REPORT 1069

ON A SOLUTION OF THE NONLINEAR DIFFERENTIAL EQUATION FOR TRANSONIC FLOW PAST A WAVE-SHAPED WALL [1]

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SUMMARY

The Prandtl-Busemann small-perturbation method is utilized to obtain the flow of a compressible fluid past an infinitely long wave-shaped wall. When the essential assumption for transonic flow (that all Mach numbers in the region of flow are nearly unity) is introduced, the expression for the velocity potential takes the form of a power series in the transonic similarity parameter. On the basis of this form of the solution, an attempt is made to solve the nonlinear differential equation for transonic flow past the wavy wall. The analysis utilized exhibits clearly the difficulties inherent in nonlinear-flow problems. The investigation has nevertheless been rigorously carried to the point where the question of the existence or nonexistence of a mixed potential flow free of discontinuities can be settled by the behavior of a single power series in the transonic similarity parameter. The calculation of the coefficients of this dominant power series has been reduced to a routine computing problem by means of recursion formulas resulting from the solution of the differential equation and the boundary condition at the wall. One of the interesting results of the analysis is the rigorous statement that the transonic similarity parameter must be less than four-thirds.

INTRODUCTION

The present report considers in detail a notion first expressed in reference 1, namely, that the Von Kármán transonic similarity rule is implicitly contained in the potential-flow solution for high-subsonic flow past a prescribed body. The calculations are very much simplified by the choice of rectangular Cartesian coordinates as independent variables. For this reason the solid boundary chosen is a two-dimensional wavy wall of small amplitude extending to infinity in both the downstream and upstream directions. The problem is first treated by means of the Prandtl-Busemann iteration method for high-subsonic undisturbed speeds. On the basis of this solution an attempt is made to solve the nonlinear differential equation for transonic flow (corresponding to the linear Tricomi equation in the hodograph plane) past a wavy wall. The analysis utilized exhibits clearly the difficulties inherent in nonlinear-flow problems. The investigation has nevertheless been rigorously carried to the point where the question of the existence or nonexistence of a mixed potential flow free of discontinuities can be settled by the behavior of a single power series in the transonic similarity parameter. The calculation of the coefficients of this dominant power series has been reduced to a routine computing problem by means of recursion formulas resulting from the solution of the differential equation and the boundary condition at the wavy wall.

The author wishes to acknowledge the invaluable aid and advice of Dr. A. Busemann of the Langley Laboratory during the writing of this report, especially with regard to the final section "General Analysis."

CALCULATION OF HIGH-SUBSONIC FLOW PAST A WAVY WALL BY MEANS OF PRANDTL-BUSEMANN ITERATION METHOD

The fundamental nonlinear differential equation for the potential flow of a compressible fluid can be written as

$$\left(\frac{c^2}{c_\infty{}^2}-M_\infty{}^2u^2\right)\phi_{XX}+\left(\frac{c^2}{c_\infty{}^2}-M_\infty{}^2v^2\right)\phi_{YY}-2M_\infty{}^2\phi_X\phi_Y\phi_{XY}=0 \tag{1}$$

where

$$\frac{c^2}{c_\infty{}^2}=1+\frac{\gamma-1}{2}M_\infty{}^2[1-(u^2+v^2)] \tag{2}$$

and

ϕ	velocity potential of flow
X, Y	rectangular Cartesian coordinates in plane of flow
u, v	fluid velocity components along X- and Y-axis, respectively
U	undisturbed stream velocity
c	local speed of sound
c_∞	speed of sound in undisturbed fluid
M_∞	Mach number of undisturbed stream (U/c_∞)
γ	ratio of specific heats at constant pressure and constant volume

The quantities ϕ, X, Y, u, and v are nondimensional with a characteristic length l as unit of length and the undisturbed stream velocity U as unit of velocity. The subscripts X and Y in equation (1) denote partial differentiation with respect to the designated variables.

In order to obtain the Prandtl-Busemann iteration equations based on small perturbations of the undisturbed stream, the assumption is made that the velocity potential ϕ can be expanded in the form

$$\phi=X+\phi_1+\phi_2+\phi_3+\ .\ .\ . \tag{3}$$

[1] Supersedes NACA TN 2383, "On a Solution of the Nonlinear Differential Equation for Transonic Flow Past a Wave-Shaped Wall" by Carl Kaplan, 1951.

For the purpose of defining and controlling the iteration procedure, the function ϕ_{n+1} and its derivatives are regarded as small compared with the preceding approximation ϕ_n and its derivatives. From equation (3) and the fact that for irrotational or potential flow $u=\phi_X$ and $v=\phi_Y$,

$$u=1+\phi_{1X}+\phi_{2X}+\phi_{3X}+\ .\ .\ .$$

and

$$v=\phi_{1Y}+\phi_{2Y}+\phi_{3Y}+\ .\ .\ .$$

When these expressions for u and v are introduced into equation (1), together with the corresponding expression for c^2/c_∞^2 given by equation (2), and the powers and products of ϕ_n and their derivatives are grouped according to the assumptions of the small-perturbation method, the following iteration equations for the first three approximations result:

$$\phi_{1xx}+\phi_{1yy}=0 \tag{4}$$

$$\phi_{2xx}+\phi_{2yy}=2M_\infty^2[(1+\sigma)\phi_{1x}\phi_{1xx}+\phi_{1y}\phi_{1xy}] \tag{5}$$

$$\begin{aligned}\phi_{3xx}+\phi_{3yy}=2M_\infty^2\Big\{&(1+\sigma)(\phi_{1xx}\phi_{2x}+\phi_{1x}\phi_{2xx})+\\ &[2\beta^2(1+\sigma)-1]\phi_{1x}\phi_{1y}\phi_{1xy}+\\ &(1+\sigma)\left[2\beta^2(1+\sigma)-\frac{3}{2}\right]\phi_{1xx}\phi_{1x}^2+\frac{1}{2}(\sigma\beta^2-1)\phi_{1xx}\phi_{1y}^2+\\ &\phi_{1xy}\phi_{2y}+\phi_{1y}\phi_{2xy}\Big\}\end{aligned} \tag{6}$$

where x and y are new independent variables defined by the transformation

$$\left.\begin{aligned}x&=X\\ y&=\beta Y\end{aligned}\right\} \tag{7}$$

and

$$\beta^2=1-M_\infty^2$$

$$\sigma=\frac{\gamma+1}{2}\frac{M_\infty^2}{\beta^2}$$

Equation (4) is a Laplace equation and equations (5) and (6) are Poisson equations where the right-hand sides contain only previously determined quantities. These equations have been treated recently in reference 2 where the particular integrals of equations (5) and (6) are given in real form. These particular integrals have been utilized in obtaining the flow over the wavy wall. Thus, the equation of the infinitely long wave-shaped wall (fig. 1) is assumed to be

$$Y=a\cos\alpha X \tag{8}$$

where

a amplitude of wave

$\alpha=\dfrac{2\pi}{\lambda}$

λ wave length

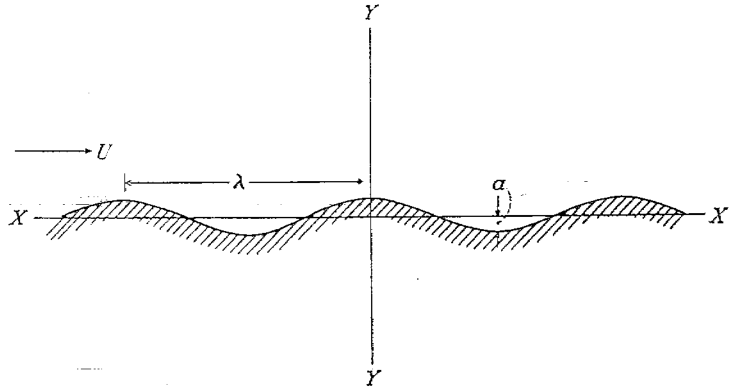

FIGURE 1.—Wave-shaped wall.

