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Static Instability of Rectangular Orthotropic Panels Subjected to Uniform In-Plane Loads and Deflection-Dependent Lateral Loads

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Summary.

Various analyses are presented for orthotropic and isotropic panels from which several elastic stability characteristics can be obtained. The loadings considered are biaxial in-plane compression and lateral loads which are dependent on the panel deflections (typically aerodynamic loadings). The panels are assumed to be resting on an elastic foundation to increase the generality of the problem.

The analyses are valid for panels whose edges experience no lateral deflection but which are in general elastically restrained against rotation.

The application of the analyses to particular configurations is shown and for certain isotropic panels correlation with experiment is found.

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1. *Introduction.*

The static stability of rectangular orthotropic panels has been the subject of many investigations when the loadings are in the plane of the panel. Corresponding analyses which include deflection – dependent lateral loadings are less widely known and the purpose of the present Report is to indicate possible theoretical approaches to the problem.

The present Report relies heavily on the theoretical approach adopted in Refs. 1, 2 which contain vibration and flutter studies for flat rectangular orthotropic panels with various degrees of rotational restraint.

In so much that the inclusion of deflection – dependent (e.g., aerodynamic) lateral loads may lead to static aeroelastic phenomena (e.g. panel divergence), reference is also made to a previous paper by the present author (Ref. 3).

2. *Analysis I.*

The configuration to be analysed consists of a flat rectangular orthotropic panel of length a, width b – Fig. 1. Uniform in-plane forces N_x, N_y, (positive in compression) and N_{xy} are applied to the panel although subsequently the in-plane shear will be discounted since the method of analysis adopted does not permit its inclusion. The panel is supported in such a way that there is no lateral deflection along the edges which are considered to be elastically restrained against rotation. The elastic restraint stiffness is assumed to be constant and equal on opposite edges.

The governing differential equation for the present problem is adapted from Refs. 4, 5 and is applicable to a plate with a small initial curvature.

$$\frac{D_x}{1-\nu_x \nu_y}\frac{\partial^4 w}{\partial x^4}+2\left(D_{xy}+\frac{\nu_y D_x}{1-\nu_x \nu_y}\right)\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{D_y}{1-\nu_x \nu_y}\frac{\partial^4 w}{\partial y^4}-$$

$$-E\Gamma\frac{\partial^6 w}{\partial x^2\,\partial y^4}+Kw+N_x\frac{\partial^2 \overline{w}}{\partial x^2}+2N_{xy}\frac{\partial^2 \overline{w}}{\partial x\,\partial y}+N_y\frac{\partial^2 \overline{w}}{\partial y^2}=\mathrm{p}. \qquad (1)$$

The term in Γ was introduced in Ref. 5 to include the effects of warping restraint of the transverse stiffeners in the y-direction which produce the orthotropicity. The above equation implicitly assumes that the principal axes of the panel are parallel to the geometric axes shown in Fig. 1, (for a more general formulation see Ref. 6).

The boundary conditions which the solution to equation (1) must satisfy are

$$\frac{D_x}{1-\nu_x \nu_y}\frac{\partial^2 w}{\partial x^2}\mp\theta_x\frac{\partial w}{\partial x}=0 \text{ and } w=0 \text{ at } \left\{\begin{matrix} x=0\\ x=a\end{matrix}\right\} \qquad (2)$$

$$\frac{D_y}{1-\nu_x \nu_y}\frac{\partial^2 w}{\partial y^2}\mp\theta_y\frac{\partial w}{\partial y}=0 \text{ and } w=0 \text{ at } \left\{\begin{matrix} y=0\\ y=b\end{matrix}\right\} \qquad (3)$$

where θ_x and θ_y are the spring constants per unit length of the rotational restraints acting at the boundaries.

An alternative form of the governing differential equation which may be deduced from equation (1) is,

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_{12} \frac{\partial^4 w}{\partial x^2\, \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} - E\Gamma \frac{\partial^6 w}{\partial x^2\, \partial y^4} + Kw +$$

$$+ N_x \frac{\partial^2 \bar{w}}{\partial x^2} + 2\, N_{xy} \frac{\partial^2 \bar{w}}{\partial x\, \partial y} + N_y \frac{\partial^2 \bar{w}}{\partial y^2} = p\,. \tag{4}$$

The form of the lateral pressure loading p is taken to represent both subsonic and supersonic flow conditions, and in a simplified form this may be written

$$p = P_1\, \bar{w} - P_2 \frac{\partial \bar{w}}{\partial x}, \tag{5}$$

where it is assumed that for subsonic flow the lateral load depends on the local displacement $\bar{w}$, and for supersonic flow on the local streamwise slope $\frac{\partial \bar{w}}{\partial x}$. These assumptions will be discussed later in Sections 4 and 5.

Substitution of eqn. (5) into eqn. (4) yields

$$\left[D_{11} \frac{\partial^4}{\partial x^4} + 2D_{12} \frac{\partial^4}{\partial x^2\, \partial y^2} + D_{22} \frac{\partial^4}{\partial y^4} \right] w - E\Gamma \frac{\partial^6 w}{\partial x^2\, \partial y^4} + Kw +$$

$$+ \left[N_x \frac{\partial^2}{\partial x^2} + 2\, N_{xy} \frac{\partial^2}{\partial x \partial y} + N_y \frac{\partial^2}{\partial y^2} \right] \bar{w} = P_1\, \bar{w} - P_2 \frac{\partial \bar{w}}{\partial x}. \tag{6}$$

In general a product solution of the form

$$w_{(x,y)} = X \left(\frac{x}{a} \right) . \, Y \left(\frac{y}{b} \right) \tag{7}$$

$$w_{0(x,y)} = X_0 \left(\frac{x}{a} \right) . \, Y_0 \left(\frac{y}{b} \right) \tag{8}$$

will not satisfy eqn. (6) since the terms $\frac{\partial^4 w}{\partial x^2\, \partial y^2}$ and $\frac{\partial^6 w}{\partial x^2\, \partial y^4}$ prevent the functions X and Y from separating. Kantorovich's method can however be used to obtain an approximate solution. Thus if $Y_0 \left(\frac{y}{b} \right)$ and $Y \left(\frac{y}{b} \right)$ are assumed functions which satisfy the boundary conditions on the edges $y = 0, b$ – see eqn. (3), and $X_0 \left(\frac{x}{a} \right)$ is also initially known or specified, then $X \left(\frac{x}{a} \right)$ is to be determined from the ordinary differential equation which results when eqns. (7) and (8) are substituted into eqn. (6) which is then multiplied by $Y \left(\frac{y}{b} \right)$ and integrated with respect to y. This procedure is essentially the first stage of a Galerkin-type analysis. The resulting ordinary differential equation is

$$\left[D_{11}X^{iv}C_0+2D_{12}\frac{a^2}{b^2}X^{ii}C_1+D_{22}\frac{a^4}{b^4}XC_2\right]-E\Gamma\frac{a^2}{b^2}X^{ii}C_2+KXC_0a^4+$$
$$+\left[N_xX^{ii}C_0+2N_{xy}\frac{a}{b}X^iC_3+N_y\frac{a^2}{b^2}XC_1\right]a^2-P_1XC_0a^4+P_2X^iC_0a^3+$$
$$+\left[N_xX_0^{ii}\bar{C}_0+2N_{xy}\frac{a}{b}X_0^i\bar{C}_3+N_y\frac{a^2}{b^2}X_0\bar{C}_1\right]a^2-P_1X_0\bar{C}_0a^4+P_2X_0^i\bar{C}_0a^3=0 \qquad (9)$$

where the primes denote differentiation with respect to x/a and

$$\left.\begin{aligned}
C_0 &= \int_0^1 Y^2\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right) & \bar{C}_0 &= \int_0^1 Y_0\left(\frac{y}{b}\right)Y\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right)\\
C_1 &= \int_0^1 Y^{ii}\left(\frac{y}{b}\right)Y\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right) & \bar{C}_1 &= \int_0^1 Y_0^{ii}\left(\frac{y}{b}\right)Y\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right)\\
C_2 &= \int_0^1 Y^{iv}\left(\frac{y}{b}\right)Y\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right) & \bar{C}_2 &= \int_0^1 Y_0^{iv}\left(\frac{y}{b}\right)Y\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right)\\
C_3 &= \int_0^1 Y^{i}\left(\frac{y}{b}\right)Y\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right) & \bar{C}_3 &= \int_0^1 Y_0^{i}\left(\frac{y}{b}\right)Y\left(\frac{y}{b}\right)d\left(\frac{y}{b}\right)
\end{aligned}\right\} \qquad (10)$$

where these primes denote differentiation with respect to $\frac{y}{b}$.

Thus, by selection of appropriate functions for $X_0\left(\frac{x}{a}\right)$, $Y_0\left(\frac{y}{b}\right)$, $Y\left(\frac{y}{b}\right)$ the problem reduces to finding the exact solution of eqn. (9) which satisfies the following boundary conditions

$$\left.\begin{aligned}
X_{(0)} &= X_{(1)} = 0\\
X_{(0}^{ii}-q_xX_{(0)}^{i} &= X_{(1)}^{ii}+q_xX_{(1)}^{i} = 0
\end{aligned}\right\} \qquad (11)$$

where $q_x = \frac{a\theta_x}{D_{11}}$ is defined as the rotational restraint coefficient. For simple supports $q_x = 0$ and for clamped supports $q_x = \infty$.

