# REPORT 1223

# THEORETICAL AND EXPERIMENTAL INVESTIGATION OF HEAT TRANSFER BY LAMINAR NATURAL CONVECTION BETWEEN PARALLEL PLATES[1]

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## SUMMARY

*Results are presented of a theoretical and experimental investigation of heat transfer involving laminar natural convection of fluids enclosed between parallel walls oriented in the direction of the body force, where one wall is heated uniformly, and the other is cooled uniformly. For the experimental work, parallel walls were simulated by using an annulus with an inner-to-outer diameter ratio near 1.*

*The results of the theoretical investigation are presented in the form of equations for the velocity and temperature profiles and the ratio of actual temperature drop across the fluid to the temperature drop for pure conduction. No experimental measurements were made of the velocity and temperature profiles, but the experimental results are compared with theory on the basis of the ratio of the actual temperature drop to the temperature drop for pure conduction. Good agreement was obtained between theory and experiment for axial temperature gradients of 40° F per foot or larger.*

## INTRODUCTION

Increased application of heat transfer to and from fluids in channels has recently required further knowledge as to the heat-transfer coefficients and temperature profiles occurring with natural convection. Turbine-blade and nuclear-reactor cooling are two of the fields concerned with this problem. Work on free-convection heat transfer over a vertical plate gave good agreement between theory and experiment. Few results have been obtained for the similar case of flow in channels. Reference 1 obtains a theoretical solution for free-convection heat transfer for fluids enclosed in channels. The reference uses a postulated velocity distribution to obtain a solution. References 2 and 3 extend this analytical work to give an exact solution of the equations in more general form for constant wall temperature and constant heat flux, respectively. Reference 3 includes the effect of forced as well as natural convection. These three references treat the case of infinite channels with the channel axis oriented in the direction of the body force and are subject to the same assumptions; namely, two-dimensional laminar flow, uniform axial temperature gradient, and constant fluid properties, except that the density is allowed to vary in the bouyancy term.

The purpose of this report is to compare the results of theory and experiment as a check on the assumptions involved in the analytical work. For mathematical simplicity, the flow between two infinite parallel plates was considered with one plate heated uniformly and the other cooled uniformly. An exact solution was obtained with the assumptions cited previously for references 1 to 3. In order to simulate the case of infinite parallel plates experimentally, an annulus formed by two concentric tubes was used. For all practical purposes, the walls of these tubes can be considered parallel if the ratio of the radii of the tubes is near 1. This radius ratio limits the spacing between the confining walls (and, consequently, the range of Grashof numbers obtainable) unless very large diameters are used.

A comparison between theory and experiment is given in this report on a heat-transfer basis alone, since no temperature or velocity profile measurements were made. The work was done at the NACA Lewis laboratory.

## ANALYSIS

Steady-state heat transfer through a fluid enclosed by two infinite parallel plates oriented in the direction of the body force is considered. One plate is heated uniformly and the other is cooled uniformly. The flow is laminar and parallel to the body force. The fluid properties are assumed constant, except that the density is allowed to vary in the buoyancy term. Viscous dissipation and work against the force field are neglected. It is further assumed that the axial temperature gradient is a constant throughout the system for any particular set of conditions.

When the aforementioned conditions are applied, the energy equation reduces to

$$\frac{k}{\rho c_p}\frac{\partial^2 t}{\partial x^2}=w\frac{\partial t}{\partial z} \tag{B1}$$

(Symbols are defined in appendix A, and a detailed discussion of the analysis is given in appendix B.)

Similarly, the Navier-Stokes' equations reduce to one equation in the $z$-direction:

$$\frac{\partial^2 w}{dx^2}=\frac{g\rho\beta}{\mu}(t_f-t) \tag{B7}$$

The reference temperature $t_f$ is taken at the center of the channel where the net viscous force is zero $\left(\frac{\partial^2 w}{\partial x^2}=0\right)$.

[1] Supersedes NACA TN 3328, "Theoretical and Experimental Investigation of Heat Transfer by Laminar Natural Convection Between Parallel Plates," by A. F. Lietze, 1954.

For a closed system of unit width with zero net mass through-flow and constant density,

$$\int_{-r}^{r} w\,dx=0$$

where $x$ is taken to be zero at the center of the channel.