The reference element of length is conveniently chosen to be $\frac{1}{\alpha}\left(=\frac{\lambda}{2\pi}\right)$. The equation of the wavy wall in nondimensional form and in terms of the variables x and y then becomes

$$y=\beta\epsilon\cos x \tag{9}$$

where, if the thickness coefficient of the wavy wall is defined as $t=\frac{a}{\lambda/2\pi}$, then $\epsilon=t$.

The expression for the velocity potential ϕ, obtained with the aid of the particular integrals given in reference 2, including terms of the order ϵ^3 and satisfying the boundary conditions at infinity and at the solid wall to the same order is as follows:

$$\begin{aligned}\phi=x+\frac{\epsilon}{\beta}e^{-y}\sin x+\frac{1}{8}\left(\frac{\epsilon}{\beta}\right)^2\{&2\sigma M_\infty^2y+[2+\sigma+\\ &(2-\sigma)\beta^2]\}e^{-2y}\sin 2x-\frac{1}{64}\left(\frac{\epsilon}{\beta}\right)^3(\{-8+M_\infty^2[24+2\sigma-\\ &11\sigma^2-\sigma(22+13\sigma)\beta^2]\}e^{-y}+M_\infty^2[\sigma(6+5\sigma)+\\ &(8+10\sigma+3\sigma^2)\beta^2]e^{-3y}+4\sigma(2+\sigma)M_\infty^4ye^{-3y})\sin x+\\ &\frac{1}{72}\left(\frac{\epsilon}{\beta}\right)^3\{[27+M_\infty^2(-25+11\sigma+4\sigma^2-9\beta^2)]e^{-3y}+\\ &6M_\infty^2\sigma(3+2\sigma+3\beta^2)ye^{-3y}+9\sigma^2M_\infty^4y^2e^{-3y}\}\sin 3x+\ .\ .\ .\end{aligned} \tag{10}$$

It is of interest to examine equation (10) when the assumptions of transonic-flow theory are introduced. These assumptions are essentially that the undisturbed flow velocity differs only slightly from the speed of sound, that the velocity component normal to the oncoming flow is small compared with the speed of sound, and that the velocity component in the direction of the oncoming flow is of the order of the critical velocity c^*. If the undisturbed stream is in the direction of the positive x-axis, then the velocity potential ϕ, referred to the critical velocity c^*, can be written as (see reference 1)

$$\phi=x+\frac{1}{\gamma+1}(1-M_\infty^2)f(x,y) \tag{11}$$

The second term on the right-hand side of this equation is the disturbance-velocity potential and implies that terms involving powers of $1-M_\infty^2$ higher than the first are to be neglected. The differential equation satisfied by the function $f(x,y)$ is obtained from equation (1) and takes the following simplified nonlinear form:

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} \tag{12}$$

The boundary conditions to be fulfilled by $f(x,y)$ are as follows:

$$\left.\begin{aligned} &\left.\begin{aligned}\frac{\partial f}{\partial x}&=-1\\ \frac{\partial f}{\partial y}&=0\end{aligned}\right\}\quad (\text{at } y=\infty)\\ &\frac{\partial f}{\partial y}=-k\sin x\quad (\text{at } y=0,\ -\infty<x<\infty)\end{aligned}\right\} \tag{13}$$

where k, the transonic similarity parameter, is

$$k=\frac{(\gamma+1)\epsilon}{(1-M_\infty^2)^{3/2}} \tag{14}$$

Thus, it is seen that f is a function of x, y, and the transonic similarity parameter k only. In the limiting case, $M_\infty\rightarrow 1$ and $\epsilon\rightarrow 0$, k retains its meaning as a transonic similarity parameter.

Note that in the Prandtl-Busemann iteration procedure the order to which the boundary condition is applied at the surface of the profile is the same as the order to which the iteration has been taken. In the transonic case, however, the surface boundary condition is always applied at the axis $y=0$. In particular, this statement of the boundary condition for transonic flow is a rigorous one in the limit as both the thickness coefficient and $1-M_\infty^2$ simultaneously approach zero but with the same distribution of slope that belongs to the family of profiles being treated. A rather complete discussion of this point is contained in reference 3.

Before application of the foregoing considerations to equation (10), it should be noted that ϕ in that equation is referred to the undisturbed stream velocity U. In order to introduce c^* as the reference velocity, both ϕ and the right-hand side of equation (10) must be multiplied by U/c^*. Now, from equation (2), with $c=c^*$ and $u^2+v^2=\frac{c^{*2}}{U^2}$, it follows that

$$M^{*2}=\frac{\frac{\gamma+1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2} \tag{15}$$

where

$$M^*=\frac{U}{c^*}$$

If terms of only the first power of $1-M_\infty^2$ are retained, it follows from equation (15) that

$$M^*=1-\frac{1}{\gamma+1}(1-M_\infty^2)+\ldots \tag{16}$$

Then, multiplying the right-hand side of equation (10) by this expression for M^*, replacing ϵ/β by $\frac{1}{\gamma+1}(1-M_\infty^2)k$ from equation (14), and neglecting all terms containing $1-M_\infty^2$ to a power higher than the first yields the following expression for ϕ:

$$\phi=x+\frac{1}{\gamma+1}(1-M_\infty^2)\left\{-x+\left(k+\frac{11}{256}k^3\right)e^{-y}\sin x+\frac{1}{16}(1+2y)k^2e^{-2y}\sin 2x+\left[-\frac{1}{256}(5+4y)\sin x+\frac{1}{288}(4+12y+9y^2)\sin 3x\right]k^3e^{-3y}\right\}+\cdots \tag{17}$$

Comparison of this equation with equation (11) shows that the result obtained by means of the Prandtl-Busemann iteration method contains implicitly the form of the solution required by transonic-flow theory. Moreover, the expression for the function $f(x,y)$, namely,

$$f(x,y)=-x+\left(k+\frac{11}{256}k^3\right)e^{-y}\sin x+\frac{1}{16}(1+2y)k^2e^{-2y}\sin 2x+\left[-\frac{1}{256}(5+4y)\sin x+\frac{1}{288}(4+12y+9y^2)\sin 3x\right]k^3e^{-3y}+\ldots \tag{18}$$

satisfies the nonlinear differential equation (12) and the boundary conditions, equation (13), to the order k^3. This fact suggests a solution for the flow over the wavy wall, in the neighborhood of Mach number unity, obtained directly from equation (12).

