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JET NOISE RADIATION FROM DISCRETE VORTICES

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SUMMARY

The modes of vibration of simple two-dimensional discrete vortices are examined and some analysis given for the acoustic radiation from such modes after a disturbance. The dominant radiation is shown to be of quadrupole type with a frequency proportional to the mean vorticity, and in a jet mixing region this frequency is expected to fall linearly with increasing vortex radius. Some speculations are made about the practical effect of forward speed on jet noise.

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CONTENTS

1 INTRODUCTION

The structure of turbulent shear layers has been the subject of much study in recent years, and there have been many observations of the existence of large coherent structures[1-9,19]. In jets these structures may carry most of the vorticity. They are small when first formed near the jet lip and then grow by amalgamation as they are convected downstream until they reach the end of the potential core. Thereafter they continue to grow in scale but slowly dissipate their energy through molecular viscosity.

In the process of evolution at sufficiently large Reynolds number, turbulent diffusion will impose small-scale irregularities on the structure, which will progressively erode the coherence of the structure. Nevertheless some coherence may be expected to be retained throughout the noise-producing regions of the jet, and as a start the acoustic radiation from simple two-dimensional vortices is studied in the following sections.

The results obtained follow closely on the analysis of Moore and Saffman[10] in which they examine the stability of vortices with elliptic cross-section in a strain field. In section 2 the motion of an unstrained circular vortex is considered, its resonances are found, and it is shown that when disturbed the vortex radiates acoustic energy at its natural frequencies. The energy radiated per unit cross-sectional area is found to increase very rapidly (like a^6 where a is the radius) with the size of the vortex. This result, however, is based on a presumption that the vorticity is kept constant as the vortex radius increases. If the mean vorticity falls, as it does for vortices growing in a jet, the result is greatly modified.

A complete analysis is not given for a vortex in a strain field because of difficulties that arise through the velocity field being unbounded at infinity. Some qualitative arguments, however, are used (in section 3) to show the general nature of the acoustic radiation under these conditions.

Section 4 consists of a general discussion in which the view is taken that the noise output of a jet could come largely from the disturbance of vortex structures, caused by general unsteadiness in the jet flow, and their resulting vibration at their natural frequency. It is shown that this has the characteristics of quadrupole radiation (or possibly higher poles from vortex harmonics, but these are small), and it is thought that similar results will hold even for the more complicated structures of a real jet. This section ends with a speculation about the effects of forward speed on jet noise, which has long been known to contain puzzling features. It is suggested that some account should be taken of the fact that, when an external stream is added, the noise-producing vortices have to swallow part of the turbulent boundary layer that has separated from the outside of the nacelle. A very crude modification of an existing formula for the velocity dependence is offered.

2 THE SOUND FIELD OF A SIMPLE VIBRATING VORTEX

2.1 Natural frequencies of an unstrained vortex

This analysis follows that of Moore and Saffman[10] for an elliptic vortex in a strain field, but is applied to a circular vortex without strain and is therefore rather simpler. Two-dimensional incompressible flow is assumed and the velocity $\underline{v}(r,\theta) \to 0$ as $r \to \infty$, where (r,θ) are polar coordinates. At equilibrium there is uniform vorticity ω within a circle of radius a and zero vorticity for $r > a$. The circulation Γ outside the vortex is given by

$$\Gamma = \pi a^2 \omega , \qquad r > a . \tag{1}$$

The perturbed velocity field $\underline{v}$ is written

$$\underline{v}(r,\theta,t) = \underline{U}(r,\theta,t) + \underline{u}(r,\theta,t) , \tag{2}$$

where $\underline{u}$ is a small time-dependent perturbation and $\underline{U}$ is defined by

$$\left.\begin{aligned} U_r &= 0 \\ U_\theta &= \frac{\Gamma r}{2\pi a^2} , \qquad r < r_v , \\ U_\theta &= \frac{\Gamma}{2\pi r} , \qquad r > r_v , \end{aligned}\right\} \tag{3}$$

where $r_v(\theta,t)$ is the radius of the vortex core following the perturbation. In the equilibrium state $r_v = a$ and $U_\theta(a,\theta) = \Gamma/(2\pi a) = \frac{1}{2}a\omega$; in general the area bounded by $r_v \left(= \int_0^{2\pi} \frac{1}{2} r_v^2 d\theta \right)$ is constant.

We take $u/(a\omega)$ to be of $O(\varepsilon)$ where ε is a small quantity such that terms of $O(\varepsilon^2)$ may be neglected. The time dependence of the motion following the perturbation is assumed to be such that we may write

$$r_v(\theta,t) = a\{1 + e^{\sigma t}f(\theta)\} \quad , \tag{4}$$

where $f = O(\varepsilon)$ and where the real part of σ will determine the growth or decay of the disturbance, and the imaginary part will determine its frequency.

This displacement must be compatible with $u_r(a,\theta,t)$, which provides one boundary condition for $\underline{u}$. The other could be derived from continuity of pressure, but it is more convenient to use

$$\underline{v} \cdot \hat{\underline{t}} \text{ continuous at } r_v \quad , \tag{5}$$

where $\hat{\underline{t}}$ is the unit vector tangential to $r = r_v(\theta)$; condition (5) is equivalent to continuity of pressure in the present problem[10].