From these three equations and the appropriate boundary conditions

$$\left(\frac{\partial t}{\partial x}\right)_{x=r}=-\frac{q''}{k} \qquad w_{x=r}=0$$

$$\left(\frac{\partial t}{\partial x}\right)_{x=-r}=-\frac{q''}{k} \qquad w_{x=-r}=0$$

the equations for the velocity and temperature profiles in dimensionless form are, respectively,

$$w^*=4v\left(\frac{\sin\frac{vx}{r}\cosh\frac{vx}{r}}{\frac{\sin v}{\sinh v}+\frac{\cosh v}{\cos v}}-\frac{\sinh\frac{vx}{r}\cos\frac{vx}{r}}{\frac{\sinh v}{\sin v}+\frac{\cos v}{\cosh v}}\right) \tag{B17}$$

$$\theta^*=-\frac{1}{2v}\left(\frac{\sinh v\cos v+\sinh\frac{vx}{r}\cos\frac{vx}{r}}{\frac{\cosh v}{\cos v}+\frac{\sin v}{\sinh v}}+\frac{\cosh v\sin v+\cosh\frac{vx}{r}\sin\frac{vx}{r}}{\frac{\sinh v}{\sin v}+\frac{\cos v}{\cosh v}}\right) \tag{B19}$$

where, by definition,

$$v=\sqrt[4]{\frac{PrGr}{64}}$$

$$Pr=\frac{c_p\mu}{k}$$

$$Gr=\frac{g\rho^2\beta\frac{\partial t}{\partial z}s^4}{\mu^2}$$

The temperature drop across the fluid from wall to wall is then found from equation (B19) when $x=r$:

$$\theta_r=-\frac{q''s}{kv}\left(\frac{\sin^2 v+\sinh^2 v}{\sinh v\cosh v+\sin v\cos v}\right) \tag{B20}$$

The temperature difference across the fluid for pure conduction is

$$(\theta_r)_c=-\frac{q''s}{k} \tag{B21}$$

The ratio of the actual temperature drop to that for pure conduction is, therefore,

$$\varphi_r=\frac{\theta_r}{(\theta_r)_c}=\frac{1}{v}\left(\frac{\sin^2 v+\sinh^2 v}{\sinh v\cosh v+\sin v\cos v}\right) \tag{B22}$$

These equations apply equally well to force fields other than gravitational, if the constant $g$ is replaced by the constant of the force field of interest.

## EXPERIMENTAL APPARATUS

A schematic diagram of the experimental equipment is shown in figure 1. In order to simulate infinite parallel plates, an annulus formed by two concentric stainless-steel tubes was used to contain the fluid. The resulting annulus was 10½ inches long with an outside diameter of 1⅜ inches and an inside diameter of 1 inch. Thus, the spacing $s$ between the walls was 3/16 inch. The tube walls were 1/16 inch thick. The outer tube was heated with a Nichrome-wire element spiral-wound around the tube, with a wire spacing of 3/32 inch, which probably gives fairly uniform heating of the hot wall. The inner tube was cooled by forced air. The cooling-air-passage length-to-diameter ratio was 84. A wire in the form of a helical spring was inserted in the cooling passage to increase the heat-transfer coefficient. This system should provide an essentially constant heat-transfer coefficient, which is necessary to provide uniform cooling of the cold wall with a linear variation of wall temperature. The cooling-air-flow rate was measured with a calibrated rotameter.

