{+0} is $\frac{1}{2}m(n-1)$ by m and C_{00} is m by m. The symmetry of the problem implies that

$$C_{0+} = C_{0-}L \tag{46}$$

where L is a $\frac{1}{2}m(n-1)$ by $\frac{1}{2}m(n-1)$ matrix of the form

$$L = \begin{pmatrix} 0 \ldots\ldots\ldots 0I \\ 0 \ldots\ldots\ldots I0 \\ \ldots\ldots \\ \ldots\ldots \\ 0I \ldots\ldots\ldots 0 \\ I0 \ldots\ldots\ldots 0 \end{pmatrix}$$

the unit matrices in L, of which there are $\frac{1}{2}(n-1)$, being $m \times m$.

Now, for symmetric lift and downwash

$$P = J\bar{P}, \tag{47}$$

where

$$\bar{P}' = \left(P_{1,\frac{1}{2}(n+1)} \ldots\ldots P_{m,\frac{1}{2}(n+1)} \ldots\ldots\ldots P_{1n} \ldots\ldots P_{mn}\right), \tag{48}$$

and $\bar{P}$ concerns only lifts on the starboard half-wing and centre-line. In (47) J is a $mn \times \frac{1}{2}m(n+1)$ matrix of the form

$$J = \begin{pmatrix} 0 \ldots\ldots\ldots 0I \\ \ldots\ldots \\ \ldots\ldots \\ 0I \ldots\ldots\ldots 0 \\ I0 \ldots\ldots\ldots 0 \\ 0I \ldots\ldots\ldots 0 \\ \ldots\ldots \\ \ldots\ldots \\ 0 \ldots\ldots\ldots 0I \end{pmatrix}$$

the unit matrices being again $m \times m$. (43) and (47) give

$$\bar{P} = -8\pi s(CJ)^{-1}W. \tag{49}$$

For anti-symmetric lift and downwash

$$P = K\bar{\bar{P}} \tag{50}$$

where

$$\bar{\bar{P}}' = \left(P_{1,\frac{1}{2}(n+3)} \ldots\ldots P_{m,\frac{1}{2}(n+3)} \ldots\ldots\ldots P_{1n} \ldots\ldots P_{mn}\right) \tag{51}$$

and $\bar{\bar{P}}$ concerns only lifts on the starboard half wing; K is a $mn \times \frac{1}{2}m(n-1)$ matrix of the form

$$K = \begin{pmatrix} 0\ldots\ldots 0-I \\ 0\ldots\ldots -I0 \\ \ldots\ldots \\ \ldots\ldots \\ -I0\ldots\ldots\ldots 0 \\ 00\ldots\ldots\ldots 0 \\ I0\ldots\ldots\ldots 0 \\ \ldots\ldots \\ \ldots\ldots \\ 0\ldots\ldots\ldots I0 \\ 0\ldots\ldots\ldots 0I \end{pmatrix}.$$

(43) and (50) give

$$-8\pi s\, W = CK\,\bar{\bar{P}}. \tag{52}$$

The first m rows of CK and hence the first m elements of W here will vanish on account of (46); this corresponds to the vanishing of the downwash on the centre-line in anti-symmetric motion. So if

$$\bar{\bar{C}} = (C_{+-}\; C_{+0}\; C_{++}) \tag{53}$$

and

$$\overline{\overline{W}}' = \left(w\left(\xi_1,\eta_{\frac{1}{2}(n+3)}\right) \ldots\ldots w\left(\xi_m,\eta_{\frac{1}{2}(n+3)}\right) \ldots\ldots w\left(\xi_1,\eta_n\right) \ldots\ldots w\left(\xi_m,\eta_n\right)\right) \tag{54}$$

then (52) gives

$$\bar{\bar{P}} = -8\pi s\,(\bar{\bar{C}}K)^{-1}\,\overline{\overline{W}}. \tag{55}$$

Equations (49) and (55) provide the required lift values for given symmetric and anti-symmetric downwash distributions.

6.2. *The Matrix Equation Connecting Lift and Downwash.*

The expression (37) for $C_{\alpha\beta}\,(\xi_r,\eta_s)$ is complicated; however the matrix C of (45) may be regarded as the sum of two matrices

$$C = \bar{C} + C_1$$

where $\bar{C}$ contains only terms arising from the double sum on the right of (37) and C_1 contains only terms involving the Kronecker delta. Then, if

$$h_\alpha(\xi) = H_\alpha\left(h_{\alpha 1} + h_{\alpha 2}\xi + \ldots\ldots + h_{\alpha m}\xi^{m-1}\right) f(\xi) \tag{57}$$

and

$$g_\beta(\eta) = G_\beta\left(g_{\beta 1} + g_{\beta 2}\eta + \ldots\ldots + g_{\beta n}\eta^{n-1}\right)\sqrt{\left(1-\eta^2\right)}, \tag{58}$$

the term $\bar{C}_{\alpha\beta}(\xi_r,\eta_s)$ arising from (37) which contributes to $\bar{C}$ can be written

$$\bar{C}_{\alpha\beta}(\xi_r,\eta_s) = (h_{\alpha 1}\cdots\cdots h_{\alpha m})\, M_{rs}\, N_{rs}\, R_{rs} \begin{pmatrix} g_{\beta 1} \\ \vdots \\ g_{\beta n} \end{pmatrix}. \qquad (59)$$

Equation (59) is strictly true only when there is no variation of the lift singularity $f(\xi)$ in the integration region considered. The equations when $f(\xi)$ does vary follow simply from those in this section, but have been omitted to save additional complication. In equation (59) M_{rs} is an $m\times q$ matrix given by

$$M_{rs} = \frac{1}{\eta_2(\xi_r,\eta_s)-\eta_1(\xi_r,\eta_s)} \begin{pmatrix} \left(\frac{W_1}{k(\zeta_1)}\cdots\cdots\frac{W_p}{k(\zeta_p)}\right) \bar{K}_1^{rs} \\ \vdots \\ \left(\frac{W_1}{k(\zeta_1)}\cdots\cdots\frac{W_p}{k(\zeta_p)}\right) \bar{K}_1^{rs} \end{pmatrix} \qquad (60)$$

where $\bar{K}_i^{rs}$ (for $i = 1,\ldots\ldots, m$) is the $p\times q$ matrix in which the element in the λ^{th} row and γ^{th} column is

$$K(x_{rs}-x_{rs,\lambda\gamma},\, y_s-y_{rs,\gamma})\, f(\xi_{rs,\lambda\gamma})\, (\xi_{rs,\lambda\gamma})^{i-1},$$

N_{rs} is the $q\times q$ diagonal matrix whose γ^{th} diagonal element is

$$\{1+\xi_M(\xi_r,\eta_s,\eta_{rs,\gamma})\}\,\sqrt{(1-\eta_{rs,\gamma}^2)}\int_{-1}^{1}\frac{p_\gamma(\phi)d\phi}{(\phi-\phi_{rs})^2}$$

and R_{rs} is the $q\times n$ matrix

$$R_{rs} = \begin{pmatrix} 1 & \eta_{rs,1}\cdots\cdots\cdots\cdots & \eta_{rs,1}^{n-1} \\ & \vdots & \\ 1 & \eta_{rs,q}\cdots\cdots\cdots\cdots & \eta_{rs,q}^{n-1} \end{pmatrix}. \qquad (61)$$

Define

$$H = \begin{pmatrix} h_{11}\cdots\cdots h_{1m} \\ \vdots \\ h_{m1}\cdots\cdots h_{mm} \end{pmatrix}, \qquad G = \begin{pmatrix} g_{11}\cdots\cdots g_{1n} \\ \vdots \\ g_{n1}\cdots\cdots g_{nn} \end{pmatrix}. \qquad (62)$$

Then from (59) and the form of the matrix C (and hence of $\bar{C}$) it follows that the elements of the $m \times n$ matrix

$$HM_{rs}N_{rs}R_{rs}G', \tag{63}$$

taken row by row and extended to become a $1 \times mn$ row matrix, give the elements of the row of $\bar{C}$ which corresponds to the point $(\bar{\xi}_r,\eta_s)$. The rows of $\bar{C}$ can then be built up of row matrices such as the one which follows from (63).

The matrix C_1 can be built up separately by using the terms involving $\delta_{\beta s}$ in (37); no particular simplification is available beyond the fact that the terms of C_1 for which $\beta \neq s$ are zero.

6.3. *Evaluation of Generalised Forces.*

The equation (42) can be written

$$Q_{uv} \doteqdot -2s\, Z_u P_v \tag{64}$$

where

$$Z_u = \Big(Z_u(\xi_1,\eta_1) \ldots\ldots Z_u(\xi_m,\eta_1) \ldots\ldots\ldots\ldots Z_u(\xi_1,\eta_n) \ldots\ldots Z_u(\xi_m,\eta_n)\Big), \tag{65}$$

and P_v is P of equation (44) for the mode v.

In view of (47) and (49), (64) may be written

$$Q_{uv} \doteqdot 32\pi s^2\, \bar{Z}_u (CJ)^{-1}\, W_v \tag{66}$$

for symmetric lift and downwash and in the form

$$Q_{uv} \doteqdot 32\pi s^2\, \bar{\bar{Z}}_u (\bar{C}K)^{-1}\, \bar{\bar{W}}_v \tag{67}$$

for anti-symmetric lift and downwash, from (50) and (55). $\bar{Z}_u$ and $\bar{\bar{Z}}_u$ are defined by

$$\bar{Z}_u = \Big(\tfrac{1}{2}Z_u\big(\xi_1,\eta_{\frac{1}{2}(n+1)}\big) \ldots \tfrac{1}{2}Z_u\big(\xi_m,\eta_{\frac{1}{2}(n+1)}\big)\; Z_u\big(\xi_1,\eta_{\frac{1}{2}(n+3)}\big) \ldots$$
$$\ldots Z_u\big(\xi_m,\eta_{\frac{1}{2}(n+3)}\big) \ldots Z_u(\xi_1,\eta_n) \ldots Z_u(\xi_m,\eta_n)\Big) \tag{68}$$

and

$$\bar{\bar{Z}}_u = \Big(Z_u\big(\xi_1,\eta_{\frac{1}{2}(n+3)}\big) \ldots Z_u\big(\xi_m,\eta_{\frac{1}{2}(n+3)}\big) \ldots Z_u(\xi_1,\eta_n) \ldots Z_u(\xi_m,\eta_n)\Big), \tag{69}$$

and W_v and $\bar{\bar{W}}_v$ are given by (44) and (54) where we take $w_v(\xi,\eta)$ to be the downwash due to $Z_v(\xi,\eta)$.