From equation (4),

$$\frac{D}{Dt}\left[r_v - a\{1 + e^{\sigma t}f(\theta)\}\right] = 0 \quad , \tag{6}$$

where D/Dt stands for the time rate of change following a particle. Now,

$$\begin{aligned}
\frac{D}{Dt} r_v &= v_r(r_v,\theta) \\
&= v_r(a,\theta) + (r_v - a)\frac{\partial}{\partial r} v_r(a,\theta) \\
&= v_r(a,\theta) + O(\varepsilon^2) \quad , \quad \text{since } (r_v - a) \text{ is } O(\varepsilon) \quad , \\
&= u_r(a,\theta) \quad .
\end{aligned}$$

Also, $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta}$, so that equation (6) becomes

$$u_r(a,\theta) = a\sigma e^{\sigma t}f(\theta) + \frac{ae^{\sigma t}}{r_v} v_\theta f'(\theta) \quad ,$$

or

$$u_r(a,\theta) = a\sigma e^{\sigma t} f(\theta) + \frac{\Gamma}{2\pi a} e^{\sigma t} f'(\theta) + O(\varepsilon^2) \quad . \tag{7}$$

The condition (5) requires continuity of $(\underline{U}\hat{\underline{t}} + \underline{u}\hat{\underline{t}})$, and in both terms $\hat{\underline{t}}$ can be taken in the θ direction; in the first term because $U_r = 0$, and in the second term because the product of $\hat{t}_r$ and u_r is of $O(\varepsilon^2)$. Since

$$\underline{U}(r_v,\theta) = \underline{U}(a,\theta) + ae^{\sigma t} f(\theta) \frac{\partial}{\partial r} \underline{U}(a,\theta) + O(\varepsilon^2) \quad , \tag{8}$$

there follows from equation (3)

$$u_{\theta 2} - u_{\theta 1} = e^{\sigma t} f(\theta) \Gamma/\pi a + O(\varepsilon^2) \quad , \qquad r = r_v \tag{9}$$

where suffix 1 denotes values inside $r = r_v$ and suffix 2 denotes values outside $r = r_v$.

A consistent inviscid approach in which ω is uniform in the two regions requires that the perturbations neither create nor destroy vorticity, so that u_1 and u_2 may be derived from velocity potentials ϕ_1 and ϕ_2 respectively, which will be defined later. The small areas crossed by the deforming boundary need not be considered in the analysis to $O(\varepsilon)$, since to this order the boundary conditions can be applied at $r = a$. Moreover, the area of the vortex remains constant in time, and we assume

$$f(\theta) = A \cos m\theta + B \sin m\theta \quad , \tag{10}$$

where m is a positive integer. Let ϕ_2 and ϕ_1 have the forms $(C \cos m\theta + D \sin m\theta) r^n e^{\sigma t}$ and $(E \cos m\theta + F \sin m\theta) r^n e^{\sigma t}$ respectively. These must satisfy

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \quad , \tag{11}$$

whence $n = \pm m$. It follows that in order to avoid singularities at $r = 0$ and $r \to \infty$, ϕ_1 and ϕ_2 must be given by

$$\left.\begin{aligned} \phi_1 &= r^m(E \cos m\theta + F \sin m\theta)e^{\sigma t} \\ \phi_2 &= r^{-m}(C \cos m\theta + D \sin m\theta)e^{\sigma t} \quad . \end{aligned}\right\} \tag{12}$$

We now obtain six linear homogeneous equations in the unknown coefficients A .. F from the boundary conditions (7) and (9): condition (7) applies to both ϕ_1 and ϕ_2 , which together with (9) gives three equations and in each of them the coefficients of $\sin m\theta$ and $\cos m\theta$ must vanish independently. If the coefficients A ... F are not all zero they can be eliminated to give an equation for the eigenvalues of σ ,

$$\frac{\Gamma^2(m - 1)^2}{\pi^2 a^2} + 4a^2\sigma^2 = 0 \quad . \tag{13}$$

Whence

$$\sigma = \pm \frac{i(m - 1)\Gamma}{2\pi a^2} = \pm \frac{i(m - 1)\omega}{2} \quad . \tag{14}$$

This result was first derived by Lord Kelvin (see Lamb[11] p.231). In terms of A , which characterises the amplitude of the disturbance, the other coefficients are given by

$$\left.\begin{aligned} B &= \mp iA \\ C &= -\frac{\Gamma a^m}{2\pi m} B \\ D &= \frac{\Gamma a^m}{2\pi m} A \\ E &= \frac{\Gamma}{2\pi m a^m} B \\ F &= -\frac{\Gamma}{2\pi m a^m} A \quad , \end{aligned}\right\} \tag{15}$$

where the result (14) has been used in expressing the first of these.

The various physical parameters can now be expressed in terms of m, ω and some arbitrary real amplitude A by taking the real part of the appropriate complex functions. There results

$$(r_v - a)/a = A \cos (\alpha t - m\theta) \quad , \tag{16}$$

which shows that the boundary of the vortex rotates with constant angular velocity α/m , and

$$\left.\begin{aligned} \phi_1 &= \left(\frac{r}{a}\right)^m \frac{\Gamma A}{2\pi m} \sin (\alpha t - m\theta) \\ \phi_2 &= -\left(\frac{a}{r}\right)^m \frac{\Gamma A}{2\pi m} \sin (\alpha t - m\theta) \quad , \end{aligned}\right\} \tag{17}$$

with

$$\alpha = \tfrac{1}{2}(m - 1)\omega = \frac{\Gamma(m - 1)}{2\pi a^2} \quad . \tag{18}$$

It is of interest, in passing, to note the values of the Reynolds stress components both inside and outside the vortex. These are (neglecting the small regions near $r = a$ that are crossed by the deforming boundary),

$$\left.\begin{aligned} \sigma_{rr} = -\rho\overline{u_r^2} &= -\frac{\rho}{2}\left(\frac{r^{m-1}\Gamma A}{2\pi a^m}\right)^2 \qquad r < a \\ &= -\frac{\rho}{2}\left(\frac{a^m \Gamma A}{2\pi r^{m+1}}\right)^2 \qquad r > a \end{aligned}\right\} \tag{19}$$

$$\sigma_{\theta\theta} = -\rho\overline{u_\theta^2} = \sigma_{rr} \quad , \tag{20}$$

$$\sigma_{r\theta} = -\rho\overline{u_r u_\theta} = 0 \quad , \tag{21}$$

where ρ is the density and the overbar denotes a time average.