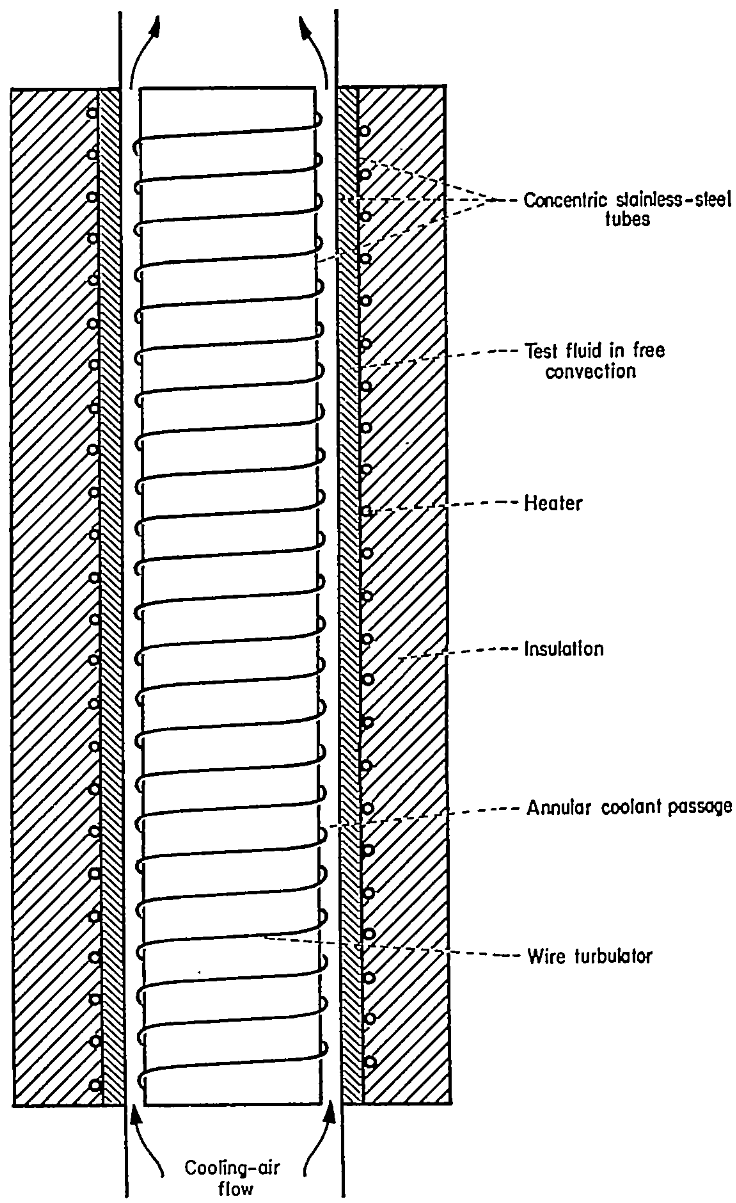


Figure 1.—Schematic diagram of experimental apparatus for measuring heat transfer by free convection in an annulus used to simulate parallel plates.

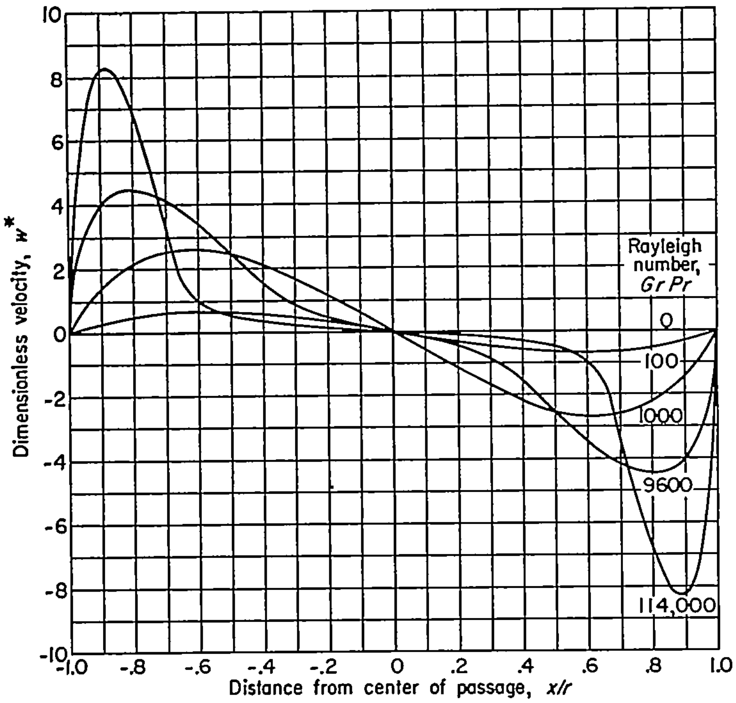


Figure 2.—Dimensionless velocity profiles for various values of product of Prandtl number and modified Grashof number.

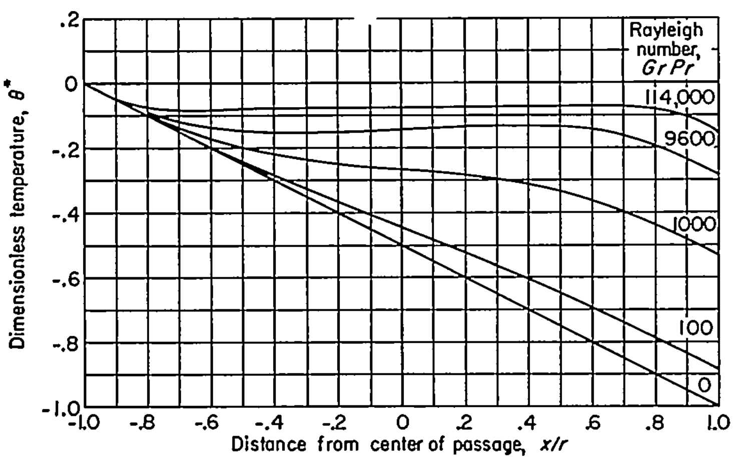


Figure 3.—Dimensionless temperature profiles for various values of product of Prandtl number and modified Grashof number.