SOLUTION OF EQUATION FOR TRANSONIC FLOW PAST A WAVY WALL

Equation (18) suggests the following form for a solution of the nonlinear differential equation (12) for transonic flow subject to the boundary conditions stated in equation (13):

$$f(x,y)=-x+\sum_{n=1}^{\infty}f_n\sin nx \tag{19}$$

where the f_n's are functions of y only. This form for $f(x,y)$ is substituted into equation (12) and repeated use made of the following identity:

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}B_{n,m}\equiv\sum_{n=0}^{\infty}\sum_{m=0}^{n}B_{n,m}$$

When the coefficients of the separate harmonic terms sin nx are placed equal to zero, the following system of nonlinear ordinary differential equations for f_n results:

$$f_n''-n^2f_n=-\frac{1}{4}n\sum_{m=0}^{n}m(n-m)f_mf_{n-m}-\frac{1}{2}n\sum_{m=0}^{\infty}(m+1)(m+n+1)f_{m+1}f_{m+n+1}\qquad (n=1,2,\ldots\infty)\tag{20}$$

Note that equation (20), with the right-hand side placed equal to zero, has a solution of the form e^{-ny}. An iteration procedure is then set up in such a manner that the highest power of k and of e^{-y} appearing in the functions f_n is equal to the number designating the iteration step. Thus, the following sets of iteration equations result:

Step 1,

$$f_1''-f_1=0\tag{21}$$

Step 2,

$$f_2''-4f_2=-\frac{1}{2}f_1^2\tag{22}$$

Step 3,

$$\left.\begin{aligned}f_1''-f_1&=-f_1f_2\\ f_3''-9f_3&=-3f_1f_2\end{aligned}\right\}\tag{23}$$

Step 4,

$$\left.\begin{aligned}f_2''-4f_2&=-\frac{1}{2}f_1^2-3f_1f_3\\ f_4''-16f_4&=-6f_1f_3-4f_2^2\end{aligned}\right\}\tag{24}$$

Step 5,

$$\left.\begin{aligned}f_1''-f_1&=-f_1f_2-3f_2f_3\\ f_3''-9f_3&=-3f_1f_2-6f_1f_4\\ f_5''-25f_5&=-10f_1f_4-15f_2f_3\end{aligned}\right\}\tag{25}$$

Step 6,

$$\left.\begin{aligned}f_2''-4f_2&=-\frac{1}{2}f_1^2-3f_1f_3-8f_2f_4\\ f_4''-16f_4&=-6f_1f_3-4f_2^2-10f_1f_5\\ f_6''-36f_6&=-15f_1f_5-\frac{27}{2}f_3^2-24f_2f_4\end{aligned}\right\}\tag{26}$$

and so forth.

The right-hand sides of these equations are known functions constructed from previously determined quantities in accord with the iteration procedure adopted.

These equations are of the second order, nonhomogeneous type with constant coefficients and are readily integrated. The resulting expressions for f_1, f_2, f_3, f_4, f_5, and f_6, with the boundary conditions (13) taken into account, are as follows:

$$f_1=\left(1+\frac{11}{256}k^2+\frac{1861}{3\times256^2}k^4\right)ke^{-y}-\left[\frac{5}{256}+\frac{2765}{9\times256^2}k^2+\left(\frac{1}{64}+\frac{33}{64\times256}k^2\right)y\right]k^3e^{-3y}-\left(-\frac{65}{576\times256}+\frac{7}{72\times256}y+\frac{9}{32\times256}y^2+\frac{1}{8\times256}y^3\right)k^5e^{-5y}+\ldots\tag{27}$$

$$f_2=\left[\frac{1}{16}+\frac{419}{72\times256}k^2+\frac{234215}{432\times256^2}k^4+\left(\frac{1}{8}+\frac{11}{1024}k^2+\frac{4085}{24\times256^2}k^4\right)y\right]k^2e^{-2y}-\left[\frac{125}{36\times256}+\frac{45895}{108\times256^2}k^2+\left(\frac{5}{256}+\frac{815}{288\times512}k^2\right)y+\left(\frac{1}{128}+\frac{11}{8192}k^2\right)y^2\right]k^4e^{-4y}-\left(-\frac{12245}{144\times256^2}-\frac{6245}{72\times256^2}y+\frac{79}{3\times256^2}y^2+\frac{47}{192\times256}y^3+\frac{1}{12\times256}y^4\right)k^6e^{-6y}+\ldots\tag{28}$$

$$f_3=\left[\frac{1}{72}+\frac{23603}{27\times256^2}k^2+\left(\frac{1}{24}+\frac{259}{72\times256}k^2\right)y+\left(\frac{1}{32}+\frac{33}{8192}k^2\right)y^2\right]k^3e^{-3y}-\left(\frac{1765}{3\times256^2}+\frac{155}{32\times256}y+\frac{117}{32\times256}y^2+\frac{1}{256}y^3\right)k^5e^{-5y}+\ldots\tag{29}$$

$$f_4=\left[\frac{7}{1536}+\frac{390547}{720\times256^2}k^2+\left(\frac{7}{384}+\frac{32947}{36\times256^2}k^2\right)y+\left(\frac{3}{128}+\frac{617}{288\times256}k^2\right)y^2+\left(\frac{1}{96}+\frac{11}{24\times256}k^2\right)y^3\right]k^4e^{-4y}-\left(\frac{35251}{90\times256^2}+\frac{2465}{576\times256}y+\frac{455}{96\times256}y^2+\frac{119}{48\times256}y^3+\frac{25}{48\times256}y^4\right)k^6e^{-6y}+\ldots\tag{30}$$

$$f_5=\left(\frac{7}{3840}+\frac{7}{768}y+\frac{25}{1536}y^2+\frac{5}{384}y^3+\frac{25}{24\times256}y^4\right)k^5e^{-5y}+\ldots\tag{31}$$

$$f_6=\left(\frac{91}{54\times2048}+\frac{91}{72\times256}y+\frac{23}{2048}y^2+\frac{13}{1024}y^3+\frac{15}{2048}y^4+\frac{9}{5120}y^5\right)k^6e^{-6y}+\ldots\tag{32}$$

Note that the functions f_1, f_2, and f_3 include the terms of equation (18), obtained from equation (10) by allowing the Mach number to approach unity.

Equations (27) to (32) may be considered to be essentially the nonlinear solution for the flow past a wavy wall of small amplitude for stream Mach numbers in the neighborhood of unity in the form of a power series in the transonic similarity parameter k. Moreover, this solution is identical with the one obtained by means of the Prandtl-Busemann iteration equations when Mach number unity is approached.

CALCULATION OF LOCAL MACH NUMBER, CRITICAL SIMILARITY PARAMETER, AND PRESSURE COEFFICIENT

GENERAL FORMULAS

From equation (11), when all terms containing $1-M_\infty^2$ to a power higher than the first are neglected, the expression for the fluid velocity referred to the critical speed of sound c^* is

$$q^2=1+\frac{2}{\gamma+1}(1-M_\infty^2)\frac{\partial f}{\partial x} \tag{33}$$

Now, from equation (2), the relation between q and the local Mach number M is

$$q^2=\frac{\frac{\gamma+1}{2}M^2}{1+\frac{\gamma-1}{2}M^2} \tag{34}$$

In the transonic approximation, the difference of any Mach number in the field of flow from unity is considered a small quantity. If terms of only the first power of $1-M^2$ are retained, equation (34) yields

$$q^2=1+\frac{2}{\gamma+1}(M^2-1)+\ldots \tag{35}$$

Hence, from equation (33),

$$1-M^2=-(1-M_\infty^2)\frac{\partial f}{\partial x} \tag{36}$$

This equation provides a means of calculating the critical value of the transonic similarity parameter; that is, the value of the parameter k for which $M=1$ at the point of maximum velocity on the boundary.

