REPORT 1325

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AVERAGE PROPERTIES OF COMPRESSIBLE LAMINAR BOUNDARY LAYER ON FLAT PLATE WITH UNSTEADY FLIGHT VELOCITY[1]

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SUMMARY

The time-average characteristics of boundary layers over a flat plate in nearly quasi-steady flow are determined. The plate may be either insulated or isothermal. The time averages are found without specifying the plate velocity explicitly except that it is positive and has an average value.

Each time average involves two groups of terms to the order considered in the report, a time average of quasi-steady terms, and terms related to the reduced frequency of the plate velocity fluctuations.

The quasi-steady terms differ from the values for steady flow at the corresponding average velocity. These differences are reinforced by the frequency dependent averages for adiabatic wall temperature and heat-transfer rate. The effects oppose one another in the case of skin friction.

The special case of harmonic velocity variation is considered, and it is found that large amplitudes accentuate the importance of the frequency-dependent terms.

Oscillating the wall to increase the heat-transfer rate is not advantageous if the power to oscillate the plate is accounted for.

INTRODUCTION

In many current problems of aerodynamics, unsteady motions of a surface are important. The accelerating and decelerating phases of missile flight and the intermittent flow in an engine during unstable combustion are examples. The nature of the boundary layer of these unsteady flows may be studied with a view to determining friction drag, surface temperature, and rate of heat transfer through the surface. Usually, the boundary layer is so thin that it responds almost instantly to temporal changes in flow conditions. Thus, the time history of the boundary layer is a succession of steady states, and such a boundary layer is called quasi-steady. If the motion involves accelerations which are particularly rapid, the quasi-steady description may require correction.

In any case, it is often desired to assess the average effect of fluctuations in flow conditions. For example, if an insulated body is in motion with a speed that varies in time about some average value, the average friction drag and the average surface temperature may differ from values appropriate to steady motion at the average speed. Another example is furnished by the speculation that the net performance of a heat exchanger could be altered by imparting an unsteady motion to the wall.

In the present study, as an idealized special case of the foregoing type of problem, a semi-infinite flat plate is assumed to be in motion parallel to its surface and normal to its leading edge with a flight velocity $U(t)$ that is always in the same direction but has a magnitude that fluctuates with time. The resulting boundary layer is assumed laminar and compressible, and the surface is either insulated or at constant temperature. The assumed Prandtl number is 0.72, which is appropriate for air under normal conditions.

The boundary layer is assumed to be nearly quasi-steady. The basic boundary-layer analysis is already available for this problem in references 1 and 2, which treat the insulated plate and the constant plate temperature cases, respectively. The velocity profile in the boundary layer is found in the form

$$\frac{u}{U}=\frac{1}{2}\left[F'(\sigma)+\zeta_0 f_0'(\sigma)+\zeta_1 f_1'(\sigma)+\ldots+\zeta_0^2 f_{00}'(\sigma)+\ldots\right] \tag{1}$$

where σ is the usual Blasius variable signifying parabolic similarity in the boundary layer. (A complete definition will appear in a subsequent section.) The function $F(\sigma)$ is the Blasius function for steady motion of a flat plate. The parameters ζ_n govern deviations from quasi-steadiness:

$$\zeta_{n-1}=\frac{X^n}{U^{n+1}}\frac{d^nU(T)}{dT^n} \qquad n=1,2,3,\ldots \tag{2}$$

In cases of flight at substantial speed, the factor U^{n+1} usually ensures that ζ_{n-1} is a rather small quantity. The functions $F(\sigma)$, $f_0(\sigma)$, and $f_1(\sigma)$ are available in reference 1. A full list of symbol notation is provided in appendix A.

If the plate surface is insulated, the temperature profile may be written as

$$\frac{\theta}{\theta_\infty}=1+\frac{\gamma-1}{2}M^2(T)r(\sigma,\zeta_0,\zeta_1,\zeta_2,\ldots) \tag{3a}$$

where the "recovery factor"

$$r=R(\sigma)+\zeta_0 r_0(\sigma)+\zeta_1 r_1(\sigma)+\ldots+\zeta_0^2 r_{00}(\sigma)+\ldots \tag{3b}$$

and the functions R, r_0, and r_1 are tabulated in reference 1. If the plate is at a constant temperature at which heat transfer takes place, reference 2 gives