Now, on a wing oscillating harmonically with deflection shape $Z_v(\xi,\eta)e^{ivt}$, the boundary condition for the downwash $w_v(\xi,\eta)$ given by the condition of tangential flow at the surface is

$$w_v(\xi,\eta) = \frac{\partial}{\partial \xi} Z_v + i\nu Z_v$$

and (66) and (67) then give

$$\left.\begin{aligned} Q_{uv} &\doteqdot 32\pi s^2\, \bar{Z}_u (CJ)^{-1} (\underline{Z}'_{v,x} + i\nu \underline{Z}'_v) \qquad \text{(symmetric)} \\ Q_{uv} &\doteqdot 32\pi s^2\, \bar{\bar{Z}}_u (\bar{\bar{C}}K)^{-1} (\underline{\underline{Z}}'_{v,x} + i\nu \underline{\underline{Z}}'_v) \qquad \text{(anti-symmetric)} \end{aligned}\right\} \tag{70}$$

where

$$\left.\begin{aligned} \underline{Z}_v &= \left(Z_v\left(\xi_1,\eta_{\frac{1}{2}(n+1)}\right) \ldots Z_v\left(\xi_m,\eta_{\frac{1}{2}(n+1)}\right) \ldots Z_v(\xi_1,\eta_n) \ldots Z_v(\xi_m,\eta_n)\right) \\ \underline{\underline{Z}}_v &= \left(Z_v\left(\xi_1,\eta_{\frac{1}{2}(n+3)}\right) \ldots Z_v\left(\xi_m,\eta_{\frac{1}{2}(n+3)}\right) \ldots Z_v(\xi_1,\eta_n) \ldots Z_v(\xi_m,\eta_n)\right) \end{aligned}\right\} \tag{71}$$

and $\underline{Z}_{v,x}$ and $\underline{\underline{Z}}_{v,x}$ are defined similarly by replacing $Z_v(\xi,\eta)$ by $\frac{\partial}{\partial x} Z_v$ on the right of (71). If

$$\left.\begin{aligned} (CJ)^{-1} &= A + iB \\ (\bar{\bar{C}}K)^{-1} &= L + iM \end{aligned}\right\} \tag{72}$$

then (70) may be written

$$\left.\begin{aligned} \frac{1}{32\pi s^2} Q_{uv} &\doteqdot \bar{Z}_u A \underline{Z}'_{v,x} - \nu \bar{Z}_u B \underline{Z}'_v + i(\bar{Z}_u B \underline{Z}'_{v,x} + \nu \bar{Z}_u A \underline{Z}'_v) \qquad \text{(symmetric)} \\ \frac{1}{32\pi s^2} Q_{uv} &\doteqdot \bar{\bar{Z}}_u L \underline{\underline{Z}}'_{v,x} - \nu \bar{\bar{Z}}_u M \underline{\underline{Z}}'_v + i(\bar{\bar{Z}}_u M \underline{\underline{Z}}'_{v,x} + \nu \bar{\bar{Z}}_u L \underline{\underline{Z}}'_v) \qquad \text{(anti-symmetric).} \end{aligned}\right\} \tag{73}$$

It should be noted that the matrices $\bar{Z}_u$ and $\bar{\bar{Z}}_u$ are found by evaluating $Z_u(\xi,\eta)$ at the lift points (ξ_α,η_β), whereas the matrices $\underline{Z}_v$, $\underline{Z}_{v,x}$, $\underline{\underline{Z}}_v$ and $\underline{\underline{Z}}_{v,x}$ are found by evaluating $Z_v(\xi,\eta)$ and its derivative at the down-wash points (ξ_r,η_s). Finally, if

$$\bar{Z} = \begin{pmatrix} \bar{Z}_1 \\ \vdots \\ \bar{Z}_j \end{pmatrix}, \quad \bar{\bar{Z}} = \begin{pmatrix} \bar{\bar{Z}}_1 \\ \vdots \\ \bar{\bar{Z}}_j \end{pmatrix}, \quad \underline{Z} = \begin{pmatrix} \underline{Z}_1 \\ \vdots \\ \underline{Z}_j \end{pmatrix}, \quad \underline{\underline{Z}} = \begin{pmatrix} \underline{\underline{Z}}_1 \\ \vdots \\ \underline{\underline{Z}}_j \end{pmatrix} \tag{74}$$

and $\underline{Z}_x$, $\underline{\underline{Z}}_x$ are similarly defined in terms of $\underline{Z}_{v,x}$, $\underline{\underline{Z}}_{v,x}$ then (73) may be written

$$\left.\begin{aligned} \frac{1}{32\pi s^2} Q &= \bar{Z} A \underline{Z}'_x - \nu \bar{Z} B \underline{Z}' + i(\bar{Z} B \underline{Z}'_x + \nu \bar{Z} A \underline{Z}') \qquad \text{(symmetric)} \\ \frac{1}{32\pi s^2} Q &= \bar{\bar{Z}} L \underline{\underline{Z}}'_x - \nu \bar{\bar{Z}} M \underline{\underline{Z}}' + i(\bar{\bar{Z}} M \underline{\underline{Z}} + \nu \bar{\bar{Z}} L \underline{\underline{Z}}') \qquad \text{(anti-symmetric)} \end{aligned}\right\} \tag{75}$$

where Q is the $(j \times j)$ matrix of generalised force coefficients Q_{uv}.

7. *Results.*

7.1. *Calculations on an Ogee Wing.*

Derivatives have been found for an ogee wing of aspect ratio 0·924 oscillating at low frequency in pitch and heave at Mach numbers in the range 1·4 to 2·6. The planform, shown in Fig. 5, has its leading edge specified by the equation

$$y = s(1{\cdot}2x_0 - 2{\cdot}4x_0^2 + 2{\cdot}2x_0^3 + 3x_0^4 - 3x_0^5),$$

where $x_0 = \bar{c}\,x/c_0$, and c_0 is the root chord. In the Mach number range in question the leading edge is wholly subsonic and the trailing edge wholly supersonic. This planform was chosen because of the availability of experimental results. In a series of wind tunnel tests[8] on a cambered model, Thompson and Fail found derivatives at low frequencies for pitch and heave. The experiment was such that the reduced frequencies in pitch and heave were different. The derivatives were assumed not to vary with frequency, since this was small, so the calculations here have also made this assumption and have been taken only at the pitching frequency of Ref. 8. The tests were carried out for a range of mean incidences and the resulting derivatives referred to body axes. Since the theory used here is linearised, and so that the direct comparison may be made, we restrict our attention to the zero mean incidence results of Ref. 8.

The ogee planform of Fig. 5 has also been tested in steady flow over the same Mach number range in a series of wind tunnel experiments performed by Taylor[9]. In addition to the cambered shape tested by Thompson and Fail, Taylor tested a plane ogee wing having the same planform.

The theoretical values of the derivatives are given in Table 1 and are shown in Figs. 6 to 10 together with the experimental results. The derivatives are defined by

$$\text{Lift} = \rho V^2 S e^{i\nu t}\left[(L_z + i\nu_0 L_{\dot{z}})\frac{\bar{c}z_0}{c_0} + (L_\theta + i\nu_0 L_{\dot{\theta}})\theta_0\right] \tag{76}$$

and

$$\text{Moment} = \rho V^2 S c_0 e^{i\nu t}\left[(M_z + i\nu_0 M_{\dot{z}})\frac{\bar{c}z_0}{c_0} + (M_\theta + i\nu_0 M_{\dot{\theta}})\theta_0\right], \tag{77}$$

where S = wing area, $\nu_0 = wc_0/V$ and the moment is measured relative to an axis distant 0·71 c_0 from the apex on the root chord. This definition, which differs from that conventionally used in flutter, is chosen to agree with Ref. 8 since the results given there are not suitable for conversion to a more conventional notation.

The variation of L_θ with Mach number is shown in Fig. 6. The theoretical values are a little low when compared with the unsteady experimental results of Ref. 8. However, it will be noted that agreement with the zero frequency results of Ref. 9 for a flat plate wing is very good, particularly at the lower Mach numbers; further, the zero frequency results for a flat plate wing are lower than those for a cambered wing. Hence it may be expected that agreement will be rather better than indicated by comparison with the $\nu \neq 0$ results of Ref. 8, if not quite as good as the agreement with the $\nu = 0$ results of Ref. 9 might lead one to hope. It should also be remarked that the 8 per cent loss in L_θ due to frequency at $M = 1{\cdot}4$, shown by the experimental results of Ref. 8, is rather more than would be expected from linearised theory, for which our results are valid.

The comparative variation of $-M_\theta$ with Mach number is shown in Fig. 7. Agreement is rather better with the zero frequency results, both for the cambered and plane wing. However, the values of $-M_\theta$ are small, being measured for an axis near the centre of pressure, and the plot tends to exaggerate the very small differences which exist.

The variation of $L_{\dot{z}}$ with Mach number is shown in Fig. 8. At vanishingly small frequency this derivative would be the same as L_θ. The theoretical results satisfy this requirement, but it can be seen that the experimental values exhibit some scatter. This apart, the same remarks apply as were made about L_θ.

The variation of $-M_{\dot{z}}$ with Mach number is shown in Fig. 9. Agreement is again good, allowing for the experimental scatter and the smallness of what is being plotted because of the closeness of the reference axis to the aerodynamic centre.

Fig. 10 shows the variation of $-M_{\dot{\theta}}$. Agreement is excellent. No plot has been made of $L_{\dot{\theta}}$, since the experimental derivatives are in some doubt, so for this derivative no independent check can be provided.

It should be emphasised that, where derivatives are calculated in this report, it has been necessary to select in advance certain values of (m,n,p,q) which specify the number of lift and integration points; the selection of these is discussed in detail elsewhere[4]. In general, when finding derivatives, their acceptability is determined either by comparison with existing results or with a repeat calculation using an increased number of points. The derivatives presented in this and the following sections have been found using only one set of (m,n,p,q) as specified in the tables. Since agreement with known results has been satisfactory, no repeat calculations using more points have been made. In general when calculations are made by the method of this report and agreement is not thought satisfactory, it is necessary to make a repeat calculation using more points.