For the case when $X_0 = Y_0 = 0$ eqn. (9) may be written as

$$X^{iv}+\pi^2\bar{A}X^{ii}+\lambda X^i-\pi^4\bar{B}X = 0 \qquad (12)$$

where

$$\bar{A} = \frac{a^2}{b^2}\left[k_x+2\frac{D_{12}}{D_{11}}\cdot\frac{C_1}{\pi^2C_0}-\frac{E\Gamma\pi^2}{D_{11}b^2}\cdot\frac{C_2}{\pi^4C_0}\right] \qquad (13)$$

$$\lambda = \frac{P_2 a^3}{D_{11}} \tag{14}$$

$$\bar{B} = -\frac{a^4}{b^4}\left[\frac{D_{22}}{D_{11}}\cdot\frac{C_2}{\pi^4 C_0}+\frac{(K-P_1)b^4}{\pi^4 D_{11}}+k_y\frac{C_1}{\pi^2 C_0}\right] \tag{15}$$

$$k_x = N_x b^2/\pi^2 D_{11} \tag{16}$$

$$k_y = N_y b^2/\pi^2 D_{11}\,. \tag{17}$$

It should be pointed out here that for the given problem the integral C_3 is zero so that the form of λ in eqn. (14) consists only of an aerodynamic term (supersonic) and N_{xy} [eqn. (9)] is not retained.

The general solution to eqn. (12) is

$$X\left(\frac{x}{a}\right) = A_1 e^{m_1 x/a}+A_2 e^{m_2 x/a}+A_3 e^{m_3 x/a}+A_4 e^{m_4 x/a} \tag{18}$$

where m_j ($j = 1,2,3,4$) satisfies the auxiliary equation

$$m^4+\pi^2 \bar{A}\, m^2+\lambda m-\pi^4 \bar{B} = 0 \tag{19}$$

and the roots of this equation are assumed to be

$$\left.\begin{aligned} m_1 &= \alpha+i\delta \\ m_2 &= \alpha-i\delta \\ m_3 &= -\alpha+\varepsilon \\ m_4 &= -\alpha-\varepsilon\,. \end{aligned}\right\} \tag{20}$$

These forms were justified in Ref. 7 which showed the following relationships to follow as a consequence,

$$\delta^2 = \frac{\lambda}{4\alpha}+\alpha^2+\pi^2\frac{\bar{A}}{2} \tag{21}$$

$$\varepsilon^2 = \frac{\lambda}{4\alpha}-\alpha^2-\pi^2\frac{\bar{A}}{2} \tag{22}$$

$$\pi^4 \bar{B} = \left(\frac{\lambda}{4\alpha}\right)^2-4\left(\alpha^2+\frac{\pi^2 \bar{A}}{4}\right)^2. \tag{23}$$

Eqn. (23) may be written as

$$\alpha^6+S_1\alpha^4+S_2\alpha^2-S_3 = 0 \tag{24}$$

where

$$\left.\begin{aligned} S_1 &= \frac{\pi^2 \bar{A}}{2} \\ S_2 &= \frac{1}{4}\left[\pi^4 \bar{B} + \left(\frac{\pi^2 \bar{A}}{2}\right)^2\right] \\ S_3 &= \left(\frac{\lambda}{8}\right)^2 \end{aligned}\right\} \tag{25}$$

and hence α can be determined for any given values of λ, $\bar{A}$, $\bar{B}$. Using eqns. (21) and (22) values of δ and ε can then be evaluated and complete results for the four roots m_j can be found by eqn. (20).

Application of the boundary conditions eqns. (11) to (18) leads to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ m_1(m_1-q_x), & m_2(m_2-q_x), & m_3(m_3-q_x), & m_4(m_4-q_x) \\ e^{m_1} & e^{m_2} & e^{m_3} & e^{m_4} \\ m_1(m_1+q_x)e^{m_1}, & m_2(m_2+q_x)e^{m_2}, & m_3(m_3+q_x)e^{m_3}, & m_4(m_4+q_x)e^{m_4} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{26}$$

and the condition for a non-trivial solution is that the determinant of the square matrix should be zero, i.e.

$$F(m_1\, m_2\, m_3\, m_4\, q_x) = 0\,. \tag{27}$$

Thus this method of analysis requires that:

(a) Spanwise mode $Y\left(\frac{y}{b}\right)$ is chosen and corresponding values of C_1/C_0 and C_2/C_0 determined using eqn. (10). Values of these parameters were obtained in Ref. 1 by the use of beam vibration modes for the mode shape $Y\left(\frac{y}{b}\right)$. The modes chosen satisfy similar boundary conditions to those for $X\left(\frac{x}{a}\right)$ in eqn. (11) but with q_x replaced by q, defined by $q_y = \frac{b\theta_y}{D_{22}}$. Thus the resulting values for C_1/C_0 and C_2/C_0 have been found in Ref. 1 for the first four modes, as functions of q_y; *see* Figs. 2a, 2b.

(b) For assumed values of λ, $\bar{A}$ and $\bar{B}$ the four roots m_1 to m_4 are obtained and, for a given value of q_x the condition that $F(m_1\, m_2\, m_3\, m_4\, q_x) = 0$, eqn. (27), can be checked.

(c) For corresponding values of λ, $\bar{A}$ and $\bar{B}$ which satisfy eqn. (27) and knowing C_1/C_0 and C_2/C_0 from (a) the critical values of k_x, k_y, P_1 and P_2 may be determined for a given panel with known values of D_{11}, D_{22}, D_{12}, Γ, K.

The typical form of the solutions is shown in Fig. 3 for the most general case. It is seen that for an assumed value of $\bar{A}$ the variation of λ with $\bar{B}$ to satisfy eqn. (27) forms a stability loop. For varying $\bar{A}$ there are a series of such loops, any point of which represents a stability condition. Clearly for each value of $\bar{A}$ there are generally two values of $\bar{B}$ for each value of λ but at one value of $\lambda = \lambda_{cr}$ these two values of $\bar{B}$ reduce to one *viz.* $\bar{B}_{cr}$.

Table 1 presents corresponding values of λ_{cr}, $\bar{B}_{cr}$ and $\bar{A}$ and is taken directly from Ref. 2. A range of values of q_x is considered.

If the possibility of supersonic flow is not to be considered then $\lambda = 0$. The corresponding values of $\bar{A}$ and $\bar{B}$ for stability are given in Table 2 which has been adapted from Ref. 1.

3. *Analysis II.*

As an alternative approach to the 'semi-exact' approach in Section 2, a more approximate method can be adopted based on the Galerkin method.

Thus if eqn. (6) is rewritten (with $w_0 = 0$) as

$$\left[D_{11}\frac{\partial^4}{\partial x^4}+2D_{12}\frac{\partial^4}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4}{\partial y^4}\right]w-E\Gamma\frac{\partial^6 w}{\partial x^2\,\partial y^4}+Kw+$$

$$+\left[N_x\frac{\partial^2}{\partial x^2}+2N_{xy}\frac{\partial^2}{\partial x\partial y}+N_y\frac{\partial^2}{\partial y^2}\right]w-P_1\,w+P_2\frac{\partial w}{\partial x}=0 \tag{28}$$

then if an approximate mode is assumed, in the form

$$w=\sum_m\sum_n A_{mn}\,F_m(x)\,G_n(y) \tag{29}$$

and substituted into eqn. (28), that equation will not be satisfied identically and there will be a net error ϕ. The Galerkin method requires that the following series of integrals be evaluated and that the resulting stability determinant formed by the coefficients of A_{mn} be equated to zero.

$$\int_0^a\int_0^b \phi\frac{\partial w}{\partial A_{mn}}\,dx\,dy=0. \tag{30}$$

If, for example, $F_m(x)=\sin\frac{m\pi x}{a}$ and $G_n(y)=\sin\frac{n\pi y}{b}$, which would correspond to a panel with simply-supported edges, the following set of equations is obtained (cf. eqns. (12) to (17))

$$\left[m^4+2\frac{a^2}{b^2}\frac{D_{12}}{D_{11}}m^2n^2+\frac{a^4}{b^4}\frac{D_{22}}{D_{11}}n^4-k_xm^2-\frac{a^2}{b^2}k_yn^2\right.$$

$$\left.+(K-P_1)\frac{a^4}{\pi^4\,D_{11}}+E\Gamma\frac{m^2n^4\pi^2a^2}{D_{11}\,b^4}\right]A_{mn}$$

$$-\frac{P_2a^3}{\pi^3\,D_{11}}\sum_r\sum_s A_{rs}\,L_{mn,rs}+2\sum_r\sum_s A_{rs}\left(\frac{a}{b}\right)k_{xy}\,P_{mn,rs}=0. \tag{31}$$

Where

$$L_{mn,rs}=\frac{4mr}{\pi\,(r^2-m^2)}\left[\neq 0\text{ if }\left\{\begin{matrix}m\pm r\text{ odd}\\ n=s\end{matrix}\right\};=0\text{ otherwise.}\right]$$

$$P_{mn,rs}=\frac{16mnrs}{\pi^2\,(r^2-m^2)\,(s^2-n^2)}\left[\neq 0\text{ if }\left\{\begin{matrix}m\pm r\text{ odd}\\ n\pm s\text{ odd}\end{matrix}\right\};=0\text{ otherwise.}\right]$$

The functions $L_{mn,rs}$ and $P_{mn,rs}$ represent the supersonic aerodynamic and shear coupling terms respectively. It should be noted that supersonic aerodynamic coupling exists only between streamwise modes with $m\pm r$ odd whereas the subsonic aerodynamic term is a direct stiffness term.

The term $k_{xy} = \dfrac{N_{xy}a^2}{D_{11}\pi^2}$ is the non-dimensional shear stress term which in the subsequent study of this section will be neglected (as in Section 2).

A study in which k_{xy} was retained is reported in Ref. 8 and the effects of the shear term were seen to be most significant for panel flutter. For static stability in the presence of aerodynamic forces it is expected that shear would continue to be most significant in determining the stability boundaries.

