REPORT 1271

ON BOATTAIL BODIES OF REVOLUTION HAVING MINIMUM WAVE DRAG[1]

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SUMMARY

The problem of determining the shape of slender boattail bodies of revolution for minimum wave drag has been reexamined. It was found that minimum solutions for Ward's slender-body drag equation can exist only for the restricted class of bodies for which the rate of change of crosssectional area at the base is zero. In order to eliminate this restriction, certain higher order terms must be retained in the drag equation and isoperimetric relations. The minimum problem for the isoperimetric conditions of given length, volume, and base area is treated as an example. According to Ward's drag equation, the resulting body shapes have slightly less drag than those determined by previous investigators.

INTRODUCTION

An approximate expression for the wave drag of slender bodies of revolution having zero rate of change of cross-sectional area at the base was first given by Von Kármán (ref. 1). By using this expression, together with the calculus of variations, several investigators (refs. 1 to 3) have determined minimum-wave-drag bodies for various isoperimetric conditions. Later, Ward (ref. 4) derived the slender-body approximation for the drag of bodies with a nonzero rate of change of cross-sectional area at the base.

In reference 5, Adams considered several minimum-wave-drag problems on the basis of Ward's equation. In each case he concluded that the minimum-drag body had zero slope at the base. This conclusion implied that the minimum shapes for Ward's equation are the same as those for Von Kármán's. Recently, Parker (ref. 6) presented a different expression for the wave drag of slender bodies and showed that the optimum body having given length and base area has a finite slope at the base. Clearly, this result is not in agreement with that obtained by Adams.

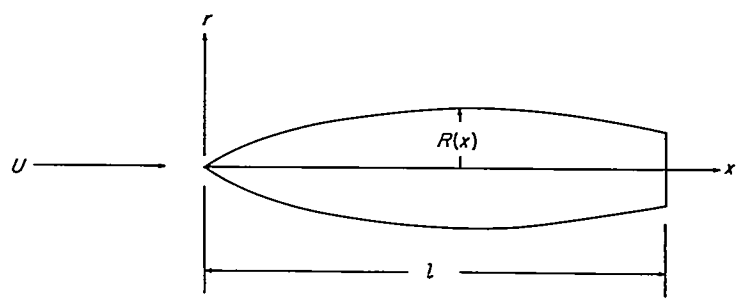
In the present report, the problem of determining minimum-drag boattail bodies of revolution on the basis of linear theory is reexamined with particular emphasis on the choice of drag equation, isoperimetric relations, and method of calculating the body shape. The minimum problem for the isoperimetric conditions of given length, volume, and base area is treated as an example.

DISCUSSION OF MINIMUM-WAVE-DRAG PROBLEM

Within the approximations of linear theory, the supersonic flow past slender bodies of revolution can be represented by a distribution of sources along the axis of the body. The wave drag D of the body may be related to the source distribution (ref. 1) by

$$\frac{4\pi D}{\rho U^2}=-\int_0^l\int_0^l f'(x)f'(\xi)\log_e|x-\xi|dx\,d\xi \tag{1}$$

provided the source distribution $f(x)$ is zero at the nose and the base (i. e., $f(0)=f(l)=0$) where ρ is the stream density and U is the stream velocity. The coordinate system is shown in the following sketch:



In the slender-body approximation, the source strength is related to the body cross-sectional-area distribution $A(x)$ by

$$f(x)=\frac{dA}{dx}=2\pi R(x)R'(x) \tag{2}$$

and the restriction that $f(l)=0$ implies that either the body is closed ($R(l)=0$) or that the body has zero slope at the base ($R'(l)=0$).

Several investigators (refs. 1 to 3) have determined minimum-wave-drag bodies for various isoperimetric conditions by applying the calculus of variations to equation (1). However, as a result of the restriction that $f(l)=0$, these shapes can be considered optimum only for the restricted class of bodies having zero rate of change of cross-sectional area at the base.

[1] Supersedes NACA Technical Note 3478 by Keith C. Harder and Conrad Rennemann, Jr., 1955.

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WARD'S DRAG EQUATION

An equation which does not have the restriction that $f(l)=0$ was proposed by Ward (ref. 4) on the basis of slender-body theory as

$$\frac{4\pi D}{\rho U^2}=-\int_0^l\int_0^l f'(x)f'(\xi)\log_e|x-\xi|dx\,d\xi+2f(l)\int_0^l f'(\xi)\log_e(l-\xi)d\xi-f^2(l)\log_e\frac{\beta R(l)}{2} \tag{3}$$

where $\beta=\sqrt{M^2-1}$ and M is the Mach number. The source strength is again related to the body geometry by equation (2).

The problem of determining the source distribution which minimizes the drag given by equation (3) for given isoperimetric conditions without specifying the value of $f(l)$ at the outset is a variable end-point problem of the calculus of variations. In appendix A this problem is considered for a general type of isoperimetric condition where it is shown that, if a mathematical minimum exists, it satisfies the condition $f(l)=0$. The significance of the mathematical solution obtained by the variational procedure warrants further consideration since the variational procedure assumes the existence of a solution at the outset and, consequently, can lead only to necessary conditions for the attainment of an extremum. Three mutually exclusive possibilities must be considered:

(1) A minimum for Ward's equation exists for the class of bodies having all values of the slope at the base and satisfies the condition $f(l)=0$.