7.2. *Calculations on a Symmetrical Tapered Wing.*

We now revert to the conventional definition of flutter derivatives, namely

$$\text{Lift} \quad = \rho V^2 S e^{i\nu t} \left[(l_z + i\nu l_{\dot{z}}) z_0 + (l_\theta + i\nu l_{\dot{\theta}}) \theta_0 \right] \tag{78}$$

and

$$\text{Moment} = \rho V^2 S \bar{c} e^{i\nu t} \left[(m_z + i\nu m_{\dot{z}}) z_0 + (m_\theta + i\nu m_{\dot{\theta}}) \theta_0 \right], \tag{79}$$

the moment being measured relative to an axis through the wing apex.

Derivatives have been found at $M = 1{\cdot}102$ and $M = \sqrt{2}$ for the symmetrical tapered planform of taper ratio 0·27 and aspect ratio 4·33 shown in Fig. 5. This planform has been used in a series of tests and calculations carried out under the aegis of the National Physical Laboratory[10]. Direct comparisons at $M = 1{\cdot}102$ are available from the exact theory of Lehrian[11], valid for hexagonal wings oscillating at low frequency in supersonic flow, and by the method of Allen and Sadler[12]. This is a supersonic theory valid for general frequency, planform and Mach number and is based on the integral equation which gives the downwash in terms of the velocity potential. This equation is solved to give the velocity potential at the vertices of a fine mesh. Derivatives calculated by this method have been given in Ref. 12.

Values of derivatives calculated at $M = 1{\cdot}102$ by the method of this report for $\nu = 0{\cdot}019$, 0·19 and 0·38, from Ref. 11 for $\nu \to 0$ and from Ref. 12 for $\nu = 0{\cdot}19$ are given in Table 2, the results from Refs. 11 and 12 being converted to the notation used here. At this Mach number the Mach lines from the apex and from the foremost point of the tips are as shown in Fig. 11a. The local lift has a discontinuity in its derivative when a Mach line is crossed, whereas the assumption has been made in Section 4.1 that the lift distribution can be approximated by a sum of chordwise and spanwise polynomials having no such discontinuity. It is seen from Fig. 11a that for nearly all chordwise sections there are two such discontinuities in the lift while for nearly all spanwise sections (that is, those having constant ξ) there are four such discontinuities. The accuracy with which the derivative can be found will reflect on the validity of the continuity assumptions of Section 4.1.

The variation of the various derivatives with frequency parameter is shown in Fig. 12. It will be noted that all derivatives vary considerably with frequency parameter since the Mach number is close to unity. In particular, for very small frequency parameters the damping derivatives $l_{\dot{\theta}}$ and $(-m_{\dot{\theta}})$ are large and negative, indicating a very severe instability at these frequencies. It will be seen that agreement between the derivatives calculated by the method of this report and the other derivatives is in all cases good. The method of Ref. 11 is exact, and the closeness of agreement with results from this is a measure of the accuracy obtained in our calculations. Results from Ref. 12 are, like those of this report, approximations to the exact solution and should be treated as such in making comparisons. The largest discrepancy is between the two values of $l_{\dot\theta}$ at $\nu = 0{\cdot}19$ which occurs, as remarked above, in a region where the derivative varies very rapidly. Expressed in terms of the value at $\nu \to 0$ the discrepancy between the two approximate values of the derivative at $\nu = 0{\cdot}19$ is only 4 per cent, while the discrepancy between each approximate value and the unknown exact value may be as low as 2 per cent. Referring to the absolute error in the derivatives which it gives for this wing, Ref. 12 suggests that an upper bound for these errors would be 5 per cent, and this combined with a comparable discrepancy in the results here could more than account for the differences which exist.

For $M = \sqrt{2}$ values of the derivatives are given by Garner, Acum and Lehrian[13]. In addition to values based on Lehrian's exact theory[11], Ref. 13 gives derivatives calculated by computer programmes based on the method of Hunt[14]. In this method a mesh is placed over the wing; for sufficiently high supersonic Mach number the velocity potential can be expressed directly in terms of the downwash, and this enables the velocity potential to be found at the vertices of the mesh. Hunt's method has been mechanised by Wicks[15] for the Deuce computer as Bristol Aircraft Programme BAC 11, and has been mechanised for the Pegasus computer by Hawker Aircraft. Hawker have developed a separate programme for infinitesimal frequency. One of the conclusions of Ref. 13 is that the Hawker Pegasus programme is marginally preferable to the Bristol Deuce programme.

Values of the derivatives calculated at $M = \sqrt{2}$ by the method of this report for $\nu = 0{\cdot}019$ and $0{\cdot}19$, from Ref. 11 for $\nu \to 0$ and from Ref. 13 (that is, the Bristol and Hawker programmes) for $\nu \to 0$, $\nu = 0{\cdot}095$ and $\nu = 0{\cdot}19$ are given in Table 3; the results from Refs. 11 and 13 are converted to our derivative notation. At $M = \sqrt{2}$ the Mach lines from the apex and from the foremost point of the tips are as shown in Fig. 11b; it will be seen that they do not interfere.

The variation of the main derivatives with frequency parameter is shown in Fig. 13; the Hawker results for $\nu \to 0$ are obtained by a completely different programme from those for $\nu = 0{\cdot}095$ and $0{\cdot}19$ and are consequently not continuously joined to these. Agreement between the derivatives calculated by the method of this report and the other derivatives is in all cases good. The greatest discrepancy is for $l_{\dot\theta}$ when ν is small; although the exact comparison at the same frequency cannot be made, this derivative appears to be of the order of 4 per cent too low. However, it should be pointed out that this disagreement is rather less than that between the Bristol and Hawker $l_{\dot\theta}$ derivatives for the higher frequency parameters.

7.3. *Calculations on a Wing with Cranked Leading Edge.*

Derivatives have been calculated at $M = \sqrt{2}$ for the wing with cranked leading edge of aspect ratio 4·58 shown in Fig. 5, which was also used in the series of tests and calculations already referred to[13]. At this Mach number the Mach lines from the apex and the leading edge discontinuity are shown in Fig. 11c. The Mach lines interfere, and at some chordwise sections there are two discontinuities in the local lift curve slope while for nearly all spanwise sections there are four such discontinuities.

Derivatives calculated by the method of this report for $\nu = 0{\cdot}01793$, by exact theory[11] for $\nu \to 0$ and by the Hawker Pegasus programme and Bristol Deuce programme for $\nu = 0{\cdot}08965$ are given in Table 4, the results from Refs. 11 and 13 having been converted to our notation. It will be seen from Table 4 that, in the limited range of frequency parameter we are considering, there is little variation of the derivatives. If we make the direct comparison between our results and the exact results of Ref. 11 it will be seen that the maximum disagreement is one of 1 per cent in $l_{\dot\theta}$.

7.4. *Calculations on a Delta Wing.*

Calculations have been made on a delta wing of aspect ratio 1·5, shown in Fig. 5, which oscillates in an airstream of Mach number 1·01. Standard methods[16] exist for calculating derivatives on delta wings with subsonic leading edges in supersonic flow, and derivatives based on Ref. 16 for $\nu \to 0$ are given in Table 5 together with derivatives calculated by the method of this report for $\nu = 0{\cdot}15$ and $0{\cdot}3$. At $M = 1{\cdot}01$

it is not possible to obtain exact comparisons at our frequency parameters from Ref. 16 since the series used there does not converge with sufficient rapidity.

Fig. 14 shows the variation of the derivatives with frequency parameter, and it will be seen that agreement is very close.

8. *Conclusions.*

A method for finding the generalised forces on wings oscillating in supersonic flow has been described. This has been programmed for the Mercury digital computer, and from calculations which have been carried out good agreement is shown with both experimental and theoretical results over a wide range of Mach numbers.

LIST OF SYMBOLS

Aerodynamic derivatives

L_z, $L_{\dot{z}}$, L_θ, $L_{\dot{\theta}}$, M_z, $M_{\dot{z}}$, M_θ, $M_{\dot{\theta}}$ are defined by equations (76) and (77)

l_z, $l_{\dot{z}}$, l_θ, $l_{\dot{\theta}}$, m_z, $m_{\dot{z}}$, m_θ, $m_{\dot{\theta}}$ are defined by equations (78) and (79)

Primes are used to denote particular examples of variables, for example x', y', ξ', η', except where they obviously indicate differentiation and matrix transposition.

A,B	Matrices defined by (72)
$c(\eta)$	Local chord $= \bar{c}\,c(\eta)$
$\bar{c}$	Mean chord
c_0	Root chord
C	Matrix of quantities $C_{\alpha\beta}(\xi_r,\eta_s)$, defined by (45) and preceding text
$C_{\alpha\beta}(\xi_r,\eta_s)$	Defined by (37)
$\bar{C},C_1$	Matrices defined by (56) and immediate text
$\bar{\bar{C}}$	Matrix defined by (53)
$f(\xi)$	Function which takes into account the chordwise singularity in lift; defined by (5)
$g_\beta(\eta)$	Interpolation function for spanwise lift distribution, defined by (12)
G	Matrix of coefficients of interpolation functions defined by (62)
G_β	Defined by (21)
$h_\alpha(\xi)$	Interpolation function for chordwise lift, defined by (9)
H	Matrix of coefficients of interpolation functions defined by (62)
H_α	Defined by (11)
$i_\alpha(\xi',\eta',\eta)$	Defined by (23)
J	Folding matrix, defined following equation (48)
$k(\zeta)$	Function which takes into account the singularity in the chordwise integration, defined by (28)
K	Folding matrix, defined following equation (51)
$K(X,Y)$	Kernel function, defined in (2)
K_0	Kernel function for two dimensional steady flow – *see* equation (13)
$l(x,y)$ or $l(\xi,\eta)$	Reduced lift
$\rho V^2 e^{ivt} l(x,y)$	Lift
$l_v(x,y)$	*See* following (38)
$\bar{l}(\xi,\eta)$	Approximation to $l(\xi,\eta)$, defined by (18)
L	Matrix defined by (72)
$L_\alpha(\xi',\eta')$	Defined by (25)