In these results, as in Moore and Saffman[10], the mode corresponding to the eigenvalue $m = 1$ is lateral translation of the vortex without deformation; the modes for $m > 1$ involve an increasing number of nodes around the vortex periphery as m increases.

2.2 The acoustic field

In section 2.1 the flow was assumed to be incompressible. We now allow the flow to be slightly compressible in the sense that we shall assume $M^2 \ll 1$, where M is the maximum Mach number of the vortex flow. The basic flow is assumed to be homentropic and the perturbations are isentropic with, as before, a velocity potential ϕ_2 in region 2. Then

$$\left(\frac{dp}{d\rho}\right)_{S \text{ constant}} = c^2 \tag{22}$$

where p and ρ are the pressure and density of the basic state respectively, S is the specific entropy and c the speed of sound. Under the conditions postulated, the speed of sound depends on r and may be shown to have the form

$$\frac{c^2(r)}{c^2(a)} = 1 + \frac{(\gamma - 1)}{2} M^2 \left\{1 - \left(\frac{a}{r}\right)^2\right\} = \hat{c}^2 \quad \text{say} , \tag{23}$$

with

$$M = U_\theta(a)/c(a) \quad . \tag{24}$$

The equations of continuity and momentum provide three equations in ϕ_2 and the perturbations in pressure and density, which may be simplified into an equation for ϕ_2. In view of the result (17), we try

$$\phi_2 = f(r) \sin (\alpha t - m\theta) \quad , \tag{25}$$

and the resulting equation for f then becomes

$$f''(r) + \left(\frac{1}{r} + \frac{M^2a^2}{r^3\hat{c}^2}\right) f'(r) + \left\{\left(\frac{\alpha}{c(a)} + \frac{Mma}{r^2}\right)^2 \frac{1}{\hat{c}^2} - \frac{m^2}{r^2}\right\} f(r) = 0 \quad . \qquad (26)$$

This equation is too complicated to possess solutions in terms of standard functions, so we apply the assumption of low Mach number. Equations (3), (18) and (24) show that

$$\frac{\alpha}{c(a)} = 0(M) \quad ,$$

and $\hat{c}^2 = 1 + 0(M^2)$, so that to $0(M^2)$ equation (26) becomes

$$f''(r) + \left(\frac{1}{r} + \frac{M^2a^2}{r^3}\right) f'(r) + \left\{\left(\frac{\alpha}{c(a)} + \frac{Mma}{r^2}\right)^2 - \frac{m^2}{r^2}\right\} f(r) = 0 \quad . \qquad (27)$$

This equation is still too complicated for standard functions to satisfy it (so far as the author is aware), but we may still neglect terms of $0(M^2)$. The coefficient of $f'(r)$ then becomes $(1/r)$, since $r > a$, but the term in curly brackets needs to be treated more carefully. We introduce the wavelength λ which must be large compared with a . In fact

$$\left.\begin{aligned} \frac{\lambda}{a} &= \frac{2\pi c}{\alpha a} = 0(M^{-1}) \quad , \\ \text{and} \quad \frac{\alpha a}{c(a)} &= \frac{2\pi a}{\lambda} + 0(M^3) \quad . \end{aligned}\right\} \qquad (28)$$

It follows that when $(r/\lambda) = 0(1)$ or greater, the term Mma/r^2 is of $0(M^2)$ times $\alpha/c(a)$ and may be neglected. On the other hand for small r , e.g. $r/\lambda = 0(M^{\frac{1}{2}})$ or less the term (Mma/r^2) makes a contribution at most of $0(M^2)$ times that of the term m^2/r^2 . It follows that

$$\left[\left(\frac{\alpha}{c(a)} + \frac{Mma}{r^2}\right)^2 - \frac{m^2}{r^2}\right] a^2 = \left[\left\{\frac{\alpha}{c(a)}\right\}^2 - \frac{m^2}{r^2}\right] a^2 + 0(M^2) \qquad (29)$$

is uniformly valid for all $r > a$. Thus under the approximation that terms of $0(M^2)$ and of $0(a/\lambda)^2$ are both negligible, equation (26) reduces to the ordinary wave equation

$$\left.\begin{aligned} f''(r) + \frac{1}{r} f'(r) + \left(k^2 - \frac{m^2}{r^2}\right) f(r) &= 0 \ , \\ \text{or} \qquad \nabla^2 \phi &= - k^2 \phi \ , \end{aligned}\right\} \qquad (30)$$

where $k^2 = \alpha^2/c^2$ is the square of the wave number in the radiation field. The above analysis justifies using equation (30) instead of the more complicated convected wave equation (23), although the 'exact' equation (25) is of some interest in itself, since it enables an estimate of errors to be made if desired.

Since the θ dependence is $\exp(- im\theta)$ in the whole flow region, the solution of (30) for outgoing waves that matches (17) is found to be

$$\phi = \mathrm{Re}\left[A_1 H_m^{(2)}(kr) e^{i(\alpha t - m\theta)}\right] \qquad (31)$$

where H_m is a Hankel function, A_1 is a constant and Re stands for the real part. For small kr(i.e. $r \ll \lambda$) , equation (31) has the asymptotic form

$$\phi \sim - \frac{A_1 (m-1)!}{\pi} \left(\frac{kr}{2}\right)^{-m} \sin\,(\alpha t - m\theta) \ , \qquad kr \ll 1 \ . \qquad (32)$$