(2) A minimum for Ward's equation exists only for the restricted class of bodies for which $f(l)=0$.

(3) No minimum exists for Ward's equation.

A single example, not satisfying the condition $f(l)=0$ but having less drag than the shape obtained by the variational procedure, is sufficient to eliminate the first possibility. Perhaps the simplest example is the cone which, for given length and base area, has less drag than the variational minimum (Von Kármán's ogive) for $\beta\frac{R(l)}{l}\geq 0.164$. However, a more illuminating example is given by the body

$$A(x)=\frac{A(l)}{l+\epsilon\log_e\frac{\epsilon}{l'}}\left[x+(l'-x)\log_e\left(1-\frac{x}{l'}\right)\right]\quad(0\leq x\leq l) \tag{4}$$

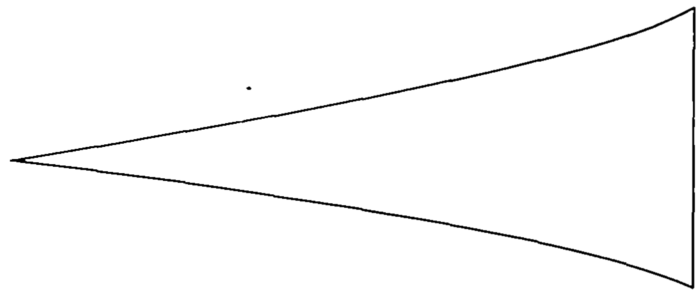
where $A(l)$ is the base area, $l'=l+\epsilon$, and ϵ is a parameter related to the slope at the base $R'(l)$ by

$$R'(l)=\frac{R(l)\log_e\frac{l'}{\epsilon}}{2\left(l+\epsilon\log_e\frac{\epsilon}{l'}\right)}$$

For this body, Ward's equation (eq. (3)) gives the result that, for small ϵ,

$$\frac{4\pi D}{\rho U^2}=-\left[\frac{A(l)}{l+\epsilon\log_e\frac{\epsilon}{l'}}\right]^2\left(\log_e\frac{\epsilon}{l'}\right)^2\left\{\log_e\left[\frac{\beta R(l)}{2l'}\left(\frac{l'}{\epsilon}\right)^{3/2}\right]+O\left(\frac{\epsilon/l'}{\log_e\frac{\epsilon}{l'}}\right)\right\}$$

which approaches minus infinity as ϵ approaches zero; that is, as the slope and curvature at the base both approach infinity. (The mathematical symbol $O(\ )$ denotes the order of a function.) The body shape for $\epsilon=0$ is shown in the following sketch:



In order to decide between the second and third possibilities, it would be necessary to prove the existence or nonexistence of a minimum solution for Von Kármán's drag equation (eq. (1)). Such a study is beyond the scope of the present report; however, it should be noted that the minimum solutions obtained for Von Kármán's equation have provided a useful guide in the search for low-drag shapes. It is in this same vein that, later in the report, minimum problems are considered on the basis of a different drag equation. Since existence proofs are not attempted, the most that can be claimed is that, if a solution exists, it must have a certain mathematical form. However, in order to avoid the repetition of this qualifying remark in the remainder of the report, solutions obtained by the calculus of variations are referred to as minimum solutions.

The problem under discussion has been previously considered by Adams (ref. 5) who correctly determined the necessary conditions for a minimum. His interpretation of these conditions was that the optimum boattail body has zero rate of change of cross-sectional area at the base. However, the proper interpretation is that, if a minimum exists, it exists only for the restricted class of bodies having zero rate of change of cross-sectional area at the base.

PARKER'S DRAG EQUATION

From the preceding discussion, it is clear that Ward's drag equation (eq. (3)) cannot be used to determine minimum-drag boattail bodies.[2] Parker (ref. 6) has shown that application of the calculus of variations to the drag equation

$$\frac{4\pi D}{\rho U^2}=\int_0^{l-\beta R(l)}\int_0^{l-\beta R(l)} f'(\xi)f'(x)\cosh^{-1}\left|\frac{(l-\xi)(l-x)-\beta^2R^2(l)}{\beta R(l)(x-\xi)}\right|dx\,d\xi \tag{5}$$

which he obtained on the basis of linear theory, yields a minimum without the restriction that $f(l)=0$ for the isoperimetric conditions of given length and base area. The body shape so determined has a finite slope at the base and less drag than the mathematical minimum for equation (3).

Equation (5) contains some higher order terms which are not included in the slender-body approximation to the drag (eq. (3)) since Parker did not make the slender-body approximation to the velocity potential in the derivation. Apparently, the additional terms are necessary in order to obtain minimum-drag shapes without the restriction that $f(l)=0$. However, it should not be inferred that equation (5) necessarily gives a better estimate of the drag of bodies satisfying the assumptions of slender-body theory than equation (3). Lighthill (ref. 8) has shown that the slender-body equations (eqs. (2) and (3)) are fully as accurate as the linearized differential equations of motion for sufficiently smooth bodies. Consequently, the slender-body results are theoretically equivalent to those obtained without making the slender-body approximation.