m		Number of chordwise lift and downwash points
M		Mach number
$\mathbf{M}$		Matrix defined by (72)
M_{rs}		Matrix defined by (60)
n		Number of spanwise lift and downwash points
N_{rs}		Diagonal matrix – *see* text preceding (61)
p		Number of chordwise integration points
$p_\gamma(\phi)$		Interpolation polynomial satisfying (34)
$P, \bar{P}, \bar{\bar{P}}$		Column matrices of values of $P_{\alpha\beta}$ defined respectively by (44), (48) and (51)
$\mathbf{P}$		Matrix of coefficients in $p_\gamma(\phi)$ defined by (88)
$P_\alpha(\eta)$		Defined by (19)
$P_{\alpha\beta}$		*See* (20)
$P_{\alpha\beta,v}$		Value of $P_{\alpha\beta}$ which arises in the approximation to $l_v(x,y)$
q		Number of points used in spanwise integration
q_u		Generalised co-ordinate
Q		Matrix of generalised force coefficients, defined by (75)
Q_u		Generalised force corresponding to co-ordinate q_u is $\rho V^2 \bar{c}^3 e^{ivt} Q_u$
Q_{uv}		Generalised force coefficients – *see* (40)
r		Suffix associated with chordwise variation of downwash point
R	=	$\sqrt{(X^2 - \beta^2 Y^2)}$
R_{rs}		Matrix defined by (61)
s		Semi-span $= s\bar{c}$
s		Suffix associated with spanwise variation of downwash point
S		Wing area
t		Reduced time
$u_\alpha(\xi',\eta',\eta)$		*See* (31)
V		Airspeed
$w(x,y)$ or $w(\xi,\eta)$		Reduced downwash
$Ve^{ivt} w(x,y)$		Downwash
$W, \bar{\bar{W}}$		Column matrices of values of $w(\xi,\eta)$ defined respectively by (44) and (54)
$W_v, \bar{\bar{W}}_v$		*See* text following (69)
W_λ		Gaussian weights associated with a p-point integration formula for $k(\zeta)$

x,y,z		Cartesian co-ordinates, referred to $\bar{c}$ as reference length
$\bar{x}(y)$		$x = \bar{x}(y)$ is the equation of the mid-chord line
x_{LE}		Value of x at leading edge
x_{rs}		Value of x at a downwash point (ξ_r,η_s)
$x_{rs,\lambda\gamma}$		Value of x at an integration point
X	=	$x'-x$
y_s		Value of y at a downwash point $(\bar{\xi}_r,\eta_s)$
$y_{rs,\gamma}$		Value of y at an integration point
Y	=	$y'-y$
$Z_1(x,y)\ldots Z_j(x,y)$		Modal deflection shapes
$Z_u,\bar{Z}_u,\bar{\bar{Z}}_u$		Row matrices whose elements are the modal deflections evaluated at the lift points. Defined by (65), (68) and (69)
$\underline{Z}_v,\underline{\underline{Z}}_v$		Row matrices whose elements are the modal deflections at the downwash points; *see* (71)
$\underline{Z}_{v,x},\underline{\underline{Z}}_{v,x}$		*See* (71) and following text
$\bar{Z},\bar{\bar{Z}},\underline{Z},\underline{Z}$		Matrices defined by (74)
$\underline{Z}_x,\underline{\underline{Z}}_x$		*See* (74) and following text
α,β		Suffices associated with chordwise and spanwise variation of lift points respectively
β	=	$\sqrt{(M^2-1)}$
γ		Suffix associated with spanwise variation of integration points
$\zeta(\xi,\eta,\xi',\eta')$		Variable for chordwise integration, defined by (27)
ζ_λ		Gaussian zeros associated with weight function $k(\zeta)$
η_s, η_β		Spanwise co-ordinates of downwash and lift points respectively
$\eta_{rs,\gamma}$	=	$\bar{\eta}_\gamma(\xi_r,\eta_s)$
$\eta_1(\xi',\eta'), \eta_2(\xi',\eta')$		Intersections of $\xi = \xi_1(\xi',\eta',\eta)$ with leading edge or tips – *see* Fig. 2
$\bar{\eta}_\gamma(\xi',\eta')$		Value of η given by (32) when $\phi = \phi_\gamma$
λ		Suffix associated with chordwise variation of integration points
ν		Frequency parameter $= \omega\bar{c}/V$ based on mean chord
ν_0		Frequency parameter $= \omega c_0/V$ based on root chord
ξ,η		Wing co-ordinates defined by (3)
ξ_α		Chordwise co-ordinate of point at which lift is evaluated
$\bar{\xi}_r$		Chordwise co-ordinate of downwash point

LIST OF SYMBOLS—*continued*

$\xi_{rs,\lambda\gamma}$		Value of ξ given by (27) when $\zeta = \zeta_\lambda$, $\eta = \eta_{rs,\gamma}$
$\xi_1(\xi',\eta',\eta)$		Equation of reversed Mach lines through (ξ',η') is $\xi = \xi_1(\xi',\eta',\eta)$
$\xi_M(\xi',\eta',\eta)$	=	$\min\{\xi_1(\xi',\eta',\eta), 1\}$
ρ		Density
ϕ		Defined by (32)
ϕ_{rs}		Value of ϕ given by (32) for point $(\bar{\xi}_r,\eta_s)$
$\phi(\xi)$		The polynomial of degree m of the set orthogonal to $f(\xi)$ over $(-1,1)$
$\psi(\eta)$		The polynomial of degree n of the set orthogonal to $\sqrt{(1-\eta^2)}$ over $(-1,1)$
ω		Angular frequency

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APPENDIX A
(see section 4.1)

The Logarithmic Singularity in the Influence Function $i_\alpha(\xi',\eta',\eta)$

The $(\eta-\eta')^2 \log|\eta-\eta'|$ term occurring in the term $i_\alpha(\xi',\eta',\eta)$ of equation (23) was derived by I. T. Minhinnick in an unpublished note; the following treatment of this singularity is based on Minhinnick's.

Watkins and Berman[5] give as an expansion for the function $K(X,Y)$ of (2)

$$K(X,Y) = e^{-ivX}\left\{\frac{2X}{R}-\frac{2ivY^2}{R}-\frac{v^2Y^2X}{\beta^2R}-v^2Y^2\log\left(\frac{X+R}{\beta|Y|}\right)+0(v^3)\right\}$$

which, expanding e^{-ivX} as a series where convenient, gives

$$K(X,Y) = \frac{2X}{R}-2iv\frac{X^2+Y^2}{R}-\frac{v^2(2M^2-1)}{\beta^2}\frac{XY^2}{R}-\frac{v^2X^3}{R}-v^2Y^2e^{-ivX}\log\left(\frac{X+R}{\beta|Y|}\right)+o(v^3). \tag{80}$$

We consider the coefficient of $Y^2 \log Y$ in the expansion of

$$I(x',Y) = \int_{x_{LE}}^{x'-\beta|Y|} f(x)\,K(x'-x,Y)\,dx \tag{81}$$

for some function $f(x)$, the integral being taken between wing leading-edge and Mach line. (81) gives

$$I(x',Y) = \int_{\beta|Y|}^{x'-x_{LE}} \left\{2X-2iv(X^2+Y^2)-\frac{v^2}{\beta^2}(2M^2-1)XY^2-v^2X^3\right\}\frac{f(x'-X)}{\sqrt{(X^2-\beta^2Y^2)}}dx +$$

$$+v^2Y^2\int_{\beta|Y|}^{x'-x_{LE}} f(x'-X)\,e^{-ivX}\log\frac{\beta}{X+\sqrt{(X^2-\beta^2Y^2)}}dx +$$

$$+v^2Y^2\log|Y|\left\{\int_{x_{LE}}^{x'} f(x)\,e^{-ivX}dx-\int_{x'-\beta|Y|}^{x'} f(x)\,e^{-ivX}dx\right\}+o(v^3). \tag{82}$$

Now $f(x'-X)$ may be expanded as a Taylor series

$$f(x'-X) = f(x')-Xf'(x')+\tfrac{1}{2}X^2f''(x')+\ldots\ldots \tag{83}$$

and the first term on the right of (82) may be written

$$\int_{\beta|Y|}^{x'-x_{LE}} \frac{1}{\sqrt{(X^2-\beta^2Y^2)}}\left\{2X-2iv(X^2+Y^2)-\frac{v^2}{\beta^2}(2M^2-1)XY^2-v^2X^3\right\}\left\{f(x')-Xf'(x')+\tfrac{1}{2}X^2f''(x')+\ldots\right\} \tag{84}$$

while the second may be written

$$\nu^2 Y^2 \left[\log \left\{ \frac{\beta}{X+\sqrt{(X^2-\beta^2 Y^2)}} \right\} \cdot \int_0^X f(x'-X)\, e^{-i\nu X} dX \right]_{\beta|Y|}^{x'-x_{LE}} +$$

$$+\nu^2 Y^2 \int_{\beta|Y|}^{x'-x_{LE}} \left\{ Xf(x') - \tfrac{1}{2}X^2(f'(x')+i\nu f(x')) + \ldots \right\} \frac{dX}{\sqrt{(X^2-\beta^2 Y^2)}}. \qquad (85)$$

In the second integral of (85), $e^{-i\nu X} f(x'-X)$ has been written as a series and integrated term by term.

The integrals in (84) and (85) are all of the form

$$J_n = \int_{\beta|Y|}^{x'-x_{LE}} \frac{X^n\, dX}{\sqrt{(X^2-\beta^2 Y^2)}},$$

which satisfy the recurrence relation

$$J_{n+2} - \frac{n+1}{n+2}\beta^2 Y^2 J_n = \frac{1}{n+2}(x'-x_{LE})^{n+1} \sqrt{\{(x'-x_{LE})^2-\beta^2 Y^2\}}$$

with

$$J_0 = \log \left\{ \frac{(x'-x_{LE})+\sqrt{\{(x'-x_{LE})^2-\beta^2 Y^2\}}}{\beta|Y|} \right\}$$

and

$$J_1 = \sqrt{\{(x'-x_{LE})^2-\beta^2 Y^2\}}.$$

Since $x'-x_{LE}>0$ for small $|Y|$, the only singularity of J_0 at $Y = 0$ arises from the term $-\log|Y|$. There is no singularity in J_1 at $Y = 0$. Hence, from the recurrence relation, the integrals $J_1, J_3, J_5, \ldots$ cannot contribute to a logarithmic singularity in $I(x',Y)$; the terms $J_0, J_2, J_4, \ldots$ do contribute and give rise to terms

$$-\log|Y| \text{ in } J_0$$

$$-\frac{1}{2}\beta^2 Y^2 \log|Y| \text{ in } J_2$$

$$-\frac{3}{8}\beta^4 Y^4 \log|Y| \text{ in } J_4$$

and so on.