From equations (17) and (32)

$$A_1 = \frac{(ka)^m \Gamma}{2^{m+1} m!} A \ . \qquad (33)$$

In the radiation field for large kr , equation (31) has the asymptotic form

$$\left.\begin{aligned} \phi &\sim \mathrm{Re}\left[A_1 \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \exp\, i(\alpha t - m\theta - kr + \tfrac{1}{2}m\pi + \tfrac{1}{4}\pi)\right] \\ &\sim \mathrm{Re}\, F(r,\theta) e^{i\alpha t} \ , \qquad \text{say} \ , \end{aligned}\right\} \qquad (34)$$

$$\left.\begin{aligned} \text{where} \qquad F(r,\theta) &= \frac{B e^{-i(m\theta + kr)}}{r^{\frac{1}{2}}} \\ \text{and} \qquad B &= A_1 \left(\frac{2}{\pi k}\right)^{\frac{1}{2}} \exp\, i(\tfrac{1}{2}m\pi + \tfrac{1}{4}\pi) \ . \end{aligned}\right\} \qquad (35)$$

It is of interest to calculate the mean radial energy flux, per unit length of the vortex, E_r , in the sound wave. This is,

$$E_r = \int_0^{2\pi} \overline{p'u_r} r d\theta \quad , \tag{36}$$

where the acoustic pressure p' is given by

$$p' = -\rho_0 \frac{\partial \phi}{\partial t} = -i\alpha\rho_0\phi \quad . \tag{37}$$

If the expression for p' is evaluated by taking the real part of the right-hand side of equation (37), the result for $m = 2$ agrees with that given by Howe[14] for a rotating elliptic vortex. From equation (34), in the far field,

$$\phi = \tfrac{1}{2}(F(r,\theta)e^{i\alpha t} + F^*(r,\theta)e^{-i\alpha t}) \quad , \tag{38}$$

where the star denotes the complex conjugate. Equations (36) to (38) yield

$$\begin{aligned} E_r &= -\frac{\rho_0\alpha}{4} \int (-iF^*F_r + iFF_r^*) r d\theta \\ &= \pi\rho_0\alpha k B\bar{B} \\ &= 2\rho_0\alpha A_1^2 \quad . \end{aligned} \tag{39}$$

After substituting from equations (18) and (33), there results

$$E_r = \frac{(m-1)\rho_0\Gamma^3}{\pi} \left\{\frac{(ka)^m A}{2^{m+1} m! a}\right\}^2 \quad . \tag{40}$$

Since $k = \alpha/c$, the quantity in curly brackets in equation (40) also depends on Γ , but it is more revealing to give the expression in terms of Mach number by equation (24). Then

$$E_r = \frac{(m-1)\rho_0\Gamma^3}{\pi}\left\{\frac{M^m(m-1)^m A}{2^{m+1} m!\, a}\right\}^2 , \tag{41}$$

and in view of the factor of M^{2m} it is evident that at low Mach number, most of the energy flux is confined to the low modes. Moreover, if we compare vortices of different size but the same vorticity ω , Γ is proportional to $a^2\omega$, and M to $a\omega$, so that E_r is proportional to a^{2m+4} . Even for the fundamental mode $m = 2$, this implies that for a given size of non-dimensional disturbance the acoustic radiation per unit area is proportional to a^6 , e.g. one vortex radiates 64 times as much energy as four vortices of half the radius for the same non-dimensional disturbance A .

The variation with Mach number is also of interest. If we assume that two jets of the same diameter but different Mach numbers contain vortices of the same size, then Γ is proportional to M , and for radiation by the fundamental mode E_r is proportional to M^7 . This is the same power law that is deduced from Lighthill's theory in two-dimensions. For modes of higher order the analysis is invalid in that only the smallest powers of M have been retained in the solution of equation (25); qualitatively, however, it is to be expected that for such modes the radiation will depend even more powerfully on M .

3 A VIBRATING VORTEX IN A STRAIN FIELD

For the unstrained vortex it was possible to calculate a matched acoustic field thanks to the simple nature of the external flow. When a strain field is present, however, the external flow is more complicated and the same procedure will not be attempted. An alternative procedure might be to use Lighthill's method[15] and evaluate an appropriate integral over the source region. This, however, is also difficult, because the unsteady flow is coherent over the whole field, so that the source region is strictly unlimited, and certainly not compact. Instead of attempting to find the acoustic field in detail, therefore, we shall be content to make a qualitative comparison with the simple vortex on the basis of the nature of the integrand in Lighthill's formulation. This is dangerous, in general, because it takes no account of possible cancellations, but in the present example it is at least unlikely that there will be greater cancellation in the field of the strained vortex than that of the unstrained vortex.

The stability of a vortex in a strain field is fully treated by Moore and Saffman[10]. They use a pure strain field, in which the steady flow behaves like

$$\left.\begin{aligned} v_x &\sim -ey \\ v_y &\sim -ex \end{aligned}\right\} \qquad x,y \to \infty , \tag{42}$$

and also generalise the theory to include simple shear. In either case the equilibrium cross-section of the vortex becomes an ellipse of which the elongation increases with the strain or shear. For sufficiently large elongation the vortex becomes unstable. Since the present interest of noise production is relevant to a vortex in a turbulent shear layer it might seem that an external shear flow would be the more appropriate. The viewpoint adopted in the present paper, however, is that the vorticity can be taken as concentrated in discrete vortices which are surrounded by potential flow, in which case each vortex is embedded in a straining irrotational flow. As a simple example, suppose the shear layer to be represented by an infinite row of equally spaced point vortices of constant strength, and consider a simple row in isolation. Then any one vortex is in the strain field of the others, which in the immediate vicinity of the vortex amounts to

$$\left.\begin{aligned} v_x &= -\pi VY/6 \\ v_y &= -\pi VX/6 \end{aligned}\right\} \tag{43}$$

where V is the strength of the equivalent vortex sheet (i.e. for the complete array of vortices v_x changes by an amount V between $y = \pm\infty$) and $X = x/d$, $Y = y/d$ where d is the spacing of the vortices, and terms of $O(X^2)$ and $O(Y^2)$ are neglected. Then each point vortex could exist in equilibrium with the imposed flow with an elliptic structure like that derived by Moore and Saffman[10], provided that the size of the vortex is small compared with the spacing d. In practice vortices are not small compared with their spacing, nor do they have such a simple structure, but these assumptions may still lead to useful results. Moore and Saffman do not give the velocity distribution explicitly for the perturbed vortices, so it is given here in the Appendix together with other results that are needed; there are a few minor changes of notation.