Provided a sufficiently large number of terms are included in the analysis the stability boundaries obtained should approximate to those obtained by more exact methods. Table 3 presents typical results obtained from Ref. 9 in connection with a study of the effects of damping on panel flutter – they can be considered here also for our present purpose. It is seen that the results in Table 3 provide stability loops of the type discussed in Section 2 and illustrated in Fig. 3.

Thus the analyses of this and the previous section have led to values of $\bar{A}$, $\bar{B}$, and λ which satisfy the given general stability problem. These values enable the buckling loads of orthotropic panels to be determined for large ranges of length-width ratio, orthotropic stiffness ratios, inplane loadings, lateral aerodynamic loadings, foundation stiffness and boundary conditions.

It is clear from the definition of $\bar{B}$ that the effect of a subsonic lateral loading P_1 is destabilising in a comparable way to k_y. Conversely the foundation stiffness K is stabilising. Further the stability loops of Table 3 (Fig. 3) show that supersonic lateral loading is stabilising.

4. *Isotropic Panels in Supersonic Flow.*

From the form of the stability loops in Table 3 (*see* Fig. 3) it is clear that the supersonic lateral loading λ is stabilising since λ increases as $\bar{B}$ increases and $\bar{B}$ is proportional to the compressive loading k_y. This stabilising influence of the supersonic airflow over the panel, on the *static* stability of the panel has been noted previously. Caution however is advised in applying this result since it is well known that supersonic airflow can initiate the phenomenon of panel flutter. A discussion of panel flutter with relevance to the effects of length-width ratio, inplane loadings etc. is contained in Ref. 10 and Figs. 4, 5 show some typical stability boundaries for isotropic panels in supersonic flow from that report. That there are static and dynamic areas of instability in the region of the pure buckling loads is clearly seen.

By definition $\lambda = P_2 a^3/D_{11}$, and for high supersonic Mach numbers (M) the accepted value of P_2 is that given by Ackeret's formula *viz.* $P_2 = 2q/\sqrt{M^2-1}$ (flow on one side of panel only) where q = dynamic pressure.

5. *Two-Dimensional Panels in Incompressible Flow.*

5.1. *Infinite Length Isotropic Panels.*

For a two-dimensional isotropic panel on an elastic foundation, eqn. (4) reduces (with $w_0 = 0$) to

$$D\frac{\partial^4 w}{\partial x^4}+Kw+N_x\frac{\partial^2 w}{\partial x^2} = P\,. \tag{32}$$

Ref. 11 has considered the dynamic stability of such an infinite length panel (i.e. $a = \infty$) for the case of $N_x = 0$ and on the assumption of a travelling wave type motion. It was shown that in practice a static instability (divergence) would occur first and from a similar analysis with the N_x term retained (Ref. 3) it can be shown that the appropriate static stability criterion is

$$\left.\begin{aligned} q_D &= \tfrac{1}{2}\left[D\left(\frac{\rho}{\mu\sigma}\right)^2+K\left(\frac{\mu\sigma}{\rho}\right)^2-N_x\right]\left(\frac{\rho}{\mu\sigma}\right), \text{ and} \\ &\left(\frac{\mu\sigma}{\rho}\right)^4+\frac{N_x}{K}\left(\frac{\mu\sigma}{\rho}\right)^2-\frac{3D}{K}=0 \end{aligned}\right\} \tag{33}$$

If $N_x = 0$ the simple result obtained is

$$q_D = 0{\cdot}878\, K^{\frac{3}{4}}\, D^{\frac{1}{4}}$$

and the obvious result is seen that with $K = 0$ the divergence speed is zero.

5.2. *Finite Length Isotropic Panels.*

For a two-dimensional isotropic panel of finite length Dugundji *et alii* (Ref. 11) have extended the previous analysis by Flax (Ref. 12) which was for a panel *not* on an elastic foundation. A panel of infinite length simply supported at a spacing a in the x direction has for its deflection curve,

$$w_{(x)} = A_m \sin \frac{m\pi x}{a}\,; \tag{34}$$

The corresponding aerodynamic pressure (Ref. 12) is

$$p = \rho U^2 \left(\frac{m\pi}{a}\right) A_m \sin \frac{m\pi x}{a} \tag{35}$$

i.e.

$$P_1 = \rho U^2 \left(m\pi/a \right) \text{ in eqn. (5), then}$$

substitution of eqns. (34) and (35) into eqn. (32) leads to the following static stability criterion

$$\bar{Q} = \frac{2qa^3}{\pi^3 D} = \frac{m^4 - \bar{N}m^2 + \bar{K}}{m} \tag{36}$$

where

$$\bar{N} = \frac{N_x a^2}{\pi^2 D}, \bar{K} = \frac{Ka^4}{\pi^4 D}.$$

The minimum dynamic pressure for divergence (static instability) occurs when

$$3m^4 - \bar{N}m^2 = \bar{K} \tag{37}$$

i.e. when

$$m^2 = + \frac{\bar{N} + \sqrt{\bar{N}^2 + 12\,\bar{K}}}{6} \tag{38}$$

for

$$N = 0 \quad m_{cr} \simeq 0{\cdot}758\, \bar{K}^{1/4} \tag{39}$$

in which case

$$Q_D = \frac{2q_D a^3}{D} \simeq 1{\cdot}756\, \bar{K}^{3/4}. \tag{40}$$

It can be shown that when $\bar{N} = 0$ and $\bar{K} < 14$ then $m_{cr} = 1$.

Alternatively if $\bar{K} = 0$, divergence always occurs in the lowest mode, $m = 1$, and

$$\bar{Q}_D = \frac{2q_D a^3}{\pi^3 D} = 1 - \bar{N}. \tag{41}$$

Therefore for $\bar{N}$ positive (i.e., N_x compressive) the divergence dynamic pressure q_D is reduced; conversely the static buckling load is reduced when there is a subsonic airflow past the panel (an opposite effect was noted for supersonic flow in Section 4).

When both $\bar{N}$ and $\bar{K}$ are zero the simple result is obtained (Refs. 12, 13).

$$Q_D = \frac{2q_D a^3}{D} = \pi^3 \simeq 31. \tag{42}$$

The above results were obtained for an infinite panel simply supported at a streamwise spacing of 'a'. For a *single* simply supported panel of chord 'a' mounted on a rigid wall or even for such a panel with clamped ends the aerodynamic pressure distribution is unlikely to be that given in eqn. (35). To analyse these two configurations Richardson (Ref. 13) assumed a chordwise distribution of sources on the panel and solved the aeroelastic problem by an iterative procedure. The result obtained for these single panels were

$$\left.\begin{aligned} Q_D &= 40 \text{ (simple supports)} \\ \text{and} \quad Q_D &= 178 \text{ (clamped ends)} \end{aligned}\right\} \tag{43}$$

To assess the accuracy of the assumed theoretical aerodynamic pressure distributions Sykes (Ref. 14) has conducted simple pressure measurement tests on a *single* half sine-wave model and on a half sine-wave of an 'infinite' array of sine waves.

The first set of tests showed similar effects as observed in Ref. 15 that the pressures are significantly lower, over the forward quarter-wave of the single half wave model, than predicted by eqn. (35). From the second set of tests results showed excellent agreement with eqn. (35).

In his subsequent analytical approach to the static deformation problem of the single simply supported panel Sykes assumed that the deformation mode could be represented by

$$w = A_1 \sin\frac{\pi x}{a} + A_2 \sin\frac{2\pi x}{a} \tag{44}$$

with a corresponding pressure distribution of

$$p = \rho U^2 \left(\frac{\pi}{a}\right) \left[A_1 f_{1(x)} + 2A_2 f_{2(x)} \right]. \tag{45}$$

For a continuous sinusoidal surface $f_{1(x)}$ and $f_{2(x)}$ would be sine functions but in Ref. 14 the functions f were determined from the above-mentioned pressure distribution tests. For a two-dimensional isotropic panel with $\Gamma = K = N_{xy} = N_y = 0$, $w_0 = 0$, one obtains from eqn. (4).

$$D\frac{d^4 w}{dx^4} + N_x \frac{d^2 w}{dx^2} = p. \tag{46}$$

If large deflection effects are considered the corresponding equation for the initial deflections w_1, under the initial compressive loading N_x^1 only is

$$D\frac{d^4 w_1}{dx^4}+N_x^1\frac{d^2 w_1}{dx^2} = 0. \tag{47}$$

Subtracting (47) from (46) leads to the well known equation

$$D\frac{d^4}{dx^4}\left[w-w_1\right]+N_x\frac{d^2 w}{dx^2}-N_x^1\frac{d^2 w_1}{dx^2} = p \tag{48}$$

The configuration being studied is shown in Fig. 6. The panel is subjected to an incompressible airflow on the upper surface only and to static air pressure on the lower surface. The panel edge members are capable of relative movement in the streamwise direction which is restrained by an elastic spring of stiffness k. The problem is to determine the variation of panel deflection with airspeed and under the given compressive loading N_x. The analysis closely follows that of Ref. 16 which considered the corresponding supersonic problem of panel flutter.

