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# Glide-Path Stability of an Aircraft under Speed Constraint

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## Summary

The motion of an aircraft operating under perfect speed constraint (e.g. by an appropriate autothrottle) is examined theoretically and it is shown that, if engine thrust acts through the aircraft centre of gravity, the aircraft will have weak flight path stability in descending flight but be unstable in climb. These effects are readily overshadowed by thrust effects in the sense that with low-slung engines the aircraft motion is destabilized and *vice versa*. Manual flight with autothrottle engaged is shown therefore to be potentially dangerous, as the strong flight-path stability possessed by the natural aircraft is suppressed and inadvertent glide-path errors are not self-correcting.

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\* Replaces R.A.E. Technical Report 71021—A.R.C. 33 027.

## 1. Introduction

In Ref. 1 Neumark had established the concept of aircraft stability under partial constraint. The most important of the cases considered then was that of rectilinear flight under pilots' elevator control. In this condition the longitudinal motion of the aircraft degenerates into a first order mode with speed as the only variable. This mode was shown to become unstable if the aircraft was flown below the so called minimum drag speed. This phenomenon became a practical handling problem with many modern aircraft in the approach and led to the development of the autothrottle as a speed stabilizing device. With the introduction of integral error signals in their control laws the modern autothrottle becomes an effective speed lock and is capable of imposing a speed constraint on the aircraft motion.

When airspeed is held constant, the phugoid mode is suppressed, as this depends on the exchange of potential and kinetic energy, i.e. of height and speed. This energy exchange mechanism of the natural aircraft also ensures stability of the flight path in elevation as a trimmed equilibrium state, which again is lost by the intervention of the autothrottle. Indeed substantial and potentially dangerous glide-path errors are occasionally observed when aircraft are manually flown in the approach with autothrottle engaged. Although normal vigilance by the pilot should ensure that such errors do not go undetected for long, occasionally pilots are distracted by other duties or for some reason or other lose concentration and it is in these circumstances that reduced aircraft stability can spell disaster. The autothrottle then appears to have replaced one form of flight hazard—that of a potential speed instability—by a possibly even more undesirable glide-path instability. This problem exists of course *only* in manually controlled flight and disappears when the elevator is automatically operated as in coupled approaches. The results of the present study therefore only apply to the former case.

The purpose of this Report is to investigate rigorously the flight-path stability of the aircraft in which speed is perfectly constrained by appropriate throttle control and to define the factors that affect this mode.

## 2. The Equations of Motion

We assume perfect speed constraint, as might be imposed by a perfect automatic throttle control. Hence  $V = \text{const.}$  Further we assume fixed elevator. From Fig. 1 we see that in trimmed flight, lift is defined by

$$L_T + T_T \sin \alpha_{ET} = W \cos \gamma_T \quad (1)$$

where  $\gamma_T$  is the trimmed flight path angle and  $\alpha_{ET}$  the incidence of the thrust axis with regard to the flight path. By definition there is also pitching moment equilibrium. Force equilibrium in the direction of flight is defined by

$$T_T \cos \alpha_{ET} = D + W \sin \gamma_T. \quad (2)$$

If the flight path  $\gamma$  deviates from the datum value  $\gamma_T$ , to prevent speed changing the autothrottle will apply an increment in thrust equal to the changed gravity component,

$$\Delta T \cos \alpha_E = (T - T_T) \cos \alpha_E = W(\sin \gamma - \sin \gamma_T). \quad (3)$$

Since speed is constant by definition and lift changes (*see* eqn. 4) only by a very small amount, we assume drag to be constant.

If the combined thrust from the engines acts along a line passing through the centre of gravity of the aircraft, the thrust increment defined in eqn. (3) would not affect pitching moment equilibrium and since there are no other contributions involved,  $\alpha$  remains constant as therefore would lift. However, if the thrust vector misses the centre of gravity by the distance  $z_E$ , Fig. 2 shows that pitching moment equilibrium demands an increment in lift.

$$\Delta L = - \Delta T \frac{z_E}{x_a} = - \frac{W}{\cos \alpha_E} (\sin \gamma - \sin \gamma_T) \frac{z_E}{x_a}, \quad (4)$$

where  $x_a$  is the distance of the aerodynamic centre from the centre of gravity (normally negative), i.e.,

$$x_a = -K_n l.$$

The total lift is then

$$L = \Delta L + L_T = W \left\{ \cos \gamma_T - \frac{T_T}{W} \sin \alpha_{ET} - \frac{z_E}{x_a} (\sin \gamma - \sin \gamma_T) \right\}. \quad (5)$$

