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An Improved Method for Calculating Generalised Airforces on Oscillating Wings in Subsonic Flow

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An Improved Method for Calculating Generalised Airforces on Oscillating Wings in Subsonic Flow

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Summary.

A lifting-surface theory of Multhopp-type has been revised to include an improved method of spanwise integration and programmed in Fortran IV. Generalised airforces have been calculated for three wings and the effects of varying the number of collocation points and the accuracy of spanwise integration are discussed.

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Detachable Abstract Cards

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1. *Introduction.*

In Ref. 1 a method for calculating generalised airforces on a thin wing oscillating harmonically at general frequencies in subsonic flow is described. This is a numerical technique which solves the integral equation by a method of collocation similar to that used originally by Multhopp[2]. Garner[3] has observed that in this approach the method of spanwise integration becomes inaccurate when the upwash points are close to the leading or trailing edge as happens when n, the number of chordwise points, is large, and has proposed an alternative technique which is significantly more accurate for large values of n. In this technique the number of integration points is increased from m, the number of spanwise upwash points, to

$$\overline{m} = q\,(m+1)-1$$

without increasing the number of collocation points. The factor q is an arbitrary positive integer which can be increased in value until the desired accuracy of integration is achieved.

In this report the method of Davies[1] has been revised to include this improvement, and reprogrammed in Fortran IV. Generalised forces have been calculated for a number of wings for various values of the parameters m, n and q. The results show improved convergence properties of the generalised force coefficients with increasing m and n, compared with the basic method of Ref. 1.

2. *Theory.*

In linearised potential flow the integral equation relating the loading $\rho_0 V^2 \lambda(x, y)\, e^{i\omega t}$ and the upwash $V \alpha(x, y)\, e^{i\omega t}$ on a thin wing oscillating harmonically in subsonic flow is

$$\alpha(x, y) = \frac{1}{4\pi} \iint_S \lambda(x_0, y_0)\, K(x-x_0, y-y_0)\, dx_0\, dy_0\,, \tag{1}$$

where S is the wing area and $K(x-x_0, y-y_0)$ is the subsonic kernel function.

The upwash $\alpha(x, y)$ is related to the wing displacement $f(x, y)$ by the equation

$$\alpha(x, y) = l \frac{\partial}{\partial x} f(x, y) + i\, \nu f(x, y)\,. \tag{2}$$

If we make the change of variable

$$\xi = \frac{1}{c(y)} [x - x_L(y)] \tag{3}$$

$$\eta = \frac{y}{s}$$

equation (1) may be written

$$\bar{\alpha}(\xi, \eta) = \frac{s}{4\pi\, l} \int_{-1}^{1} \frac{c(y_0)}{l}\, d\eta_0 \int_0^1 \bar{\lambda}(\xi_0, \eta_0)\, \bar{K}(\chi, \mu)\, d\xi_0 \tag{4}$$

where

$$\bar{\alpha}(\xi,\eta) = \alpha(x,y)\,e^{ivx/l}$$

$$\bar{\lambda}(\xi,\eta) = \lambda(x,y)\,e^{ivx/l}$$

$$\hat{K}(\chi,\mu) = l^2 K(x-x_0, y-y_0)\,e^{iv(x-x_0)/l} \tag{5}$$

$$\chi = \frac{x-x_0}{l}$$

$$\mu = \frac{y-y_0}{l}$$

and by equations (6) and (7) of Ref. 1

$$\hat{K}(\chi,\mu) = \int_{\left(\frac{-\chi+MR}{1-M^2}\right)}^{\infty} \frac{e^{-iv\tau}\,d\tau}{(\tau^2+\mu^2)^{3/2}} + \frac{M(M\chi+R)}{R(\chi^2+\mu^2)} \exp\left\{\frac{-iv(-\chi+MR)}{1-M^2}\right\} \tag{6}$$

and

$$R = \sqrt{\chi^2+(1-M^2)\,\mu^2}\,. \tag{7}$$

We assume that the loading distribution $\bar{\lambda}(\xi_0,\eta_0)$ is the product of a regular function and the weighting factor $\sqrt{\frac{1-\xi_0}{\xi_0}}\sqrt{1-\eta_0^2}$ so that it has the correct singular behaviour at the wing edges. Furthermore we assume that the regular function can be approximated, with good accuracy, by a double polynomial in ξ_0 and η_0 of degree $n-1$ in ξ_0 and degree $m-1$ in η_0. This approximation to $\bar{\lambda}(\xi_0,\eta_0)$ may be written in the form

$$\bar{\lambda}(\xi_0,\eta_0) = \sum_{i=1}^{n}\sum_{j=1}^{m} h_i^{(n)}(\xi_0)\,g_j^{(m)}(\eta_0)\,\bar{\lambda}(\xi_i,\eta_j) \tag{8}$$

where $h_i^{(n)}(\xi_0)$ is an interpolation function which is unity at the point $\xi_i^{(l)}$ and zero at the other $(n-1)$ loading points, and which is the product of a polynomial of degree $(n-1)$ in ξ_0 and the weighting function $\sqrt{\frac{1-\xi_0}{\xi_0}}$. Thus

$$h_i^{(n)}(\xi_0) = \prod_{r=1}^{n}{}' \frac{(\xi_0-\xi_r)}{(\xi_i-\xi_r)} \sqrt{\frac{\xi_i^{(l)}}{1-\xi_i^{(l)}}} \sqrt{\frac{1-\xi_0}{\xi_0}} \tag{9}$$

where the dash ′ indicates that the factor $i = r$ is to be omitted in the product. The points $\xi_i^{(l)}$ are given by

$$\xi_i^{(l)} = \tfrac{1}{2}-\tfrac{1}{2}\cos\left(\frac{2i-1}{2n+1}\pi\right) \quad i = 1,2,\ldots,n. \tag{10}$$

Similarly $g_j^{(m)}(\eta_0)$ is an interpolation function which is unity at the point n_j and zero at the other $(m-1)$ loading points, and which is the product of a polynomial of degree $(m-1)$ in η_0 and the weighting function $\sqrt{1-\eta_0^2}$. Thus

$$g_j^{(m)}(\eta_0) = \prod_{r=1}^{m}{}' \frac{(\eta_0-\eta_r)}{(\eta_j-\eta_r)} \frac{\sqrt{1-\eta_0^2}}{\sqrt{1-\eta_j^2}} \tag{11}$$

where the dash indicates that the factor $i = r$ is to be omitted in the product. The points η_j are given by

$$\eta_j = \cos\left(\frac{j\pi}{m+1}\right) \quad j = 1, 2, \ldots, m. \tag{12}$$

It must be borne in mind that the $\bar{\lambda}(\xi_0, \eta_0)$ given by equation (8) is only an approximation to the true loading $\bar{\lambda}(\xi_0, \eta_0)$ although no attempt is made to distinguish between the two quantities.