CALCULATION OF THE CRITICAL VALUE OF k

For the family of wavy walls of small amplitude (including the limiting case of vanishingly small amplitude) with $M=1$ at $x=0$ and $y=0$, equation (36) yields the following relation for the determination of the critical value of k:

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=0\\y=0}}=0 \tag{37}$$

By means of equation (19) together with the expression for f_1 to f_6 given by equations (27) to (32), equation (37) yields the following power series, exact to seven terms, for the determination of the critical value of k for the family of wavy profiles:

$$k+\frac{1}{8}k^2+\frac{25}{384}k^3+\frac{337}{9216}k^4+\frac{4043}{256\times 576}k^5+\frac{359381}{270\times 256^2}k^6+\ldots=1 \tag{38}$$

The procedure adopted in order to estimate the critical value of k is as follows: From equation (38) the value of k can be found for 2, 3, 4, 5, 6, and 7 terms. These values of k are, respectively, 1, 0.8990, 0.8644, 0.8504, 0.8424, and 0.8377. The last two values indicate the approach to the asymptotic value of k, that is, the value of k when the number of terms in equation (38) is infinite. If the values of k approach smoothly to the asymptote, the estimated critical value $k=0.8377$ is very nearly correct.

Suppose now that both sides of equation (36) are divided by $[(\gamma+1)\epsilon]^{2/3}$. Then, since

$$k=\frac{(\gamma+1)\epsilon}{(1-M_\infty^2)^{3/2}}$$

equation (36) can be written as

$$\frac{1-M^2}{[(\gamma+1)\epsilon]^{2/3}}=-k^{-2/3}\frac{\partial f}{\partial x} \tag{39}$$

The right-hand side of equation (39) is a function of x, y, and the parameter k only and is characteristic of the entire family of boundary profiles. For the family of wavy walls with the critical value assigned to k, equation (39) evaluated at the wall becomes

$$\frac{1-M^2}{[(\gamma+1)\epsilon]^{2/3}}=1.1253(1-0.8536\cos x-0.0989\cos 2x-0.0299\cos 3x-0.0122\cos 4x-0.0038\cos 5x-0.0017\cos 6x) \tag{40}$$

Table I lists the values of $\frac{1-M^2}{[(\gamma+1)\epsilon]^{2/3}}$ for values of x between $-\pi$ and π and figure 2 shows the corresponding curve.

CALCULATION OF THE PRESSURE COEFFICIENT

Bernoulli's theorem for a compressible fluid assumes the following differential form along a streamline:

$$d(p-p_\infty)+\frac{1}{2}\rho U^2 d\left(\frac{q^2}{U^2}-1\right)=0 \tag{41}$$

where

p pressure in fluid
ρ density of fluid
q speed of fluid

and the subscript ∞ denotes the quantity in the undisturbed fluid. Now, from equation (2) the relation between the nondimensional speed q/U and the local Mach number M is given by

$$\frac{q^2}{U^2}=\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M^2}\frac{M^2}{M_\infty^2} \tag{42}$$

In accordance with the assumption of transonic-flow theory that all Mach numbers in the flow differ only slightly from unity, equation (42) becomes

$$\frac{q^2}{U^2}-1=\frac{2}{\gamma+1}[1-M_\infty^2-(1-M^2)]+\ldots \tag{43}$$

where powers of $1-M_\infty^2$ and $1-M^2$ higher than the first have been neglected. Then with the pressure coefficient defined as

$$C_{p,M_\infty}=\frac{p-p_\infty}{\frac{1}{2}\rho_\infty U^2}$$

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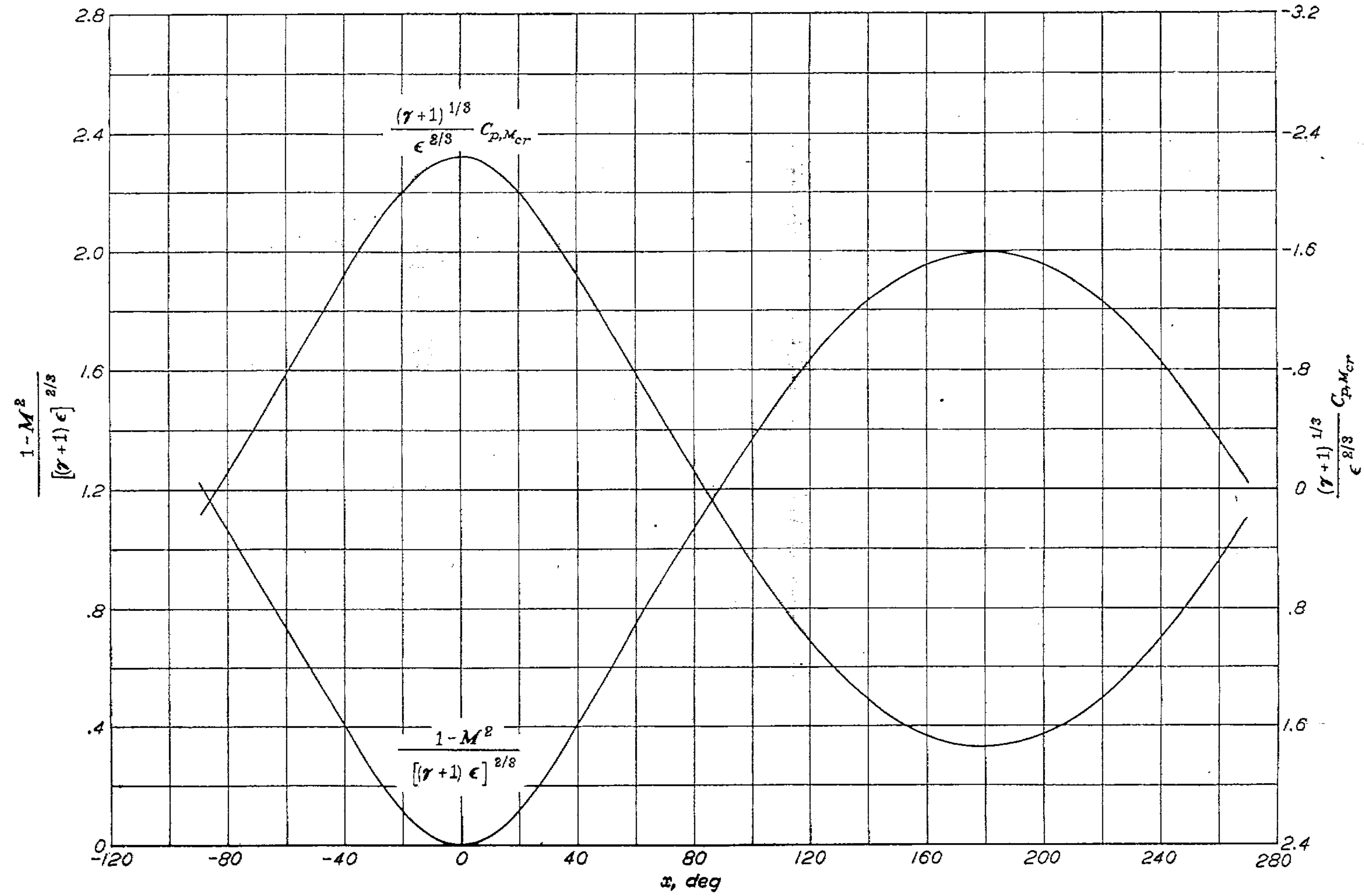