Since the first term of (85) does not contribute a $Y^2 \log|Y|$ term to a logarithmic singularity, the terms which do contribute to the $Y^2 \log|Y|$ term of (82) *via* (84) and (85) may be singled out. They give a contribution

$$\{i\nu(M^2+1)f(x')+(M^2-1)f'(x')\}\, Y^2 \log|Y|.$$

There remains the third term of (82), which gives rise to a term

$$\nu^2 Y^2 \log|Y| \int_{x_{LE}}^{x'} f(x)\, e^{-i\nu X}\, dx$$

and the coefficient of $Y^2 \log|Y|$ in $I(x',Y)$ is, from (82), thus

$$i\nu(M^2+1) f(x') + (M^2-1) f'(x') + \nu^2 \int_{x_{LE}}^{x'} f(x)\, e^{-i\nu(x'-x)}\, dx. \qquad (86)$$

So, from (23) and (86), the result (25) for $L_\alpha(\xi',\eta')$ follows on replacing $f(x)$ by $h_\alpha(\xi)/H_\alpha$ and changing the variable from x to ξ. Since we are expanding for small $|Y|$, the upper limit in (23) becomes ξ'.

APPENDIX B

(*see Section* 4.4)

Evaluation of the Integrals $\int_{-1}^{1} \frac{p_\gamma(\phi)}{(\phi-\phi_{rs})^2}\, d\phi$

We have defined q interpolation polynomials $p_\gamma(\phi)$ of degree $(q-1)$ by (34). If

$$p_\gamma(\phi) = p_{\gamma 1} + p_{\gamma 2}\,\phi + \ldots\ldots + p_{\gamma q}\,\phi^{q-1}$$

then, in matrix form

$$\begin{pmatrix} \int_{-1}^{1} \frac{p_1(\phi)}{(\phi-\phi_{rs})^2}\, d\phi \\ \vdots \\ \int_{-1}^{1} \frac{p_q(\phi)}{(\phi-\phi_{rs})^2}\, d\phi \end{pmatrix} = P \begin{pmatrix} \int_{-1}^{1} \frac{d\phi}{(\phi-\phi_{rs})^2} \\ \vdots \\ \int_{-1}^{1} \frac{\phi^{q-1}\, d\phi}{(\phi-\phi_{rs})^2} \end{pmatrix}, \qquad (87)$$

where

$$P = \begin{pmatrix} p_{11} \cdots\cdots\cdots p_{1q} \\ \vdots \\ p_{q1} \cdots\cdots\cdots p_{qq} \end{pmatrix}. \tag{88}$$

The integrals in the column matrix on the right of (87) may be evaluated as follows. Let

$$I_\gamma = \int_{-1}^{1} \frac{\phi^\gamma d\phi}{(\phi-\phi_{rs})^2}.$$

Then the I_γ satisfy a recurrence relation

$$I_\gamma - 2\phi_{rs} I_{\gamma-1} + \phi_{rs}^2 I_{\gamma-2} = \left\{ \begin{array}{ll} 0 & (\gamma \text{ odd}) \\ \dfrac{2}{\gamma-1} & (\gamma \text{ even}) \end{array} \right\} (\gamma \geqslant 2)$$

with

$$I_0 = \frac{-2}{1+\phi_{rs}^2}, I_1 = \log\left|\frac{1-\phi_{rs}}{1+\phi_{rs}}\right| - \frac{2\phi_{rs}}{1-\phi_{rs}^2}.$$

The solution of this recurrence relation for $\gamma \geqslant 2$ is

$$I_\gamma = -\frac{2\phi_{rs}^\gamma}{1-\phi_{rs}^2} + \gamma\, \phi_{rs}^{\gamma-1} \log\left|\frac{1-\phi_{rs}}{1+\phi_{rs}}\right| + 2(\gamma-1)\, \phi_{rs}^{\gamma-2} + \tfrac{2}{3}(\gamma-3)\, \phi_{rs}^{\gamma-4} +$$

$$+ \cdots\cdots + \left\{ \begin{array}{ll} \dfrac{2}{\gamma-1} & (\gamma \text{ even}) \\ \dfrac{4\phi_{rs}}{\gamma-2} & (\gamma \text{ odd}) \end{array} \right\}.$$

TABLE 1

Theoretical Derivatives for a Pitching and Heaving Ogee Wing

$m = 5, \quad n = 10, \quad p = 5, \quad q = 11$

M	ν_0	L_z	L_θ	$-M_z$	$-M_\theta$	$L_{\dot{z}}$	$L_{\dot{\theta}}$	$-M_{\dot{z}}$	$-M_{\dot{\theta}}$
1·4	0·14	−0·0011	0·7356	0·0000	0·0207	0·7358	0·7249	0·0207	0·0409
1·8	0·12	−0·0007	0·6679	0·0000	0·0172	0·6681	0·6246	0·0172	0·0337
2·2	0·11	−0·0004	0·6140	0·0000	0·0134	0·6141	0·5488	0·0134	0·0293
2·6	0·10	−0·0002	0·5717	0·0000	0·0108	0·5718	0·4885	0·0108	0·0258

TABLE 2

Theoretical Derivatives for a Symmetrical Tapered Wing, $M = 1 \cdot 102$

Method	ν	l_z	l_θ	$-m_z$	$-m_\theta$	$l_{\dot{z}}$	$l_{\dot{\theta}}$	$-m_{\dot{z}}$	$-m_{\dot{\theta}}$
This report, $m = 4$, $n = 10$, $p = 4$, $q = 10$	0·019	0·0028	3·5362	0·0025	2·5630	3·5344	−4·7869	2·5611	−4·2577
	0·19	0·1941	2·8831	0·1578	1·9127	2·7382	−2·8551	1·7690	−2·1504
	0·38	0·2612	2·1287	0·1128	1·2986	1·8275	0·0381	1·0330	0·7564
Reference 11	→0	0	3·5173	0	2·5746	3·5173	−4·7964	2·5746	−4·3957
Reference 13	0·19	0·2013	2·89	0·1570	1·87	2·74	−3·05	1·73	−2·13

TABLE 3

Theoretical Derivatives for a Symmetrical Tapered Wing, $M = \sqrt{2}$

Method	ν	l_z	l_θ	$-m_z$	$-m_\theta$	$l_{\dot{z}}$	$l_{\dot{\theta}}$	$-m_{\dot{z}}$	$-m_{\dot{\theta}}$
This report $m = 4, n = 8$ $p = 4, q = 8$	0·019	0·0004	1·8974	0·0003	1·4617	1·8972	0·5339	1·4614	0·4732
	0·19	0·0333	1·8620	0·0310	1·4229	1·8412	0·5655	1·4014	0·5104
Hawker programme for $\nu \to 0$	$\to 0$	0	1·8964	0	1·4730	1·8964	0·5570	1·4730	0·4913
Hawker programme for $\nu \neq 0$	0·095	0·0090	1·9020	0·0086	1·4854	1·8963	0·5280	1·4795	0·4748
	0·19	0·0348	1·8876	0·0331	1·4692	1·8653	0·5416	1·4454	0·4908
BAC programme	0·095	0·0085	1·8807	0·0081	1·4740	1·8755	0·5568	1·4685	0·4955
	0·19	0·0331	1·8671	0·0315	1·4590	1·8466	0·5688	1·4376	0·5095
Reference 11	$\to 0$	0	1·8928	0	1·4678	1·8928	0·5558	1·4678	0·4895

TABLE 4

Theoretical Derivatives for a Wing with Cranked Leading Edge

Method	ν	l_z	l_θ	$-m_z$	$-m_\theta$	$l_{\dot{z}}$	$l_{\dot{\theta}}$	$-m_{\dot{z}}$	$-m_{\dot{\theta}}$
This report $m = 4, n = 8,$ $p = 4, q = 8$	0·01793	0·0003	1·9409	0·0003	1·5987	1·9407	0·5982	1·5985	0·5816
Reference 11	$\to 0$	0	1·9349	0	1·6037	1·9349	0·6042	1·6037	0·5797
Hawker programme for $\nu \neq 0$	0·08965	0·0087	1·9383	0·0088	1·6141	1·9325	0·5791	1·6077	0·5652
BAC programme	0·08965	0·0083	1·9194	0·0084	1·6045	1·9140	0·6003	1·5983	0·5786

TABLE 5

Theoretical Derivatives for a Delta Wing, $M = 1{\cdot}01$

Method	ν	l_z	l_θ	$-m_z$	$-m_\theta$	$l_{\dot{z}}$	$l_{\dot{\theta}}$	$-m_{\dot{z}}$	$-m_{\dot{\theta}}$
This report, $m = 3, n = 7$ $p = 3, q = 7$	0·15	0·0027	1·1350	0·0041	1·5531	1·1149	1·9821	1·5206	3·0034
	0·3	−0·0057	1·1276	−0·0104	1·5451	1·0919	2·0950	1·4893	3·1903
Reference 17	$\to 0$	0	1·1718	0	1·5624	1·1718	2·0100	1·5624	3·0149