Having already dealt with the pitch freedom, the only freedom to be treated for stability analysis is vertical motion. This is defined by forces normal to the flight path. If we define  $\dot{w}$  as the acceleration normal to the flight path,

$$m\dot{w} = -L + mg \cos \gamma - T \sin \alpha_E \quad (6)$$

where with eqn. (3)

$$T = T_T + \Delta T = T_T + W(\sin \gamma - \sin \gamma_T).$$

Introducing eqn. (5) for  $L$  we then get

$$m\dot{w} = mg \left\{ -\cos \gamma_T + \frac{T_T}{W} \sin \alpha_{ET} + \frac{z_E}{x_a} (\sin \gamma - \sin \gamma_T) + \cos \gamma - \frac{T_T}{W} \sin \alpha_E - (\sin \gamma - \sin \gamma_T) \sin \alpha_E \right\}. \quad (7)$$

Writing

$$\gamma = \gamma_T + \Delta\gamma$$

we get

$$\cos \gamma = \cos(\gamma_T + \Delta\gamma) = \cos \gamma_T \cos \Delta\gamma - \sin \gamma_T \sin \Delta\gamma.$$

If we define  $w$  as the increment in vertical velocity with respect to the datum value and linearize,

$$\sin \Delta\gamma = \Delta\gamma = -\frac{w}{V},$$

and

$$\cos \gamma = \cos \gamma_T \cos \Delta\gamma + \sin \gamma_T \frac{w}{V}.$$

Similarly

$$(\sin \gamma - \sin \gamma_T) = -\frac{w}{V} \cos \gamma_T.$$

Eqn. (7) now reduces to

$$\begin{aligned} \frac{\dot{w}}{V} = & -\cos \gamma_T + \frac{T_T}{W} \sin \alpha_{ET} - \frac{z_E}{x_a} \frac{w}{V} \cos \gamma_T + \cos \gamma_T \cos \Delta\gamma \\ & + \sin \gamma_T \frac{w}{V} - \frac{T_T}{W} \sin \alpha_E + \frac{w}{V} \cos \gamma_T \sin \alpha_E. \end{aligned} \quad (8)$$

We can simplify

$$\sin \alpha_E - \sin \alpha_{ET} = \alpha_E - \alpha_{ET} = \Delta\alpha.$$

This is the increment in incidence associated with pitch response to  $\Delta T$  as considered in eqn. (4).

By considering again pitching moment equilibrium,  $\Delta\alpha$  can be expressed as

$$\Delta\alpha = -\frac{\Delta T}{W} \frac{z_E}{x_a} \frac{C_{LT}}{\partial C_L / \partial \alpha}, \quad (9)$$

$C_{LT}$  being trimmed  $C_L$ .

Taking  $\Delta T$  from (3) and  $T_T$  from (2) we get finally

$$-\frac{T_T}{W} \Delta\alpha = -\frac{\omega}{V} \cos \gamma_T \left( \frac{D}{W} + \sin \gamma_T \right) \frac{z_E}{x_a} \frac{C_{LT}}{\partial C_L / \partial \alpha},$$

or simplified,

$$-\frac{T_T}{W} \Delta\alpha = -\frac{\omega}{V} \left( \frac{1}{L/D} + \sin \gamma_T \right) \frac{z_E}{x_a} \cos \gamma_T \frac{C_{LT}}{\partial C_L / \partial \alpha}.$$

We can now introduce this expressions into eqn. (8) and get,

$$\begin{aligned} \frac{\dot{\omega}}{g} = & -\cos \gamma_T (1 - \cos \Delta\gamma) - \frac{\omega}{V} \left( \frac{1}{L/D} + \sin \gamma_T \right) \frac{z_E}{x_a} \cos \gamma_T \frac{C_{LT}}{\partial C_L / \partial \alpha} \\ & - \frac{z_E}{x_a} \frac{\omega}{V} \cos \gamma_T + \sin \gamma_T \frac{\omega}{V} + \frac{\omega}{V} \cos \gamma_T \sin \alpha_E. \end{aligned} \quad (10)$$

Expanding

$$\cos \Delta\gamma = \cos \left( \frac{\omega}{V} \right) = 1 - \frac{1}{2} \left( \frac{\omega}{V} \right)^2 + \dots$$

and retaining only the first two terms, eqn. (10) takes the final form

$$\begin{aligned} \dot{\omega} = g \left\{ \frac{\omega}{V} \left[ \sin \gamma_T - \frac{z_E}{x_a} \cos \gamma_T \left( 1 + \left( \frac{1}{L/D} + \sin \gamma_T \right) \frac{C_{LT}}{\partial C_L / \partial \alpha} \right) \right. \right. \\ \left. \left. + \cos \gamma_T \alpha_E \right] - \cos \gamma_T \frac{1}{2} \left( \frac{\omega}{V} \right)^2 \right\} \end{aligned} \quad (11)$$