FIGURE 2.—Distribution of $\frac{1-M^2}{[(\gamma+1)\epsilon]^{2/3}}$ and $\frac{(\gamma+1)^{1/3}}{\epsilon^{2/3}} C_{p,M_{cr}}$ at the surface for the family of wave-shaped walls.

equation (41) takes the following form for transonic flow:

$$dC_{p,M_\infty}=\frac{2}{\gamma+1}\, d\,[1-M^2-(1-M_\infty^2)] \tag{44}$$

After integration,

$$C_{p,M_\infty}=\frac{2}{\gamma+1}\,[1-M^2-(1-M_\infty^2)]$$

or, with the aid of equation (36),

$$C_{p,M_\infty}=-\frac{2}{\gamma+1}\,(1-M_\infty^2)\left(1+\frac{\partial f}{\partial x}\right) \tag{45}$$

The right-hand side of this equation can be considered to be the first term in a power-series development of C_{p,M_∞} in $1-M_\infty^2$. In particular, when the local Mach number first attains unity, then $\frac{\partial f}{\partial x}=0$ at $x=0$, $y=0$ and equation (45) becomes

$$C_{p,M_{cr}}=-\frac{2}{\gamma+1}\,(1-M_{cr}^2) \tag{46}$$

a result valid in the transonic range only. Again, if the thickness coefficient approaches zero as $M_\infty\to 1$ and $M=1$ at $x=0$, $y=0$, then equation (46) shows that the slope $\frac{dC_{p,M_\infty}}{d(1-M_\infty^2)}$ at the critical value $M_\infty=M_{cr}\to 1$ is a constant $-\frac{2}{\gamma+1}$, independent of the particular family of profiles treated. This result is valid whether the approach to Mach number unity is made from the subsonic or supersonic region.

If both sides of equation (45) are multiplied by $\frac{(\gamma+1)^{1/3}}{\epsilon^{2/3}}$, then

$$\frac{(\gamma+1)^{1/3}}{\epsilon^{2/3}}\, C_{p,M_\infty}=-2k^{-2/3}\left(1+\frac{\partial f}{\partial x}\right) \tag{47}$$

where the right-hand side depends only on x, y, and the parameter k and is characteristic of the family of boundary profiles treated. For the family of wavy walls with the critical value of k chosen, equation (47) takes the following form at the solid surface ($y=0$):

$$\frac{(\gamma+1)^{1/3}}{\epsilon^{2/3}}\, C_{p,M_{cr}}=-2.2507(0.8536\cos x+0.0989\cos 2x+0.0299\cos 3x+0.0122\cos 4x+0.0038\cos 5x+0.0017\cos 6x) \tag{48}$$

Table I lists the values of $\frac{(\gamma+1)^{1/3}}{\epsilon^{2/3}} C_{p,M_{cr}}$ for values of x between $-\pi$ and π and figure 2 shows the corresponding curve.

TABLE I.—VALUES OF $\frac{1-M^2}{[(\gamma+1)\epsilon]^{2/3}}$ AND $\frac{(\gamma+1)^{1/3}}{\epsilon^{2/3}}C_{p,M_{cr}}$ AT THE SURFACE FOR THE FAMILY OF WAVE-SHAPED WALLS

x (deg)	$\frac{1-M^2}{[(\gamma+1)\epsilon]^{2/3}}$	$\frac{(\gamma+1)^{1/3}}{\epsilon^{2/3}}C_{p,M_{cr}}$
−180	1.996929	1.743190
−175	1.991505	1.738342
−170	1.987188	1.723708
−160	1.957430	1.664191
−150	1.906695	1.562722
−140	1.834890	1.419111
−130	1.743433	1.236197
−120	1.634666	1.018664
−110	1.510815	.770962
−100	1.373602	.496536
−95	1.300524	.350379
−90	1.224843	.199018
−85	1.146916	.043163
−80	1.067146	−.116375
−70	.903656	−.443357
−60	.737110	−.776449
−50	.569667	−1.111335
−40	.404761	−1.441145
−30	.250223	−1.750223
−20	.119935	−2.010799
−10	.031487	−2.187694
−5	.007977	−2.234715
0	.000007	−2.250654
5	.007977	−2.234715
10	.031487	−2.187694
20	.119935	−2.010799
30	.250223	−1.750223
40	.404761	−1.441145
50	.569667	−1.111335
60	.737110	−.776449
70	.903656	−.443357
80	1.067146	−.116375
85	1.146916	.043163
90	1.224843	.199018
95	1.300524	.350379
100	1.373602	.496536
110	1.510815	.770962
120	1.634666	1.018664
130	1.743433	1.236197
140	1.834890	1.419111
150	1.906695	1.526722
160	1.957430	1.664191
170	1.987188	1.723708
180	1.996929	1.743190

GENERAL ANALYSIS

An examination of the expressions for f_1 to f_6 given by equations (27) to (32) shows that the general form of f_n is

$$f_n=\sum_{p=0}^{\infty} e^{-(2p+n)y} \sum_{q=0}^{\substack{n-1\\2p+n-2}} y^q \sum_{r=p}^{\infty} A_{q}^{n}{}_{r}^{p}\, k^{n+2r} \qquad (n=1,2,\ldots\infty) \qquad (49)$$

where, if $p=0$, the upper limit of q is $n-1$, and, if $p\neq 0$, the upper limit of q is $2p+n-2$. The four-labeled coefficients $A_{q}^{n}{}_{r}^{p}$ are real numbers calculated from recursion formulas obtained from the system of differential equations (20) and the boundary condition at the surface of the wavy wall. The boundary condition at $y=\infty$ is automatically satisfied by the form of f_n; whereas the boundary condition at the wall takes the form

$$(f_n')_{y=0}=\begin{matrix}-k & (n=1)\\ 0 & (n\neq 1)\end{matrix} \qquad (50)$$

Inserting the expression for f_n given by equation (49) into equation (50) yields immediately the following results:

$$A_{0}^{1}{}_{0}^{0}=1$$

and

$$\sum_{p=0}^{r}(2p+n)A_{0}^{n}{}_{r}^{p}=\sum_{p=0,1}^{r}A_{1}^{n}{}_{r}^{p} \qquad (n=1,2,\ldots\infty) \qquad (51)$$

where, if $n=1$, the lower limit of p on the right-hand side is unity and, if $n\neq 1$, the lower limit of p is zero. Also, if $n=1$, the upper limit r of p goes from 1 to ∞ and, if $n\neq 1$, r goes from 0 to ∞.