If we substitute the series in equation (8) into equation (4), we get

$$\bar{\alpha}(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\lambda}(\xi_i, \eta_j) \int_{-1}^{1} \frac{g_j^{(m)}(\eta_0)}{(\eta-\eta_0)^2} I_i^{(n)}(\eta, \eta_0, \xi)\, d\eta_0 \tag{13}$$

where

$$I_i^{(n)}(\eta, \eta_0, \xi) = \frac{s}{4\pi l^2} c(y_0)(\eta-\eta_0)^2 \int_0^1 h_i^{(n)}(\xi_0) \hat{K}(\chi, \mu)\, d\xi_0 . \tag{14}$$

Equation (13) is the basic equation relating the upwash and approximate loading. This equation cannot be satisfied exactly all over the wing. However it can be satisfied at a number mn of upwash points and a set of mn simultaneous equations obtained for the values of the loading function $\bar{\lambda}(\xi_i, \eta_j)$ at the loading points, in temrs of the upwash at the upwash points. The $\bar{\lambda}(\xi_i, \eta_j)$ can be determined accurately, only if the coefficients of the set of simultaneous equations are evaluated accurately, and this depends on the accuracy of evaluation of the chordwise and spanwise integrals involved.

The loading points are given by

$$x_{ij}^{(l)} = c(y_j)\, \xi_i^{(l)} + x_L(y_j) \quad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m \end{array} \tag{15}$$

$$y_j = s\, \eta_j$$

and the upwash points are given by

$$x_{kr}^{(w)} = c(y_r)\, \xi_k^{(w)} + x_L(y_r) \quad \begin{array}{l} k = 1, 2, \ldots, n \\ r = 1, 2, \ldots, m \end{array} \tag{16}$$

$$y_r = s\, \eta_r$$

where $\qquad \xi_k^{(w)} = 1 - \xi_{n-k+1}^{(l)} .$

The chordwise integral in equation (14) is evaluated numerically. Special care is required when $(\eta-\eta_0)$ is small and ξ_0 is close to the upwash point ξ, since the integrand changes rapidly in this region. It can be shown that if $\chi > 0$, i.e. ξ_0 is head of the upwash point ξ, then

$$\lim_{(\eta-\eta_0)\to 0} (\eta-\eta_0)^2 \hat{K}(\chi, \mu) = 2 \tag{17}$$

and if $\chi < 0$ (ξ_0 aft of upwash point ξ) then

$$\lim_{(\eta-\eta_0)\to 0} (\eta-\eta_0)^2 \hat{K}(\chi, \mu) = 0\,. \tag{18}$$

There is thus a discontinuity in the integrand at $(\eta-\eta_0) = 0$ and an extremely rapid variation when $(\eta-\eta_0)$ is small. In performing the numerical integration under these circumstances the chord has been divided into two regions one ahead of and the other aft of the upwash point. Each region is divided into intervals, the length of which varies directly with its distance from the upwash point, and over each interval a low order Gaussian quadrature is taken. The size of the intervals will depend on the value of $(\eta-\eta_0)$, but they can be chosen to give any desired degree of accuracy.

The spanwise integral in equation (13) can be evaluated approximately by using the interpolation formula

$$g_j^{(m)}(\eta_0)\, I_i^{(n)}(\eta, \eta_0, \xi) = \sum_{p=1}^{\bar{m}} g_j^{(m)}(\bar{\eta}_p)\, I_i^{(n)}(\eta, \bar{\eta}_p, \xi) \quad g_p^{(\bar{m})}(\eta_0) \tag{19}$$

and integrating each term obtained by putting this series into equation (13). In equation (19) $g_p^{(\bar{m})}(\eta_0)$ is an interpolation function of degree $(\bar{m}-1)$ of the same kind as $g_j^{(m)}(\eta_0)$,

$$\begin{aligned} \bar{\eta}_p &= \cos\left(\frac{p\pi}{\bar{m}+1}\right) \quad p = 1, 2, \ldots, \bar{m} \\ \bar{m} &= q(m+1)-1 \end{aligned} \tag{20}$$

and q is an integer. When $q = 1$, $\bar{m} = m$ and the formula in equation (19) reduces to the basic Multhopp formula. For q greater than 1, $\bar{m} > m$ and so there are more terms in the approximation given by equation (19), the approximation will therefore be better and the accuracy of the integration obtained, when this series is substituted in equation (13), can be expected to improve. The higher the value of q the greater is the expected accuracy. This is the essence of the improvement as proposed by Garner[3].

The accuracy of the representation of equation (13) can be improved if the lowest order logarithmic term is removed from $I_i^{(n)}(\eta, \eta_0, \xi)$ and treated separately and the remainder is dealt with using the interpolation formula.

We write

$$\begin{aligned} g_j^{(m)}(\eta_0)\, I_i^{(n)}(\eta, \eta_0, \xi) &= g_j^{(m)}(\eta)\, \frac{\sqrt{1-\eta_0^2}}{\sqrt{1-\eta^2}} F_i(\eta, \xi)(\eta-\eta_0)^2 \log|\eta-\eta_0| + \\ &+ \left[g_j^{(m)}(\eta_0)\, I_i^{(n)}(\eta, \eta_0, \xi) - g_j^{(m)}(\eta)\, \frac{\sqrt{1-\eta_0^2}}{\sqrt{1-\eta^2}} F_i(\eta, \xi)(\eta-\eta_0)^2 \log|\eta-\eta_0| \right] \end{aligned} \tag{21}$$

and the expression in [ ] is approximated by an $\bar{m}$-point interpolation formula, so that

$$g_j^{(m)}(\eta_0)\, I_i^{(n)}(\eta, \eta_0, \xi) = g_j^{(m)}(\eta)\, F_i(\eta, \xi)\, \frac{\sqrt{1-\eta_0^2}}{\sqrt{1-\eta^2}} (\eta-\eta_0)^2 \log|\eta-\eta_0| +$$

$$+ \sum_{p=1}^{\bar{m}} \Bigg[g_j^{(m)}(\bar{\eta}_p)\, I_i^{(n)}(\eta, \bar{\eta}_p, \xi) -$$

$$- g_j^{(m)}(\eta) \frac{\sqrt{1-\bar{\eta}_p{}^2}}{\sqrt{1-\eta^2}} F_i(\eta, \xi)(\eta-\bar{\eta}_p)^2 \log|\eta-\bar{\eta}_p| \Bigg] g_p^{(\bar{m})}(\eta_0) \tag{22}$$

where $F_i(\eta, \xi)$ is the coefficient of the lowest-order logarithmic term and is defined in equation (60) of Ref. 1.