Since the strained vortex has an elliptic cross-section it becomes convenient to express the velocities in terms of elliptic coordinates as used by Moore and Saffman (see Appendix). We now consider the acoustic radiation according to Lighthill's method, which gives the radiation in terms of an area integral (for our two-dimensional problem) usually expressed in rectangular coordinates. The integral is then rewritten in elliptic coordinates which enables the form of the results to be seen more clearly.

3.1 Acoustic radiation

For a fixed source frequency α, Lighthill's method in two-dimensions leads to an expression[13] for the far-field density fluctuation $\rho'(\underline{x},\alpha)$ of the form

$$\rho'(\underline{x},\alpha) \sim (k_0x)^{-\frac{1}{2}} e^{ik_0x} \int S(\underline{y},\alpha) e^{-ik_0y_r} d\underline{y} \ , \qquad (44)$$

where $\underline{x}$ and $\underline{y}$ are rectangular coordinates, the integral over $\underline{y}$ extends over the source region,

S is the source strength,

k_0 is the wave number α/c,

and y_r is the coordinate in the direction of $\underline{x}$.

S will have the form $\rho v_1 v_2$ (where $\underline{v}$ is the velocity vector), and if we choose axes such that $\underline{x} = (x_1,0)$, then equation (44) leads to

$$\rho'(x_1,0,\alpha) \sim (k_0x)^{-\frac{1}{2}} \frac{\left(-k_0^2\right)}{c^2} \rho \iint v_1^2 e^{-ik_0y_1} dy_1 dy_2 \ . \qquad (45)$$

Where the wavelength is large compared with the size of the source region, $k_0y_1 \ll 1$ and $\exp(-ik_0y_1) \approx 1$. This should be an acceptable approximation for region 1, but not for region 2 both because of the phase changes and the departure of the external flow field from that assumed in section 3 above, as mentioned earlier. For region 1 the integral in equation (45) becomes, for the x_1 direction (i.e. in the direction of the major axis of the ellipse)

$$I_1 = \int_0^{\xi_0} \int_0^{2\pi} (v_\xi r_0 \sinh \xi \cos \eta - v_\eta r_0 \cosh \xi \sin \eta)^2 d\eta d\xi \quad . \tag{46}$$

It has already been assumed in the form of equation (45) that the integrand of equation (46) has a single Fourier component with frequency α , and this is true for the largest unsteady term, which derives from the product of the time-dependent part and the time-independent part in squaring the expression within brackets; v_ξ and v_η are given in equation (A-6). There results

$$|\rho'(x,0,\alpha)|_{I_1} = |\rho'(y,0,\alpha)|_{I_1} = \frac{\omega^4 r_0^4 A\rho}{c^4 (k_0 r)^{\frac{1}{2}}} f_1(\xi_0) \tag{47}$$

where r is the distance away of the observer, and $f_1(\xi_0)$ is a function of ξ_0 that can be evaluated, which, however, it is pointless to quote when we cannot complete the integration satisfactorily for the outer region. Here $\xi = \xi_0$ defines the equilibrium boundary of the vortex, as $r = a$ did for the circular vortex.

In region 2 the appropriate integration will be limited in practice by the spacing of the vortices, d . Equation (A-7) shows that it will also contain a term depending explicitly on the strain field e ; since, however, ξ_0 depends on e the final result for ρ' will have the form

$$\rho' = \frac{\omega^4 r_0^4 A\rho}{c^4 (k_0 r)^{\frac{1}{2}}} f_2\left(\frac{d}{r_0} , \frac{e}{\omega}\right) \tag{48}$$

which may be taken to incorporate equation (47).

Apart from the strain ratio (e/ω), equation (48) leads to a similar form for the acoustic output as before in equations (40) and (41). For the model of an infinite row of line vortices, the ratio e/ω is determined by the size and spacing. From equations (42) and (43),

$$e = \frac{\pi V}{6d} \quad . \tag{49}$$

Also Vd must equal the total vorticity in each line vortex, i.e. $\pi ab\omega$, where a and b are the major and minor axes of the ellipse; thus

$$\frac{e}{\omega} = \frac{\pi^2 ab}{6d^2} \quad . \tag{50}$$

As d reduces, e/ω increases towards the critical value of about 0.15 at which the vortex becomes unstable, and at the same time the frequency α reduces towards zero. The spacing corresponding to marginal stability is given by equation (50) with $e/\omega \approx 0.15$ and $a \approx 2.9b$, i.e. $d \approx 2a$. Since in this model the ellipses are aligned with their major axes in the direction of the row (see Appendix) the critical condition is reached when the ellipses are just about touching each other (see also Moore and Saffman[16]). In a practical shear layer starting from a splitter plate the vortices would be expected to grow (on average) with distance from the splitter plate and are oblique to the direction of the shear layer, see e.g. Brown and Roshko[8], and Damms and Küchemann[17]. In addition, the vortex structure may be more stable than that of vortices formed from constant vorticity as here. Nevertheless, we should expect the same general properties to hold.

4 GENERAL DISCUSSION

The foregoing analysis suggests that stable vortices will vibrate at their resonant frequencies if disturbed, and that in doing so they will radiate sound at the same frequency. There is now much evidence that large vortices do develop in the mixing region of a jet[1-9], and since there are many disturbances present in such a region it must be expected that sound will be radiated in accordance with their natural frequencies. In particular, at least at low Mach numbers, the fundamental mode of deformation will be dominant and the familiar quadrupole radiation will result.