In order to find the recursion formulas for the coefficients $A_{q}^{n}{}_{r}^{p}$, the expression for f_n given by equation (49) is substituted into the system of differential equations (20). The calculation is facilitated by the introduction of the following notations:

$$A_p^n=\sum_{q=0}^{\substack{n-1\\2p+n-2}} y^q \sum_{r=p}^{\infty} A_{q}^{n}{}_{r}^{p}\, k^{2r} \qquad (52)$$

and

$$A^{m,n}_p=\sum_{q=0}^{p} A_q^m A_{p-q}^n \qquad (53)$$

where the quantities $A^{m,n}_p$ arise from the multiplication of the two infinite series $\sum_{p=0}^{\infty}A_p^m e^{-2py}$ and $\sum_{p=0}^{\infty}A_p^n e^{-2py}$. Note that the quantities $A^{m,n}_p$ are symmetric with respect to the upper labels m and n. Then

$$f_n=k^n e^{-ny}\sum_{p=0}^{\infty}A_p^n e^{-2py} \qquad (54)$$

When this expression for f_n is substituted into equation (20) and repeated use is made of the identity

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}B_{m,n}\equiv\sum_{n=0}^{\infty}\sum_{m=0}^{n}B_{m,n}$$

the exponential terms in y can be eliminated, and the following recursion formulas result:

$$-2n(A_0^n)'+(A_0^n)''=-\frac{1}{4}n\sum_{m=1}^{n-1}m(n-m)A^{m,n-m}_0$$

$$(p=0;\ n=2,3,\ldots\infty) \qquad (55)$$

and

$$4p(p+n)A_p^n-2(2p+n)(A_p^n)'+(A_p^n)''=$$

$$-\frac{1}{4}n\sum_{m=1}^{n-1}m(n-m)A^{m,n-m}_p-\frac{1}{2}n\sum_{m=1}^{p}m(n+m)k^{2m}A^{m,n+m}_{p-m}$$

$$(p=1,2,\ldots\infty;\ n=1,2,\ldots\infty) \qquad (56)$$

Note that these recursion formulas still contain powers of y and k. Both y and k can be eliminated for given values of p and recursion formulas containing only the coefficients $A_{q}^{n}{}_{r}^{p}$ thus obtained. For example, consider equation (55) for which $p=0$. By repeated use of the identity

$$\sum_{\alpha=0}^{n}\sum_{\beta=0}^{\alpha}B_{\alpha,\beta}\equiv\sum_{\beta=0}^{n}\sum_{\alpha=\beta}^{n}B_{\alpha,\beta}$$

the following recursion formula is obtained:

$$-2n(q+1)A_{q+1}^{n}{}_{r}^{0}+(q+1)(q+2)\delta^{n}_{q}{}^{-2}A_{q+2}^{n}{}_{r}^{0}=$$

$$-\frac{1}{4}n\sum_{t=0}^{q}\sum_{m=q-t}^{n-2-t}(m+1)(n-m-1)\sum_{s=0}^{r}A_{q-t}^{m+1}{}_{s}^{0}A_{t}^{n-m-1}{}_{r-s}^{0}$$

$$(n=2,3,\ldots\infty;\ q=0,1,2,\ldots(n-2);$$

$$r=0,1,2,\ldots\infty) \qquad (57)$$

where

$$\delta^{n-2}_{q}=\begin{matrix}0\\1\end{matrix}\qquad\begin{matrix}(q=n-2)\\(q\neq n-2)\end{matrix}$$

A number of interesting relations can be easily obtained from this recursion formula. Thus, for $q=n-2$ and $r=0$,

$$2(n-1)A^{n\,0}_{n-1\,0}=\frac{1}{4}\sum_{m=0}^{n-2}(n-m-1)(m+1)A^{n-1-m\,0}_{n-2-m\,0}\,A^{m+1\,0}_{m\,0}$$

Then

$$\left.\begin{matrix} A^{2\,0}_{1\,0}=\frac{1}{8} & A^{5\,0}_{4\,0}=\frac{25}{48\times 128} \\ A^{3\,0}_{2\,0}=\frac{1}{32} & A^{6\,0}_{5\,0}=\frac{9}{5120} \\ A^{4\,0}_{3\,0}=\frac{1}{96} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{matrix}\right\} \tag{58}$$

From these numerical values the general form is found to be

$$A^{n\,0}_{n-1\,0}=\frac{n^{n-2}}{n!4^{n-1}}$$

Similarly, for $q=n-2$ and $r=1$,

$$A^{n\,0}_{n-1\,1}=\frac{n^{n-2}}{(n-1)!4^{n-1}}A^{1\,0}_{0\,1}$$

In a corresponding manner, more complicated expressions can be obtained for $A^{n\,0}_{n-1\,r}$ when $r=2, 3, \ldots$.

Note that in the expression for the disturbance-potential function $f(x,y)$, infinite series of the type

$$\sum_{n=1}^{\infty}y^{n-1}e^{-ny}k^{n+2}\sin nx\ A^{n\,0}_{n-1\,1}$$

occur. The ratio of the $(n+1)$st and nth terms is

$$kye^{-y}\frac{\sin (n+1)x}{\sin nx}\,\frac{A^{n+1\,0}_{n\,1}}{A^{n\,0}_{n-1\,1}}$$

In order for the series to converge, the limit of this ratio as $n\rightarrow\infty$ must be less than unity. Thus, if the maximum value of $ye^{-y}\left(=\frac{1}{e}\right)$ is inserted and $x=0$, Cauchy's ratio test yields

$$\frac{k}{e}\,\frac{e}{4}<1 \text{ or } k<4 \tag{59}$$

Other infinite series occur in the expression for $f(x,y)$ which diverge for values of the transonic similarity parmeter k considerably less than 4. Thus, consider the recursion equation (57) and take $q=0$ and $r=0$. Then

$$-2nA^{n\,0}_{1\,0}+2\delta^{n-2}_{0}A^{n\,0}_{2\,0}=-\frac{1}{4}n\sum_{m=0}^{n-2}(m+1)(n-m-1)A^{m+1\,0}_{0\,0}\,A^{n-m-1\,0}_{0\,0}$$

and

$$A^{2\,0}_{1\,0}=\frac{1}{8}$$

$$6A^{3\,0}_{1\,0}=2A^{3\,0}_{2\,0}+3A^{2\,0}_{0\,0}$$

$$4A^{4\,0}_{1\,0}=A^{4\,0}_{2\,0}+3A^{3\,0}_{0\,0}+2(A^{2\,0}_{0\,0})^2$$

$$10A^{5\,0}_{1\,0}=2A^{5\,0}_{2\,0}+10A^{4\,0}_{0\,0}+15A^{2\,0}_{0\,0}\,A^{3\,0}_{0\,0}$$

$$12A^{6\,0}_{1\,0}=2A^{6\,0}_{2\,0}+15A^{5\,0}_{0\,0}+24A^{2\,0}_{0\,0}\,A^{4\,0}_{0\,0}+\frac{27}{2}(A^{3\,0}_{0\,0})^2$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

Also from equation (57),

$$4A^{3\,0}_{2\,0}=A^{2\,0}_{1\,0}$$

$$8A^{4\,0}_{2\,0}=3A^{4\,0}_{3\,0}+4A^{2\,0}_{1\,0}\,A^{2\,0}_{0\,0}+3A^{3\,0}_{1\,0}$$

$$20A^{5\,0}_{2\,0}=6A^{5\,0}_{3\,0}+15A^{2\,0}_{1\,0}\,A^{3\,0}_{0\,0}+15A^{3\,0}_{1\,0}\,A^{2\,0}_{0\,0}+10A^{4\,0}_{1\,0}$$

$$8A^{6\,0}_{2\,0}=2A^{6\,0}_{3\,0}+8A^{2\,0}_{1\,0}\,A^{4\,0}_{0\,0}+9A^{3\,0}_{1\,0}\,A^{3\,0}_{0\,0}+8A^{4\,0}_{1\,0}\,A^{2\,0}_{0\,0}+5A^{5\,0}_{1\,0}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

and

$$30A^{5\,0}_{3\,0}=12A^{5\,0}_{4\,0}+15A^{3\,0}_{2\,0}\,A^{2\,0}_{0\,0}+10A^{4\,0}_{2\,0}+15A^{2\,0}_{1\,0}\,A^{3\,0}_{1\,0}$$

$$12A^{6\,0}_{3\,0}=4A^{6\,0}_{4\,0}+9A^{3\,0}_{2\,0}\,A^{3\,0}_{0\,0}+8A^{4\,0}_{2\,0}\,A^{2\,0}_{0\,0}+5A^{5\,0}_{2\,0}+8A^{2\,0}_{1\,0}\,A^{4\,0}_{1\,0}+\frac{9}{2}(A^{3\,0}_{1\,0})^2$$

$$48A^{6\,0}_{4\,0}=20A^{6\,0}_{5\,0}+24A^{4\,0}_{3\,0}\,A^{2\,0}_{0\,0}+15A^{5\,0}_{3\,0}+27A^{3\,0}_{2\,0}\,A^{3\,0}_{1\,0}+24A^{4\,0}_{2\,0}\,A^{2\,0}_{1\,0}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