If we insert equation (22) into equation (13) we get

$$\bar{\alpha}(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\lambda}(\xi_i, \eta_j)\, P_{ij}(\xi, \eta) \tag{23}$$

where

$$P_{ij}(\xi, \eta) = g_j^{(m)}(\eta) \frac{F_i(\eta, \xi)}{\sqrt{1-\eta^2}} \Bigg[\int_{-1}^{1} \log|\eta-\eta_0| \sqrt{1-\eta_0^2}\, d\eta_0$$

$$- \sum_{p=1}^{\bar{m}} (\eta-\bar{\eta}_p)^2 \log|\eta-\bar{\eta}_p| \sqrt{1-\bar{\eta}_p{}^2} \int_{-1}^{1} \frac{g_p^{(\bar{m})}(\eta_0)}{(\eta-\eta_0)^2} d\eta_0 \Bigg]$$

$$+ \sum_{p=1}^{\bar{m}} g_j^{(m)}(\bar{\eta}_p)\, I_i^{(n)}(\eta, \bar{\eta}_p, \xi) \int_{-1}^{1} \frac{g_p^{(\bar{m})}(\eta_0)}{(\eta-\eta_0)^2} d\eta_0 \tag{24}$$

Equation (23) is a set of mn simultaneous equations which can be solved for the unknown loading $\bar{\lambda}(\xi_i, \eta_j)$ and hence the loading distribution. To obtain the generalised airforce for a given mode of oscillation this loading distribution is integrated as shown in Ref. 1. The generalised aerodynamic force at time t in the mode

$$Z_p(x, y, t) = l f_p(x, y)\, b_{p0}\, e^{i\omega t} \tag{25}$$

is given by

$$Q_p(t) = \rho V^2 s l^2 \sum_{q=1}^{k} Q_{pq}\, b_{q0}\, e^{i\omega t} \tag{26}$$

where

$$Q_{pq} = \frac{1}{sl} \int \int_S f_p(x_0, y_0)\, \bar{\lambda}_q(x_0, y_0)\, dx_0\, dy_0 \,. \tag{27}$$

In flutter theory it is convenient to write

$$Q_{pq} = Q'_{pq} + i\, v\, Q''_{pq} \tag{28}$$

where Q'_{pq} and Q''_{pq} are real numbers.

In equation (26) $f_p(x, y)$ $p = 1, 2, \ldots, k$ defines the shapes of the k independent modes of displacement and $\rho V^2 \lambda_q(x, y) e^{i\omega t}$ is the loading distribution on the wing corresponding to the harmonic oscillation

$$Z_q(x, y, t) = l f_q(x, y)\, e^{i\omega t} \,. \tag{29}$$

The integrand in equation (27) is approximated similarly to $\bar{\lambda}(x_0, y_0)$ in equation (8) and a matrix equation is obtained for the k^2 generalised airforce coefficients Q_{pq}. This is derived in Ref. 1 as

$$[Q] = [f]\,[B] \left[\frac{\Lambda^{++} \pm \Lambda^{+-}}{2} \right]^{-1} [D]\,[\alpha] \tag{30}$$

where $[Q]$ is a square matrix, of order $k \times k$, with the element

$$Q_{pq}$$

in the p^{th} row and q^{th} column.

In equation (30) the positive sign is used if the spanwise loading (or displacement mode) is symmetrical about $\eta = 0$ and the negative sign is used when the loading is anti-symmetrical. Moreover this equation only applies if $[f]$ and $[\alpha]$ are both symmetrical or both anti-symmetrical, and m the number of spanwise collocation points is even.

The calculation of the matrix of generalised airforce coefficients $[Q]$ from equation (30) has been programmed in Fortran IV for the ICL 1907 computer. In its present form values of $m \leqslant 32$, $n \leqslant 10$, $\frac{1}{2} mn \leqslant 32$ and $\bar{m} \leqslant 160$ can be used, without exceeding a store capacity of 32K. A typical running time on the ICL 1907 computer is 35 minutes when $m = 16$, $n = 4$ and $q = 9$.

3. *Discussion.*

To assess the effect of increasing the number of spanwise integration points, generalised forces have been calculated for three wings oscillating in heave and pitching about an axis through the wing apex. The wings considered are shown in Fig. 1. They are a circular wing (aspect ratio $4/\pi$), a rectangular wing of aspect ratio 2 and a tapered swept back wing of aspect ratio 2 (called wing E). These wings have already been the subject of an investigation in Ref. 4. The frequency parameter is based on the quoted reference length of each wing.

For each wing at any specified values of m and n the parameter q was increased until the coefficients in the generalised force matrices converged to three significant figures. Better agreement than this can be achieved by further increase in q but this requires much greater computing time. The converged results obtained for the three wings for a selection of values of m and n, are presented in Tables 1–4, where the coefficients for $q = 1$ are also shown in parentheses. As pointed out in Section 2, when $q = 1$ the method is identical with that of Davies[1]. (In tables 1–4 the sub-scripts of the generalised force Q_{pq} are defined as, $p = 1, 2$ means lift, pitching moment; $q = 1, 2$ means plunging, pitching motion.)

In Fig. 2 the effect of increasing q is shown for wing E with $n = 4$ and $m = 16$. For convenience of presentation the values shown have been normalised by dividing by the value of the coefficient at $q = 7$. For this wing the central kink has been rounded for $|\eta| \leqslant 0{\cdot}19509$ using the equation proposed by Garner in Ref. 3. It may be seen that adequate convergence has been achieved for $q = 7$. Similar results, were obtained for the rectangular and circular wings, but the rate of convergence was greater. In general it may be stated that the rate of convergence with q, increased as m increased, and decreased as n increased. For values of $m = 4$ and $n = 6$ rapid variations occurred in the coefficients at low values of q and quite high values of q were necessary to achieve the required degree of convergence.