If Δ is the relative movement of the panel edges towards each other when the aerodynamic forces act, then

$$N_x^1-N_x = k\Delta \tag{49}$$

and if lateral deflections are not excessive we may write

$$\Delta = -(N_x^1-N_x)\frac{a}{Eh}+\tfrac{1}{2}\int_0^a\left[\left(\frac{dw}{dx}\right)^2-\left(\frac{dw_1}{dx}\right)^2\right]dx \tag{50}$$

therefore

$$(N_x^1-N_x) = \psi\left(\frac{Eh}{2a}\right)\int_0^a\left[\left(\frac{dw}{dx}\right)^2-\left(\frac{dw_1}{dx}\right)^2\right]dx \tag{51}$$

where

$$\psi = \left[1+(Eh/ka)\right]^{-1}. \tag{52}$$

Eqns. (48) and (51) are to be solved for the boundary conditions

$$\left.\begin{aligned} w_{(0)} = w_{(a)} &= 0 \\ \frac{d^2w}{dx^2}(o) = \frac{d^2w}{dx^2}(a) &= 0 \end{aligned}\right\} \tag{53}$$

Hence, using eqns. (44) and (45), with the measured functions f_1, f_2, in a two mode Galerkin analysis and with

$$w_1 = B_1 \sin\frac{\pi x}{a} \tag{54}$$

yields the following expression for A_1 for example,

$$A_1 = \frac{B_1\,(1-S)\,(4-S-\psi\chi-{\cdot}421\,\bar{Q})}{(1-S-\psi\chi-{\cdot}896\,\bar{Q})\,(4-S-\psi\chi-{\cdot}421\,\bar{Q})-{\cdot}00447\,\bar{Q}^2} \tag{55}$$

where

$$S = \bar{N}^1 \quad (\textit{see}\ \text{eqn. (36)}) \qquad \bar{Q} = 2qa^3/\pi^3 D$$

$$\chi = \frac{1}{\psi}\left[\bar{N}_x^1 - \bar{N}_x\right]\left(\frac{a^2}{\pi^2 D}\right) = \frac{1}{\psi}\left[\bar{N}^1 - \bar{N}\right]$$

which from eqn. (51)

$$= \frac{3(1-\nu^2)}{h^2}\left[B_1^2-(A_1^2+4A_2^2)\right].$$

Therefore for given values, of the parameters ψ, B_1, N_x' it is possible to evaluate A_1, A_2 and hence N_x from eqn. (51) and to examine the panel static deflection as a function of airspeed.

If a single degree of freedom analysis using A_1 and f_1 only is pursued the result is

$$A_1 = B_1\,(1-S)/(1-S-\psi\chi-0{\cdot}896\,\bar{Q}) \tag{56}$$

where

$$\chi = \frac{3\,(1-\nu^2)}{h^2}\left[B_1^2 - A_1^2\right].$$

If f_1 is made equal to $\sin\dfrac{\pi x}{a}$ the corresponding result for a simply supported panel in the infinite array is

$$A_1 = B_1\,(1-S)/(1-S-\psi\chi-\bar{Q}). \tag{57}$$

For divergence, $A_1 \to \infty$ which occurs if there is no spring restraint, i.e., $\psi = 0$, when the denominators of eqns. (55), (56) and (57) are zero, *viz.* when (if $S = 0$); $\bar{Q} = 1{\cdot}11$, $1{\cdot}12$, or $1{\cdot}0$ respectively. Thus for the finite panel aerodynamics the critical dynamic pressure is higher than for the assumed sinusoidal pressure distribution of the infinite panel array.

The application of eqns. (55) – (57) to an experiment was attempted in Ref. 14 and the results are shown in Fig. 7. The theoretical and experimental results show similar trends.

It is not surprising that in Ref. 14 or in Refs. 17 and 18 panel divergence was not experienced since very large deflection effects due to plate and aerodynamic behaviour could be introduced. However, the results of Ref. 18 are of interest in that at certain airspeeds the panel, which had some initial curvature, experienced significant and increasing deflections, appearing to have the form of a divergence – *see* Fig. 8.

From a series of such curves for different tensile loads an attempt was made to estimate the corresponding critical dynamic pressures q_D. These are shown in Fig. 9 and are seen to follow a linear variation which when extrapolated predicts the buckling load in the absence of an airflow. This linear relationship has been shown theoretically in eqn. (41).

6. *Three-Dimensional Panels in Incompressible Flow.*

6.1. *Orthotropic Panels of Finite Aspect Ratio.*

For a panel of finite aspect ratio with simply supported edges, the assumed mode of deformation is

$$w = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{58}$$

and eqn. (12) becomes (with $\Gamma = K = 0$)

$$m^4 - \bar{A}\, m^2 - \bar{B} = 0 \tag{59}$$

with

$$\bar{A} = \frac{a^2}{b^2}\left[k_x - 2n^2 \frac{D_{12}}{D_{11}}\right] \tag{60}$$

and

$$\bar{B} = \frac{-a^4}{b^4}\left[\frac{D_{22}\, n^4}{D_{11}} - \frac{P_1\, b^4}{\pi^4 D_{11}} - k_y\, n^2\right]. \tag{61}$$

Eqn. (59) yields the results shown in Table 2 for $q_x = 0$. Since for a two-dimensional panel there is little error in assuming that

$$P_1 = \rho U^2 \left(\frac{m\pi}{a}\right) \tag{62}$$

the same assumption will be made here for the three-dimensional panel. This should yield a conservative result – *see* Fig. 7. From eqn. (59)

$$\frac{m^4}{\beta^4} + \frac{2dm^2n^2}{\beta^2} + n^2 = \frac{b^2}{\pi^2 D_{22}}\left[\frac{N_x}{\beta^2}\left(\frac{D_{22}}{D_{11}}\right)^{\frac{1}{2}} m^2 + N_y\,.\,n^2 + \frac{2qb^2 m}{\pi a}\right] \tag{63}$$

where

$$\beta = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}, \qquad d = D_{12}/\sqrt{D_{11} D_{22}}, \qquad q = \tfrac{1}{2}\rho U^2 .$$

Eqn. (63) reduces for an isotropic plate with $N_y = 0$, to eqn. (36) – (with $K = 0$). The form of (63) is identical to that used by Wittrick in Ref. 19.

It remains to find the values of N_x, N_y and q which combine to produce instability. Following the procedure of Ref. 19 it is now assumed that the values of N_y and q are known and the critical value of N_x is determined from eqn. (63) rearranged, thus,

$$\frac{b^2 N_x}{\pi^2 \sqrt{D_{11} D_{22}}} = \frac{m^2}{\beta^2} + 2dn^2 + \frac{\beta^2 n^4}{m^2} - \frac{b^2 N_y}{\pi^2 D_{22}} \cdot \frac{\beta^2 n^2}{m^2} - \frac{2q\beta^2 b^4}{\pi a m}. \tag{64}$$

Differentiating (64) with respect to n gives

$$\frac{b^2}{\pi^2\sqrt{D_{11}D_{22}}}\frac{\partial N_x}{\partial n} = \frac{2\beta^2 n}{m^2}\left[2\left(n^2+\frac{dm^2}{\beta^2}\right)-\frac{b^2 N_y}{\pi^2 D_{22}}\right] \tag{65}$$

which was obtained in Ref. 19. Hence if

$$N_y < 2\pi^2\left(\frac{D_{22}}{b^2}+\frac{D_{12}}{a^2}\right) \tag{66}$$

the minimum N_x occurs when $n = 1$ and eqn. (64) may then be written,

$$\frac{b^2 N_x}{\pi^2\sqrt{D_{11}D_{22}}} = 2d+\frac{m^2}{\beta^2}+\frac{\beta^2}{m^2}\left[1-\frac{b^2 N_y}{\pi^2 D_{22}}-\frac{2qb^4 m}{\pi a}\right]. \tag{67}$$

If the square-bracketed term is greater than zero, i.e. the following inequality is satisfied

$$\left[N_y+2q\,D_{22}\frac{\pi b^2 m}{a}\right] < \frac{\pi^2 D_{22}}{b^2} \tag{68}$$

then eqn. (67) may be written as

$$\bar{k} = \left[\left(\frac{m}{\gamma}\right)+\left(\frac{\gamma}{m}\right)\right]^2 \tag{69}$$

with

$$\bar{k} = 2+\frac{\frac{b^2 N_x}{\pi^2}-2\,D_{12}}{\left[D_{11}D_{22}\left(1-\frac{b^2 N_y}{\pi^2 D_{22}}-2q\frac{b^4 m}{\pi a}\right)\right]^{\frac{1}{2}}} \tag{70}$$

$$\gamma = \frac{a}{b}\left[\frac{D_{22}}{D_{11}}\left(1-\frac{b^2 N_y}{\pi^2 D_{22}}-2q\frac{b^4 m}{\pi a}\right)\right]^{1/4}. \tag{71}$$

Ref. 19 has shown that eqn. (69) yields a stability curve identical with that for an isotropic plate under biaxial loading provided the definitions of eqns. (70) and (71) are considered. Therefore the conclusions of Ref. 19 apply here, *viz.* a single curve relating the parameters $\bar{k}$ and γ serves to specify the buckling load N_x in each of the following cases (i) orthotropic plate with all edges simply supported subjected to compressive forces N_x and N_y on its edges and a subsonic airflow of dynamic pressure q across one surface. N_y and q are known and related by inequality (68). $\bar{k}$ and γ are given by eqns. (70), (71). (ii) The same as (i) but with the ends ($x = 0, a$) clamped and the sides simply supported.