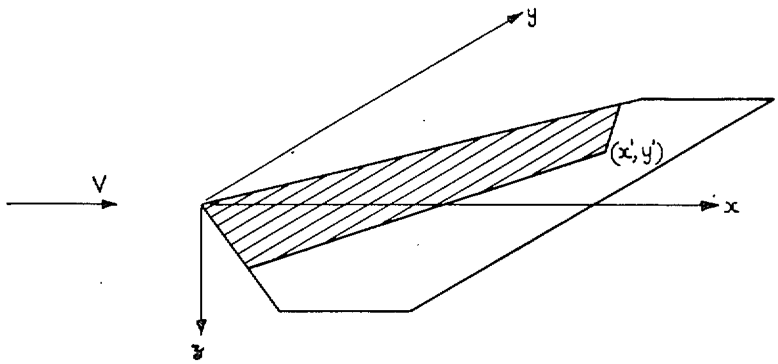

FIG. 1. Wing planform and co-ordinate system

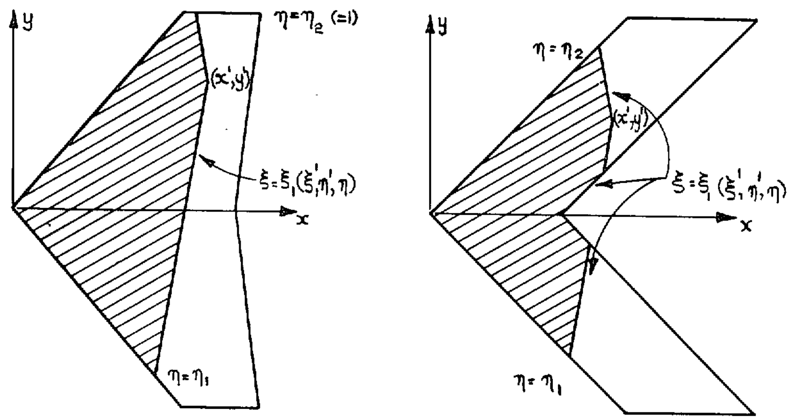

FIG. 2. Integration areas for equation (4)

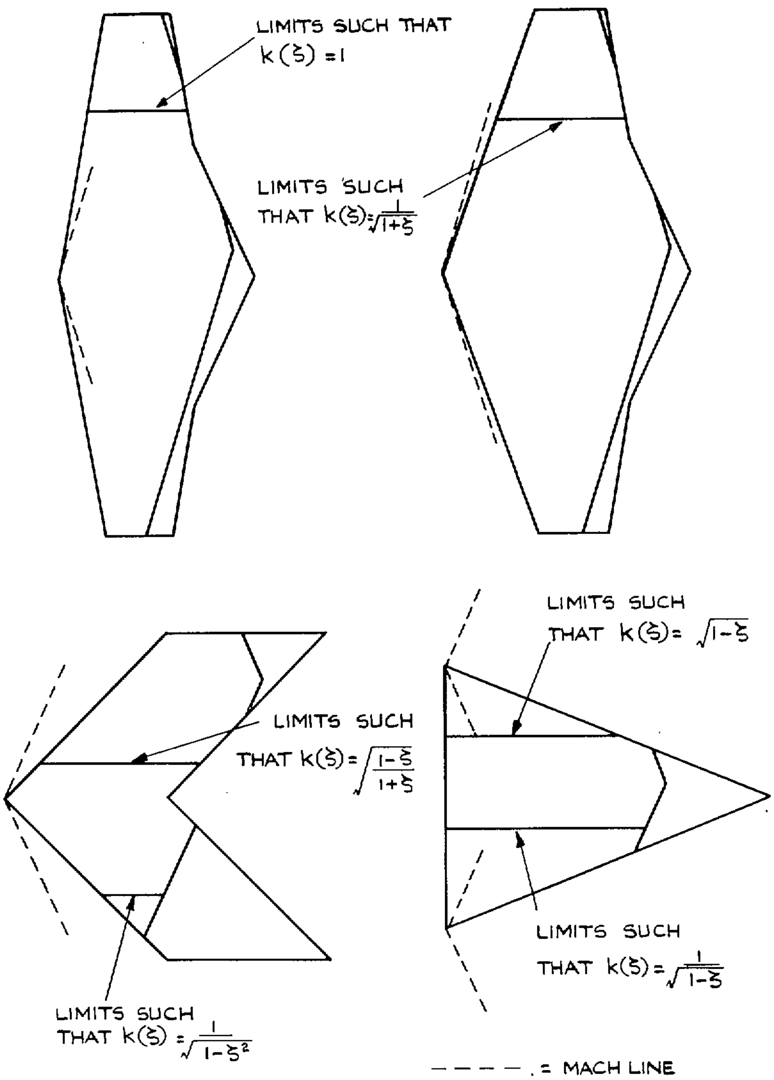

FIG. 3. Chordwise integration limits and values of $k(\zeta)$ for various planforms

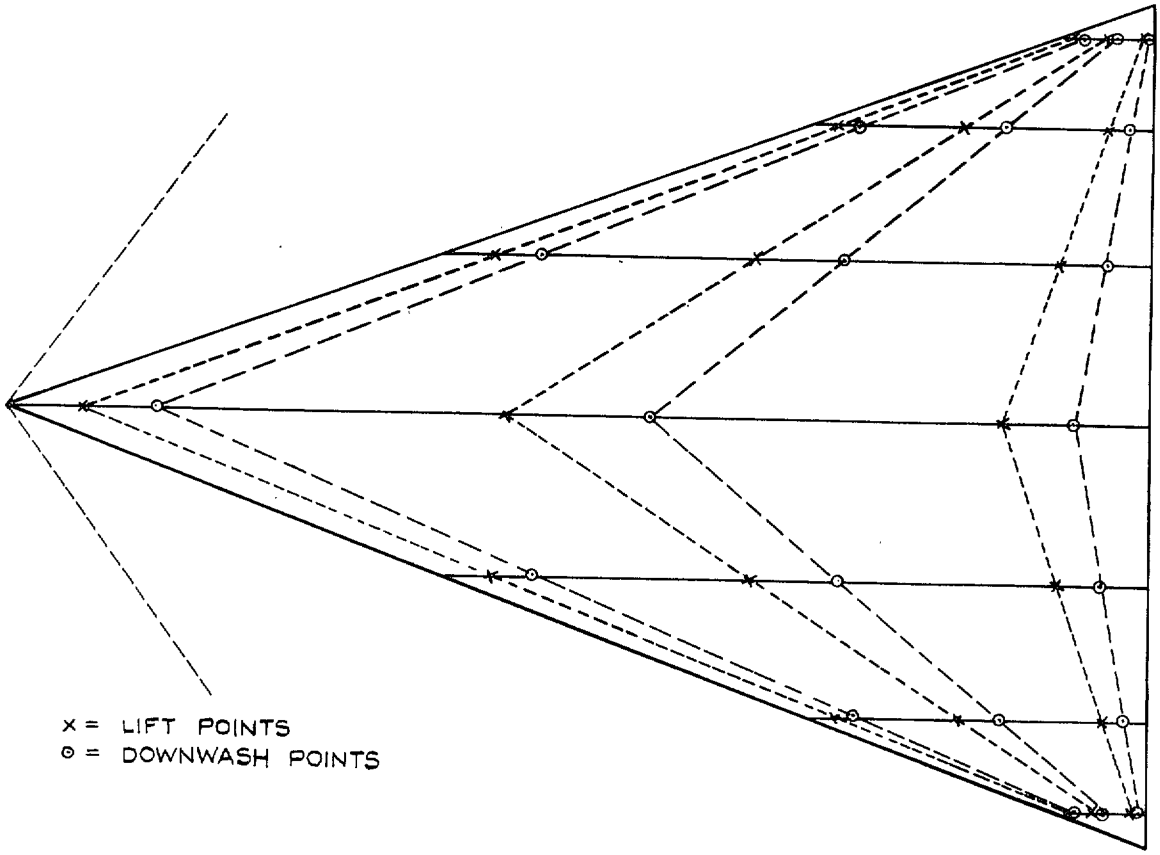

FIG. 4. Lift and downwash points for a delta wing with subsonic leading edge, $m = 3$, $n = 7$