or

$$\dot{\omega} = g \cos \gamma_T \left\{ \frac{\omega}{V} \left[ \tan \gamma_T + \alpha_E - \frac{z_E}{x_a} \left( 1 + \frac{C_{LT}}{\partial C_L / \partial \alpha} \left( \frac{1}{L/D} + \sin \gamma_T \right) \right) \right] - \frac{1}{2} \left( \frac{\omega}{V} \right)^2 \right\} \quad (12)$$

since

$$\left( \frac{1}{L/D} + \sin \gamma_T \right) \frac{C_{LT}}{\partial C_L / \partial \alpha} \ll 1$$

except when  $\sin \gamma_T$  is large, we can in the context of the landing approach ignore this term and simplify to get,

$$\dot{w} = g \cos \gamma_T \left\{ \frac{w}{V} \left( \tan \gamma_T + \alpha_E - \frac{z_E}{x_a} \right) - \frac{1}{2} \left( \frac{w}{V} \right)^2 \right\} \quad (13)$$

This expression is the final formulation of the equations of motion describing the stability of an aircraft flown under perfect speed constraint.

### 3. Stability Analysis

Equation (13) is a non-linear first order differential equation. Being of rather simple form we shall be able to solve it analytically. However, for most purposes a linearized solution will suffice, if stability in the vicinity of a given datum condition only is of interest. Ignoring the quadratic term we get a first order root,

$$\lambda = \frac{g}{V} \cos \gamma_T \left( \tan \gamma_T + \alpha_{ET} - \frac{z_E}{x_a} \right). \quad (14)$$

Introducing the c.g. margin  $K_n = -x_a/l$  this can be written in the form

$$\lambda = \frac{g}{V} \cos \gamma_T \left( \tan \gamma_T + \alpha_E - \frac{z_E}{l} \frac{1}{K_n} \right). \quad (15)$$

If we restrict interest to small values of the glidepath inclination  $\gamma_T$  we can combine

$$\tan \gamma_T + \alpha \approx \theta_{ET}.$$

where  $\theta_{ET}$  is the pitch attitude of the engine thrust axis in the datum flight condition. This reduces the number of factors in eqn. (14) to two, i.e.

$$\lambda = \frac{g}{V} \left( \theta_{ET} - \frac{z_E}{x_a} \right). \quad (16)$$

We note that positive attitude  $\theta_{ET}$  is destabilizing and so is a thrust line passing below the aircraft centre of gravity. ( $z_E > 0$ ). The latter effect is amplified if the c.g.—margin (i.e.  $x_a$ ) is reduced.

Conversely negative pitch attitude (a relatively unlikely case except in steep approaches) is stabilizing as are high engine installations.

For the case where thrust acts precisely through the centre of gravity ( $z_E = 0$ ) eqn. (16) has been evaluated in Fig. 3 for the range of speed and  $\theta_{ET}$  relevant to the landing approach. Only in the rather unlikely case that the attitude of the engine axis is negative, implying negative aircraft attitude, can the speed constrained aircraft be expected to be flight path stable, otherwise a degree of instability is to be expected. However, for practical purposes the motion can be considered as neutrally stable, since with the very small values of  $\theta_{ET}$  typical for this flight condition, the time constants involved are of the order of hundreds of seconds. Since the ratio  $z_E/x_a$  can easily be larger than unity, that is much larger than practical values of  $\theta_{ET}$ , especially in flight with small longitudinal stability, the effect of engine thrust axis offset is potentially more powerful and is more likely to lead to noticeable stability or instability, depending on the sign of  $z_E$ .

Figure 4 has been prepared to illustrate the effect of varying flight path angle or the associated pitch attitude  $\theta_E$ . The normal acceleration increment defined by eqn. (6) has been plotted against pitch attitude  $\theta_E$  for various values of  $\theta_{ET}$  (assuming  $z_E = 0$ ).

Positive  $\Delta n$  implies a tendency to increase  $\gamma$  and hence  $\theta_E$ . Therefore the intersection of each curve with the negative abscissa defines a point of stable equilibrium, and with the positive axis an unstable equilibrium condition.

To complete the analysis we return to eqn. (13) and treat it in its complete form. To permit separation of the variables  $w$  and  $t$  we write it as

$$\frac{d\omega}{\omega \left\{ 2V \left( \tan \gamma_T + \alpha_E - \frac{z_E}{x_a} \right) - \omega \right\}} = \frac{g}{2V^2} \cos \gamma_T dt \quad (17)$$

writing

$$A = 2V \left( \tan \gamma_T + \alpha_E - \frac{z_E}{x_a} \right) \quad (18)$$

and

$$B = \frac{g}{2V^2} \cos \gamma_T$$

we integrate both sides of eqn. (17)

$$\int_{w_0}^w \frac{d\omega}{\omega(A - \omega)} = B \int_{t_0}^t dt \quad (19)$$

when  $w = w_0$  at  $t = t_0$  are the initial conditions.