In Figs. 3 and 4 the values of the coefficients, converged with respect to q, are plotted for different values of m, the number of spanwise collocation points. These results apply to wing E with $n = 4$. It may be seen that at $m = 20$ the coefficients have still not converged with respect to m; however as m increases, the differences between the coefficients for $q = 1$ and the coefficients which have converged with respect to q decrease. This is consistent with the results for $m = 32$, $n = 2$, see Tables 2 and 3 where the differences in the respective coefficients are very small indeed. These trends are also true for the rectangular and circular wings, although for the former at $m = 8$ the coefficients seemed to have achieved adequate convergence with respect to m.

The effect of varying n, the number of chordwise points, is shown in Fig. 5, where again the results are appropriate to wing E. From the figure, it may be seen that for the vibration modes considered little difference exists between the coefficients calculated for $n = 4$ and $n = 6$. This conclusion is also true for the rectangular and circular wings.

4. *Conclusions.*

The differences between the generalised force coefficients for $q = 1$ and the values, converged with respect to q, decrease as m increases. In some instances the value of $\bar{m} = 118$ (the number of spanwise integration stations) has been necessary to achieve coefficients agreeing to three significant figures.

One of the features of the method is that the accuracy is improved without increasing the number of collocation stations, so that large matrices may be avoided; however for wings with kinks it appears desirable to use as many spanwise collocation stations as possible to achieve sufficient convergence with respect to m.

For the rigid body modes considered 4 chordwise points seem to be sufficient to achieve convergence with respect to n. In general values of $n \geqslant 4$ and $m \geqslant 8$ are desirable.

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LIST OF SYMBOLS

$b_q(t) = b_{q0}\, e^{i\omega t}$	Generalised co-ordinate
B	Matrix defined in equation (106) of Ref. 1
$c(y)$	Wing chord at station y
D	Matrix defined in equation (117) of Ref. 1
$F_i(\eta, \xi)$	Coefficient of lowest order logarithmic singularity
$f_p(x, y)$	Shape of p^{th} mode of oscillation
f	Matrix defined in equation (104) of Ref. 1
$g_j^{(m)}(\eta_0), g_p^{(\bar{m})}(\eta_0)$	Interpolation functions (spanwise)
$h_i^{(n)}(\xi_0)$	Interpolation function (chordwise)
$I_i^{(n)}(\eta, \eta_0, \xi)$	Defined in equation (14)
$\hat{K}(\chi, \mu)$	Defined in equation (5)
l	Typical dimension of wing
$l(x, y)$	Loading function
m	Number of spanwise points on complete wing (an even integer)
$\bar{m}$	Number of spanwise integration points on complete wing
M	Mach number of free stream
n	Number of chordwise points
$P_{ij}(\xi, \eta)$	Defined in equation (24)
q	Factor which controls the number of spanwise integration points $= (\bar{m}+1)/(m+1)$
$Q_p(t)$	Generalised aerodynamic force at time t in the p^{th} mode of oscillation, defined in equation (26)
$Q_{pq} = Q'_{pq} + i\nu Q''_{pq}$	Generalised aerodynamic force coefficient
$\left\{ \begin{array}{l} Q'_{pq} \\ Q''_{pq} \end{array} \right.$	Defined in equation (28)
Q	Matrix of generalised force coefficients defined in equation (30)
R	Defined in equation (7)
s	Semi-span of wing
$z(x, y, t)$	Vertical displacement of a point x, y on the surface of the wing at time t
t	Time
V	Free stream speed
$x = x_L(y)$	Equation of leading edge of wing
$x_{i,j}^{(l)}, y_j$	Loading points defined in equation (15)
$x_{k,r}^{(w)}, y_r$	Upwash points defined in equation (16)

LIST OF SYMBOLS—*continued*

$\alpha(x, y)\ Ve^{i\omega t}$		Upwash at point $(x, y, 0)$
$\bar{\alpha}(\xi, \eta)$	=	$\alpha(\xi, \eta)\, e^{i\nu x/l}$
α		Matrix defined in equation (70) of Ref. 1
η_j		Spanwise points
$\Lambda^{++}, \Lambda^{+-}$ etc		Matrices defined in equation (69) of Ref. 1
$\lambda(x, y) = \dfrac{1}{\rho_0 V^2}\, l(x, y)$		Reduced loading function
$\bar{\lambda}(\xi_0, \eta_0)$	=	$\lambda(\xi_0, \eta_0)\, e^{i\nu x_0/l}$
μ, χ		Defined in equation (5)
ξ, η		Defined in equation (3)
$\nu = \dfrac{\omega l}{V}$		Frequency parameter
ρ_0		Free-stream density
ω		Circular frequency

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TABLE 1

Generalised Force Coefficients, Rectangular Wing, $M = 0{\cdot}8$, $\nu = 1{\cdot}0$.

Q'_{11}

n \ m	4		8	
2	−0·907674	(−0·909338)	−0·907485	(−0·909589)
4	−0·911858	(−0·846775)	−0·910964	(−0·904891)
6	−0·911655	(−0·793711)	−0·911696	(−0·871175)

Q''_{11}

n \ m	4		8	
2	3·107482	(3·101992)	3·107375	(3·115865)
4	3·263704	(3·205223)	3·262977	(3·257133)
6	3·263986	(3·159129)	3·263840	(3·227808)

Q'_{12}

n \ m	4		8	
2	3·030681	(3·037286)	3·030651	(3·045620)
4	3·320289	(3·285888)	3·319703	(3·322331)
6	3·321592	(3·236433)	3·321988	(3·294025)

Q''_{12}

n \ m	4		8	
2	3·242613	(3·236881)	3·242235	(3·250193)
4	3·328148	(3·181039)	3·326152	(3·303562)
6	3·326222	(3·089413)	3·324775	(3·238557)

Q'_{21}

n \ m	4		8	
2	−0·827062	(−0·806057)	−0·826967	(−0·825951)
4	−0·968607	(−0·904942)	−0·967904	(−0·957794)
6	−0·968408	(−0·863401)	−0·967889	(−0·929210)

Q''_{21}

n \ m	4		8	
2	0·776051	(0·761983)	0·776031	(0·777995)
4	0·849240	(0·830738)	0·849088	(0·839298)
6	0·848143	(0·839101)	0·846711	(0·838832)

Q'_{22}

n \ m	4		8	
2	0·434314	(0·430592)	0·434348	(0·439316)
4	0·498781	(0·513825)	0·498964	(0·497586)
6	0·498487	(0·533450)	0·497570	(0·507025)

Q''_{22}

n \ m	4		8	
2	1·879400	(1·840344)	1·879243	(1·879765)
4	2·196302	(2·073197)	2·194847	(2·172617)
6	2·195252	(2·000122)	2·193455	(2·118965)

TABLE 2

Generalised Force Coefficient Q'_{pq}, Wing E, $M = 0{\cdot}781$, $\nu = 1{\cdot}0$.