The following relations are obtained from the supplementary equations (51):

$$\begin{matrix} 2A^{2\,0}_{0\,0}=A^{2\,0}_{1\,0} & 5A^{5\,0}_{0\,0}=A^{5\,0}_{1\,0} \\ 3A^{3\,0}_{0\,0}=A^{3\,0}_{1\,0} & 6A^{6\,0}_{0\,0}=A^{6\,0}_{1\,0} \\ 4A^{4\,0}_{0\,0}=A^{4\,0}_{1\,0} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{matrix}$$

From these relations, together with the ones listed in equation (58),

$$\begin{matrix} A^{2\,0}_{1\,0}=\frac{1}{8} & A^{5\,0}_{1\,0}=\frac{7}{768} \\ A^{3\,0}_{1\,0}=\frac{1}{24} & A^{6\,0}_{1\,0}=\frac{7\times 13}{72\times 256} \\ A^{4\,0}_{1\,0}=\frac{7}{384} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{matrix}$$

and lead to the general rule

$$A_{1\,0}^{n\,0}=\frac{\{3n-5\}}{n!4^{n-1}} \qquad (n=2, 3, \ldots \infty) \tag{60}$$

From equation (51), $nA_{0\,0}^{n\,0}=A_{1\,0}^{n\,0}$; therefore,

$$A_{0\,0}^{n\,0}=\frac{\{3n-5\}}{nn!4^{n-1}} \qquad (n=2, 3, \ldots \infty) \tag{61}$$

where, by definition,

$$\{3n-5\}=1\times4\times7\times10\times13\times \ldots \times(3n-5)$$

In the expression for the local Mach number in the field of flow given by equation (36), the expression $\partial f/\partial x$ occurs. This expression, with $x=0$, contains infinite series of the type

$$F=\sum_{n=1}^{\infty} nk^{n}e^{-ny}A_{0\,0}^{n\,0} \tag{62}$$

and

$$G=\sum_{n=2}^{\infty} nk^{n}e^{-ny}A_{1\,0}^{n\,0} \tag{63}$$

If the maximum value of $e^{-ny}(=1)$ and the expressions given by equations (60) and (61) are inserted, the Cauchy ratio test shows that both series converge for $k<\frac{4}{3}$. In particular, the series expression for F evaluated at the surface $(y=0)$ can be expressed in closed form. Thus,

$$F=k+\sum_{n=2}^{\infty}\frac{\{3n-5\}}{n!4^{n-1}}k^{n}=2-2\left(1-\frac{3}{4}k\right)^{2/3} \tag{64}$$

The graph of F against k is a semicubical parabola with the cusp point at $k=\frac{4}{3}$ and $F=2$. With the restriction that the transonic similarity parameter k be positive and that one and only one value of k correspond to a given value of F, the permissible values of k and F are confined to the part of the parabola lying between the origin (0,0) and the apex (4/3,2). The power-series expression for G evaluated at the surface can also be expressed in closed form; namely,

$$G=\sum_{n=2}^{\infty}\frac{\{3n-5\}}{(n-1)!4^{n-1}}k^{n}=\frac{k}{\left(1-\frac{3}{4}k\right)^{1/3}}-k \tag{65}$$

This expression, together with the one for F, shows clearly that the parameter k cannot be equal to but must be less than four-thirds.

A close examination of the recursion formulas (55) and (56) discloses the important fact that each one of the manifold of power series in k that appear in the functions f_n can be expressed in terms of the members of a single dominant set of power series. This dominant set consists of one power series in k from each f_n, namely, the one multiplied by e^{-ny} only. According to equation (49), the members of the dominant set of power series are given by

$$S_n=\sum_{r=0}^{\infty} A_{0\,r}^{n\,0}k^{n+2r} \qquad (n=1, 2, \ldots \infty) \tag{66}$$

Several examples are now given to illustrate this important observation. Consider the series that belong to the set

$$\sum_{r=0}^{\infty} A_{1\,r}^{n\,0}k^{n+2r} \qquad (n=2, 3, \ldots \infty)$$

From the recursion formulas,

$$\sum_{r=0}^{\infty} A_{1\,r}^{2\,0}k^{2r+2}=\frac{1}{8}S_1^2$$

$$\sum_{r=0}^{\infty} A_{1\,r}^{3\,0}k^{2r+3}=\frac{1}{96}S_1^3+\frac{1}{2}S_1S_2$$

$$\sum_{r=0}^{\infty} A_{1\,r}^{4\,0}k^{2r+4}=\frac{1}{512}S_1^4+\frac{1}{16}S_1^2S_2+\frac{3}{4}S_1S_3+\frac{1}{2}S_2^2$$

. . . . .

Consider now the set

$$\sum_{r=0}^{\infty} A_{2\,r}^{n\,0}k^{n+2r} \qquad (n=3, 4, \ldots \infty)$$

Then

$$\sum_{r=0}^{\infty} A_{2\,r}^{3\,0}k^{2r+3}=\frac{1}{32}S_1^3$$

$$\sum_{r=0}^{\infty} A_{2\,r}^{4\,0}k^{2r+4}=\frac{1}{128}S_1^4+\frac{1}{4}S_1^2S_2$$

. . . . .

Consider the set

$$\sum_{r=1}^{\infty} A_{0\,r}^{n\,1}k^{n+2r} \qquad (n=1, 2, \ldots \infty)$$

From the recursion formula (56),

$$\sum_{r=1}^{\infty} A_{0\,r}^{1\,1}k^{2r+1}=-\frac{3}{256}S_1^3-\frac{1}{8}S_1S_2$$

$$\sum_{r=1}^{\infty} A_{0\,r}^{2\,1}k^{2r+2}=-\frac{17}{12\times256}S_1^4-\frac{7}{96}S_1^2S_2-\frac{1}{4}S_1S_3$$

$$\sum_{r=1}^{\infty} A_{0\,r}^{3\,1}k^{2r+3}=-\frac{163}{256^2}S_1^5-\frac{43}{1024}S_1^3S_2-\frac{33}{256}S_1^2S_3-\frac{3}{32}S_1S_2^2-\frac{3}{8}S_1S_4$$

. . . . .

Consider the set—

$$\sum_{r=1}^{\infty} A^{n\,1}_{1\,r}\, k^{n+2r} \qquad (n=1, 2, \ldots \infty)$$

Then

$$\sum_{r=1}^{\infty} A^{1\,1}_{1\,r}\, k^{2r+1} = -\frac{1}{64} S_1^{\,3}$$

$$\sum_{r=1}^{\infty} A^{2\,1}_{1\,r}\, k^{2r+2} = -\frac{3}{256} S_1^{\,4} - \frac{1}{8} S_1^{\,2} S_2$$

$$\sum_{r=1}^{\infty} A^{3\,1}_{1\,r}\, k^{2r+3} = -\frac{15}{2048} S_1^{\,5} - \frac{57}{512} S_1^{\,3} S_2 - \frac{9}{32} S_1^{\,2} S_3 - \frac{3}{16} S_1 S_2^{\,2}$$

. . . . .