The solution of eqn. (17) is

$$\left( -\frac{1}{A} \log \frac{A - w}{w} \right) - \left( -\frac{1}{A} \log \frac{A - w_0}{w_0} \right) = Bt$$

or

$$\log \frac{\frac{A}{w_0} - 1}{\frac{A}{w} - 1} = ABt$$

from

$$\frac{\frac{A}{w_0} - 1}{\frac{A}{w} - 1} = e^{ABt} = e^{\lambda t}$$

we get finally

$$w = \frac{A}{1 + \frac{w_0}{e^{\lambda t}} \left( \frac{A}{w_0} - 1 \right)}, \quad (20)$$

with

$$\lambda = AB = \frac{g}{V} \cos \gamma_T \left( \tan \gamma_T + \alpha_E - \frac{z_E}{x_a} \right) \quad (21)$$

which is identical to the root for the linearized solution eqn. (14).

For the simple case when  $z_E$  and  $\alpha_E$  are assumed to be zero, eqn. (20) can be readily evaluated. Since then the vertical velocity in the datum flight condition  $w_T = -V \tan \gamma_T$  and  $A = -2w_T$ , eqn. (20) reduces to

$$\frac{w(t)}{w_T} = \frac{2}{1 - \frac{1 + 2w_T/w_0}{e^{\lambda t}}}$$

with the results shown in Fig. 5.

The presentation has been nondimensionalised to give it generality. As would be expected from Fig. 4, the response after a flight-path disturbance can take three different forms depending on the regime into which the aircraft is placed by the initial disturbance, i.e. on  $\gamma_0$  (or  $w_0$ ).

We note that in regimes I and II the aircraft converges towards  $\gamma_T$ , the rate of convergence being faster in recoveries from dives ( $\gamma < \gamma_{TS}$ ). We note from equation (1) that in the absence of thrust effects, there are two possible trim conditions, which define flight-path angles of equal magnitude but opposite sign, i.e.  $\gamma_{TS} = -\gamma_{TU}$ . If the aircraft is initially at a climb angle greater than  $\gamma_{TU}$ , the (unstable) equilibrium climb angle, it will diverge, the rate of divergence increasing with time. It should be noted that the time scales shown in Fig. 5 are arbitrary and can be shifted to place any desired initial value of  $\gamma_0$  at  $t = 0$ .

The effect of thrustline offset on the flight-path stability of an aircraft operating with perfect speed constraint is illustrated in Fig. 6. We see that even a small amount of thrust offset below the aircraft centre of gravity will make  $\gamma_S$  into an unstable equilibrium condition. One notes that there will again be a secondary stable equilibrium state well to the left of the range illustrated at  $\gamma \approx 24^\circ$ , toward which the aircraft will diverge if disturbed to a glide-path angle  $\gamma < \gamma_S$ .

If the thrustline passes above the aircraft centre of gravity the result is a substantial increase in stability about the trim point  $\gamma_S$ . The trim constants relevant to the three cases shown in Fig. 6, assuming  $V = 140$  knots, would be:

$\frac{z_E}{x_a}$	$\lambda$	$\tau$ (seconds)
-0.25	+0.0221 s <sup>-1</sup>	-45 s    unstable
0	-0.0119 s <sup>-1</sup>	+83.5 s    stable
+0.25	-0.0459 s <sup>-1</sup>	+21.8 s    stable

In the latter case with  $z_E/x_a = +0.25$  quite a substantial degree of glide-path stability is obtained so that with this configuration speed constraint need not be an unacceptable mode of operation even in the approach.

#### 4. Discussion and Conclusions

In this report the stability characteristics of aircraft operating under perfect speed constraint have been investigated. Detailed analysis has shown that there are three factors which determine the exact degree of flight path stability pertaining in this case, namely the flight path angle itself, the incidence of the engine thrust axis with respect to the flight path and the distance by which the thrust axis misses the centre of gravity of the aircraft. The first two are in fact additive so that we can combine them and



reduce the parameters effectively involved to two, the pitch attitude  $\theta_E$  of the engine thrust axis in the given datum flight condition and the distance  $z_E$  by which the total installed thrust operated on by the autothrottle misses the aircraft centre of gravity. Negative pitch attitude is stabilizing and so are engines installed above the aircraft centre of gravity and vice versa. Of these parameters engine thrust line offset is potentially by far the most powerful and likely to determine the issue. This effect is further amplified when longitudinal stability is weak.