Q'_{11}

n \ m	4	8	16	20	32
2	−0·827506 (−0·934366)	−0·811734 (−0·706996)	−0·749726 (−0·702871)		−0·728112 (−0·724363)
4	−0·950711 (−0·916643)	−0·859544 (−0·832675)	−0·765942 (−0·806057)	−0·746586 (−0·771905)	
6	−0·949330 (−0·865850)	−0·855514 (−0·809402)			

Q'_{12}

n \ m	4	8	16	20	32
2	1·972108 (1·876491)	1·818014 (1·880741)	1·809549 (1·817447)		1·807117 (1·806195)
4	1·926759 (2·091089)	2·008788 (2·114697)	2·092249 (2·116399)	2·103017 (2·117231)	
6	1·925111 (2·086863)	2·011948 (2·128904)			

Q'_{21}

n \ m	4	8	16	20	32
2	−1·111408 (−1·331276)	−1·032332 (−0·968355)	−0·979353 (−0·939129)		−0·963566 (−0·961716)
4	−1·335816 (−1·385361)	−1·197778 (−1·233347)	−1·125693 (−1·208346)	−1·109624 (−1·163757)	
6	−1·336038 (−1·318736)	−1·195005 (−1·205901)			

Q_{22}

n \ m	4	8	16	20	32
2	1·744480 (1·413432)	1·647012 (1·723001)	1·628670 (1·652272)		1·619948 (1·617650)
4	1·450787 (1·555235)	1·662274 (1·717684)	1·784162 (1·732222)	1·806660 (1·773646)	
6	1·449719 (1·594404)	1·668103 (1·734161)			

Generalised Force Coefficients Q''_{pq}, Wing E, $M = 0{\cdot}781$, $\nu = 1{\cdot}0$.

Q''_{11}

n \ m	4		8		16		20		32	
2	2·716533	(2·684381)	2·559904	(2·472499)	2·472881	(2·45442)			2·444178	(2·439311)
4	2·803900	(2·739156)	2·694333	(2·679141)	2·630925	(2·667747)	2·616108	(2·643563)		
6	2·803568	(2·700773)	2·692734	(2·669947)						

Q''_{12}

n \ m	4		8		16		20		32	
2	5·226087	(5·195853)	4·886855	(4·615787)	4·667439	(4·541635)			4·595619	(4·583603)
4	5·619109	(5·247196)	5·233187	(5·087345)	4·971280	(5·037454)	4·915570	(4·968307)		
6	5·619243	(5·120835)	5·228686	(5·029553)						

Q''_{21}

n \ m	4		8		16		20		32	
2	2·920050	(2·788900)	2·729385	(2·697467)	2·652202	(2·626375)			2·627589	(2.624076)
4	2·971529	(2·847397)	2·883881	(2·865443)	2·859773	(2·867085)	2·853869	(2·866644)		
6	2·974221	(2·826999)	2·887071	(2·850640)						

Q''_{22}

n \ m	4		8		16		20		32	
2	6·298656	(6·327156)	5·856056	(5·720367)	5·662199	(5·572229)			5·602625	(5·594226)
4	6·822974	(6·608707)	6·479161	(6·475459)	6·313135	(6·423697)	6·273403	(6·356554)		
6	6·827414	(6·453029)	6·484123	(6·391376)						

TABLE 4

Generalised Force Coefficients, Circular Wing, $M = 0, \nu = 0{\cdot}001$.

$Q'_{1,1}$

n \ m	4	16	20	Exact[5]
2	−0·000002 (−0·000002)	−0·000002 (−0·000002)	−0·000002 (−0·000002)	0
4	−0·000002 (−0·000002)	−0·000002 (−0·000002)	−0·000002 (−0·000002)	
6				

$Q''_{1,1}$

n \ m		4	16	20	Exact[5]
2	2·75902	(2·829085)	2·804184 (2·810027)	2·805876 (2·809800)	2·812
4	2·760125	(2·827152)	2·806514 (2·817019)	2·808388 (2·815049)	
6					

$Q'_{1,2}$

n \ m	4	16	20	Exact[5]
2	2·757900 (2·829083)	2·804182 (2·810025)	2·805874 (2·809798)	2·812
4	2·760123 (2·827150)	2·806512 (2·817017)	2·808386 (2·815047)	
6				

$Q''_{1,2}$

n \ m		4	16	20	Exact[5]
2	6·569262	(6.765574)	6·591353 (6·599187)	6·592302 (6·594280)	6·578
4	6·561578	(6·605034)	6·573905 (6·622234)	6·575802 (6·607673)	
6					

$Q'_{2,1}$

n \ m	4	16	20
2	−0·000003 (−0·000003)	−0·000003 (−0·000003)	−0·000003 (−0·000003)
4	−0·000003 (−0·000003)	−0·000003 (−0·000003)	−0·000003 (−0·000003)
6			

$Q''_{2,1}$

n \ m		4	16	20
2	1·278801	(1·310910)	1·324803 (1·330346)	1·326532 (1·330103)
4	1·296473	(1·317437)	1·342294 (1·344511)	1·343909 (1·348857)
6				

$Q'_{2,2}$

n \ m	4	16	20
2	1·278798 (1·310907)	1·324800 (1·330343)	1·326529 (1·330100)
4	1·296470 (1·317434)	1·342291 (1·344508)	1·343909 (1·348854)
6			

$Q''_{2,2}$

n \ m		4	16	20
2	5·959463	(6·140625)	5·987708 (5·996250)	5·988874 (5·990846)
4	5·964913	(5·937610)	5·982859 (6·032238)	5·984707 (6·021710)
6				