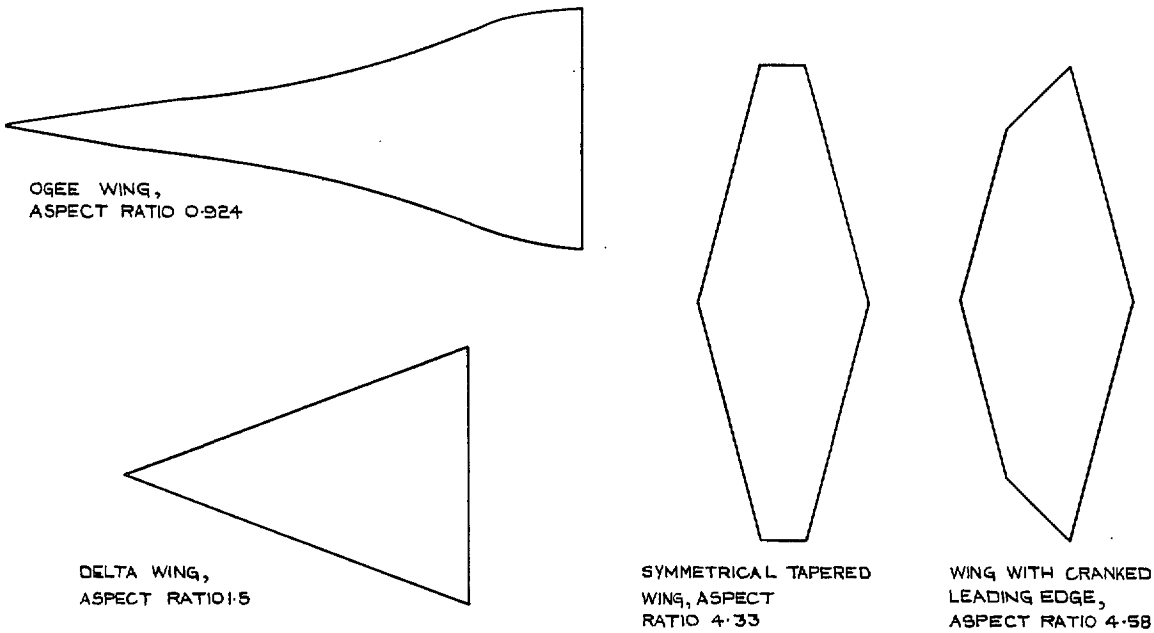

FIG. 5. Wing planforms used in derivative calculations

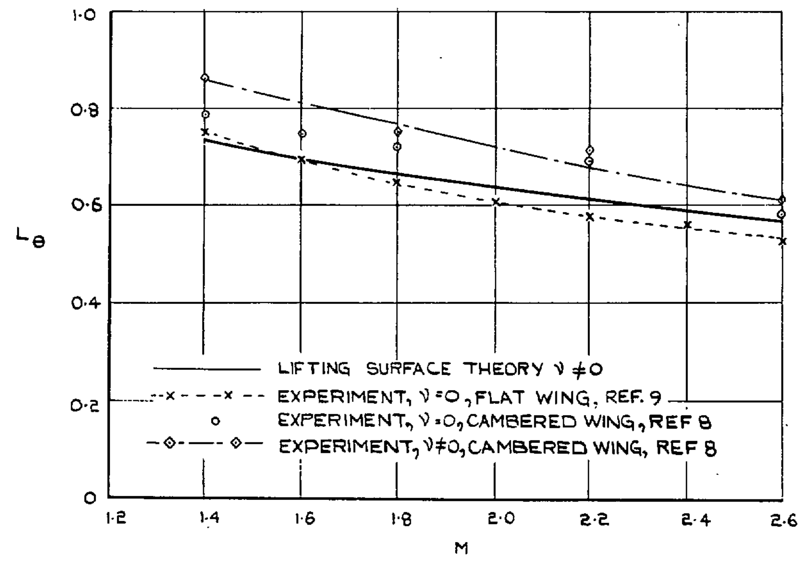

FIG. 6. Variation of L_θ with M for an ogee wing

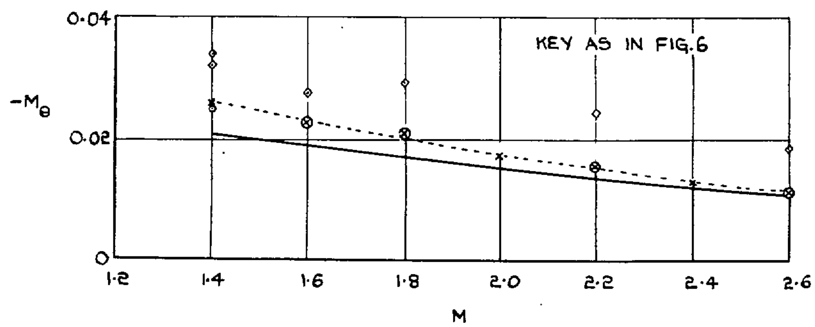

FIG. 7. Variation of $-M_\theta$ with M for an ogee wing

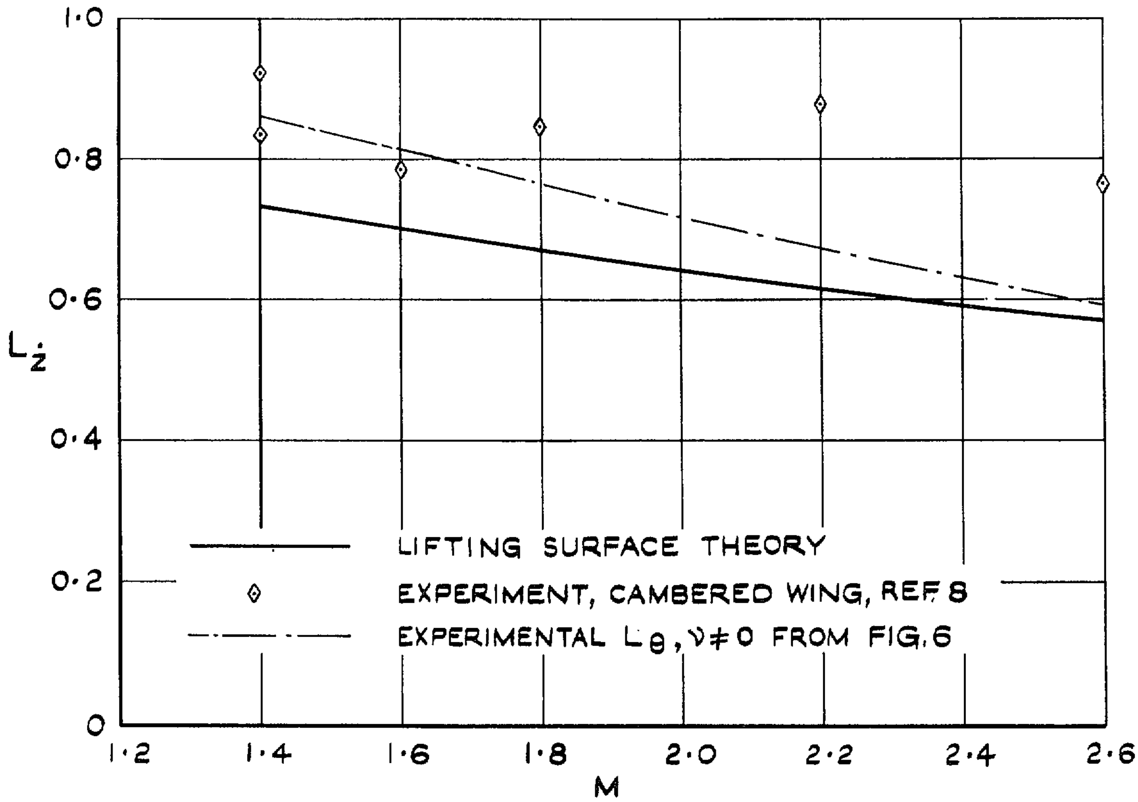

FIG. 8. Variation of $L_{\dot{z}}$ with M for an ogee wing

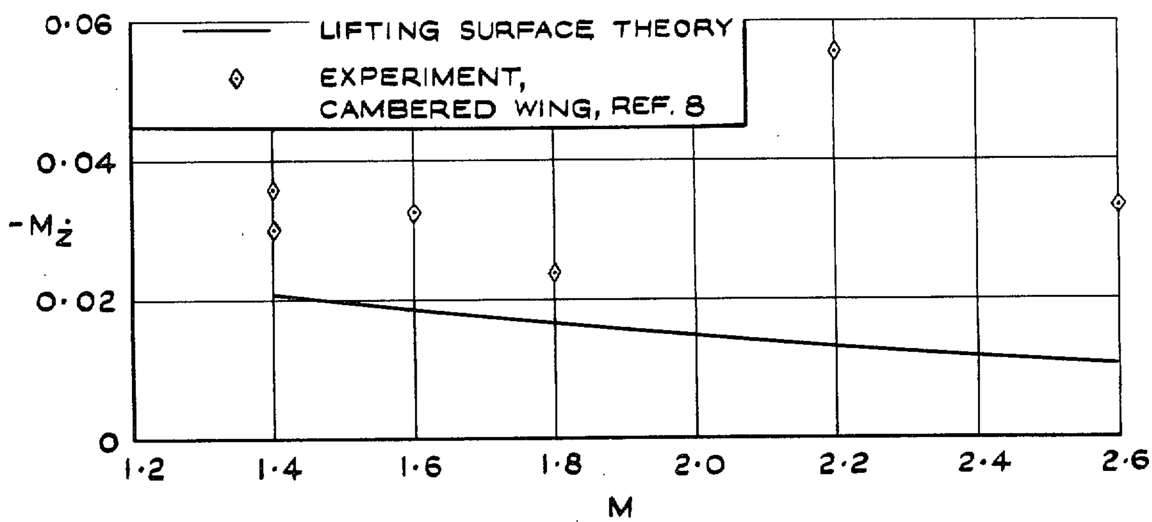

FIG. 9. Variation of $-M_{\dot{z}}$ with M for an ogee wing

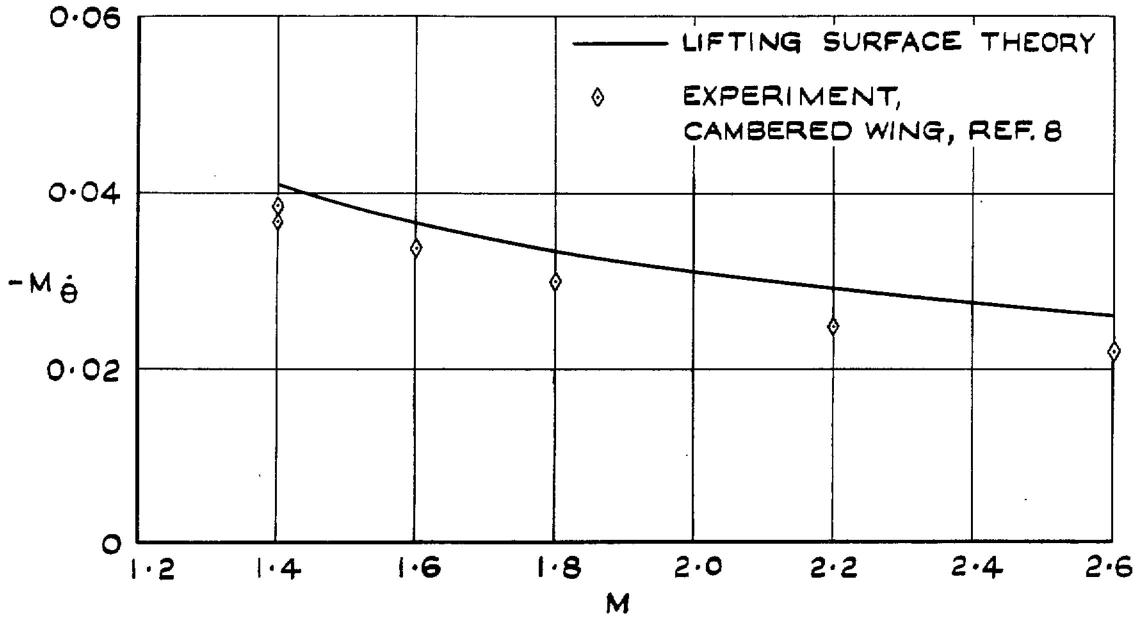

FIG. 10. Variation of $-M_{\dot{\theta}}$ with M for an ogee wing