ISOPERIMETRIC CONDITIONS

The isoperimetric conditions most commonly considered have been those of fixed length, volume, and base area. In order to carry out the mathematical details of determining the source strength which minimizes the drag, the isoperimetric conditions must be directly related to the source strength. The simplest relations would appear to be those given by slender-body theory. However, in order to carry out the analysis on the basis of Parker's equation, certain higher order terms must be retained in the isoperimetric relations. In particular, the limits of integration in the isoperimetric relations must be the same as those in the drag equation. Furthermore, the analysis can sometimes be simplified by including certain additional higher order terms in the integrand of the isoperimetric relations.

The relation between the isoperimetric conditions and source strength used in the example to be treated in the present report is obtained by approximately satisfying the boundary conditions on a cone passing through the nose and base. The linear-theory expression for $A'(x)$ is

$$A'(x)=\int_0^{x-\beta r}\frac{(x-\xi)f'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \tag{6}$$

where r is on the body surface. Equation (6) is approximately satisfied by evaluating the integral on the cone $r=R(l)\,\frac{x}{l}$. Then,

$$A'(x)\approx\int_0^{x\left[1-\beta\frac{R(l)}{l}\right]}\frac{(x-\xi)f'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2\frac{R^2(l)}{l^2}x^2}} \tag{7}$$

from which, with $\delta=\beta\,\frac{R(l)}{l}$,

$$A(x)\approx\int_0^{x(1-\delta)}f'(\xi)\sqrt{(x-\xi)^2-\delta^2x^2}\,d\xi$$

and

$$A(l)\approx\int_0^{l(1-\delta)}f'(\xi)\sqrt{(l-\xi)^2-\delta^2l^2}\,d\xi \tag{8}$$

where use has been made of the condition $f(0)=0$. Similarly, the volume V is given approximately by

$$V\approx\frac{1}{2}\int_0^{l(1-\delta)}(l-\xi)f'(\xi)\sqrt{(l-\xi)^2-\delta^2l^2}\,d\xi \tag{9}$$

In the derivation of equations (8) and (9) from equation (7), terms of the order of $\delta^2\log_e\delta$ have been neglected. The slender-body approximation to the isoperimetric relations is obtained by equating δ to zero in equations (8) and (9).

CALCULATION OF BODY SHAPE

When the calculus of variations is applied to the drag equation and isoperimetric relations, the resulting source strength for minimum drag contains several constants to be determined from the isoperimetric conditions. The calculation of these constants and the body shape can be treated independently of the minimization process.

Since higher order terms have been retained in the drag equation and isoperimetric relations, the question arises as to whether similar terms should be retained in the body-shape calculation. Theoretically, the inclusion of these terms does not affect the accuracy of the result for shapes satisfying the assumptions of slender-body theory. Even so, it is interesting to compare the various body shapes obtained from the source distribution

$$f(\xi)=K\sqrt{\xi[l+\beta R(l)-\xi]} \tag{10}$$

found by Parker to give minimum drag for the isoperimetric conditions of given length and base area. In equation (10), K is a constant to be determined from the isoperimetric

[2] Essentially the same arguments can be used to show that Lighthill's drag equation (ref. 7), which was derived for slender shapes with discontinuities in slope, cannot be used to determine minimum-drag bodies with corners.

conditions. The numerical value of K depends on the method used to calculate the body shape.

Parker calculated the body shape from this source distribution without making the slender-body approximation by numerically solving the integral equation

$$\pi R^2(x)=\int_0^{x-\beta R(x)} f'(\xi)\sqrt{(x-\xi)^2-\beta^2R^2(x)}\,d\xi \tag{11}$$

The body shape calculated from equation (10) by means of the slender-body expression $A'(x)=f(x)$ is

$$A(x)=\frac{Kl^2}{2(1+c)^2}\left[t\sqrt{1-t^2}+\cos^{-1}(-t)\right] \quad (-1\leqq t\leqq c) \tag{12}$$

where

$$\frac{x}{l}=\frac{1+t}{1+c},\ \frac{Kl^2}{2(1+c)^2}=\frac{A(l)}{c\sqrt{1-c^2}+\cos^{-1}(-c)},\ \text{and } c=\frac{l-\beta R(l)}{l+\beta R(l)}.$$

When the expression $A'(x)=f(x)$ is altered to take partially into account the fact that a given point on the body is influenced only by sources in the upstream Mach cone by equating

$$\frac{dA(x)}{dx}=f\left(x\left[1-\beta\frac{R(l)}{l}\right]\right) \tag{13}$$

the body shape is given by

$$A(x)=\frac{l^2K}{4c(1+c)}\left[t\sqrt{1-t^2}+\cos^{-1}(-t)\right] \quad (-1\leqq t\leqq 2c-1) \tag{14}$$

where

$$t=\frac{2cx}{l}-1$$

and

$$\frac{l^2K}{4c(1+c)}=\frac{A(l)}{2(2c-1)\sqrt{c(1-c)}+\cos^{-1}(1-2c)}$$

The body shapes calculated by means of equations (11), (12), and (14) for $\beta\frac{R(l)}{l}=0.2$ are compared in figure 1. The differences between the shapes are small even for this rather large value of $\beta\frac{R(l)}{l}$. For smaller values of $\beta\frac{R(l)}{l}$, the differences are even less and the shapes become coincident as $\beta\frac{R(l)}{l}$ approaches zero. Similar body-shape calculations based on equations (2) and (13) for the source strength that is derived in the example (fixed volume, length, and base area) were performed for $\beta\frac{R(l)}{l}=0.05$ and several values of $\beta^2\frac{V}{l^3}$, where V is the volume. This comparison is not presented since the body shapes obtained by the two methods are almost identical. Evidently, the difference in body shape is appreciable only for shapes that cannot be considered slender. Consequently, the simpler slender-body relation is preferable.