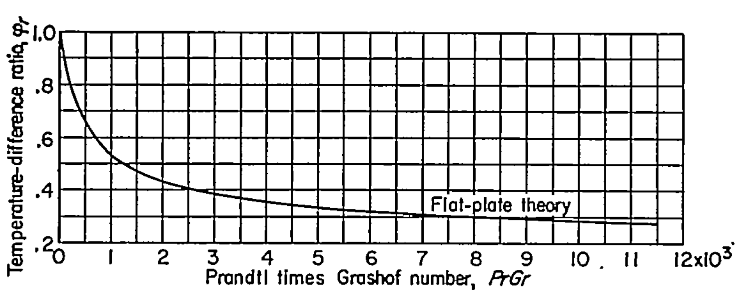


Figure 4.—Theoretical variation of ratio of actual temperature difference across fluid to temperature difference for pure conduction with product of Prandtl number and modified Grashof number.

The electrical power was supplied by a transformer with a variable secondary voltage and rated at 1 kva at maximum voltage. The electrical power input was measured with a calibrated ammeter and voltmeter. All temperatures were

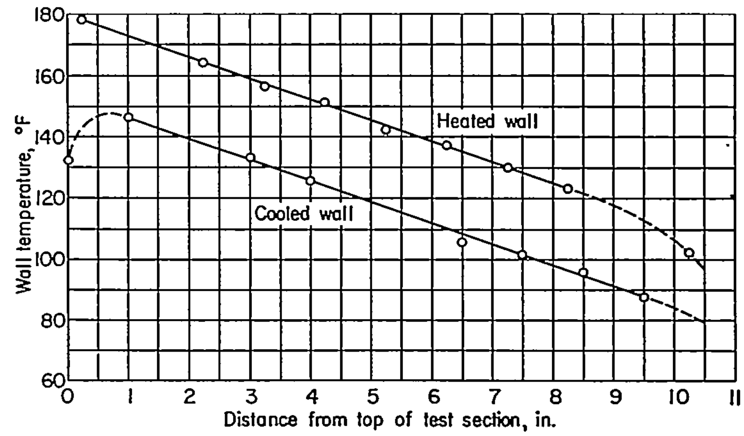


Figure 5.—Temperature measurements of heated and cooled walls of test section. Heat flux, 1300 Btu per hour per square foot; Prandtl times Grashof number, 3480; temperature-differance ratio, 0.389.

measured with iron-constantan thermocouples and a self-balancing potentiometer. The thermocouples used to measure wall temperatures were set into slots milled axially into the tube walls. The thermocouple leads were enclosed in Inconel tubing laid in the slots. The thermocouple junctions were covered with insulating cement, and the slots were then filled with silver solder and finished off flush with the tube surface.

## RESULTS AND DISCUSSION

### ANALYTICAL

The dimensionless velocity and temperature, which can be calculated from equations (B17) and (B19), respectively, are shown graphically in figures 2 and 3 for a few values of the product of Prandtl and modified Grashof numbers. The equations show that the dimensionless velocity and temperature depend only on this product and the position in the channel. The Grashof number resulting from the analysis differs from the conventional Grashof number insofar as the temperature difference usually appearing is replaced by the temperature gradient and the characteristic length appears as the fourth power instead of the cube. The product of Prandtl number and Grashof number is sometimes referred to as the Rayleigh number.

The velocity profiles of figure 2 show the point of maximum velocity moving closer to the wall as the Rayleigh number is increased. It is felt that this fact, in combination with the stabilizing influence of the wall, would tend to inhibit the occurrence of turbulent flow.

The contribution of convective flow in the heat-transfer process can best be seen from the ratio of the actual temperature drop across the channel to the temperature drop for pure conduction. This ratio, which is the same as the ratio of the molecular to the apparent conductivity, can be obtained from equation (B22) and is plotted in figure 4. For large values of $PrGr$, the actual temperature drop across the fluid is less than $\frac{1}{3}$ the temperature drop for pure conduction.

### EXPERIMENTAL

A typical plot of measured wall temperatures for large temperature gradients is shown in figure 5. With uniform heating of the hot wall and constant heat-transfer coefficient on the cold wall, the temperature gradients on the hot and cold walls are constant and equal, except for the extreme ends of the test section. Data were not obtained to verify the assumption of uniform temperature gradient throughout the fluid.

For small temperature gradients the wall temperatures were linear, but the gradient on the heated wall was larger than that on the cooled wall. The assumption of uniform temperature gradient throughout the system is, therefore, invalid for small temperature gradients. With water as the test fluid and with $\frac{3}{16}$-inch spacing between the plates, the minimum value for which the assumption of uniform temperature gradient is valid is approximately 40° F per foot. Additional data are required to determine this limit more accurately. The data presented in this report are for temperature gradients of 40° F per foot or greater.