[1] Supersedes NACA Technical Note 3886, "Average Properties of Compressible Laminar Boundary Layer on Flat Plate with Unsteady Flight Velocity," by Franklin K. Moore and Simon Ostrach, 1956.

$$\Theta \equiv \frac{\theta-\theta_\infty}{\theta_w-\theta_\infty}=H(\sigma)+\frac{U^2}{2c_p(\theta_w-\theta_\infty)}S(\sigma)+$$
$$\zeta_0\left[h_0(\sigma)+\frac{U^2}{2c_p(\theta_w-\theta_\infty)}s_0(\sigma)\right]+$$
$$\zeta_1\left[h_1(\sigma)+\frac{U^2}{2c_p(\theta_w-\theta_\infty)}s_1(\sigma)\right]+\cdots+$$
$$\zeta_0^2\left[h_{00}(\sigma)+\frac{U^2}{2c_p(\theta_w-\theta_\infty)}s_{00}(\sigma)\right]+\cdots \tag{4}$$

and provides the functions H, S, h_0, s_0, h_1, and s_1.

A study of the time-average properties of the flow represented by equations (1) to (4), when the speed of the plate[2] U fluctuates with time, is described herein. The functions f_{00}, r_{00}, h_{00}, and s_{00} are required for this purpose, and, not being available in references 1 and 2, are determined herein.

A previous report (ref. 3) studies the average rate of heat transfer from an oscillating flat plate, but differs from the present study in that the plate is doubly infinite and there is no net motion through the surrounding fluid; on the average, the plate is at rest. Appropriate comparisons are made herein between the results of the present study and those of reference 3.

DERIVATION OF FORMULAS FOR AVERAGE PROPERTIES OF BOUNDARY LAYERS

The stream function ψ for nearly quasi-steady flows is defined in references 1 and 2 as

$$\psi=\sqrt{C\nu_\infty UX}\,f(\sigma,\ \zeta_0,\ \zeta_1,\ \zeta_2,\ .\ .\ .) \tag{5}$$

where

$$\sigma \equiv \frac{Y}{2}\sqrt{\frac{U}{C\nu_\infty X}} \tag{6}$$

The coordinates X and Y measured in a system with its origin fixed at the leading edge of the plate are related to those (x,y) in a coordinate system which is stationary in the fluid by (see sketches (a) and (b)):

$$X=x+\int_0^t U\,dt \tag{7}$$

$$Y=\int_0^y \frac{\rho}{\rho_\infty}\,dy \tag{8}$$

Equation (8) is employed to make the momentum equation independent of the energy equation. The velocity in the X,Y-system is related to that in the x,y-system by

$$u(X,Y,T)=u_a(x,y,t)+U(t) \tag{9}$$

The constant C is the proportionality factor in the assumed viscosity-temperature variation

$$\mu=\mu_\infty C\frac{\theta}{\theta_\infty}=\rho_\infty\nu_\infty C\frac{\theta}{\theta_\infty} \tag{10}$$

[2] In the present problem, if compressibility is important, unsteadiness is restricted to enter only through motion of the plate. Thus, in a corresponding wind-tunnel test the model position would be varied mechanically and the tunnel flow would be held constant. The analyses of references 1 and 2 for a compressible fluid would not apply for a fixed model in a pulsing flow.

This constant may be evaluated by matching equation (10) with the Sutherland formula at some appropriate point, for example, at the wall. Thus

$$C=\sqrt{\frac{\theta_w}{\theta_\infty}}\left(\frac{\theta_\infty+216^\circ\ \mathrm{R}}{\theta_w+216^\circ\ \mathrm{R}}\right) \tag{11}$$

The unsteady boundary-layer characteristics can be determined from equations (1) to (6).