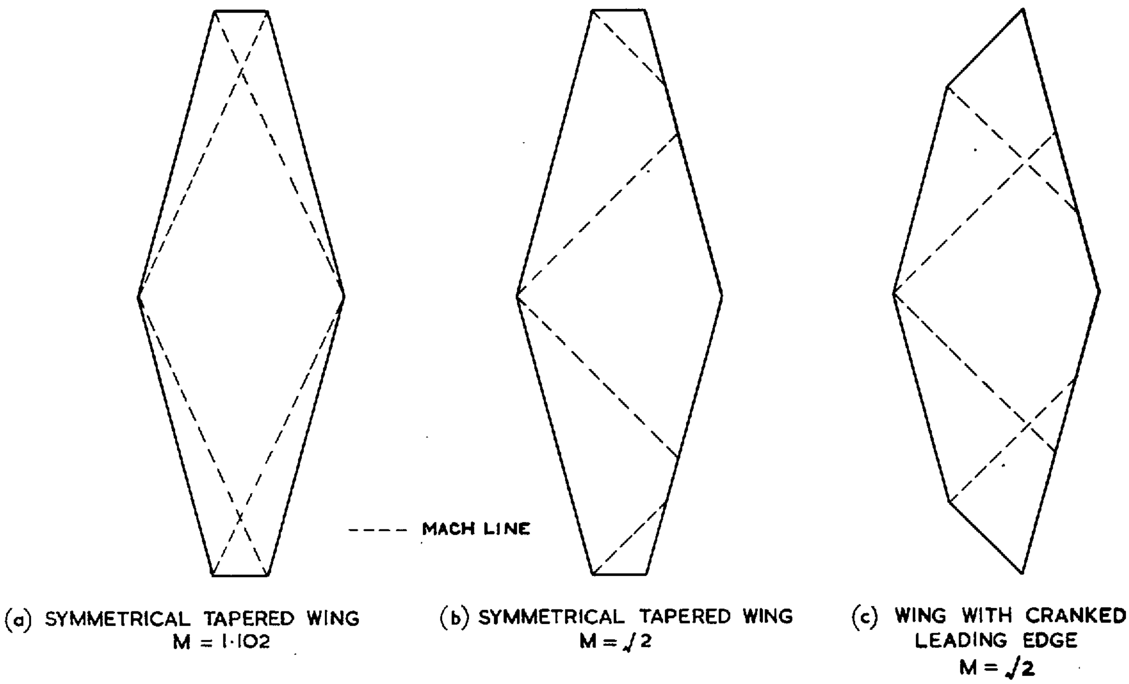

FIG. 11 (a to c). Mach line interference on wings with supersonic leading edges

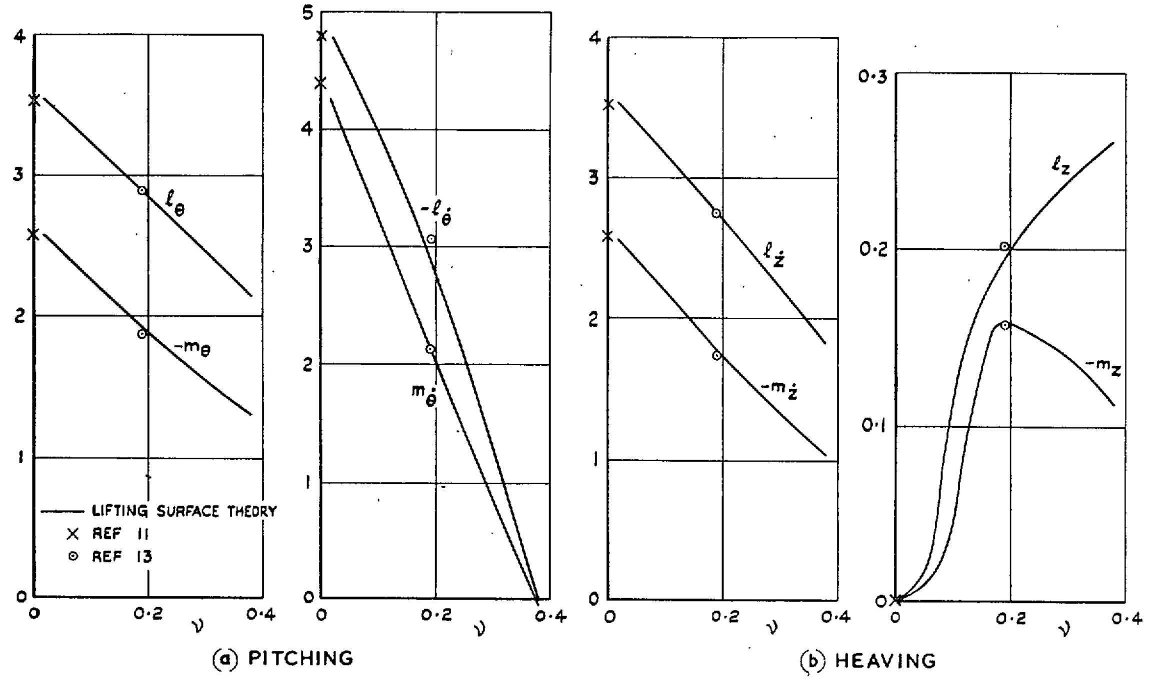

FIG. 12 (a and b). Variation of derivatives with frequency for a symmetrical tapered wing at $M = 1{\cdot}102$

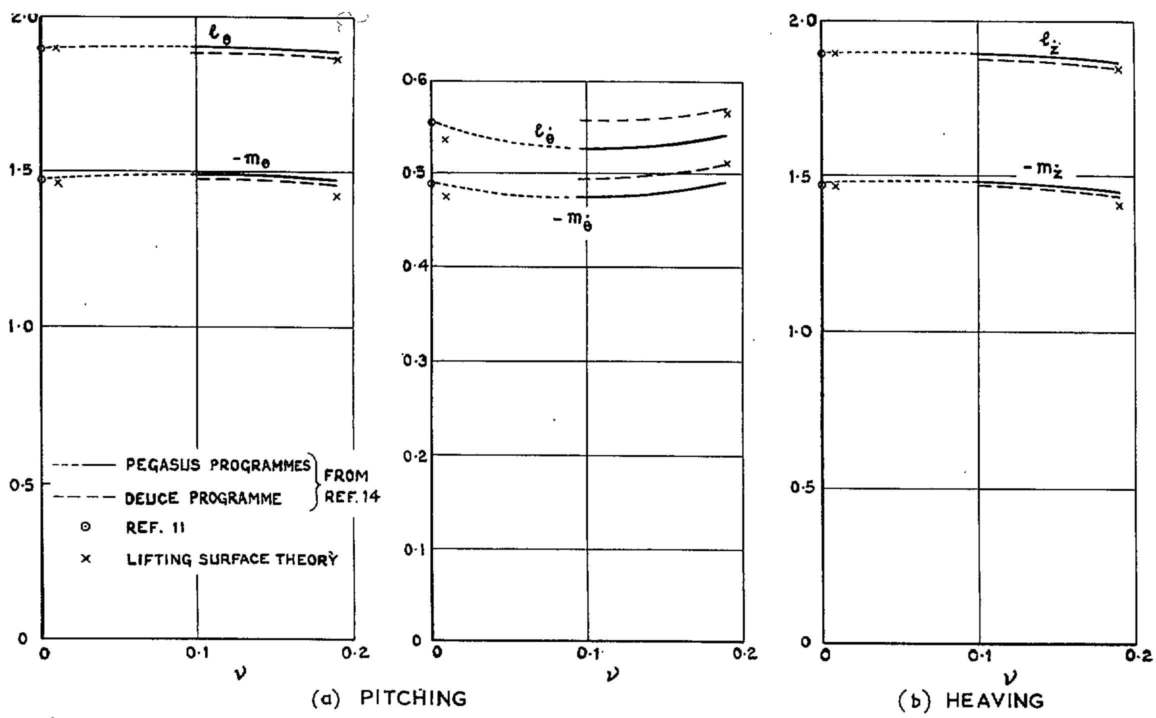

FIG. 13 (a and b). Variation of derivatives with frequency on a symmetrical tapered wing at $M = \sqrt{2}$

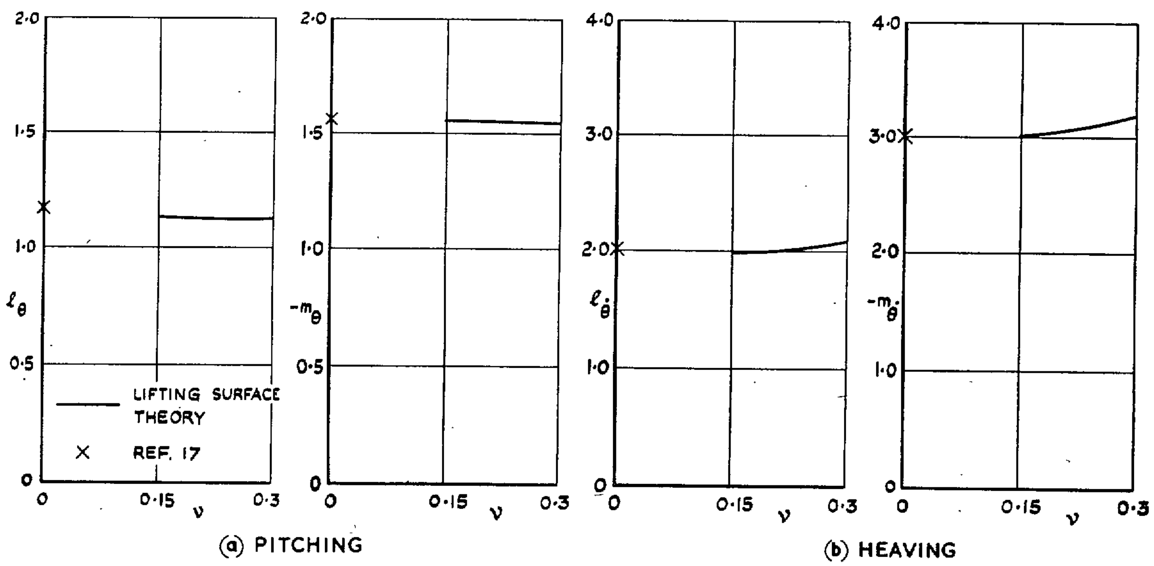

FIG. 14 (a and b). Variation of derivatives with frequency for a delta wing at $M = 1{\cdot}01$

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