In configurations with high-set engines there may be sufficient glide path stability in flight under speed lock to make the engagement of such an autothrottle mode entirely acceptable, even in otherwise unaided manual approaches. However, for the aircraft with engines installed in line with or below the centre of gravity, the action of the autothrottle will deprive the aircraft of glide path stability and its use must be treated with some caution.

None of this applies of course in fully coupled approaches when the glide path is automatically controlled by signals routed to the elevator. Equally, if the pilot engages the pitch attitude hold mode of the autopilot, as is frequently done in airline practice, this also restores glidepath stability. However, if the autothrottle is used in a purely manual approach with an aircraft and where this leads to marginal or even negative glidepath stability, there is then the potential danger that during a period of inattention or distraction of the pilot the aircraft may experience major changes in its glidepath. It would appear that when good guidance—either visual or from ILS is available, such upsets will not go undetected for long, but when such guidance is absent, lack of inherent glide path stability is a serious hazard and the use of autothrottle in such circumstances may be illadvised. Since in an aircraft suffering from speed-instability, the pilot will wish to be assisted by the autothrottle, it may be desirable to modify the control law of the device for use in manual flight so that it is restricted to the function of speed stabilization without speed hold constraint. To optimise performance in fully automatic flight, the speed hold facility can be readily restored by linking it to the engagement of the autopilot.

## LIST OF SYMBOLS

$A$	See equation (15)
$C_L$	Lift coefficient
$D$	Aircraft drag
$g$	Gravitational acceleration
$K_n = -\frac{x_a}{l}$	cg—margin
$L$	Lift
$l$	Aircraft longitudinal reference length
$m$	Aircraft mass
$\Delta n$	Increment in acceleration normal to flight path
$t$	Time
$T$	Thrust
$V$	True airspeed
$W$	Aircraft weight
$w$	Vertical velocity normal to flight path
$x_a$	Distance of aerodynamic centre from cg (normally negative)
$z_E$	Thrustline offset from aircraft cg
$\alpha_E$	Angle of incidence of the thrust vector
$\gamma$	Flight-path angle (positive for climb)
$\lambda$	Stability root
$\tau$	First order response time constant
$\theta_E$	Pitch attitude angle of thrust vector

## REFERENCE

<i>No.</i>	<i>Author</i>	<i>Title, etc.</i>
1	S. Neumark .. .. .	Problems of longitudinal stability below minimum drag speed and theory of stability under constraint. ARC R & M 2983 (1953)

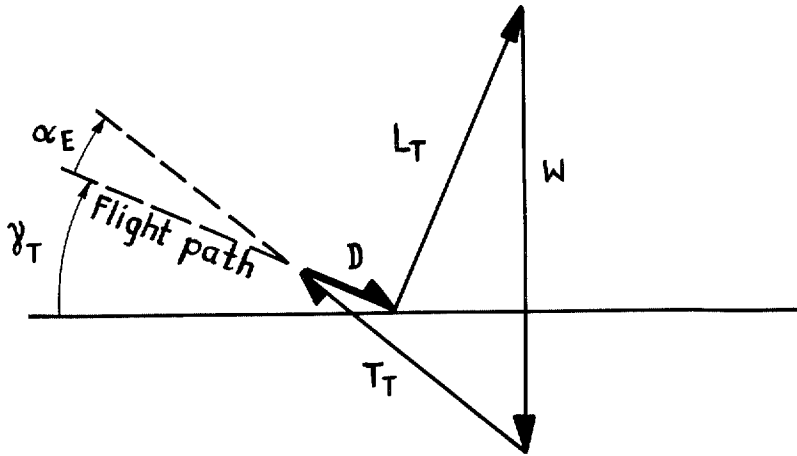


FIG. 1. Lift and drag equilibrium in steady rectilinear flight.

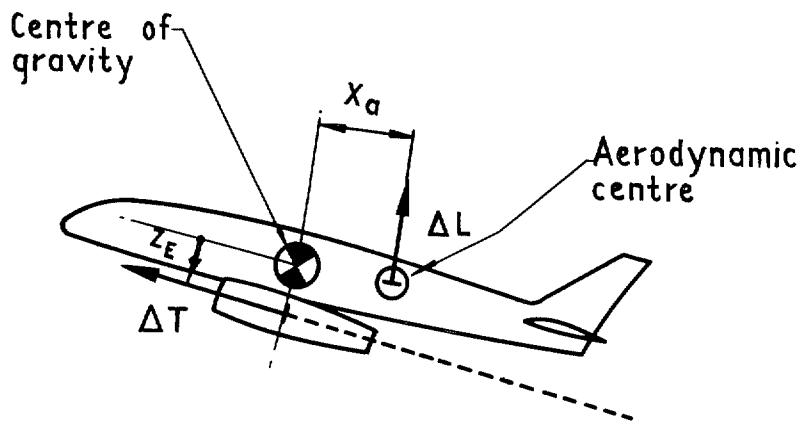


FIG. 2. Pitching moment equilibrium between thrust and lift increments.