It is of interest to examine the purely aeroelastic behaviour of the finite aspect ratio panel and for this purpose eqn. (63) will be used with $N_x = N_y = 0$. Thus

$$\frac{2qb^4 m}{\pi^3 aD_{22}} = \frac{m^4}{\beta^4}+\frac{2dm^2n^2}{\beta^2}+n^4 \tag{72}$$

and it is clear that the minimum q occurs when $n = 1$. Minimising q with respect to m yields the condition

$$\frac{m^2_{cr}}{\beta^2} = \frac{(d^2+3)^{\frac{1}{2}}-d}{3}. \tag{73}$$

Hence for a given orthotropic plate the critical divergence dynamic pressure q_D can be obtained.
For the isotropic plate the appropriate results are

$$m^2_{cr} = a^2/3b^2 \tag{74}$$

$$2q_D b^3/\pi^3 D = 16\sqrt{3}/9 \simeq 3{\cdot}08. \tag{75}$$

The similarity in form between eqns. (75) and (41) should be noted. From eqn. (74) $m_{cr} < 1$ when $b^2 > \frac{a^2}{3}$ but since the lowest permissible value of m is unity then for quite small values of panel aspect ratio $\left(\frac{b}{a}\right)$ the value $m = 1$ would apply and the corresponding result from eqn. (72) for the isotropic plate is

$$\bar{Q}_D = \frac{2q_D a^3}{\pi^3 D} = \left(1+\frac{a^2}{b^2}\right)^2. \tag{76}$$

It is clear that when $b \to \infty$ i.e. a two-dimensional panel, eqn. (76) reduces to give the same result as eqn. (41) *viz.*

$$\bar{Q}_D = 1. \tag{77}$$

In comparison the result shown in eqn. (75) obviously only applies for very low aspect ratio panels where $b^2 < a^2/3$. Actually as only integer values of m are acceptable the result shown in eqn. (75) is a lower bound on the value of q_D and only reached asymptotically at large values of a/b or at values of a/b yielding an integer value of m from eqn. (74).

6.2. *Isotropic Panels of very Low Aspect Ratio.*

For a panel of very low aspect ratio with simply supported edges the assumed mode of deformation is taken in Ref. 20 as

$$w = A \sin\frac{\pi y}{b} . e^{-i2\pi x/l} \tag{78}$$

where l is the longitudinal wavelength.

A more precise formulation of the aerodynamic problem is given in Ref. 20 than was assumed in Sections 5.2 and 6.1 for both the compressible and the incompressible flow cases and an aerodynamic integral $F(\eta)$ was obtained which had to be evaluated numerically for various values of $\eta = \left(\frac{\pi b}{l}\right)^2 . (1-M^2)$. The limiting value of $\eta \to 0$ corresponds to very low aspect ratio theory. For divergence in the absence of in-plane loads the criterion obtained was

$$\frac{2q_D b^3}{\pi^3 D} = \frac{1}{\pi F(\eta)}\left[\left(\frac{2b}{l}\right)^2 + 2 + \left(\frac{l}{2b}\right)^2\right]. \tag{79}$$

The corresponding aeroelastic stability boundary of $q_D b^3/2\pi^2 D \sim M$ is shown in Fig. 10 for very low aspect ratio panels. At $M = 0$ the curve shown yields the result

$$\frac{2q_D b^3}{\pi^3 D} \simeq 4{\cdot}45\,. \tag{80}$$

Thus the approximate aerodynamic theory used in Section 6.1 and leading to eqn. (75) has given a conservative result compared with the more precise theory of this Section which gives eqn. (80). This conservatism was anticipated in Section 6.1 in view of the results shown in Fig. 7.

If the above, single, low aspect ratio panel is one of an infinite array in the spanwise direction it is found from Ref. 20 that the corresponding divergence condition for $M = 0$ is given by

$$\frac{2qb^3}{\pi^3 D} = \left[1+\left(\frac{2b}{l}\right)^2\right]^{\frac{1}{2}}\left[\left(\frac{2b}{l}\right)^2+2+\left(\frac{l}{2b}\right)^2\right]. \tag{81}$$

The minimum value q_D occurs when $\left(\frac{2b}{l}\right)^2 = \frac{2}{3}$ in which case

$$2q_D b^3/\pi^3 D \simeq 5{\cdot}4\,. \tag{82}$$

Thus the critical dynamic pressure is higher than in the corresponding single panel case.

It may probably be reasonably inferred from these results that where the panel being analysed (in the more general case of $N_x \neq 0$, $N_y \neq 0$ and $q \neq 0$) is simply supported and part of an infinite array then the results quoted in previous sections are conservative in that the destabilising effect of the lateral subsonic aerodynamic pressures would not be as severe as given.

7. *Conclusions.*

Alternative methods of analysis have been given by which the static stability of orthotropic panels subjected to in-plane loads and lateral aerodynamic pressures may be determined.

It has been shown that the inclusion of both supersonic and subsonic aerodynamic forces involves no great additional complexity.

Supersonic loading has been shown to be stabilising but subsonic loading was destabilising.

Comparisons have been made where possible with existing solutions in the literature and also with some experimental data.

8. *Acknowledgements.*

Acknowledgement is made to the Director, NASA Research Centre, for making available the data shown in Table 3. The data shown in Tables 1 and 2 are likewise taken from NASA TN.D. – 3500 and D – 2815 respectively.

LIST OF SYMBOLS

a,b	Panel dimensions in x and y directions respectively
$\bar{A},\bar{B}$	Parameters defined in eqns. (13), (15), (60), (61)
A_r	Generalised co-ordinates in displacement function of x, eqn. (18), etc.
A_{mn}	Generalised co-ordinates in displacement function of x, y, eqn. (29)
B_r	Generalised co-ordinate for initial displacement function, eqn. (54).
$C_s,\bar{C}_s$	Integrals defined in eqn. (10)
d	$D_{12}/\sqrt{D_{11}D_{22}}$
D_{ij}	Flexural and twisting rigidities of orthotropic panel relative to $x-y$ axes
D	Flexural rigidity of isotropic plate $= Eh^3/12(1-\nu^2)$
E	Young's Modulus
$f_{1(x)},f_{2(x)}$	Pressure functions, eqn. (45)
$F_{(x)}, G_{(y)}$	Modal functions of x, y
h	Panel thickness
k_x	$N_x b^2/\pi^2 D_{11}$ – dimensionless stress resultant in x direction
k_y	$N_y b^2/\pi^2 D_{11}$ – dimensionless stress resultant in y direction
k	In-plane elastic spring stiffness of panel edge members, eqn. (49)
$\bar{k}$	Parameter defined in eqn. (69)
K	Elastic foundation stiffness, eqn. (1) etc.
$\bar{K}$	$Ka^4/\pi^4 D$ dimensionless foundation stiffness parameter
l	Longitudinal wavelength of very low aspect ratio panel, eqn. (78)
$L_{mn,rs}$	Generalized supersonic airforce terms, eqn. (31)
m,n,r,s	Mode numbers
M	Mach number
N_x,N_y,N_{xy}	Midplane stress resultants (positive for inplane compression)
$\bar{N};\bar{N}'$	$N_x a^2/\pi^2 D; N'_x a^2/\pi^2 D$
p	Lateral pressure loading
$P_{mn,rs}$	Shear coupling terms, eqn. (31)
P_1,P_2	Assumed subsonic and supersonic aerodynamic pressure coefficients, eqn. (5)
$q;q_D$	Dynamic pressure; divergence dynamic pressure
q_x,q_y	Rotational restraint coefficient along x, y edges of panel
$\bar{Q}$	$2q\,a^3/\pi^3 D$, dimensionless dynamic pressure
Q_D	$\pi^3\,\bar{Q}_D$
S	$\bar{N}'$, dimensionless initial compressive load

S_r	Parameter defined in eqn. (25)
T	Tensile load in x-direction
U	Airspeed
w_0	Initial deflection before loading
w_1	Initial deflection under initial compressive loading N'_x, eqn. (47)
w	Additional deflection due to lateral and in-plane loading
$\bar{w}$	$w+w_0$ = total deflection
x,y	Streamwise and lateral co-ordinates (Fig. 1)
X,Y	Displacement functions in w in x, y directions
X_0, Y_0	Displacement functions in w_0 in x, y directions
z	Normal panel co-ordinate (Fig. 1)
$\alpha,\delta,\varepsilon$	Parameters in eqns. (20) to (23)
β	$(a/b)\,(D_{22}/D_{11})^{\frac{1}{4}}$, eqn. (63)
γ	Parameter defined in eqn. (71)
Γ	Warping restraint parameter, eqn. (1)
θ_x,θ_y	Spring constants per unit length of edge rotational restraints
ν	Poisson's ratio
λ	Supersonic aerodynamic pressure parameter, eqn. (14)
ρ	Air density
Δ	Relative movement of panel edges, eqn. (49)
ψ	Parameter defined in eqn. (52)
χ	Parameter defined in eqn. (55)
η	$(1-M^2)\,(\pi b/l)^2$, parameter defined in Section 6.2.
σ	Panel mass per unit area
μ	$\rho l/2\pi\sigma$

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TABLE 1

Static Stability Solutions for Supersonic and Subsonic Flow.

$\bar{A}$	$q_x = 0$		$q_x = 40$		$q_x = \infty$	
	λ_{cr}	$\bar{B}_{cr}$	λ_{cr}	$\bar{B}_{cr}$	λ_{cr}	$\bar{B}_{cr}$
−30	4119	190·5	4348	215·0	4470	226·0
−20	2608	113·5	2839	134·5	2949	144·0
−10	1330	53·50	1560	71·00	1655	78·50
−5	794·6	30·05	1022	45·50	1107	51·50
−4	697·1	25·75	923·4	41·00	1006	46·50
−3	603·1	21·80	828·2	36·50	908·6	41·69
−2	512·6	18·00	736·5	32·20	814·5	37·20
−1	426·0	14·30	648·3	27·90	723·8	32·49
0	343·3	10·75	563·8	23·75	636·6	28·25
1	264·9	7·500	483·0	20·02	553·0	24·00
2	190·9	4·375	406·3	16·25	473·3	19·75
3	121·8	1·375	333·9	12·50	397·6	15·75
4	57·98	−1·412	265·8	9·000	326·1	12·00
5	0	−4	202·4	5·625	258·9	8·250
6	51·27	−6·400	144·1	2·250	196·5	4·750
7	94·54	−8·625	91·17	−0·900	138·9	1·250
8	127·5	−10·69	44·08	−4·062	86·74	−2·217
9	145·5	−12·75	3·373	−7·170	40·27	−5·614
10	137·3	−16·25	30·27	−10·30	0	−9
11	99·85	−22·50	56·07	−13·60	33·48	−12·50
12	51·01	−29·37	73·14	−17·00	59·51	−16·00
13	0	−36	80·73	−21·00	77·40	−19·85
14	49·67	−42·40	78·58	−25·69	86·54	−24·10
15	96·08	−48·55	67·60	−31·20	86·74	−29·00
16	137·7	−54·50	49·96	−37·45	78·60	−34·70
17	172·7	−60·39	28·40	−44·30	63·69	−41·20
18	199·0	−66·25	5·411	−51·56	44·20	−48·32
19	212·8	−72·60	17·04	−59·16	22·34	−55·97
20	208·2	−81·00	37·48	−67·05	0	−64

TABLE 2

Static Stability Solutions for Subsonic Flow only.