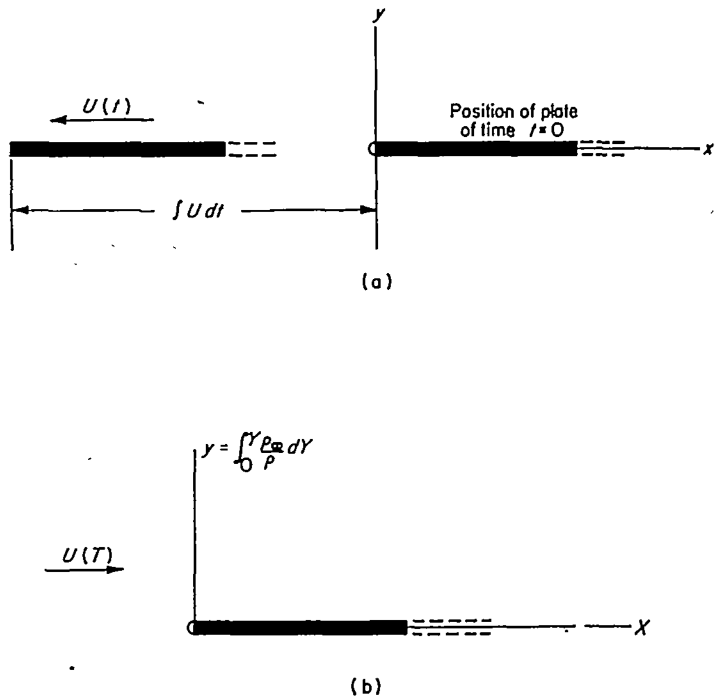
FORM OF UNSTEADY BOUNDARY-LAYER CHARACTERISTICS

Insulated plate.—The unsteady boundary-layer characteristics from which the average properties will be determined are presented in this and the subsequent section in a more extended form than in references 1 and 2. The wall shear stress $\tau_w=\left(\mu\frac{\partial u}{\partial y}\right)_w$ may be obtained in the following dimensionless form from equations (1), (8), and (10), and from the state equation for constant pressure ($\rho\theta$=Constant):

$$C_f\equiv\frac{\tau_w}{\frac{1}{2}\rho_\infty U^2}=\frac{1}{2}\sqrt{\frac{C\nu_\infty}{UX}}\,[F''(0)+\zeta_0 f_0''(0)+\zeta_1 f_1''(0)+\zeta_0^2 f_{00}''(0)+\ldots] \tag{12}$$

The displacement thickness δ^* may be defined[3] as

$$\delta^*\equiv\int_0^\infty\left(1-\frac{\rho}{\rho_\infty}\frac{u}{U}\right)dy \tag{13}$$



(a) Coordinates fixed in fluid at rest.

(b) Coordinates fixed in plate.

[3] The steady-flow definition of displacement thickness is adopted herein. Actually, the displacement effect of the boundary layer is not properly represented by this definition if the flow is unsteady. However, in reference 4 it is shown that the steady expression is part of the correct one, and because this expression yields the correct quasi-steady result its calculation is therefore warranted.

Upon application of equation (1), the state equation ($p\theta$=Constant), and equations (3) into (13) there results

$$\delta^*=\sqrt{\frac{C\nu_\infty X}{U}}\left\{\lim_{\sigma\to\infty}(2\sigma-F)-[\zeta_0 f_0(\infty)+\zeta_1 f_1(\infty)+\zeta_0^2 f_{00}(\infty)+\ldots]+m\left[\int_0^\infty R\,d\sigma+\zeta_0\int_0^\infty r_0\,d\sigma+\zeta_1\int_0^\infty r_1\,d\sigma+\zeta_0^2\int_0^\infty r_{00}\,d\sigma+\ldots\right]\right\} \tag{14}$$

where

$$m=(\gamma-1)M_\infty^2 \tag{15}$$

Isothermal plate.—The dimensionless wall-shear-stress, or local-skin-friction coefficients for the case of heat transfer at the plate, is identical in form with its counterpart for an insulated plate (eq. (12)). The constant C is evaluated by assuming that θ_w in equation (11) is the adiabatic wall temperature for the insulated plate and the maintained isothermal temperature in the heat-transfer case.

The displacement thickness for this case is obtained from equation (1), the state equation, and equations (4) and (13) and is given by

$$\delta^*=\sqrt{\frac{C\nu_\infty X}{U}}\left\{\lim_{\sigma\to\infty}(2\sigma-F)-[\zeta_0 f_0(\infty)+\zeta_1 f_1(\infty)+\zeta_0^2 f_{00}(\infty)+\ldots]+m\left[\int_0^\infty S\,d\sigma+\zeta_0\int_0^\infty s_0\,d\sigma+\zeta_1\int_0^\infty s_1\,d\sigma+\zeta_0^2\int_0^\infty s_{00}\,d\sigma+\ldots\right]+2\Phi\left[\int_0^\infty H\,d\sigma+\zeta_0\int_0^\infty h_0\,d\sigma+\zeta_1\int_0^\infty h_1\,d\sigma+\zeta_0^2\int_0^\infty h_{00}\,d\sigma+\ldots\right]\right\} \tag{16}$$