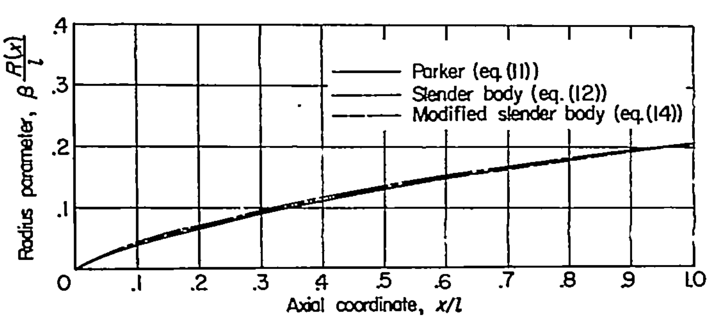


FIGURE 1.—Comparison of body shapes calculated from source ditrsibution given by equation (10) by various methods for $\beta\frac{R(l)}{l}=0.2$.

The discussion concerning the inclusion of higher order terms is briefly summarized as follows: Higher order terms must be retained in the drag equation in order to obtain the minimum-drag boattail body; having done this, higher order terms must also be retained in the isoperimetric relations in order to perform the analysis. Once the source strength for minimum drag has been determined within several undetermined constants, higher order terms need not be retained in the calculation of the shape and drag of bodies satisfying the assumptions of slender-body theory.

PROBLEM OF LENGTH, VOLUME, AND BASE AREA

The problem of determining the body shape that gives minimum wave drag for fixed length, volume, and base area is treated in order to illustrate the ideas developed in the preceding sections. The minimum-drag body having given length and base area or given length and volume can be obtained as special cases of the problem under consideration.

The source distribution for minimum drag is obtained by applying the calculus of variations to equations (5), (8), and (9), and as shown in appendix B, this leads to the source distribution

$$f(\xi)=(a+b\xi)\sqrt{\xi[l-\xi+\beta R(l)]} \tag{15}$$

where a and b are constants to be determined from the isoperimetric conditions.

As discussed in the previous section, the body shape is determined on the basis of the slender-body equations. Integration of $A'(x)=f(x)$ gives

$$A(x)=\frac{l^2}{(1+c)^2}\left\{\frac{A}{2}\left[t\sqrt{1-t^2}+\cos^{-1}(-t)\right]-\frac{B}{3}(1-t^2)^{3/2}\right\}$$
$$(-1\leqq t\leqq c) \tag{16}$$

where

$$\frac{x}{l}=\frac{1+t}{1+c} \qquad c=\frac{l-\beta R(l)}{l+\beta R(l)} \qquad B=\frac{l+\beta R(l)}{2}\,b$$

and

$$A=a+\frac{l+\beta R(l)}{2}\,b$$

The base area is given by

$$A(l)=\frac{l^2}{(1+c)^2}\left\{\frac{A}{2}\left[c\sqrt{1-c^2}+\cos^{-1}(-c)\right]-\frac{B}{3}(1-c^2)^{3/2}\right\} \tag{17}$$

The volume is obtained from equation (16) as

$$V=\frac{l^3}{2(1+c)^3}\left\{A\left[c\cos^{-1}(-c)+\sqrt{1-c^2}-\frac{1}{3}(1-c^2)^{3/2}\right]-\frac{B}{4}\left[\frac{2c}{3}(1-c^2)^{3/2}+c\sqrt{1-c^2}+\cos^{-1}(-c)\right]\right\} \tag{18}$$

Equation (16) for $A(x)$ (with the use of eqs. (17) and (18)) reduces to the minimum-drag body shape given by Haack (ref. 3) and Adams (ref. 5) when $c=1$. In this case, $A(x)$ is given by

$$A(x)=\frac{A(l)}{\pi}\left[t\sqrt{1-t^2}+\cos^{-1}(-t)\right]+\frac{8}{3\pi}\left[\frac{2V}{l}-A(l)\right](1-t^2)^{3/2}\quad(-1\leq t\leq 1)\tag{19}$$

where

$$\frac{x}{l}=\frac{1+t}{2}$$

This body was obtained by Haack on the basis of Von Kármán's drag equation (eq. (1)) and by Adams on the basis of Ward's drag equation and is referred to in the remainder of the report as the Haack-Adams body.

The constants A and B in the equation for the body shape (eq. (16)) are determined from equations (17) and (18) in terms of l, β, $R(l)$, and V. The solution may be expressed as

$$\beta^2A=A_1-A_2\beta^2\frac{V}{l^3}\tag{20}$$

and

$$\beta^2B=B_1-B_2\beta^2\frac{V}{l^3}\tag{21}$$

where A_1, A_2, B_1, and B_2 are functions of $\beta\frac{R(l)}{l}$. Values of A_1, A_2, B_1, and B_2 are given in table I for values of $\beta\frac{R(l)}{l}$ between 0.01 and 0.10.