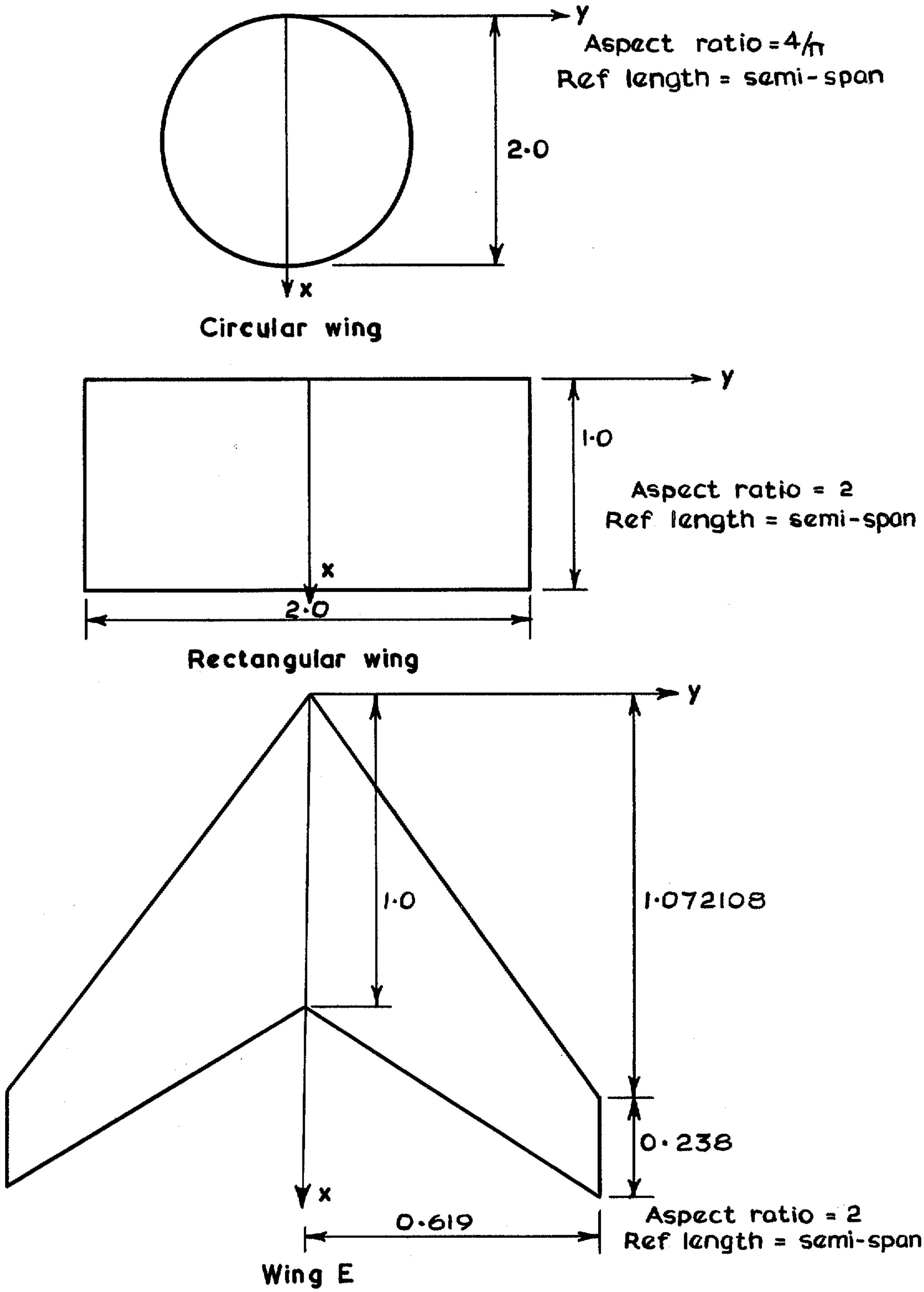

FIG. 1. Wing planforms.

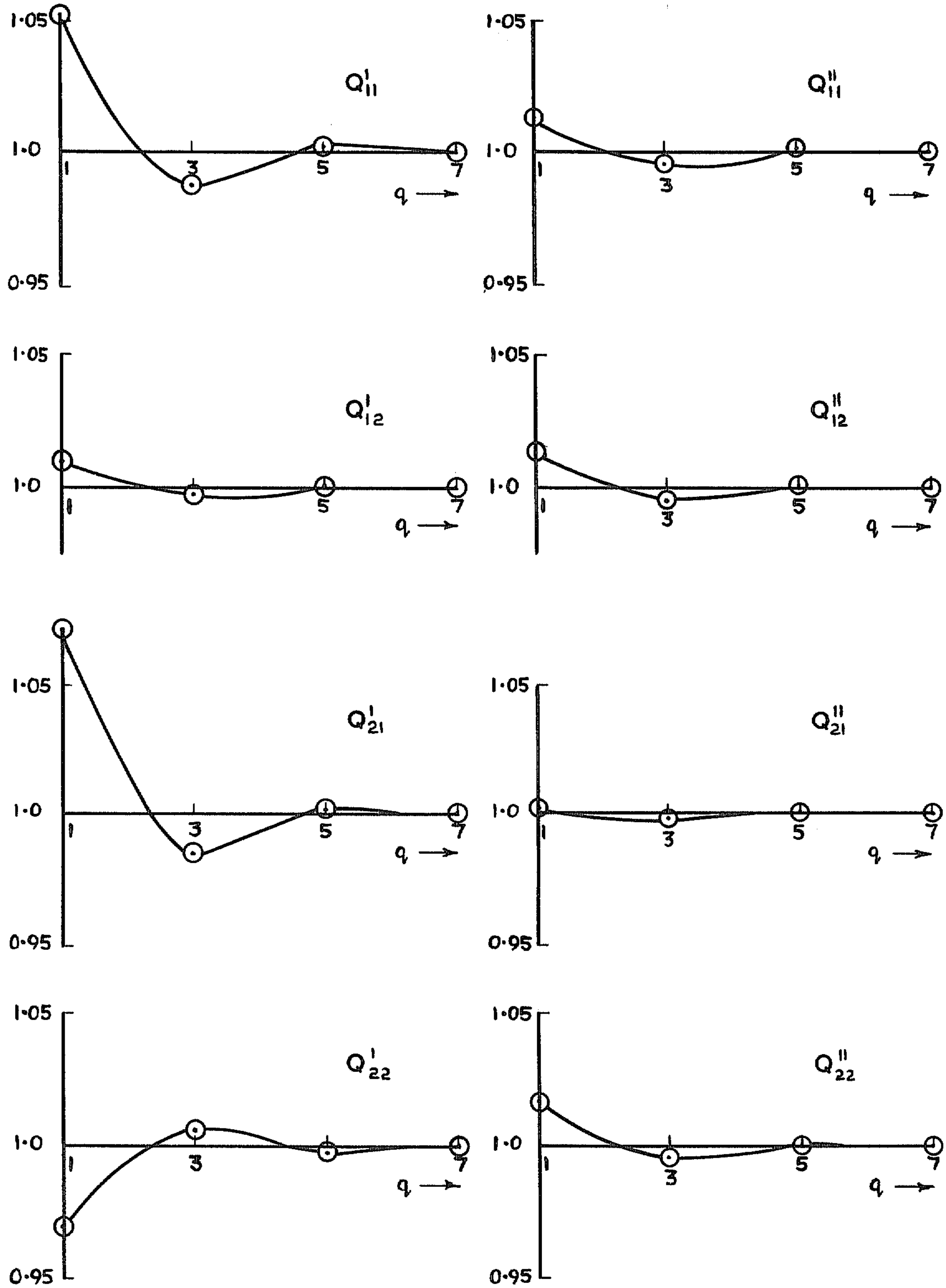

FIG. 2. Effect of increasing q on normalised, generalised force coefficients, wing E, $n = 4, m = 16$, $M = 0{\cdot}781$, $\nu = 1{\cdot}0$.