where

$$\Phi=\frac{\theta_w-\theta_\infty}{\theta_\infty} \tag{17}$$

The local rate of heat transfer is given by

$$q=-k\left(\frac{\partial\theta}{\partial y}\right)_w \tag{18}$$

Using the definition of the Prandtl number, the state equation, and equations (4), (8), and (10) in equation (18) yields

$$q=-\frac{c_p}{2Pr}(\theta_w-\theta_\infty)\sqrt{\frac{CU\mu_\infty\rho_\infty}{X}}\left\{H'(0)+\frac{m}{2\Phi}S'(0)+\zeta_0\left[h_0'(0)+\frac{m}{2\Phi}s_0'(0)+\ldots\right]+\zeta_1\left[h_1'(0)+\frac{m}{2\Phi}s_1'(0)+\ldots\right]+\zeta_0^2\left[h_{00}'(0)+\frac{m}{2\Phi}s_{00}'(0)+\ldots\right]\right\} \tag{19}$$

TIME AVERAGES FOR ARBITRARY VELOCITY FLUCTUATIONS

It is now assumed that the velocity fluctuates in a periodic but otherwise arbitrary manner, that is,

$$U(t)=U_m g(\omega t) \tag{20}$$

where ω is the frequency of the fluctuations and g is an arbitrary positive function so that

$$\bar{g}\equiv\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\,d\tau=1 \tag{21}$$

where

$$\tau=\omega t \tag{22}$$

Substituting equations (20) and (22) into equation (2) yields

$$\zeta_0=\Omega\frac{g'(\tau)}{g^2};\qquad \zeta_1=\Omega^2\frac{g''(\tau)}{g^3} \tag{23}$$

with the frequency parameter Ω given by

$$\Omega=\frac{X\omega}{U_m} \tag{24}$$

Therefore, for arbitrary fluctuations of the velocity, the average local skin friction coefficient to order Ω^2 is obtained by applying equations (20), (22), (23), and (24) to equation (12) and integrating as in equation (21). Since for a periodic function,

$$\overline{g^n g'}=0 \tag{25}$$

and also

$$\overline{g^{-n}g''}=n\overline{g^{-(n+1)}(g')^2} \tag{26}$$

the following equation results:

$$\overline{C_f}\equiv\frac{1}{2\pi}\int_0^{2\pi}\frac{2\tau_w\,d\tau}{\rho_\infty U_m^2}=\frac{1}{2}\sqrt{\frac{C\nu_\infty}{XU_m}}\left\{F''(0)\overline{g^{3/2}}+\Omega^2\left[f_{00}''(0)+\frac{3}{2}f_1''(0)\right]\overline{g^{-5/2}(g')^2}\right\} \tag{27}$$

According to the remarks in the previous section, equation (27) holds for both the insulated and isothermal plates where only the value of C differs.

In a similar manner the time averages of equations (14), (16), and (19) are

$$\overline{\delta^*}=\sqrt{\frac{C\nu_\infty X}{U_m}}\left(\overline{g^{-1/2}}\lim_{\sigma\to\infty}(2\sigma-F)+m^*\overline{g^{3/2}}\int_0^\infty R\,d\sigma+\Omega^2\left\{\left[\frac{7}{2}f_1(\infty)+f_{00}(\infty)\right]\overline{g^{-9/2}(g')^2}+m^*\left(\frac{3}{2}\int_0^\infty r_1\,d\sigma+\int_0^\infty r_{00}\,d\sigma\right)\overline{g^{-5/2}(g')^2}\right\}\right) \tag{28}$$

for the displacement thickness of the insulated plate;

$$\overline{\delta^*}=\sqrt{\frac{C\nu_\infty X}{U_m}}\left(\overline{g^{-1/2}}\lim_{\sigma\to\infty}(2\sigma-F)+m^*\overline{g^{3/2}}\int_0^\infty S\,d\sigma+2\Phi\overline{g^{-1/2}}\int_0^\infty H\,d\sigma+\Omega^2\left\{-\left[\frac{7}{2}f_1(\infty)+f_{00}(\infty)\right]\overline{g^{-9/2}(g')^2}+m^*\left(\frac{3}{2}\int_0^\infty s_1\,d\sigma+\int_0^\infty s_{00}\,d\sigma\right)\overline{g^{-5/2}(g')^2}+2\Phi\left(\frac{7}{2}+\int_0^\infty h_1\,d\sigma+\int_0^\infty h_{00}\,d\sigma\right)\overline{g^{-9/2}(g')^2}\right\}\right) \tag{29}$$