Because of the end effects on the flow and because of the heat losses from the ends of the test section, it was impossible to obtain a heat balance between the electrical heat input and the enthalpy rise of the coolant. Measurements of the radial temperature drop through the insulation surrounding the heater indicated a negligible radial heat loss. In that section of the tube where the wall temperature varies linearly, there is no net axial conduction in the wall. Hence, the uniform heat flux (as calculated by the electrical heat input) that enters the wall is transmitted directly to the fluid. The heat loss is limited to the ends of the test section and, therefore, does not affect the results of the test.

The calculated temperature drop through the walls containing the water was negligible compared with the temperature drop across the water, and, therefore, the measured wall temperatures were taken to be the temperature of the surface adjoining the water.

The temperature drop across the fluid for pure conduction $\Delta t_c$ was calculated from the equation for an annulus given in reference 4:

$$\Delta t_c = q''_o \frac{r_o \log_e(r_o/r_i)}{k}$$

where $q''_o$ was calculated from measurements of the electrical heat input.

The physical properties used in evaluating Prandtl number, Grashof number, and the temperature drop for pure conduction were obtained from reference 5 for saturated liquid water, except for the coefficient of thermal expansion $\beta$. Values of the coefficient of thermal expansion of water were taken from reference 6 and are plotted in figure 6. These values are mean values for 1° temperature change. Inasmuch as $\beta$ is a function of temperature, different mean values would be obtained for larger temperature increments. The properties were all evaluated at the bulk temperature of the water for each test. For flat plates, the bulk temperature is the center-line temperature halfway between the ends.

Figure 7 shows the ratio of the actual temperature drop across the fluid to the temperature drop for pure conduction plotted against the product of Prandtl and Grashof numbers. The data points on the figure are the experimental results taken with water as the test fluid, and the curve is the result of analysis taken from figure 4.

#### COMPARISON BETWEEN ANALYSIS AND EXPERIMENT

Inasmuch as no temperature measurements were made within the fluid, a comparison between theory and experiment can be made only for the over-all heat-transfer results. Except for the random scatter of the heat-transfer data shown in figure 7, good agreement is obtained between theory and experiment. Inasmuch as the same assumptions were used in references 2 and 3, the agreement shown here lends support to the more general analytical treatment given in these references.

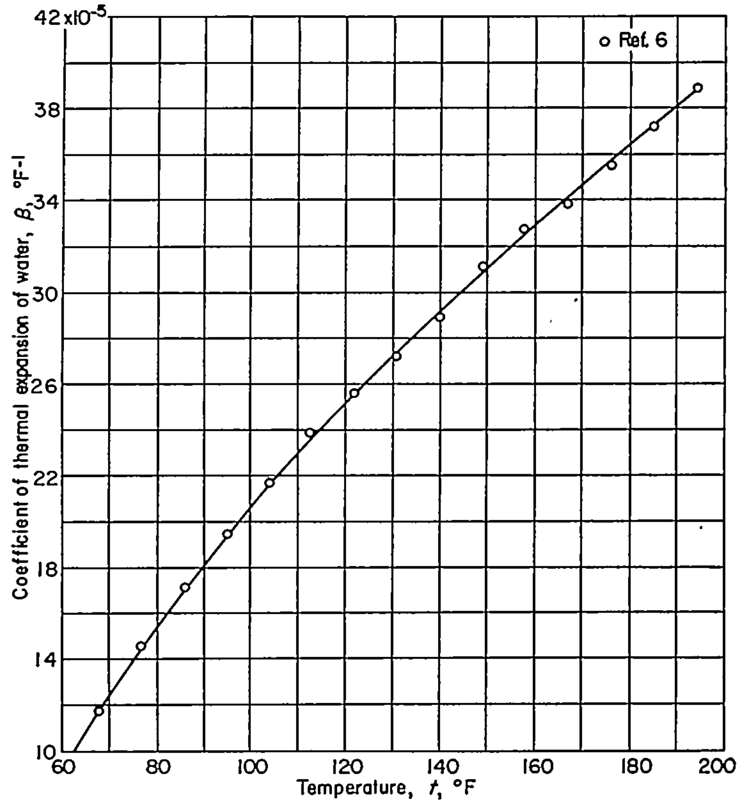


Figure 6.—Variation of coefficient of thermal expansion of water with temperature for 1° change in temperature.

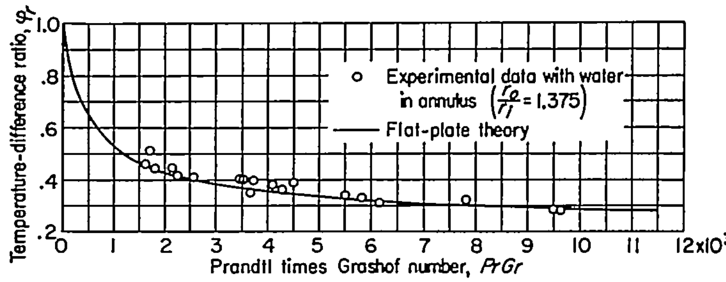


Figure 7.—Variation of ratio of actual temperature difference across fluid to temperature difference for pure conduction with product of Prandtl number and modified Grashof number.