Other examples are

$$\sum_{r=2}^{\infty} A^{1\,2}_{3\,r}\, k^{2r+1} = -\frac{1}{2048} S_1^{\,5}$$

$$\sum_{r=2}^{\infty} A^{1\,2}_{2\,r}\, k^{2r+1} = -\frac{3}{8192} S_1^{\,5} - \frac{3}{256} S_1^{\,3} S_2$$

$$\sum_{r=2}^{\infty} A^{1\,2}_{1\,r}\, k^{2r+1} = \frac{3}{8192} S_1^{\,5} - \frac{7}{1536} S_1^{\,3} S_2 - \frac{1}{16} S_1 S_2^{\,2} - \frac{1}{64} S_1^{\,3} S_2$$

$$\sum_{r=2}^{\infty} A^{1\,2}_{0\,r}\, k^{2r+1} = \frac{61}{256\times 576} S_1^{\,5} + \frac{1}{384} S_1^{\,3} S_2 - \frac{1}{48} S_1 S_2^{\,2} + \frac{1}{256} S_1^{\,3} S_2 - \frac{1}{8} S_2 S_3$$

The recognition of the existence of a dominant set of power series in k represents a major reduction in the complexity of the present problem. Thus, consider the array of infinite power series contained in equation (66):

$$\left.\begin{aligned}
S_1 &= A^{1\,0}_{0\,0} k + A^{1\,0}_{0\,1} k^3 + A^{1\,0}_{0\,2} k^5 + A^{1\,0}_{0\,3} k^7 + \ldots \\
S_2 &= A^{2\,0}_{0\,0} k^2 + A^{2\,0}_{0\,1} k^4 + A^{2\,0}_{0\,2} k^6 + A^{2\,0}_{0\,3} k^8 + \ldots \\
S_3 &= A^{3\,0}_{0\,0} k^3 + A^{3\,0}_{0\,1} k^5 + A^{3\,0}_{0\,2} k^7 + A^{3\,0}_{0\,3} k^9 + \ldots \\
S_4 &= A^{4\,0}_{0\,0} k^4 + A^{4\,0}_{0\,1} k^6 + A^{4\,0}_{0\,2} k^8 + A^{4\,0}_{0\,3} k^{10} + \ldots \\
S_5 &= A^{5\,0}_{0\,0} k^5 + A^{5\,0}_{0\,1} k^7 + A^{5\,0}_{0\,2} k^9 + A^{5\,0}_{0\,3} k^{11} + \ldots \\
&. \; . \; . \; . \; .
\end{aligned}\right\} \tag{67}$$

An examination of equations (27) to (32) shows that the coefficients $A^{n\,0}_{0\,r}$ of the series S_n appear to be positive and monotonically decreasing. The series formed from the first column on the right-hand side of equation (67) therefore dominates the series formed from succeeding columns. Moreover, the first-column series $\sum_{n=1}^{\infty} A^{n\,0}_{0\,0} k^n$ has a radius of convergence $k=\frac{4}{3}$ (see equation (61)). The radii of convergence of the other columnar series therefore are either equal to or greater than four-thirds. Similarly, an examination of the series S_n shows that the coefficients in each column on the right-hand side of equation (67) also seem to form positive and monotonically decreasing sequences. This behavior means that S_1 is the dominant series of the set S_n and, in fact, of the aggregate of power series in k in the expression for the disturbance potential $f(x,y)$.

Consider now the series consisting of the first terms of the odd-labeled series $S_1, S_3, S_5, \ldots$, that is,

$$\sum_{n=0}^{\infty} A^{2n+1\,0}_{0\,0}\, k^{2n+1} \tag{68}$$

According to the theory of power series (and it can be easily verified), the radius of convergence of this power series is still $k=\frac{4}{3}$. Now, a comparison of corresponding terms with the dominant series S_1 shows that

$$A^{1\,0}_{0\,1} = \frac{11}{256} > A^{3\,0}_{0\,0} = \frac{1}{72}$$

and

$$A^{1\,0}_{0\,2} = \frac{1861}{3\times 256^2} > A^{5\,0}_{0\,0} = \frac{7}{15\times 256}$$

Thus, if in general $A^{1\,0}_{0\,n} > A^{2n+1\,0}_{0\,0}$, then the radius of convergence of the dominant series S_1 can be less than the radius of convergence of the comparison series given by expression (68) and therefore may conceivably be equal to the critical value $k_{cr}=0.8377$. Moreover, it would then follow that the original Prandtl-Busemann small-perturbation method is valid for purely subsonic flows only. This conclusion would not invalidate other approaches to the transonic-flow problem (reference 4).

Unfortunately, the coefficients $A^{1\,0}_{0\,n}$ do not conform to any apparent or superficial law, but perhaps a careful study of the recursion formulas (55) and (56) and the supplementary or boundary relations (51) will yield a rigorous proof of the foregoing statements. Otherwise, it remains to calculate a reasonable number of the coefficients $A^{1\,0}_{0\,n}$. For this purpose the development of complete recursion formulas similar to equation (57) for the required values of p is worth while. Thus, for $p=1$, the recursion formula is

$$\begin{aligned}
&4(n+1)A^{n\,1}_{q\,r+1} - 2(n+2)(q+1)\delta^n_q A^{n\,1}_{q+1\,r+1} \\
&+(q+1)(q+2)\delta^{n-1,n}_q A^{n\,1}_{q+2\,r+1} = \\
&\qquad -\frac{1}{4} n\,\delta^n_q \sum_{t=0}^{q} \sum_{m=q-t}^{n-1-t} \Bigg[(m+1)(n-m-1) \\
&\qquad \sum_{s=0}^{r} A^{m+1\,0}_{q-t\,s} A^{n-m-1\,1}_{t\,r-s+1} + m(n-m) \sum_{s=0}^{r} A^{m\,1}_{q-t\,s+1} A^{n-m\,0}_{t\,r-s}\Bigg] - \\
&\qquad \frac{1}{2} n(n+1) \sum_{s=0}^{r} A^{1\,0}_{0\,s} A^{n+1\,0}_{q\,r-s}
\end{aligned}$$

$$(n=1,2,\ldots\infty;\ q=0,1,2,\ldots n;\ r=0,1,2,\ldots\infty) \tag{69}$$

where

$$\delta^{n-1,n}_q = \begin{matrix} 0 \\ 1 \end{matrix} \qquad \begin{matrix} (q=n-1 \text{ or } n) \\ (q\neq n-1 \text{ or } n) \end{matrix}$$

Finally, it may be of interest to give the general formula for the Mach number distribution at the surface of the wavy wall. Thus, by means of equations (36) and (49),

$$\frac{M^2-1}{1-M_\infty^2}=-1+\sum_{n=1}^{\infty} k^n \sum_{m=0}^{\left[\frac{n}{2}\right]} (n-2m) \cos (n-2m)x \sum_{p=0}^{m} A^{n-2m}_{0}{}^{p}_{m} \tag{70}$$

where $\left[\frac{n}{2}\right]$ denotes the integral part of $n/2$.

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LANGLEY FIELD, VA., *April 26, 1951.*

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