for the displacement thickness of the isothermal plate; and

$$\overline{q}=-\frac{c_p\theta_\infty\Phi}{2Pr}\sqrt{\frac{C\mu_\infty\rho_\infty U_m}{X}}\left(H'(0)\overline{g^{1/2}}+\frac{m^*}{2\Phi}S'(0)\overline{g^{5/2}}+\Omega^2\left\{\left[\frac{5}{2}h_1'(0)+h_{00}'(0)\right]\overline{g^{-7/2}(g')^2}+\frac{m^*}{2\Phi}\left[\frac{1}{2}s_1'(0)+s_{00}'(0)\right]\overline{g^{-3/2}(g')^2}\right\}\right) \tag{30}$$

for the average local heat-transfer rate of an isothermal plate where

$$m=m^*g^2$$

and

$$m^*=\frac{(U_m)^2}{c_p\theta_\infty} \tag{31}$$

The average value of the adiabatic wall temperature (i. e., for the insulated plate) is also obtained in a similar manner from equation (3) and is given by

$$\frac{\overline{\theta_w}}{\theta_\infty}=1+\frac{m^*}{2}\left\{R(0)\overline{g^2}+\Omega^2\left[r_1(0)+r_{00}(0)\right]\overline{g^{-2}(g')^2}\right\} \tag{32}$$

SECOND-ORDER SOLUTIONS

In order to determine the average properties of unsteady boundary layers to order Ω^2, it is evident from equations (27) to (32) that the second-order (in ζ_0) solutions (denoted by double-zero subscript) must be known. These solutions are determined in subsequent sections by extending the results of references 1 and 2.

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The partial differential equations describing the unsteady flow and heat transfer of compressible viscous fluids are given in reference 1 as

$$\psi_{YT}+\psi_Y\psi_{XY}-\psi_X\psi_{YY}=U'(T)+C\nu_\infty\psi_{YYY} \tag{33}$$

and

$$\theta_T+\psi_Y\theta_X-\psi_X\theta_Y=\frac{C\nu_\infty}{Pr}\theta_{YY}+\frac{C\nu_\infty}{c_p}(\psi_{YY})^2 \tag{34a}$$

In reference 2 it was found more convenient to write equation (34a) as

$$\Theta_T+\psi_Y\Theta_X-\psi_X\Theta_Y=\frac{C\nu_\infty}{Pr}\Theta_{YY}+\frac{C\nu_\infty}{c_p(\theta_w-\theta_\infty)}(\psi_{YY})^2 \tag{34b}$$

where

$$\Theta=\frac{\theta-\theta_\infty}{\theta_w-\theta_\infty} \tag{35}$$

The appropriate boundary conditions on ψ are

$$\psi_Y(X,\infty,T)=\psi_Y(O,Y,T)=U(T) \tag{36a}$$

$$\psi_Y(X,O,T)=\psi(X,O,T)=0 \tag{36b}$$

For the case of an insulated surface as treated in reference 1, the boundary conditions on θ are

$$\theta(X,\infty,T)=\theta(O,Y,T)=\theta_\infty \tag{37a}$$

and

$$\theta_Y(X,O,T)=0 \tag{37b}$$

For an isothermal plate, equations (37) are replaced (see ref. 2) by

$$\Theta(X,\infty,T)=\Theta(O,Y,T)=0 \tag{38a}$$

and also

$$\Theta(X,O,T)=1 \tag{38b}$$

SOLUTIONS

The methods of solutions for the insulated and isothermal plate are identical to those in references 1 and 2, respectively, but, in each case, the method is extended herein to yield second-order results.