Mode Type	$q_x = 0$		Mode Type	$q_x = 40$		Mode Type	$q_x = \infty$	
	$\bar{A}$	$\bar{B}$		$\bar{A}$	$\bar{B}$		$\bar{A}$	$\bar{B}$
*S*1	−60	61	*S*1	−55·56	+63·140	*S*1	−38·480	49·020
	−20	21		−3·32	8·096		−2·034	7·646
	−4	5		−·76	5·187		−0·076	5·233
	0	1		·690	3·511		+0·613	4·371
	4	−3		3·980	−·430		4·000	0
	5	−4		8·892	−7·136		10·000	−9·000
*A*2	5	−4	*A*2	9·054	−7·280	*A*2	10·000	−9·000
				9·500	−9·340		10·232	−10·160
	6	−8		11·01	−16·41		11·048	−14·270
	8	−16		12·32	−22·66		12·096	−19·610
	10	−24		14·134	−31·55		14·024	−29·660
	12	−32		16·716	−44·89		16·092	−40·880
	13	−36		18·876	−57·08		20·000	−64·000
*S*3	13	−36	*S*3	19·836	−68·54	*S*3	20·000	−64·00
				20·34	−73·43		20·460	−68·57
	20	−81		25·84	−129·6		25·780	−125·70
	25	−144		31·44	−192·7		34·000	−225·00
*A*4	25	−144	*A*4	31·28	−191·3	*A*4	34·00	−225·0
				33·92	−234·6		36·08	−261·3
	30	−224		47·56	−483		52·00	−576·0
	—	—	*S*5	52	−593·3	*S*5	52·00	−576·0
	—	—		68·92	−1064·0		74·00	−1225·0
	—	—	*A*6	74·08	−1253	*A*6	74·00	−1225·0
	—	—		96·20	−2134		100·00	−2304·0
	—	—	*S*7	100·6	−2346	*S*7	100·00	−2304·0
	—	—		123·6	−3591		130·00	−3969·0

Am refers to asymmetric mode of mode number *m* having $(m-1)$ lines of zero deflection.

Sm refers to symmetric mode of mode number *m* having $(m-1)$ lines of zero deflection.

TABLE 3

Static Stability Solutions for Supersonic and Subsonic Flow.

A	$q_x = 0$			$q_x = 43$				$q_x = \infty$			
	λ	$\bar{B}_1$	$\bar{B}_2$	$\bar{A}$	λ	$\bar{B}_1$	$\bar{B}_2$	$\bar{A}$	λ	$\bar{B}_1$	$\bar{B}_2$
0	0	1·00	16·00	9·4	0	−12·19	−11·48	15·55	0	−37·11	−25·63
	100	1·51	15·87		2	−12·18	−11·49		20	−36·94	−25·79
	200	3·16	15·39		4	−12·15	−11·52		40	−36·38	−26·31
	300	6·57	13·42		6	−12·09	−11·58		60	−35·31	−27·32
	340	9·70	11·81		8	−11·95	−11·71		80	−32·91	−29·62
	343·3	10·75	10·75		8·5	−11·83	−11·83		82	−31·60	−31·60
−10·89	0	11·89	59·56	−1·68	0	4·57	39·07	−5·17	0	12·39	63·88
	300	13·34	60·02		200	5·81	38·98		400	15·87	64·14
	600	17·79	61·32		400	9·78	38·51		600	27·34	64·01
	1200	38·03	64·69		600	17·66	36·51		1000	38·00	61·97
	1400	52·02	62·42		710	28·01	30·45		1100	47·21	57·87
	1420	58·2	58·2		715	29·23	29·23		1125	52·00	52·00
−21·78	0	22·78	103·12	−25·5	0	27·66	131·10	−27	0	35·69	159·86
	1000	32·48	108·13		1000	36·54	135·66		1000	43·53	164·08
	2000	63·55	122·23		2000	64·44	148·73		2000	67·76	176·42
	2600	97·22	131·76		3000	118·76	165·22		3000	111·55	195·10
	2800	115·21	131·18		3200	136·74	165·35		3700	161·29	205·81
	2850	124·59	126·40		3300	151·02	160·62		3900	185·99	201·17
	2855	125·49	125·49		3310	156·0	156·0		3925	195·0	195·0

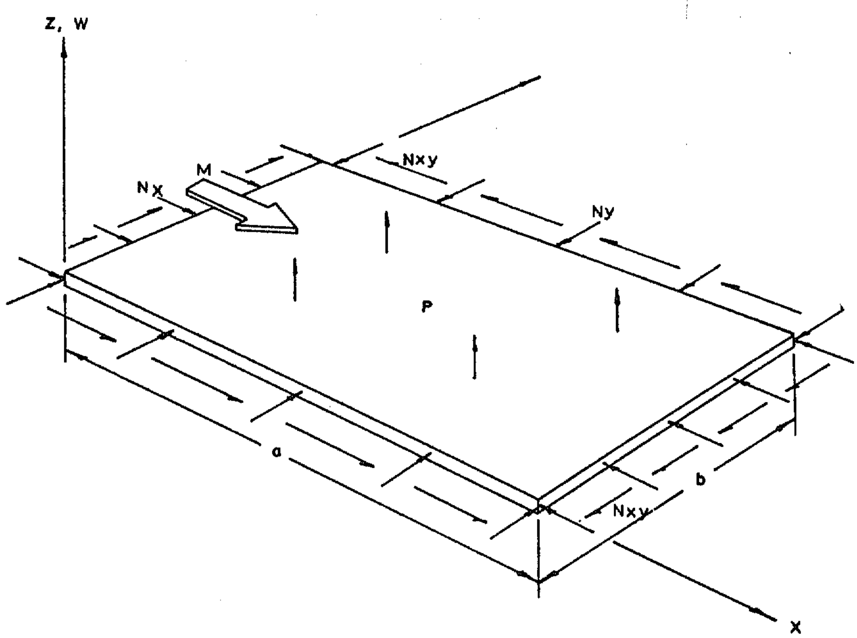
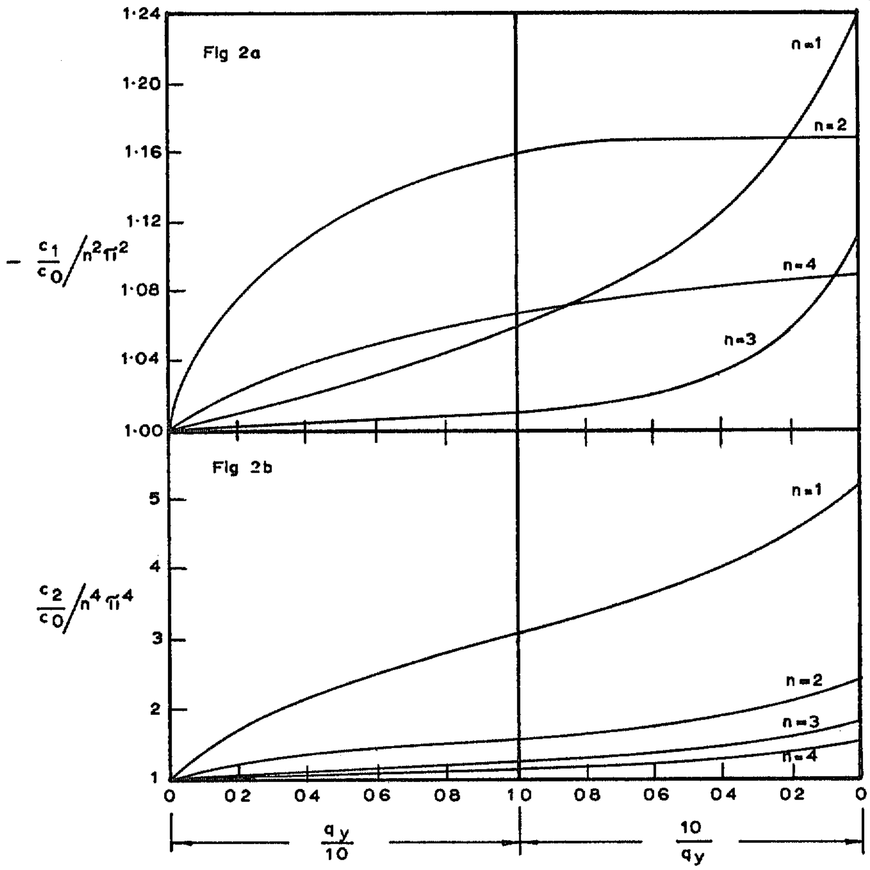


FIG. 1. Panel geometry co-ordinate system and flow direction.



FIGS. 2a & b. Variation of C_1/C_0 and C_2/C_0 with rotational restraint coefficient q_y for first four modes in y-direction. (Ref. 1).