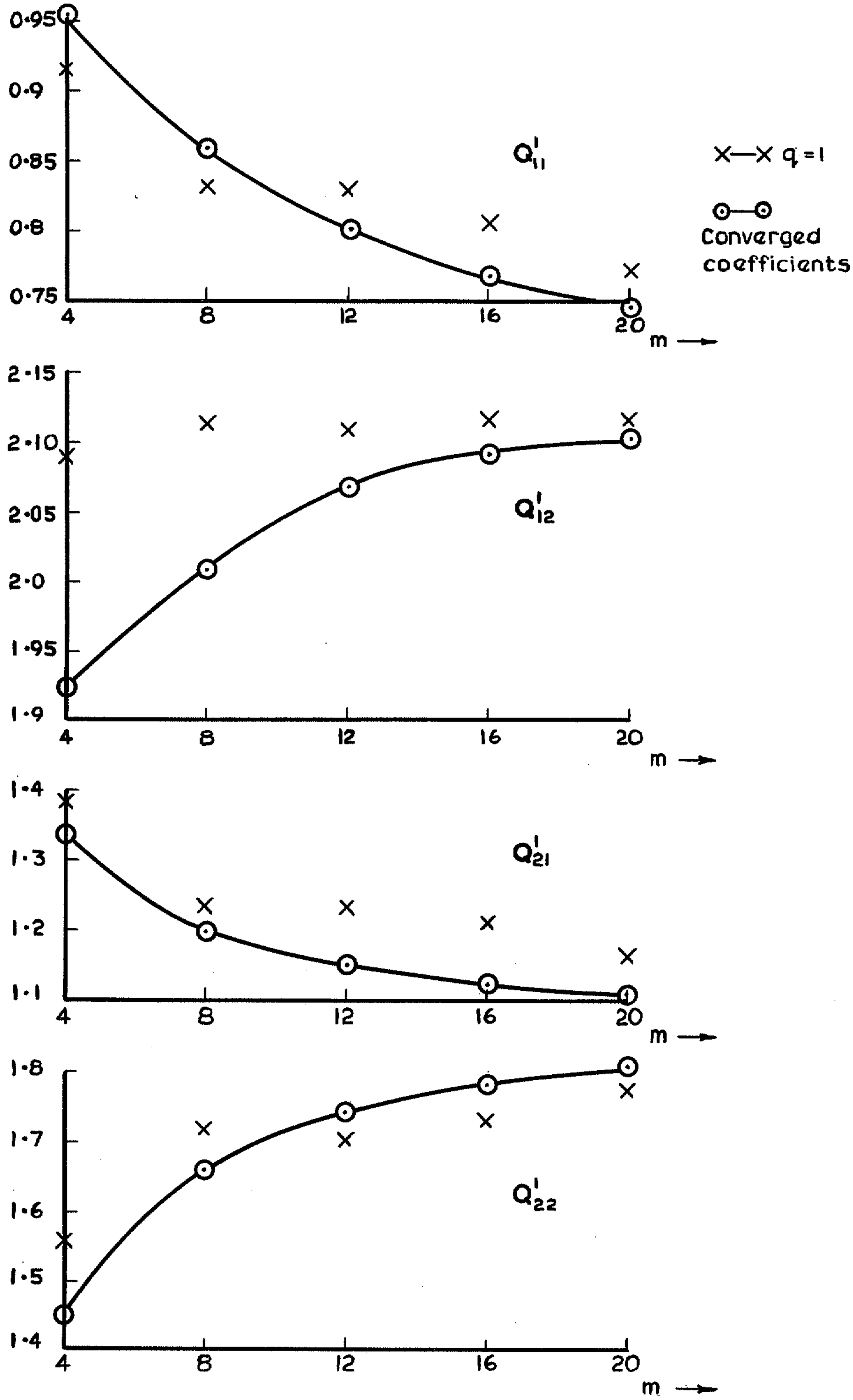

FIG. 3. Effect of increasing m on generalised force coefficients Q'_{pq}, wing E, $n = 4$, $m = 0{\cdot}781$, $\nu = 1{\cdot}0$.