In a real jet the structure of the vortices is complicated and, no doubt, subject to considerable scatter. For a round jet they may be nearly axisymmetric when they first form close to the lip of the jet, but if so they soon distort into unsymmetric three-dimensional shapes. It is hard to imagine such vortices having simple well-defined natural frequencies, but locally at least they may do so, and the picture of acoustic radiation from resonant modes may not be far from the truth on a local basis, which will merge into an overall sound field through time averaging. It may also be noted that Michalke and Fuchs[21] have shown that axisymmetric modes are the most powerful in noise radiation from a round jet.

The vortices themselves go through an evolutionary process as they are convected downstream, and this process will also emit sound. In a simple shear layer, they increase in size and spacing but decrease in number combining with each other as they move downstream[8], and a similar development must be expected in the mixing region of a round jet. There, however, they must reach a maximum strength at the end of the potential core, since beyond this point there is little scope for further increase in strength and they must gradually weaken through turbulent diffusion and decay, although their volume continues to increase.

The manner of growth of the vortices is a subject for argument. Damms and Küchemann[17] use an inviscid model with all the vorticity concentrated in a vortex sheet. The vortex sheet then rolls up into an array of vortex cores and each core continues to grow by winding in more sheet uniformly in time in a manner based on one of the Mangler-Weber solutions[18]. This process cannot continue indefinitely, and it is supposed that when the cores become overcrowded the weaker ones become elongated and through turbulent diffusion merge into a thick sheet which then carries on feeding the stronger vortices that continue to grow. This is what appears to happen according to films taken of the Brown and Roshko experiments, but Damms and Küchemann also discuss other methods of vortex amalgamation such as vortex pairing observed by Laufer[7]. Numerical calculations using point vortices also exhibit these effects[19].

Moore and Saffman[15] discuss the process more generally. They argue that the amalgamation cannot be inviscid, in the sense of two regions of vorticity merging by combining their volume with the same mean vorticity, since this would violate the similarity law in which the size and spacing grow uniformly with x, the downstream distance. The violation arises because the circulation around a vortex is proportional to ωr_0^2 (where ω is the mean vorticity and r_0 the mean radius), so if r_0 grows like x and ω remains constant the spacing must increase like x^2 to preserve a constant mean velocity jump across the mixing layer. Alternatively, if similarity is preserved the mean vorticity must decrease linearly with x.

In fact, this arises naturally in the model of Damms and Küchemann, since the average vorticity in the Mangler-Weber solution is inversely proportional to the distance from the centre of the core. Thus as the cores grow their mean vorticity reduces at the correct rate for overall similarity to be preserved. On the other hand, the loss of cores by elongation and subsequent enrollment into stronger cores offends against Moore and Saffman's stability result which suggests that such elongated cores should be unstable. As mentioned above, however, other types of amalgamation cited by Damms and Küchemann might be possible.

Moore and Saffman argue that the vortices grow by ingesting fluid some of which is irrotational. They consider an array of initially circular vortices of constant vorticity and show that they will disintegrate if $r > 0.3\ell$, where r is the radius and ℓ the spacing of the vortices; this result is confirmed by numerical analysis. Thus, Moore and Saffman argue that vortices in a shear layer do disintegrate when they get too close to a more powerful neighbour, which then ingests the resulting mixture of rotational and irrotational fluid.

This argument is not wholly convincing, however. In the first place, the films of Brown and Roshko's experiment and of Laufer's experiment show no sign of vortex disintegration taking place. In addition the stability limit found by Moore and Saffman applies to a vortex with a core of uniform vorticity, and the corresponding limit might be quite different either for a rolled-up vortex sheet or a smooth vortex with $\omega \sim 1/r$. A solution for the stability of such a vortex ($\omega \sim 1/r$) is not quite so easy to find as for one in which ω is constant since the velocity perturbations within the vortex are no longer irrotational. On general grounds, however, one might expect a vortex with initial vorticity $\sim 1/r$ to be rather more stable in a strain field than one with constant vorticity, since the greater concentration towards the centre will tend to keep the central regions circular. This is also true of the rolled-up sheet which remains circular near its centre.

One feature that is accepted here is that on average the mean vorticity of real vortices formed in a turbulent mixing layer is inversely proportional to the radius of the vortex. Equations (14) and (A-9) then imply that the natural frequencies of the vortex are also inversely proportional to its radius, although again this is not proved since the vortex structure is different from that of the analysis. If this result is accepted, the further inference follows that the frequency of the radiated sound will vary inversely as the vortex radius and hence inversely as the distance from the nozzle lip as long as the similarity behaviour remains approximately true. This is in general accord with observations. If it can be shown that vortices do disintegrate under the high strain fields of a typical jet, then this mechanism would deserve further study since it would clearly be accompanied by a radiation of acoustic energy.

In section 2 it was noted that, for a vortex of constant ω , the acoustic energy radiated per unit area is proportional to r_0^6 , where r_0 is the radius of the vortex. This result is considerably modified when applied to a particular vortex as it grows in the jet mixing region. In that case Γ is proportional to r_0 instead of r_0^2 and the maximum Mach number M remains constant, so that equation (41) would give the result that the energy radiated (per unit length of a two-dimensional vortex) is directly proportional to the radius; thus the energy radiated per unit area would actually fall linearly with r_0 as the vortex grows in size. This is a consequence of the vortex being diluted with irrotational flow as it increases in size. Equation (41) does not strictly apply to a vortex growing in this way, however, so these inferences should be treated with caution.