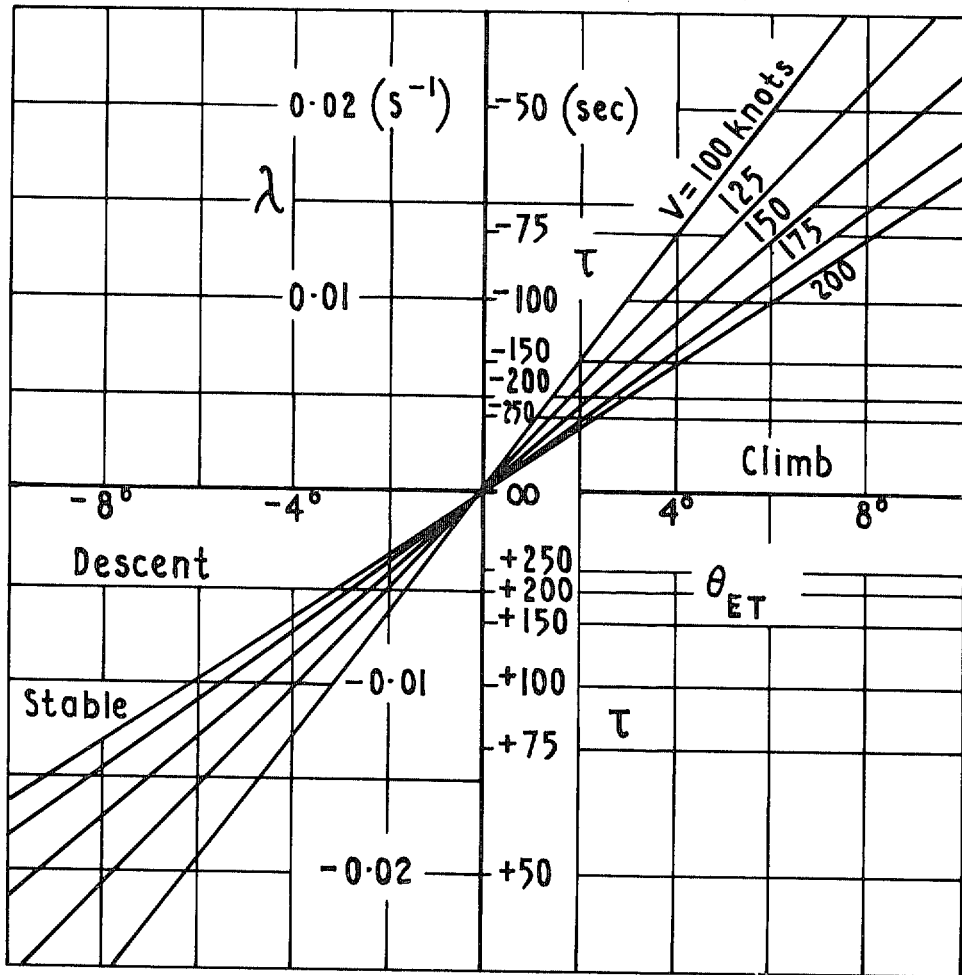


FIG. 3. Glide path stability of the speed constrained aircraft as a function of speed and trimmed engine attitude angle  $\theta_{ET}$ .