The effect of the curvature of the walls on the experimental results is unknown; but this effect is believed to be minimized, because the data are in the form of the temperature ratio with this effect included in both the numerator and the denominator. A theoretical investigation for an annulus is necessary to permit a comparison between theory and experiment with larger diameter ratios for the annulus.

In order to determine the upper limit for which the analysis is valid, it is necessary to obtain data at higher values of the product of Prandtl number and modified Grashof number.

## CONCLUSIONS

For the range of conditions investigated, the simplifying assumptions used in the analysis for free convection in channels oriented in the direction of the body force are reasonable and lead to accurate quantitative results.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, *September 8, 1954*

# APPENDIX A

## SYMBOLS

The following symbols are used in this report:

| | |
|---|---|
| $A_1, A_2 \ldots$ | constants |
| $a$ | $\sqrt[4]{\dfrac{g\rho^2\beta c_p}{\mu k}\dfrac{\partial t}{\partial z}}$, ft$^{-1}$ |
| $B_1, B_2 \ldots$ | constants |
| $C$ | constant of integration |
| $c_p$ | specific heat at constant pressure, Btu/(lb)(°F) |
| $D$ | operator, $\partial/\partial x$ |
| $F_z$ | body force in $z$-direction, lb/(hr$^2$)(sq ft) |
| $Gr$ | modified Grashof number, $\dfrac{g\rho^2\beta\dfrac{\partial t}{\partial z}s^4}{\mu^2}$, dimensionless |
| $g$ | acceleration due to gravity, $4.17\times10^8$ ft/hr$^2$ |
| $i$ | $\sqrt{-1}$ |
| $k$ | thermal conductivity, Btu/(hr)(sq ft)(°F/ft) |
| $Pr$ | Prandtl number, $c_p\mu/k$, dimensionless |
| $p$ | pressure, lb/sq ft |
| $q''$ | heat flux at wall, Btu/(hr)(sq ft) |
| $q_o''$ | heat flux at outer wall of annulus, Btu/(hr)(sq ft) |
| $r$ | 1/2 plate spacing, $s/2$ |
| $r_i$ | inside radius of annulus, ft |
| $r_o$ | outside radius of annulus, ft |
| $s$ | plate spacing, ft |
| $t$ | temperature, °F |
| $t_f$ | reference temperature, °F |
| $t_r$ | temperature at $x=r$, °F |
| $t_{-r}$ | temperature at $x=-r$, °F |
| $u$ | $ax/\sqrt{2}$, dimensionless |
| $v$ | $ar/\sqrt{2}=\sqrt[4]{\dfrac{PrGr}{64}}$ |
| $w$ | velocity, ft/hr |
| $w^*$ | dimensionless velocity, $\dfrac{w\rho c_p s\dfrac{\partial t}{\partial z}}{q''}$ |
| $x$ | transverse coordinate, ft |
| $z$ | longitudinal coordinate, ft |
| $\beta$ | coefficient of thermal expansion, °F$^{-1}$ |
| $\theta$ | $t-t_{-r}$, °F |
| $\theta_r$ | $t_r-t_{-r}$, °F |
| $(\theta_r)_c$ | temperature difference across fluid for pure conduction |
| $\theta^*$ | dimensionless temperature, $\dfrac{(t-t_{-r})k}{q''s}$ |
| $\mu$ | viscosity, lb/(hr)(ft) |
| $\rho$ | density, lb/cu ft |
| $\varphi_r$ | ratio of actual temperature difference across fluid to temperature difference for pure conduction |

# APPENDIX B

## DETAILS OF ANALYSIS

For the conditions of the problem stated in the ANALYSIS, the general energy equation reduces to

$$\frac{k}{\rho c_p}\frac{\partial^2 t}{\partial x^2}=w\frac{\partial t}{\partial z} \tag{B1}$$

Similarly, the Navier-Stokes' equations for incompressible flow reduce to

$$\frac{\partial p}{\partial x}=0$$

$$F_z-g\frac{\partial p}{\partial z}+\mu\frac{\partial^2 w}{\partial x^2}=0 \tag{B2}$$