An important practical consideration as regards jet noise is the effect of forward speed. This has recently been discussed by Cocking and Bryce[20] in relation to their measurements on a jet in a wind tunnel. In their discussion they introduce the relative velocity, $v_{rel} = v_j - v_\infty$, where v_j is the jet velocity and v_∞ that of the free stream. One crude approach to jet noise in flight is to argue that v_{rel} is all that matters in the mixing region, so that, for example, the acoustic intensity radiated behaves like v_{rel}^8. Cocking and Bryce argue that the length of the potential core increases in flight thereby increasing the volume from which noise is being radiated, and on this basis they derive different laws for the noise radiation depending on the assumptions used for the size of eddies. For example, if it is assumed that eddy volume $\sim y^3$ where y is the radial width of the mixing region, they find

$$\text{noise} \propto \rho_0 V_{rel}^{6.5} V_j^{1.5} d^2/c^5 \qquad (51)$$

where d is the jet diameter. Their experimental results show an even slower dependence on V_{rel}, roughly like $V_{rel}^5 V_j^3$.

We may speculate on the process of vortex formation when V_∞ is non zero. In practice there will be a thick turbulent boundary layer on the outside of the nacelle. The velocity gradients in the viscous inner part of this layer will be very steep and it is probably reasonable to assume an effective slip velocity V_1 say (the velocity at the edge of the viscous sub-layer) where V_1 is about $0.6V_\infty$. The eddies may therefore be expected to form with vorticity that matches a velocity jump of $V_j - V_1$ (rather than $V_j - V_\infty$), but as they evolve they will ingest fluid from the outer turbulent boundary layer some of which will have weak negative vorticity relative to their positive primary vorticity. The distribution of this negative vorticity will be variable but since it seems hardly likely that it will *reduce* the noise output, one might perhaps ignore it as a first approximation. The velocity terms in equation (51) should then be modified to $(V_j - V_1)^{6.5} V_j^{1.5}$, which would perhaps give a slightly better agreement with experiment. The subject is however, very complicated and needs further study.

5 CONCLUSIONS

It has been shown how stable vortices radiate sound in accordance with their natural frequencies when disturbed. At low Mach number the fundamental mode is the most important, and the radiation from this mode is typical of quadrupole radiation. It is also noted that the larger eddies radiate sound much more efficiently per unit area than do smaller eddies for the same mean vorticity, although this result is greatly modified if one compares vortices at different stages of development in the same mixing region since their mean vorticity reduces with size.

A discussion of the process of eddy formation and growth (following Moore and Saffman) concludes that the vortices become relatively weaker as they grow and this results in a reduction of their natural frequencies, in accordance with observations of jet noise. A speculation as to the effect of forward speed suggests that V_{rel} should be replaced by $V_j - V_1$ where V_1 is the velocity at the edge of the viscous sub-layer in the turbulent boundary layer outside the nacelle.

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Appendix

THE STABILITY SOLUTION OF MOORE AND SAFFMAN[10] FOR ELLIPTIC VORTICES

Elliptic coordinates (ξ,η) are used such that

$$\left.\begin{aligned} x &= r_0 \cosh\xi \cos\eta \\ y &= r_0 \sinh\xi \sin\eta \quad . \end{aligned}\right\} \tag{A-1}$$

In steady equilibrium the vortex boundary is defined by

$$\xi = \xi_0 \tag{A-2}$$

and within this boundary there is uniform vorticity ω, with zero vorticity outside. Moore and Saffman show the vortex to be stable if the eccentricity of the ellipse is less than some critical value. In terms of ξ_0, the greater the value of ξ_0 the smaller the eccentricity and the greater the stability, with the critical value being given by $\xi_0 \approx 0.36$.

The perturbation of the boundary ξ_0 is defined by

$$\xi_v = \xi_0 + e^{\sigma t} f(\eta) \quad , \tag{A-3}$$

with

$$h^2 f(\eta) = (A \cos m\eta + B \sin m\eta) r_0^2 \tag{A-4}$$

where A and B are constants and h is the line element of the elliptic coordinates given by

$$h^2 = r_0^2(\sinh^2\xi + \sinh^2\eta) \quad . \tag{A-5}$$

Inside the vortex (region 1) the velocity components are then,

$$
\left.
\begin{aligned}
v_\xi &= \frac{\omega r_0^2}{h}\left[\frac{(\sinh^2\xi_0 \cosh^2\xi \sin 2\eta - \cosh^2\xi_0 \sinh^2\xi \sin 2\eta)}{2(\sinh^2\xi_0 + \cosh^2\xi_0)} + e^{\sigma t}(mE \sinh m\xi \cos m\eta + mF \cosh m\xi \sin m\eta)\right]_{\xi < \xi_0} \\
v_\eta &= \frac{\omega r_0^2}{h}\left[\frac{(\sinh^2\xi_0 \sinh 2\xi \cos^2\eta + \cosh^2\xi_0 \sinh 2\xi \sin^2\eta)}{2(\sinh^2\xi_0 + \cosh^2\xi_0)} + e^{\sigma t}(-mE \cosh m\xi \sin m\eta + mF \sinh m\xi \cos m\eta)\right]_{\xi < \xi_0} .
\end{aligned}
\right\} \quad \text{(A-6)}
$$

Outside the vortex (region 2) they are,

$$
\left.
\begin{aligned}
v_\xi &= -\frac{\omega r_0^2}{h}\left[\frac{e^{2\xi}}{4}\left\{1 - e^{-4(\xi-\xi_0)}\right\}\frac{e}{\omega}\sin 2\eta + me^{\sigma t - m\xi}(C \cos m\eta + D \sin m\eta)\right]_{\xi > \xi_0} \\
v_\eta &= \frac{\Gamma}{2\pi h} - \frac{\omega r_0^2}{h}\left[\frac{e^{2\xi}}{4}\left\{1 + e^{-4(\xi-\xi_0)}\right\}\frac{e}{\omega}\cos 2\eta + me^{\sigma t - m\xi}(C \sin m\eta - D \cos m\eta)\right]_{\xi > \xi_0} .
\end{aligned}
\right\} \quad \text{......(A-7)}
$$

The boundary conditions provide the relations for the constants A to F .