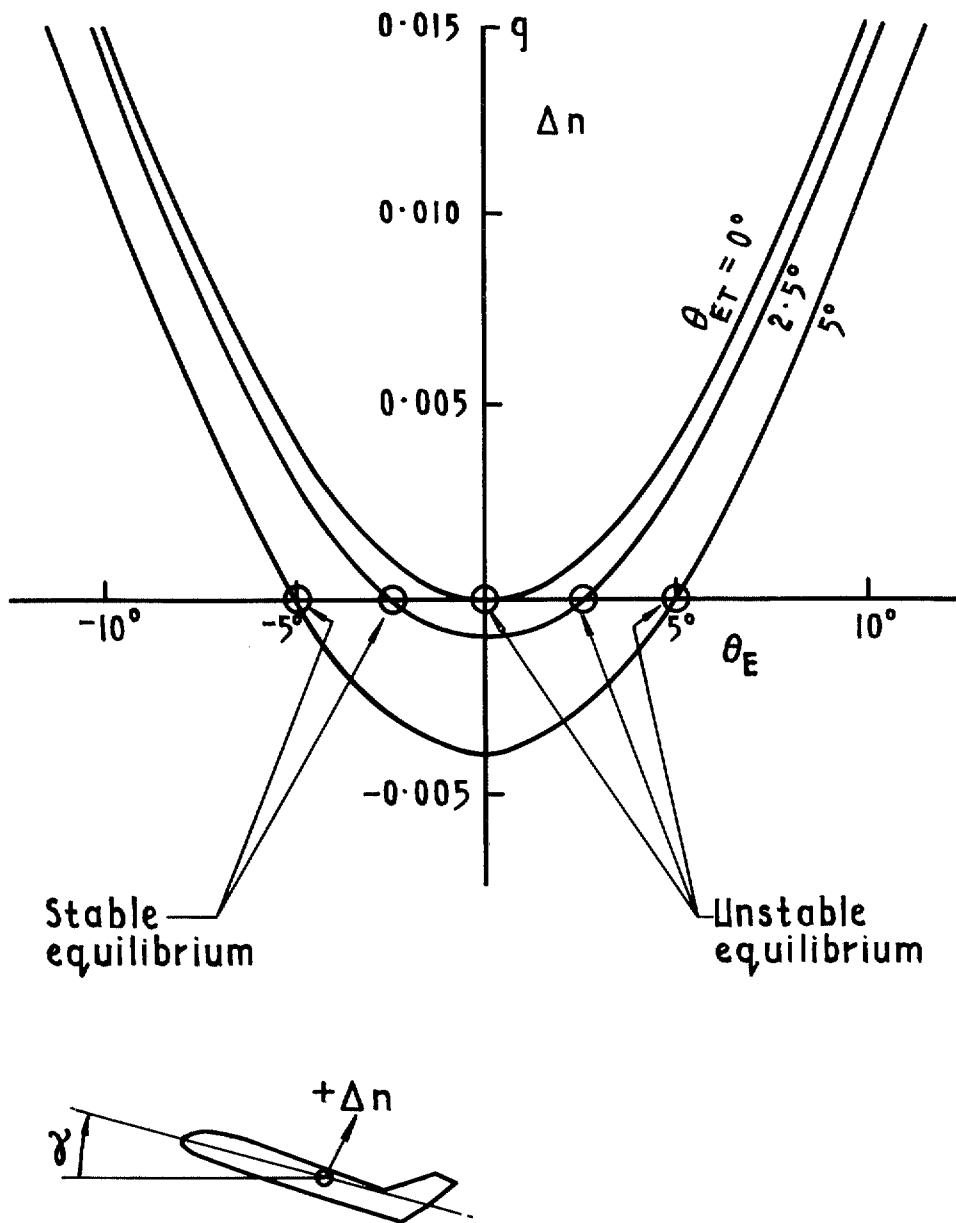


FIG. 4. Acceleration normal to the flight path for an aircraft flying at fixed speed and elevator.