Insulated plate.—For the insulated plate the boundary-value problem is defined by equations (33), (34a), (36), and (37). For nearly quasi-steady flows the stream function given by equation (5) and the temperature function given by equation (3a) are substituted into equations (33), (34a), (36), and (37) to yield

$$f_{\sigma\sigma\sigma}+ff_{\sigma\sigma}=-8\frac{XU'}{U^2}+2\left(2\frac{XU'}{U^2}+X\sum_{n=0}^{\infty}f_{\sigma\zeta_n}\zeta_{nX}\right)f_\sigma+2\frac{XU'}{U^2}\sigma f_{\sigma\sigma}-2Xf_{\sigma\sigma}\sum_{n=0}^{\infty}f_{\zeta_n}\zeta_{nX}+4\frac{X}{U}\sum_{n=0}^{\infty}f_{\sigma\zeta_n}\zeta_{nT} \tag{39}$$

$$r_{\sigma\sigma}+Prfr_\sigma+\frac{Pr}{2}(f_{\sigma\sigma})^2=4Pr\left(2\frac{XU'}{U^2}r+\frac{\sigma}{2}\frac{XU'}{U^2}r_\sigma+\frac{X}{U}\sum_{n=0}^{\infty}r_{\zeta_n}\zeta_{nT}+\frac{X}{2}f_\sigma\sum_{n=0}^{\infty}r_{\zeta_n}\zeta_{nX}-\frac{X}{2}r_\sigma\sum_{n=0}^{\infty}f_{\zeta_n}\zeta_{nX}\right) \tag{40}$$

$$f_\sigma(\infty,\zeta_n)=2;\qquad f_\sigma(0,\zeta_n)=f(0,\zeta_n)=0 \tag{41}$$

$$r_\sigma(0,\zeta_n)=r(\infty,\zeta_n)=0 \tag{42}$$

Introduction of equation (2) makes equations (41) and (42) self-consistent (i. e., functions of σ and ζ_n only). For nearly quasi-steady flows, $\zeta_n\ll 1$ (see refs. 1 or 2); therefore, the functions f and r can be expanded as follows:

$$f(\sigma,\zeta_n)\equiv F(\sigma)+\zeta_0f_0(\sigma)+\zeta_1f_1(\sigma)+\ .\ .\ .\ +\zeta_0^2f_{00}(\sigma)+\ .\ .\ .\ +\zeta_0\zeta_1f_{01}(\sigma)+\ .\ .\ . \tag{43}$$

$$r(\sigma,\zeta_n)\equiv R(\sigma)+\zeta_0r_0(\sigma)+\zeta_1r_1(\sigma)+\ .\ .\ .\ +\zeta_0^2r_{00}(\sigma)+\ .\ .\ .\ +\zeta_0\zeta_1r_{01}(\sigma)+\ .\ .\ . \tag{44}$$

Substitution of equations (42) and (44) into equations (39) to (42) and collection of terms independent of ζ and those multipled by ζ_0 and ζ_1 yield the three sets of equations solved in reference 1. Since for the present purposes the next higher order terms of equations (43) and (44) are needed (see eqs. (27) to (32), e. g.), the terms multiplied by ζ_0^2 are collected and yield

$$f_{00}'''+Ff_{00}''-4F'f_{00}'+5F''f_{00}=-2f_0'(2-f_0')+f_0''(2\sigma-3f_0) \tag{45}$$

and

$$r_{00}''+Pr(Fr_{00}'-4F'r_{00})=Pr[2\sigma r_0'+2f_0'r_0-5R'f_{00}-3r_0'f_0-F''f_{00}''-\tfrac{1}{2}(f_0'')^2] \tag{46}$$

in addition to

$$f_{00}'(\infty)=f_{00}'(0)=f_{00}(0)=0 \tag{47}$$

and

$$r_{00}'(0)=r_{00}(\infty)=0 \tag{48}$$

The function F and its derivatives are available in reference

5, and f_0,r_0 and their derivatives are presented in reference 1. Hence, the second-order functions f_{00} and r_{00} for the insulated plate are defined by equations (45) to (48).

Isothermal plate.—Equation (45) holds for the isothermal plate as well as for the insulated one. However, for the isothermal plate, equations (34b) and (38) replace equations (34a) and (37). The temperature function given by equation (4) is obtained by letting

$$\Theta=h(\sigma,\zeta_n)+\frac{U^2(T)}{2c_p(\theta_w-\theta_\infty)}\,s(\sigma,\zeta_n)$$

where

$$h(\sigma,\zeta_n)=H(\sigma)+\zeta_0h_0(\sigma)+\zeta_1h_1(\sigma)+\ .\ .\ .\ +\zeta_0^2h_{00}(\sigma)+\ .\ .\ .+\zeta_0\zeta_1h_{01}(\sigma)+\ .\ .\ .$$

and

$$s(\sigma,\zeta_n)\equiv S(\sigma)+\zeta_0s_0(\sigma)+\zeta_1s_1(\sigma)+\ .\ .\ .\ +\zeta_0^2s_{00}(\sigma)+\ .\ .\ .\ +\zeta_0\zeta_1s_{01}(\sigma)$$