$$\left.\begin{aligned}
A\{m \tanh 2\xi_0 - 2 \cosh m\xi_0(\cosh m\xi_0 - \sinh m\xi_0)\} &= 2\frac{\sigma}{\omega}B \\
E &= Be^{-m\xi_0}/m \\
F &= \omega bE/\sigma \\
C &= -\tfrac{1}{2}E(e^{2m\xi_0} - 1) \\
D &= -\omega bE(e^{2m\xi_0} - 1)/2\sigma \\
\text{where} \quad b(\xi_0) &= \frac{m^2 e^{-\xi_0(m+2)}\left(1 + e^{4\xi_0} - 2e^{2m\xi_0}\right)}{4 \cosh 2\xi_0}
\end{aligned}\right\} \qquad \text{(A-8)}$$

and for the frequency α ,

$$\alpha^2 = -\sigma^2 = \frac{\omega^2}{4}\left\{(m \tanh 2\xi_0 - 1)^2 - (\cosh 2m\xi_0 - \sinh 2m\xi_0)^2\right\} . \qquad \text{(A-9)}$$

We have already noted that as ξ_0 increases the ellipse becomes more circular, and in the limit $\xi_0 \to \infty$ it is easy to see that equation (A-9) agrees with equation (18). The coefficients A to F have been made non-dimensional unlike the corresponding coefficients in Moore and Saffman[10].

SYMBOLS

A-F	non-dimensional amplitude coefficients
E_r	mean radial flux of acoustic energy/unit length of vortex
M	Mach number - specifically the greatest Mach number in the vortex flow
S	source strength (equation (44))
$\underline{U}$	steady state velocity
a	vortex radius; major axis of ellipse
a_2, a_4	coefficients defined by equation (26)
b	minor axis of ellipse
c	speed of sound
d	vortex spacing; jet diameter
e	strain parameter (equation (42))
$f(\theta)$	mode shape
k	wave number
m	mode number
p	pressure
r	radial coordinate; r_0 vortex radius; r_1 vortex spacing
t	time
$\underline{u}$	time-dependent velocity perturbation
$\underline{v}$	time-dependent velocity
Γ	circulation
α	frequency given by equation (18)
η	elliptic coordinate
θ	angular coordinate
λ	wavelength
ξ	elliptic coordinate
ρ	density
σ	rate of growth (equation (4))
ϕ	velocity potential of disturbance
ω	vorticity

REFERENCES

No.	Author	Title, etc.
1	S.C. Crow F.H. Champagne	Orderly structure in jet turbulence. J. Fluid Mech. 48, 3, 547-591 (1971)
2	J.C. Lau M.J. Fisher H.V. Fuchs	The intrinsic structure of turbulent jets. J. Sound and Vibration, 22, 4, 379-406 (1972)
3	H.V. Fuchs	Space correlations of the fluctuating pressure in subsonic turbulent jets. J. Sound and Vibration, 23, 1, 77-99 (1972)
4	P.A. Lush	The pressure and velocity fields of convected vortices. (Letter) J. Sound and Vibration, 27, 2, 266-270 (1973)
5	H.V. Fuchs	Comments on 'the pressure and velocity fields of convected vortices'. (Letter) J. Sound and Vibration, 30, 2, 249-251 (1973)
6	I.J. Wygnanski F.H. Champagne	On transition in a pipe. Part 1 The origin of puffs and slugs in the flow of a turbulent slug. J. Fluid Mech., 59, 2, 281-335 (1973)
7	J. Laufer R.E. Kaplan W.T. Chu	On the generation of jet noise. Paper No.21 AGARD CP-131 (1974)
8	G.L. Brown A. Roshko	On density effects and large structure in turbulent mixing layers. J. Fluid Mech., 64, 4, 775-816 (1974)
9	A. Michalke	The instability of free shear layers. Prog. Aeron. Sci., 12, 213-243 (1972)
10	D.W. Moore P.G. Saffman	Structure of a line vortex in an imposed strain. Aircraft wake turbulence. (Editors J.H. Olsen, A. Goldberg and M. Rogers) 339-354, Plenum Press (1971)
11	H. Lamb	Hydrodynamics. Sixth edition p.231, Cambridge University Press (1932)

REFERENCES (concluded)

No.	Author	Title, etc.
12	L.D. Landau E.M. Lifshitz	Fluid Mechanics (chapter 8) Pergamon (1959)
13	D.G. Crighton	Basic principles of aerodynamic noise generation. Prog. in Aero. Sciences, 16, 1, 31-96 (1975)
14	M.S. Howe	Contributions to the theory of aerodynamic sound, with application to excess jet noise and the theory of the flute. J. Fluid Mech., 71, 4, 625-673 (1975)
15	M.J. Lighthill	Sound generated aerodynamically Proc. Roy. Soc., A267, 147-182 (1962)
16	D.W. Moore P.G. Saffman	The density of organised vortices in a turbulent mixing layer. J. Fluid Mech., 69, 3, 465-473 (1975)
17	Susan M. Damms D. Küchemann	On a vortex sheet model for the mixing between two parallel streams. 1 Description of the model and experimental evidence. Proc. Roy. Soc., A339, 451-461 (1974) RAE Technical Report 72139 (ARC 34267) (1972)
18	K.W. Mangler J. Weber	The flow field near the centre of a rolled-up vortex sheet. J. Fluid Mech., 30, 1, 177-196 (1967) RAE Technical Report 66324 (ARC 28827) (1966)
19	P.O.A.L. Davies J.C. Hardin J.P. Mason	A calculation of jet noise from a potential flow model. AIAA paper 75-441 (1975)
20	B.J. Cocking W.D. Bryce	Subsonic jet noise in flight based on some recent wind-tunnel results. AIAA paper 75-462 (1975)
21	A. Michalke H.V. Fuchs	Description of turbulence and noise of an axisymmetric shear flow. DLR-FB, 74-50 (1974)

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