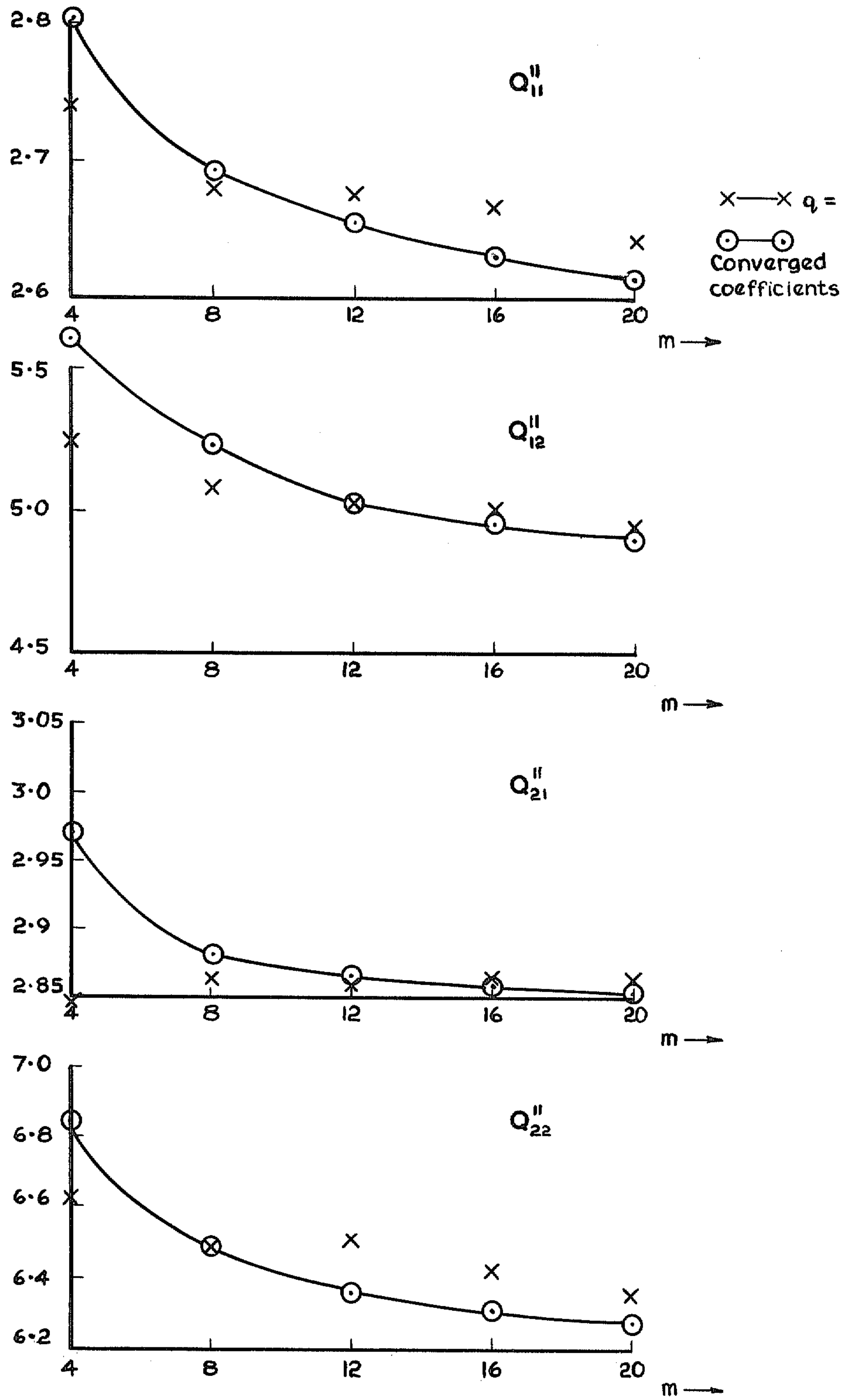

FIG. 4. Effect of increasing m on generalised force coefficients Q''_{pq}, wing E, $n = 4$, $m = 0{\cdot}781$, $\nu = 1{\cdot}0$.

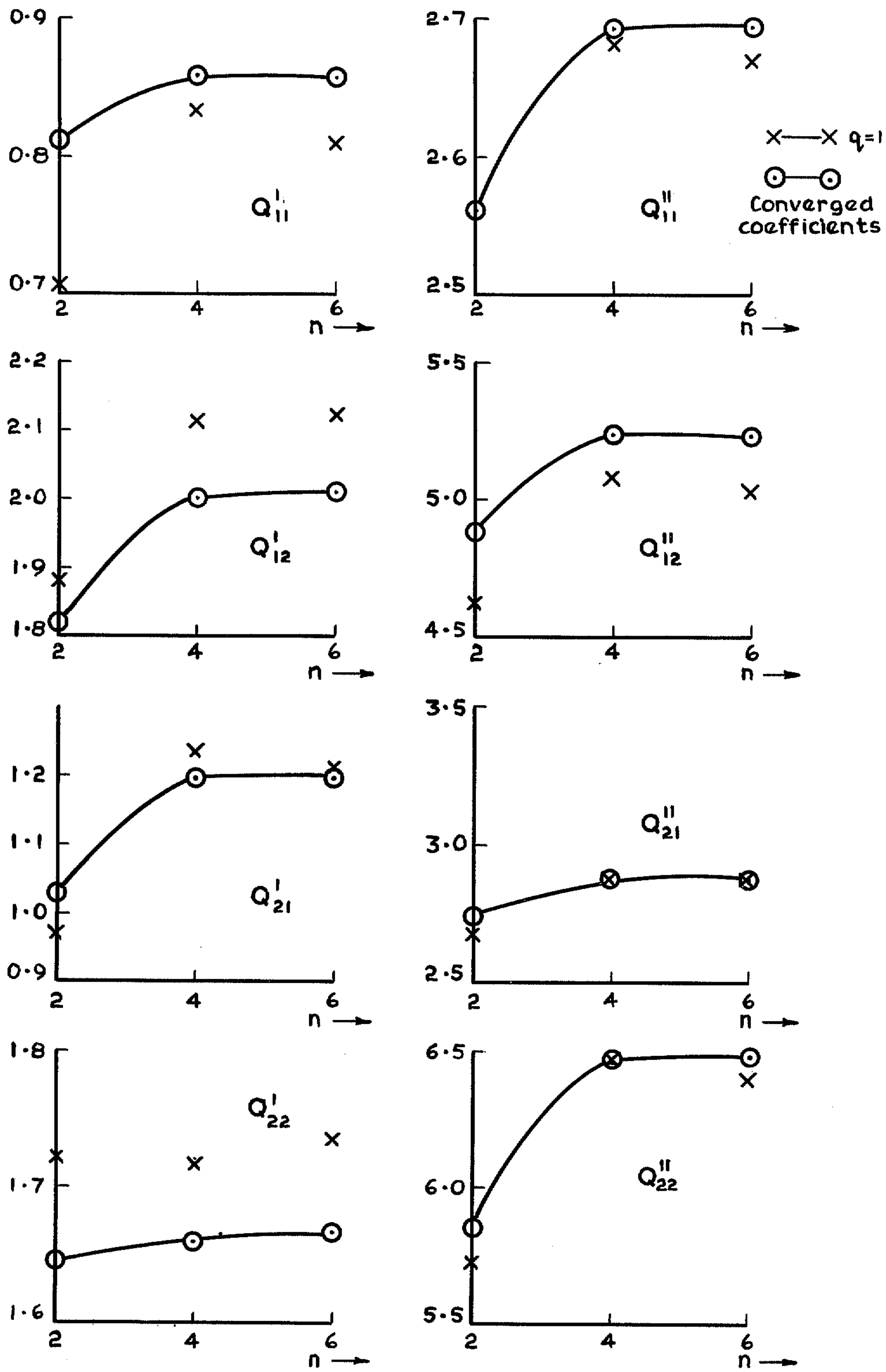

FIG. 5. Effect of increasing n on generalised force coefficients, wing E, $m = 8$, $M = 0{\cdot}781$, $\nu = 1{\cdot}0$.

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