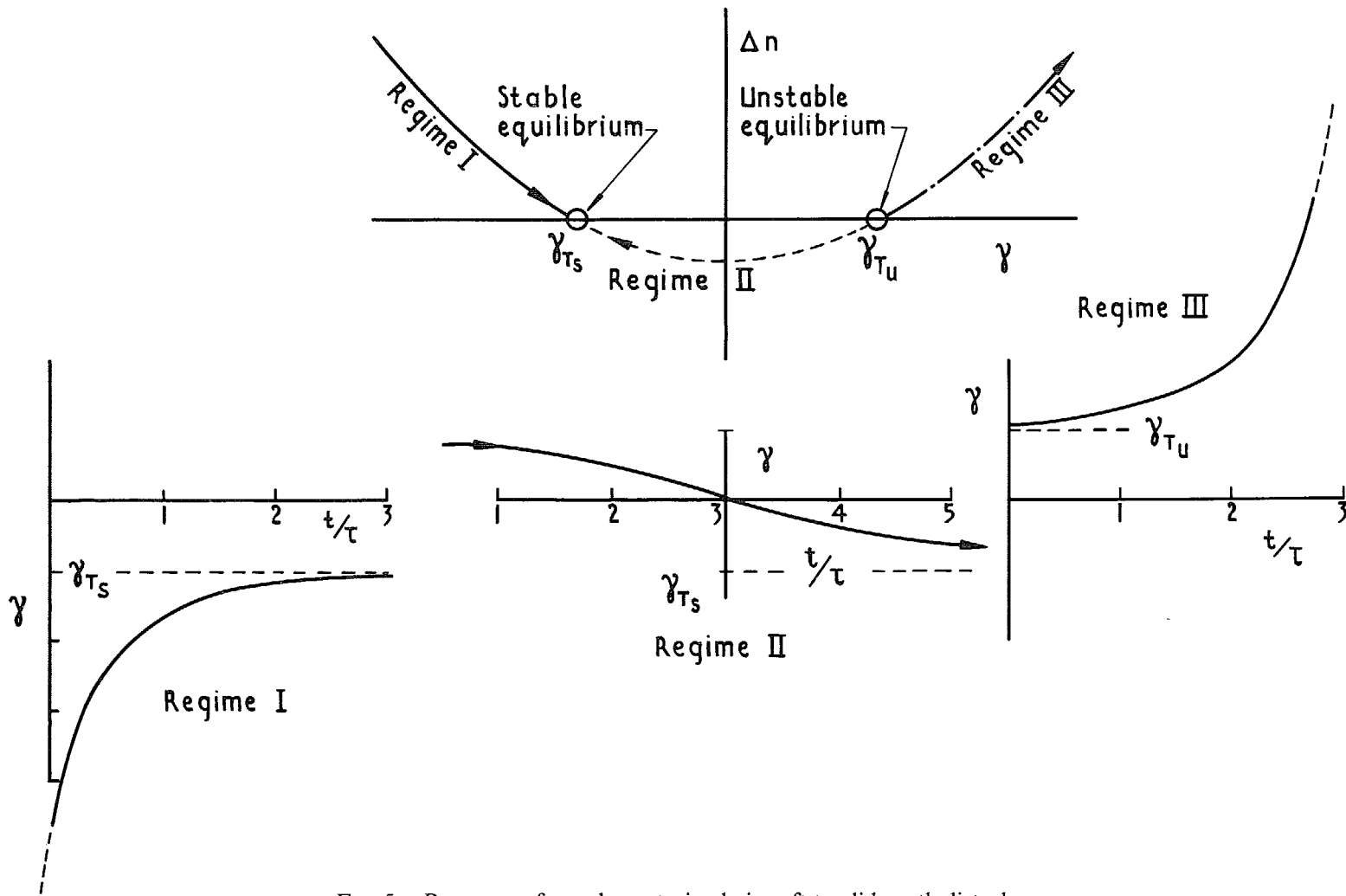


FIG. 5. Response of speed constrained aircraft to glide path disturbance.

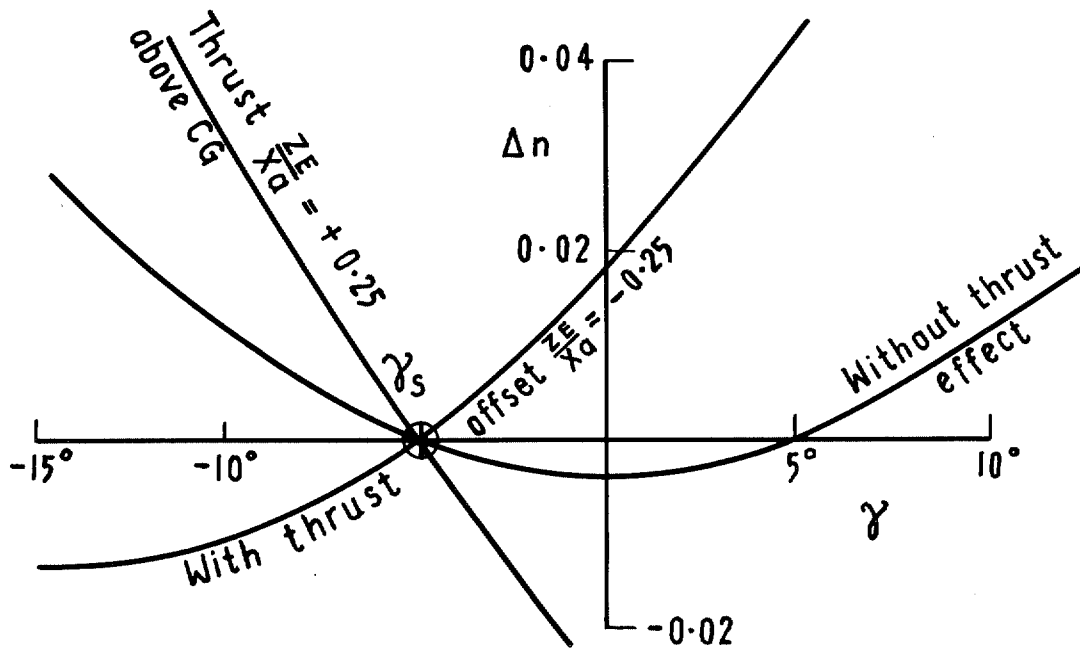


FIG. 6. Illustrating the effect of engine thrust acting below and above the aircraft CG on flight path stability under speed constraint.