Substituting equation (4) into equations (34b) and (38) yields those equations treated in reference 2 independent of ζ and to order ζ_0 and ζ_1. To order ζ_0^2 the following equations are obtained:

$$h_{00}''+Pr(Fh_{00}'-4F'h_{00})=Pr(2\sigma h_0'-3f_0h_0'-5H'f_{00}+2h_0f_0'-8h_0) \tag{49}$$

$$s_{00}''+Pr(Fs_{00}'-4F's_{00})=Pr\left[2\sigma s_0'-5S'f_{00}-3f_0s_0'+2s_0f_0'-F''f_{00}''-\frac{1}{2}\,(f_0'')^2\right] \tag{50}$$

$$h_{00}(\infty)=h_{00}(0)=0 \tag{51}$$

$$s_{00}(\infty)=s_{00}(0)=0 \tag{52}$$

The functions on the right-hand sides of equations (49) and (50) are given in references 1, 2, and 5.

Equations (45) to (52) thus define four boundary-value problems for the functions f_{00}, r_{00}, h_{00}, and s_{00}. These problems are solved for a Prandtl number of 0.72 by a numerical integration method described in references 1 and 2. The functions are presented in table I.

RESULTS FOR AVERAGE PROPERTIES OF UNSTEADY BOUNDARY LAYER

The following formulas are obtained by substituting the results of references 1 and 2 and those of the previous section into equations (27), (28), (29), (30), and (32), respectively:

$$\overline{C}_f=(0.6640)\sqrt{\frac{C\nu_\infty}{XU_m}}\,[\overline{g^{3/2}}-(1.306)\,\Omega^2\overline{g^{-5/2}g'^2}+\ .\ .\ .] \tag{53}$$

For the insulated plate case,

$$\overline{\delta^*}=(1.721)\sqrt{\frac{C\nu_\infty X}{U_m}}\,\{\overline{g^{-1/2}}+(0.645)\,m^*\overline{g^{3/2}}+\Omega^2[(1.986)\overline{g^{-9/2}g'^2}+(3.420)m^*\overline{g^{-5/2}g'^2}]+\ .\ .\ .\} \tag{54}$$

$$\frac{\overline{\theta}_w}{\theta_\infty}=1+(0.8480)\,\frac{m^*}{2}\,[\overline{g^2}+(5.051)\,\Omega^2\overline{g^{-2}g'^2}+\ .\ .\ .] \tag{55}$$

and for the isothermal plate case,

$$\overline{\delta^*}=(1.721)\sqrt{\frac{C\nu_\infty X}{U_m}}\,\{(1+1.13\ \Phi)\overline{g^{-1/2}}+(0.1673)m^*\overline{g^{3/2}}+\Omega^2[(1.986+1.095\ \Phi)\ \overline{g^{-9/2}g'^2}+(0.3112)\ m^*\overline{g^{-5/2}g'^2}]+\ .\ .\ .\} \tag{56}$$

$$\overline{q}=(0.8211)\ \frac{c_p\theta_\infty}{2}\sqrt{\frac{C\mu_\infty\rho_\infty U_m}{X}}\,\{\overline{g^{1/2}}\Phi-(0.424)\ m^*\overline{g^{5/2}}-\Omega^2[(0.8350)\Phi\overline{g^{-7/2}g'^2}+(0.7574)m^*\overline{g^{-3/2}g'^2}]+\ .\ .\ .\} \tag{57}$$

DISCUSSION OF QUASI-STEADY TERM

The leading terms (i. e., terms independent of Ω) of equations (53) to (57) reflect the nonlinear dependence of the physical quantities on U in a quasi-steady situation. Under the restriction that g is positive, all these leading terms are, of course, positive.