The body force under the action of gravity is

$$F_z=-g\rho \tag{B3}$$

Since the pressure is independent of $x$, then $\dfrac{\partial p}{\partial z}$ is independent of $x$ and can be evaluated at any value of $x$. It is convenient, however, to evaluate $\dfrac{\partial p}{\partial z}$ where $\dfrac{\partial^2 w}{\partial x^2}=0$. Then, from equations (B2) and (B3),

$$\frac{\partial p}{\partial z}=-\rho_f \tag{B4}$$

where the subscript $f$ denotes a reference condition where $\dfrac{\partial^2 w}{\partial x^2}=0$. Combining equations (B2), (B3), and (B4) gives

$$g\rho\left(1-\frac{\rho_f}{\rho}\right)=\mu\frac{\partial^2 w}{\partial x^2} \tag{B5}$$

From the definition of the coefficient of thermal expansion $\beta$,

$$\frac{\rho_f}{\rho}=1+\beta(t-t_f) \tag{B6}$$

Combining (B5) and (B6) yields

$$\frac{d^2w}{dx^2}=\frac{\rho g\beta}{\mu}(t_r-t) \qquad \text{(B7)}$$

The reference temperature $t_r$ is a function of $z$ and varies linearly with $z$ as does $t$, according to the assumption of constant temperature gradient. The velocity $w$ is therefore only a function of $x$, and its derivative is written as a total derivative.

Differentiating equation (B7) with respect to $x$ gives

$$\frac{d^3w}{dx^3}=-\frac{g\rho\beta}{\mu}\frac{\partial t}{\partial x} \qquad \text{(B8)}$$

Differentiating with respect to $x$ again gives

$$\frac{d^4w}{dx^4}=-\frac{g\rho\beta}{\mu}\frac{\partial^2 t}{\partial x^2} \qquad \text{(B9)}$$

Combining equations (B1) and (B9) yields

$$\frac{d^4w}{dx^4}=-\frac{g\rho^2\beta c_p}{\mu k}\frac{\partial t}{\partial z}w \qquad \text{(B10)}$$

Let $a^4=\frac{g\rho^2\beta c_p}{\mu k}\frac{\partial t}{\partial z}$. Then

$$\frac{d^4w}{dx^4}+a^4w=0$$

The solution of this homogeneous equation can be obtained by using the operator $D$:

$$(D^4+a^4)w=0$$

$$(D^2+ia^2)(D^2-ia^2)w=0$$

$$(D+i\sqrt{i}\,a)(D-i\sqrt{i}\,a)(D+\sqrt{i}\,a)(D-\sqrt{i}\,a)w=0$$

but $\sqrt{i}=\frac{i+1}{\sqrt{2}}$ and $i\sqrt{i}=\frac{i-1}{\sqrt{2}}$. Therefore,

$$\left(D+\frac{i-1}{\sqrt{2}}a\right)\left(D-\frac{i-1}{\sqrt{2}}a\right)\left(D+\frac{i+1}{\sqrt{2}}a\right)\left(D-\frac{i+1}{\sqrt{2}}a\right)w=0 \qquad \text{(B11)}$$

$$\left(D+\frac{i-1}{\sqrt{2}}a\right)w=0$$

$$\frac{dw}{dx}+\frac{i-1}{\sqrt{2}}aw=0$$

$$\frac{dw}{w}+\frac{i-1}{\sqrt{2}}a\,dx=0$$

$$\log w+\frac{i-1}{\sqrt{2}}ax=C$$

$$e^{C-\frac{i-1}{\sqrt{2}}ax}=w$$

$$w=e^Ce^{-\frac{i-1}{\sqrt{2}}ax}=A_1e^{-\frac{iax}{\sqrt{2}}}e^{\frac{ax}{\sqrt{2}}} \qquad \text{(B12)}$$

Similarly, from equation (B11),

$$w=A_2e^{+\frac{iax}{\sqrt{2}}}e^{-\frac{ax}{\sqrt{2}}} \qquad \text{(B13)}$$

$$w=A_3e^{-\frac{iax}{\sqrt{2}}}e^{-\frac{ax}{\sqrt{2}}} \qquad \text{(B14)}$$

$$w=A_4e^{\frac{iax}{\sqrt{2}}}e^{\frac{ax}{\sqrt{2}}} \qquad \text{(B15)}$$

The general solution is equal to the sum of equations (B12) to (B15), or

$$w=e^{-iu}(A_1e^u+A_3e^{-u})+e^{iu}(A_2e^{-u}+A_4e^u)$$

where $u=\frac{ax}{\sqrt{2}}$.