A direct comparison of the drag of the body of the present report with that of the Haack-Adams body is made on the basis of Ward's drag equation. From equation (3), the drag of the source distribution given by equation (15) is

$$\frac{D}{\rho U^2}=\frac{1}{8\pi}\frac{l^2}{(1+c)^2}\Big(A^2\{[\cos^{-1}(-c)]^2+2c\sqrt{1-c^2}\cos^{-1}(-c)-(1-c^2)\}-4AB(1-c^2)[c+\sqrt{1-c^2}\cos^{-1}(-c)]+\frac{B^2}{2}[(2-5c^2)(1-c^2)+2c\sqrt{1-c^2}(2c^2-1)\cos^{-1}(-c)+(\cos^{-1}(-c))^2]+2(A+Bc)^2(1-c^2)\log_e[4(1+c)]\Big)\tag{22}$$

TABLE I

COEFFICIENTS OF EQUATIONS (20) AND (21)

$\beta\frac{R(l)}{l}$	$\beta^2A=A_1-A_2\beta^2\frac{V}{l^3}$ (eq. (20))		$\beta^2B=B_1-B_2\beta^2\frac{V}{l^3}$ (eq. (21))	
	A_1	A_2	B_1	B_2
0.10	0.080572	0.76164	0.26804	18.0291
.09	.064852	.66854	.21938	18.1466
.08	.050946	.57666	.17531	18.2780
.07	.038807	.48644	.13590	18.4255
.06	.028390	.39843	.10121	18.5919
.05	.019652	.31338	.071357	18.7809
.04	.012554	.23231	.046443	18.9973
.03	.0070607	.15669	.026623	19.2482
.02	.0031450	.088843	.012091	19.5439
.01	.00079066	.032849	.0031002	19.8686

The drag of the Haack-Adams body is

$$\frac{D}{\rho U^2}=\frac{2l^2}{\pi}\left\{9\frac{A^2(l)}{l^4}+32\frac{V}{l^3}\left[\frac{V}{l^3}-\frac{A(l)}{l^2}\right]\right\}\tag{23}$$

The form of equations (20), (21), and (22) indicates that $\frac{\beta^4D}{\rho U^2l^2}$ is a function of $\beta^2\frac{V}{l^3}$ and $\beta\frac{R(l)}{l}$. In figure 2, the drag of the Haack-Adams body (eq. (23)) and the drag given by equation (22) are plotted on a logarithmic scale for several values of $\beta\frac{R(l)}{l}$. To help orient the reader, several body shapes are shown for $\beta=1$, $\frac{R(l)}{l}=0.05$. The drag given by equation (22) is somewhat less than that of the Haack-Adams body for most values of $\beta^2\frac{V}{l^3}$. For example, for $\beta\frac{R(l)}{l}=0.05$ and $\beta^2\frac{V}{l^3}=0.01$, which represents a fuselage-type shape, the body given by equation (16) has approximately 7½ percent less drag than the Haack-Adams body. Each drag curve begins at a particular value of $\beta^2\frac{V}{l^3}>0$; for a given value of $\beta\frac{R(l)}{l}$, smaller values of $\beta^2\frac{V}{l^3}$ give rise to negative body areas.

For a given value of $\beta\frac{R(l)}{l}$, the slope at the base of the body is positive for small values of $\beta^2\frac{V}{l^3}$ and is negative for large values. The two drag curves become nearly tangent at intermediate values of $\beta^2\frac{V}{l^3}$ for which the body slope at the base is near zero. Actually, the Haack-Adams body must have less drag for this condition since this body gives minimum drag for Ward's equation for the class of bodies which have zero slope at the base.

The value of $\beta^2\frac{V}{l^3}$ for minimum drag is obtained, for a given value of $\beta\frac{R(l)}{l}$, by equating B to zero in equations (17) and (18). This procedure gives the optimum body having a given length and base area.

In figure 3 the body shape of the present report is compared with the Haack-Adams body for $\beta\frac{R(l)}{l}=0.05$ and $\beta^2\frac{V}{l^3}=0.003$ and 0.02. The bodies are plotted to an expanded vertical scale (expanded 5 times) to illustrate the differences which for the most part are small. The most significant difference occurs near the base for the larger values of $\beta^2\frac{V}{l^3}$ where the body given by equation (16) does not exhibit the reflex shape of the afterbody characteristic of the Haack-Adams body.

The effect of Mach number on body shape is illustrated in figure 4 where the optimum shapes (vertical scale enlarged 2½ times) for $\frac{R(l)}{l}=0.05$ and $\frac{V}{l^3}=0.02$ are compared for $M=\sqrt{2}$ and $M=\sqrt{5}$. The body shape of the present report exhibits a small dependence on Mach number, whereas the Haack-Adams body is independent of Mach number.