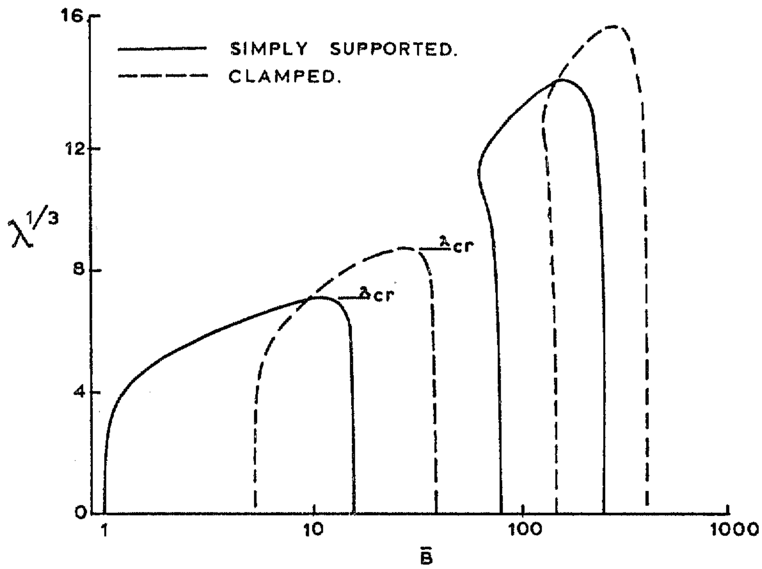


FIG. 3. First two stability loops for clamped and simply-supported panels. $\bar{A} = 0$.

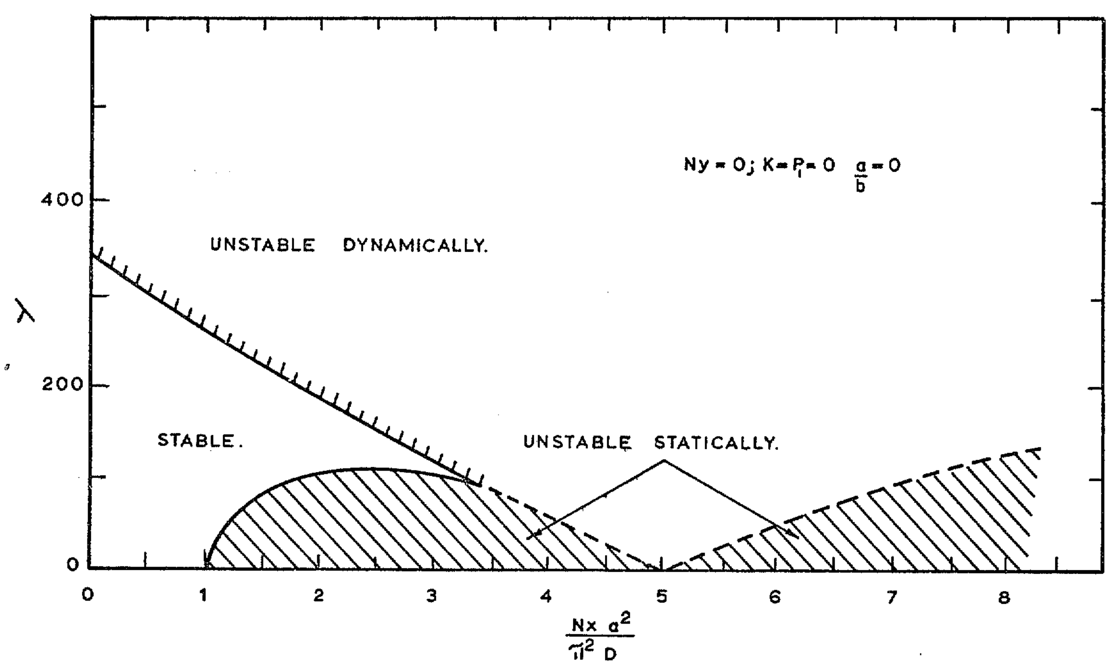


FIG. 4. Stability boundaries for isotropic panels of infinite aspect ratio in supersonic flow. (Ref. 10).

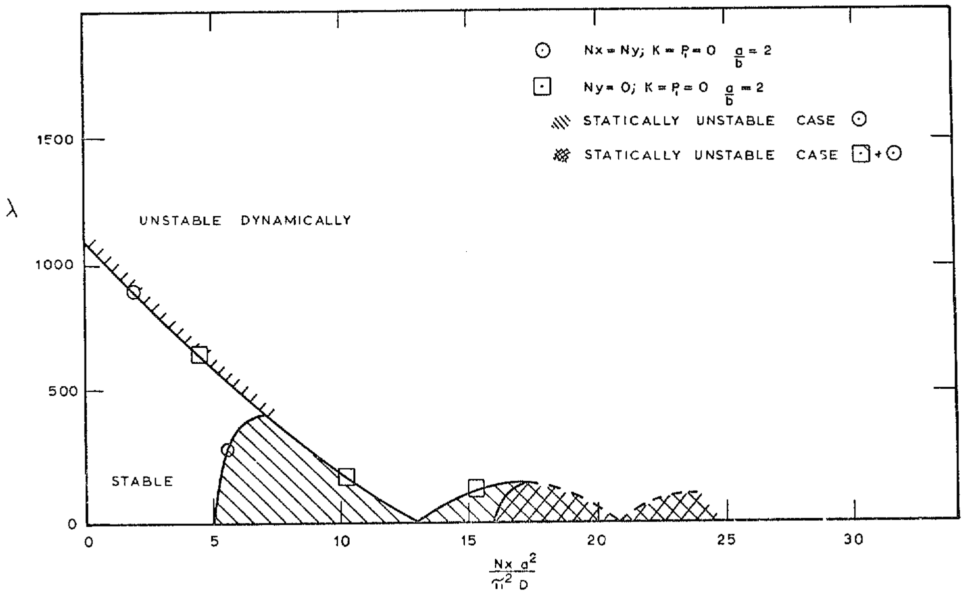


FIG. 5. Stability boundaries for isotropic panels of finite aspect ratio in supersonic flow. (Ref. 10).

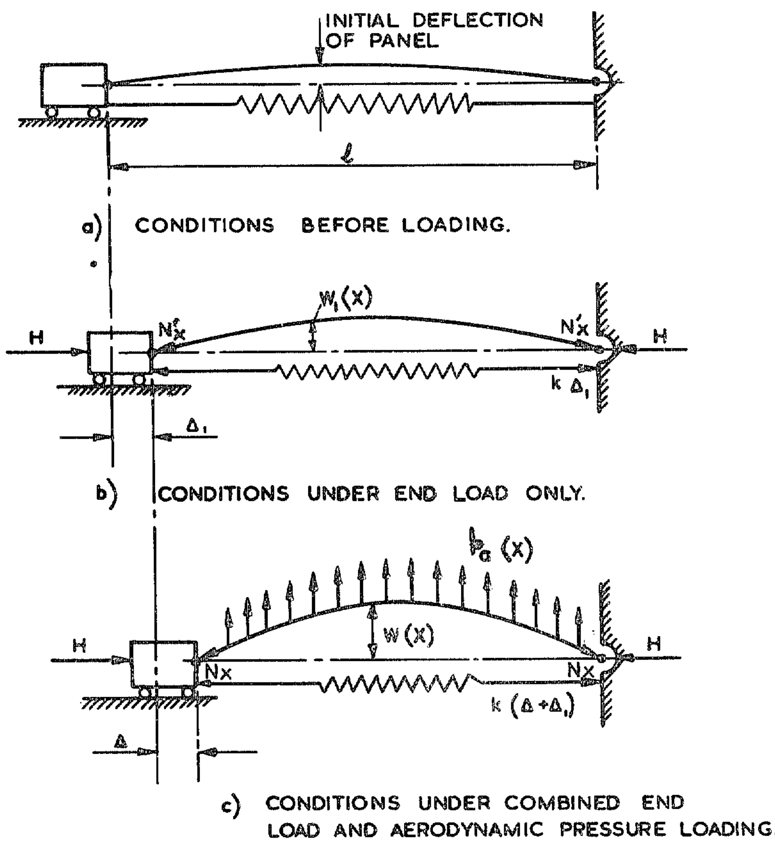


FIG. 6. Forces and displacements of single simply-supported panel. (Ref. 14).

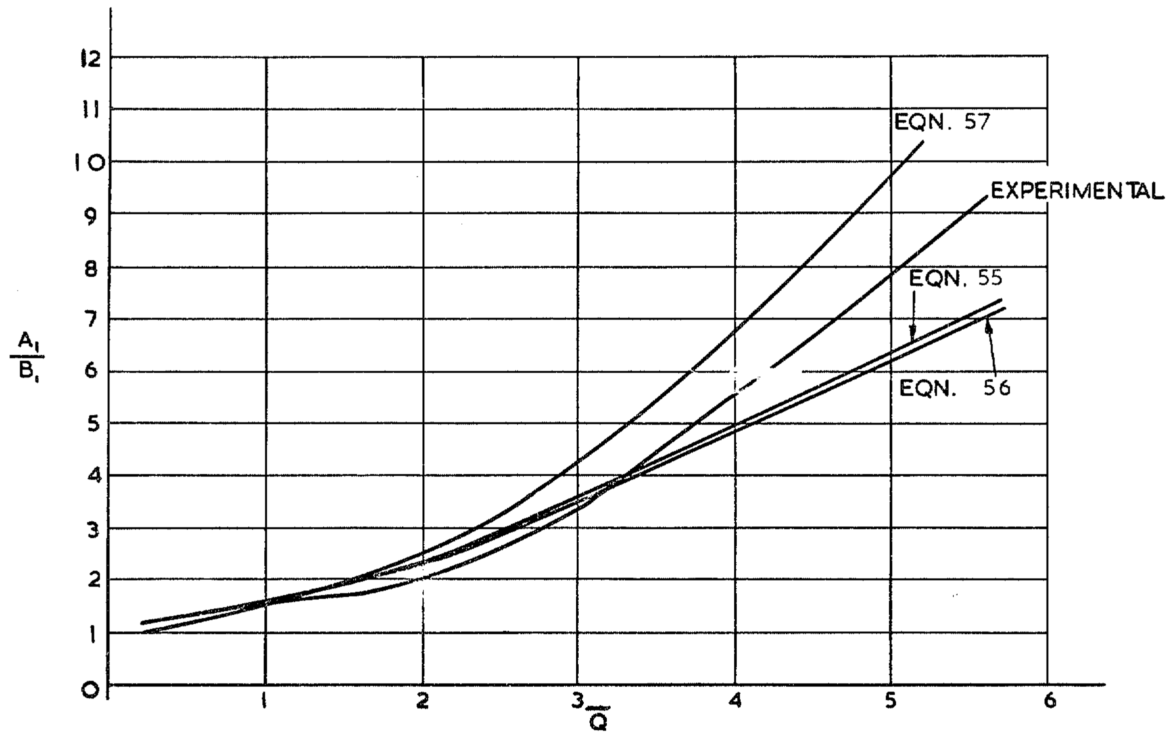


FIG. 7. Variation of panel deflection ratio with dynamic-pressure parameter. (Ref. 14).