Using the Euler formulas

$$e^{-iu}=\cos u-i\sin u$$

$$e^{iu}=\cos u+i\sin u$$

and the conversion formulas

$$e^u=\cosh u+\sinh u$$

$$e^{-u}=\cosh u-\sinh u$$

and letting

$$B_1=A_1+A_4+A_3+A_2$$

$$B_2=A_1+A_4-A_3-A_2$$

$$B_3=i(A_2-A_3+A_4-A_1)$$

$$B_4=i(A_4-A_1-A_2+A_3)$$

the velocity equation becomes

$$w=B_1\cos u\cosh u+B_2\cos u\sinh u+B_3\sin u\cosh u+B_4\sin u\sinh u \qquad \text{(B16)}$$

The constants of equation (B16) can be evaluated from the boundary conditions. There are four boundary conditions required to determine the four constants. The velocity at the bounding walls must be zero, while the boundary conditions on the temperature are given by the temperature gradients. It can be shown from equation (B1) that, for the case of interest here, the temperature gradients at the two walls must be equal. Integrating equation (B1) gives

$$\frac{k}{\rho c_p}\left[\left(\frac{\partial t}{\partial x}\right)_{x=r}-\left(\frac{\partial t}{\partial x}\right)_{x=-r}\right]=\frac{\partial t}{\partial z}\int_{-r}^{r}w\,dx$$

For a system closed at both ends, there is no net mass through-flow; therefore, with constant density,

$$\int_{-r}^{r}w\,dx=0$$

Consequently, the transverse temperature gradients at the two walls must be equal. The boundary conditions are expressed mathematically as

$$\left(\frac{\partial t}{\partial x}\right)_{x=r}=-\frac{q''}{k}$$

$$\left(\frac{\partial t}{\partial x}\right)_{x=-r}=-\frac{q''}{k}$$

$$w_{x=r}=0$$

$$w_{x=-r}=0$$

When these boundary conditions are imposed on equations (B8) and (B16), the constants of equation (B16) are found to be

$$B_1=0$$

$$B_2=-\frac{4q''v}{\rho c_p s\frac{\partial t}{\partial z}}\,\frac{\sin v\cosh v}{(\sin v\cos v+\cosh v\sinh v)}$$

$$B_3=\frac{4q''v}{\rho c_p s\frac{\partial t}{\partial z}}\,\frac{\cos v\sinh v}{(\sin v\cos v+\cosh v\sinh v)}$$

$$B_4=0$$

Substituting these values for the constants of equation (B16) gives the equation for the velocity in dimensionless form:

$$w^*=4v\left(\frac{\sin\frac{vx}{r}\cosh\frac{vx}{r}}{\frac{\sin v}{\sinh v}+\frac{\cosh v}{\cos v}}-\frac{\sinh\frac{vx}{r}\cos\frac{vx}{r}}{\frac{\sinh v}{\sin v}+\frac{\cos v}{\cosh v}}\right)\tag{B17}$$

Differentiating equation (B17) and combining with equation (B7) result in an equation for the temperature in dimensionless form:

$$\frac{tk}{q''s}=\frac{t_r k}{q''s}-\frac{1}{2v}\left(\frac{\sin\frac{vx}{r}\cosh\frac{vx}{r}}{\frac{\sinh v}{\sin v}+\frac{\cos v}{\cosh v}}+\frac{\sinh\frac{vx}{r}\cos\frac{vx}{r}}{\frac{\sin v}{\sinh v}+\frac{\cosh v}{\cos v}}\right)\tag{B18}$$

The actual temperature cannot be determined from this equation, because the reference temperature $t_r$ is also a function of $z$.

The difference between the wall temperature and the temperature at any point in the fluid at the same value of $z$ can be determined from equation (B18). From the definition of $\theta^*$, equation (B18) becomes

$$\theta^*=-\frac{1}{2v}\left(\frac{\sinh v\cos v+\sinh\frac{vx}{r}\cos\frac{vx}{r}}{\frac{\cosh v}{\cos v}+\frac{\sin v}{\sinh v}}+\frac{\cosh v\sin v+\cosh\frac{vx}{r}\sin\frac{vx}{r}}{\frac{\sinh v}{\sin v}+\frac{\cos v}{\cosh v}}\right)\tag{B19}$$

The temperature difference wall-to-wall can be found from equation (B19) when $x=r$:

$$\theta_r=-\frac{q''s}{kv}\left(\frac{\sin^2 v+\sinh^2 v}{\sinh v\cosh v+\sin v\cos v}\right)\tag{B20}$$

The temperature difference across the fluid for pure conduction is

$$(\theta_r)_c=-\frac{q''s}{k}\tag{B21}$$

The ratio of the actual temperature drop to the temperature drop for pure conduction is

$$\varphi_r=\frac{\theta_r}{(\theta_r)_c}=\frac{1}{v}\left(\frac{\sin^2 v+\sinh^2 v}{\sinh v\cosh v+\sin v\cos v}\right)\tag{B22}$$

From the definitions of $v$ and $a$,

$$v^4=\frac{1}{64}\left(\frac{g\rho^2\beta\frac{\partial t}{\partial z}s^4}{\mu^2}\right)\left(\frac{c_p\mu}{k}\right)=\frac{GrPr}{64}$$

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