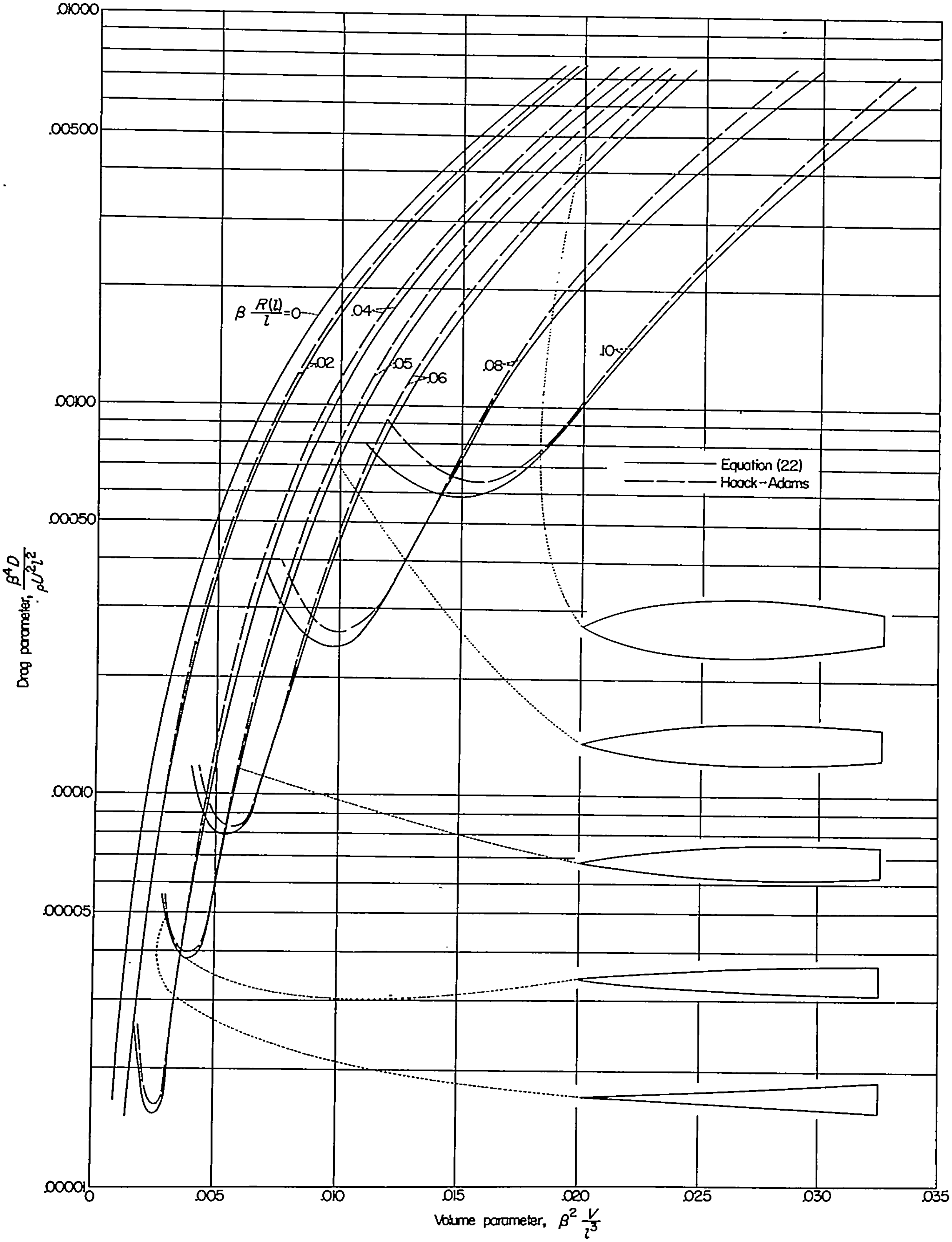


FIGURE 2.—Comparison of drag given by Ward's equation for Haack-Adams body and body given by equation (22). Body shapes shown are for $\beta\frac{R(l)}{l}=0.05$.

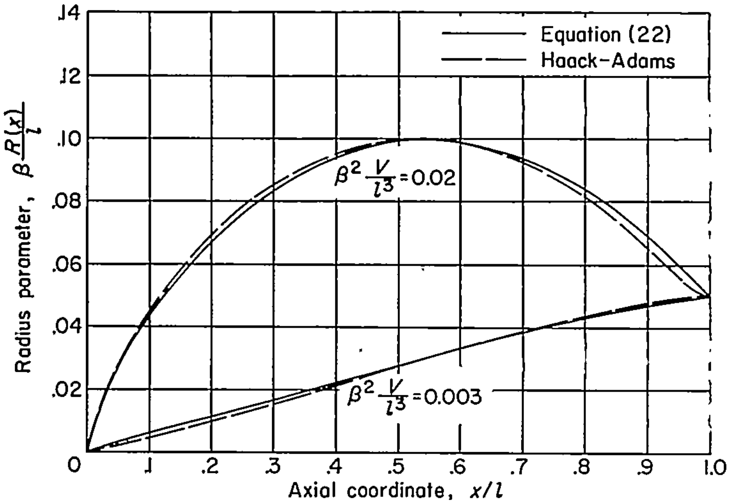


FIGURE 3.—Comparison of Haack-Adams body with that given by equation (22) for $\beta\frac{R(l)}{l}=0.05$.