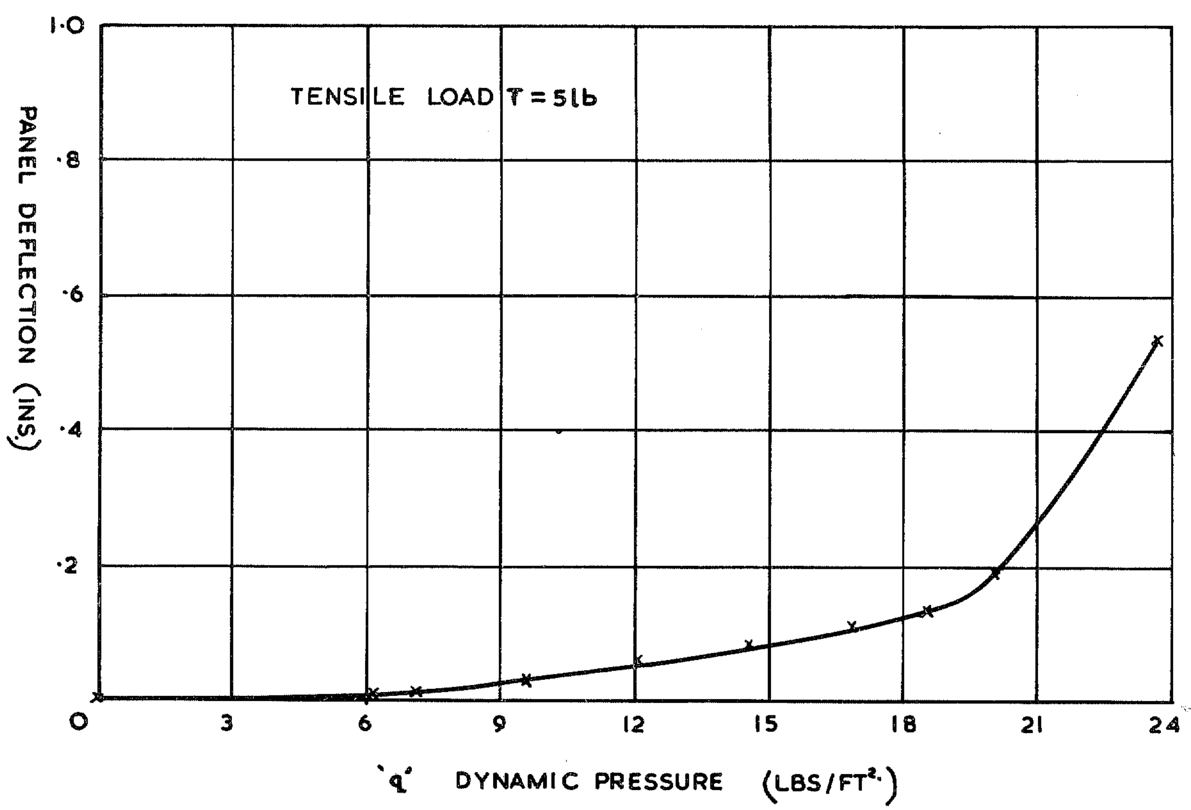


FIG. 8. Panel deflection variation with dynamic pressure for a constant tensile load. (Ref. 18).

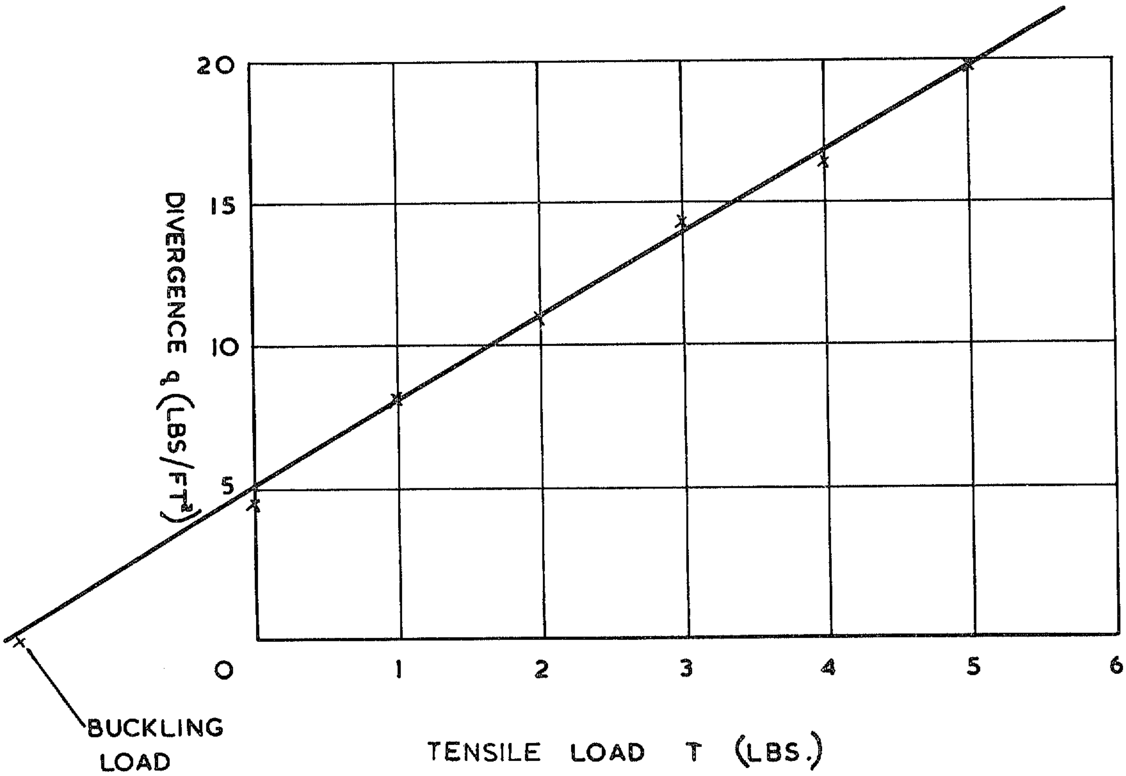


FIG. 9. Variation of divergence dynamic pressure with tensile load. (Ref. 18).

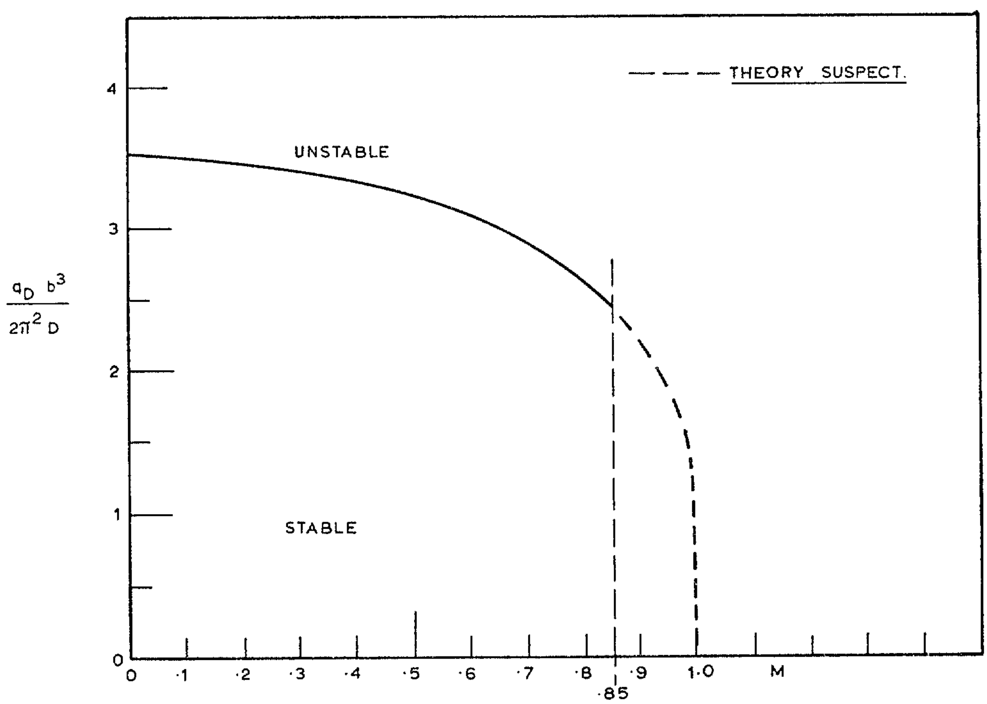


FIG. 10. Stability boundary for very low aspect ratio panel. (Ref. 20).

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