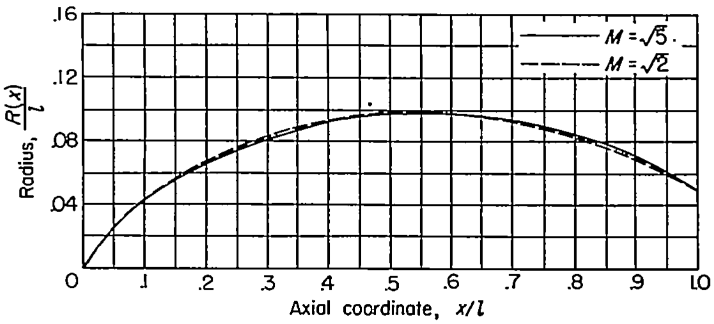


FIGURE 4.—Effect of Mach number on body shape given by equation (22) for $\frac{V}{l^3}=0.02$ and $R=\frac{R(l)}{l}=0.05$.

CONCLUSIONS

The problem of determining the shape of slender boattail bodies of revolution for minimum wave drag has been reexamined and the following conclusions are indicated:

1. Minimum solutions for Ward's drag equation can exist only for the restricted class of bodies for which the rate of change of cross-sectional area at the base is zero.

2. In order to eliminate this restriction, certain higher order terms must be retained in the drag equation and isoperimetric relations. However, higher order terms need not be retained in the calculation of drag and body shape from the source distribution.

3. Adams in NACA Technical Note 2550 correctly determined the necessary conditions for a minimum for Ward's drag equation. His interpretation of these conditions was that the optimum boattail body has zero rate of change of cross-sectional area at the base. However, the proper interpretation is that, if a minimum exists, it exists only for the restricted class of bodies having zero rate of change of cross-sectional area at the base.

4. Application of the ideas expressed in conclusion 2 to the minimum problem of given length, volume, and base area led to body shapes which have slightly less drag than the Haack-Adams body.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *June 8, 1955.*

APPENDIX A

APPLICATION OF CALCULUS OF VARIATIONS TO WARD'S DRAG EQUATION

In this appendix the calculus of variations is applied to Ward's drag equation (eq. (3)) for a general type of isoperimetric condition to determine the source strength for minimum drag when the source strength at the base $f(l)$ is not specified at the outset. From equation (3), Ward's drag equation is

$$\frac{4\pi D}{\rho U^2}=-\int_0^l\int_0^l f'(x)f'(\xi)\log_e|x-\xi|dx\,d\xi+$$
$$2f(l)\int_0^l f'(\xi)\log_e(l-\xi)d\xi-f^2(l)\log_e\beta\frac{R(l)}{2} \quad \text{(A1)}$$

The usual isoperimetric conditions considered may be related to the source strength by expressions of the form

$$I_i=\int_0^l f(\xi)g_i(\xi)d\xi \quad \text{(A2)}$$

For example, $g(\xi)=1$ for fixed base area and $g(\xi)=(l-\xi)$ for fixed volume. In the subsequent analysis, it is assumed that one of the isoperimetric conditions is that of fixed base area. This assumption simplifies the analysis without restricting its generality.

In the derivation of equation (A1) it is assumed that

$$f(0)=0 \quad \text{(A3)}$$

Equation (A3) gives one of the end-point conditions to be satisfied by the minimizing source distribution. At the other end point, $x=l$, the value of $f(x)$ is not prescribed.

The source distribution for minimum drag is obtained by considering the variation of the function

$$J=\frac{4\pi D}{\rho U^2}+\sum\lambda_i I_i \quad \text{(A4)}$$

where the Lagrange multipliers λ_i are determined from the isoperimetric conditions. The variation of equation (A4) is obtained by considering the one-parameter family of comparison functions (see ref. 9, for example)

$$f(x)=F(x)+\epsilon\eta(x) \quad \text{(A5)}$$

where $F(x)$ is the function which minimizes equation (A4), ϵ is the parameter of the family, and $\eta(x)$ is an arbitrary function within the condition

$$\eta(0)=0 \quad \text{(A6)}$$

This condition arises since all the comparison functions must satisfy the same end-point condition as the minimizing function. Since no end-point condition is prescribed at $x=l$, $\eta(l)$ is arbitrary.

From equations (A4) and (A5), J is a function of ϵ and the source strength for minimum drag is determined from the condition

$$\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0}=0=\int_0^l \eta(\xi)\left[-\frac{d}{d\xi}\int_0^l \frac{F(x)dx}{x-\xi}+\sum\frac{\lambda_i}{2}g_i(\xi)\right]d\xi+$$

$$\eta(l)\left[\lim_{\xi\to l}F(l)\log_e\frac{2(l-\xi)}{\beta R(l)}\right] \qquad \text{(A7)}$$

where the equation has been simplified by several integrations by parts and use of equations (A3) and (A6).

Since equation (A7) must hold for all choices of $\eta(\xi)$ consistent with equation (A6), it must in particular hold for those choices of $\eta(\xi)$ for which $\eta(l)=0$. For such $\eta(\xi)$,

$$\int_0^l \eta(\xi)\left[-\frac{d}{d\xi}\int_0^l \frac{F(x)dx}{x-\xi}+\sum\frac{\lambda_i}{2}g_i(\xi)\right]d\xi=0$$

and from the basic lemma of the calculus of variations (ref. 8),

$$\frac{d}{d\xi}\int_0^l \frac{F(x)dx}{x-\xi}=\sum\frac{\lambda_i}{2}g_i(\xi) \qquad \text{(A8)}$$

With this result, and for general $\eta(\xi)$ once again, that is, $\eta(l)$ not necessarily equal to zero, the end-point condition obtained is that

$$\lim_{\xi\to l}\left[F(l)\log_e\frac{2(l-\xi)}{\beta R(l)}\right]=0 \qquad \text{(A9)}$$

In order to satisfy this condition, $F(l)$ must equal zero. Consequently, the body shapes which give a mathematical minimum for Ward's equation, if they exist, must have zero rate of change of cross-sectional area at the base.

APPENDIX B

APPLICATION OF CALCULUS OF VARIATIONS TO PARKER'S DRAG EQUATION

The source distribution for minimum drag for the isoperimetric condition of given length, volume, and base area is obtained by considering the variation of the function

$$J=\frac{4\pi D}{\rho U^2}+\lambda_1 A(l)+\lambda_2 V \qquad \text{(B1)}$$

where D, $A(l)$, and V are given by equations (5), (8), and (9), respectively, and λ_1 and λ_2 are Lagrange multipliers. By proceeding in the same manner as in appendix A, the variation of equation (B1) is obtained by considering the one-parameter family of comparison functions.

$$f(x)=F(x)+\epsilon\eta(x) \qquad \text{(B2)}$$

where $F(x)$ is the function which minimizes equation (B1), ϵ is the parameter of the family, and $\eta(x)$ is an arbitrary function within the condition $\eta(0)=0$.

The source strength for minimum drag is then determined from the condition

$$\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0}=0=-\int_0^{l-\beta R(l)}\eta(\xi)\frac{d}{d\xi}\left\{2\int_0^{l-\beta R(l)}\frac{F(x)}{x-\xi}\frac{\sqrt{(l-\xi)^2-\beta^2R^2(l)}}{\sqrt{(l-x)^2-\beta^2R^2(l)}}dx+\left[\lambda_1+\frac{\lambda_2}{2}(l-\xi)\right]\sqrt{(l-\xi)^2-\beta^2R^2(l)}\right\}d\xi+$$

$$\eta(l-\beta R(l))\lim_{\xi\to l-\beta R(l)}\left\{2\int_0^{l-\beta R(l)}\frac{F(x)}{x-\xi}\frac{\sqrt{(l-\xi)^2-\beta^2R^2(l)}}{\sqrt{(l-x)^2-\beta^2R^2(l)}}dx+\left[\lambda_1+\frac{\lambda_2}{2}(l-\xi)\right]\sqrt{(l-\xi)^2-\beta^2R^2(l)}\right\} \qquad \text{(B3)}$$

where the equation has been simplified by several integrations by parts and use of the conditions

$$F(0)=\eta(0)=0$$

Since equation (B3) must hold for all choices of $\eta(\xi)$ consistent with $\eta(0)=0$, it must in particular hold for those choices of $\eta(\xi)$ for which $\eta[l-\beta R(l)]=0$. For such $\eta(\xi)$,

$$\int_0^{l-\beta R(l)}\eta(\xi)\frac{d}{d\xi}\left\{2\int_0^{l-\beta R(l)}\frac{F(x)}{x-\xi}\frac{\sqrt{(l-\xi)^2-\beta^2R^2(l)}}{\sqrt{(l-x)^2-\beta^2R^2(l)}}dx+\left[\lambda_1+\frac{\lambda_2}{2}(l-\xi)\right]\sqrt{(l-\xi)^2-\beta^2R^2(l)}\right\}d\xi=0$$

and from the basic lemma of the calculus of variations,

$$2\int_0^{l-\beta R(l)}\frac{F(x)}{x-\xi}\frac{\sqrt{(l-\xi)^2-\beta^2R^2(l)}}{\sqrt{(l-x)^2-\beta^2R^2(l)}}dx+\left[\lambda_1+\frac{\lambda_2}{2}(l-\xi)\right]\sqrt{(l-\xi)^2-\beta^2R^2(l)}=N \qquad \text{(B4)}$$

where N is a constant. With this result, and for general $\eta(\xi)$, the end-point condition is obtained as

$$\lim_{\xi\to l-\beta R(l)}\left\{2\int_0^{l-\beta R(l)}\frac{F(x)}{x-\xi}\frac{\sqrt{(l-\xi)^2-\beta^2R^2(l)}}{\sqrt{(l-x)^2-\beta^2R^2(l)}}dx+\left[\lambda_1+\frac{\lambda_2}{2}(l-\xi)\right]\sqrt{(l-\xi)^2-\beta^2R^2(l)}\right\}=0 \tag{B5}$$

Since equation (B4) must hold for all values of ξ, and in particular for $\xi\to l-\beta R(l)$, from equation (B5), $N=0$. Hence, $\sqrt{(l-\xi)^2-\beta^2R^2(l)}$ can be canceled from each term of equation (B4) and the following integral equation is obtained for the source strength:

$$2\int_0^{l-\beta R(l)}\frac{F(x)}{x-\xi}\frac{dx}{\sqrt{(l-x)^2-\beta^2R^2(l)}}=-\lambda_1-\frac{\lambda_2}{2}(l-\xi) \tag{B6}$$

The solution of equation (B6) satisfying the condition $F(0)=0$ is

$$F(\xi)=f(\xi)=(a+b\xi)\sqrt{\xi[l-\xi+\beta R(l)]} \tag{B7}$$

where a and b are constants related to λ_1 and λ_2.

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