Obviously, in the completely steady insulated case, when $g\equiv1$, all the leading terms in the braces equal 1. In the steady isothermal case, of course, only the factors involving g became unity. In the general quasi-steady case, when only the restriction that $\overline{g}=1$ is applied (eq. (21)), the magnitudes of the leading terms may be compared with unity (the steady case) using a special case of the Schwartz inequality (ref. 6), namely,

$$\overline{g^{2m}}\ \overline{g^{2n}}\geq\overline{g^{m+n}}^2 \tag{58}$$

If the choice $m,n=0,1$ is made, equation (58) yields

$$\overline{g^2}\geq1 \tag{59a}$$

For $m,n=0,1/2$, the result is

$$\overline{g^{1/2}}\leq1 \tag{59b}$$

For m, $n=3/4$, $1/4$, $\overline{g^{3/2}}\ \overline{g^{1/2}}\geq\overline{g}=1$ and, therefore, in view of equation (59b), yields

$$\overline{g^{3/2}}\geq1 \tag{59c}$$

If $m=n=1/4$, $\overline{g^{1/2}}\ \overline{g^{-1/2}}\geq1$, which, together with equation (59b), yields

$$\overline{g^{-1/2}}\geq1 \tag{59d}$$

If $m,n=5/4,1/4$, then $\overline{g^{5/2}}\ \overline{g^{1/2}}\geq\overline{g^{3/2}}^2$ and, therefore, in view of equations (59b) and (59c), yields

$$\overline{g^{5/2}}\geq1 \tag{59e}$$

Insulated surface.—Inspection of equations (53) to (55), together with equations (59), shows that for the insulated surface case, the quasi-steady values of $\overline{C}_f$, $\overline{\delta^*}$, and $\overline{\theta}_w$ are always greater than or equal to the corresponding quantity for uniform flow at the average velocity U_m.

The foregoing effects are clarified by considering the average velocity profile. For example, suppose that for half of the time, $g=1/2$, and for the rest of the time, $g=3/2$, so that

$\bar{g}=1$ (fig. 1(a)); for incompressible flow, the average quasi-steady velocity profile is

$$\frac{\bar{u}}{U_m}=\frac{1}{4}\left[\frac{1}{2}F'\left(\frac{\sigma_m}{\sqrt{2}}\right)+\frac{3}{2}F'\left(\sqrt{\frac{3}{2}}\,\sigma_m\right)\right] \tag{60}$$

The result appears in figure 1 (b); the dash-dot lines represent the instantaneous quasi-steady profiles at the low and high velocities, the dash line represents the Blasius profile for the average (as if $g=1$), and the solid line is the average profile from equation (60). The velocity from equation (60) is greater than the Blasius value near the wall and less than that value far from the wall. These profile differences are more pronounced if the variation of stream velocity is more extreme; if $g=1/2$ for nine-tenths of the time and if $g=11/2$ for the rest of the time (fig. 1(c)), then

$$\frac{\bar{u}}{U_m}=\frac{1}{4}\left[\frac{9}{10}F'\left(\frac{\sigma_m}{\sqrt{2}}\right)+\frac{11}{10}F'\left(\sigma_m\sqrt{\frac{11}{2}}\right)\right] \tag{61}$$

The result appears in figure 1 (d). The deviation of the average profile from the Blasius shape clearly implies increased skin friction, and also suggests an increase of δ^*.

Note that the effects just discussed can be determined without fully specifying the function g. All that is needed is a specification of the proportion of time during which the various velocity values apply; in effect, a probability density distribution for velocity is sufficient. In deriving equations (60) and (61), the illustrative velocity functions were chosen for simplicity; of course, the present analysis would not apply near the sudden velocity changes which were postulated.

Heat transfer.—If there is heat transfer at constant wall temperature, the quasi-steady value of $\overline{\delta^*}$ from equation (56) is greater than the Blasius value unless Φ is nearly at its maximum negative value, −1, which corresponds to a wall temperature of absolute zero. The heat-transfer rate $\bar{q}$ must be regarded as a function of Φ and m^* (eq. (56)). The finer cross-hatched plane of figure 2 denotes q as a function of Φ and m^* for the average $U_m(g\equiv 1)$. Positive q denotes heat flow out of the surface into the gas. If g is not always 1, then the first term of equation (57), independent of m^*, tends to diminish the magnitude of quasi-steady heat-transfer rate. The second term, proportional to m^* and independent of Φ, always provides less heat flow out of the surface. The result is sketched as the coarser crosshatched plane of figure 2.

Figure 2 indicates that variable stream velocity results in increased heat transfer to the wall only in the case of a cooled ($q<0$) wall, and even then only if m^* has a value so that

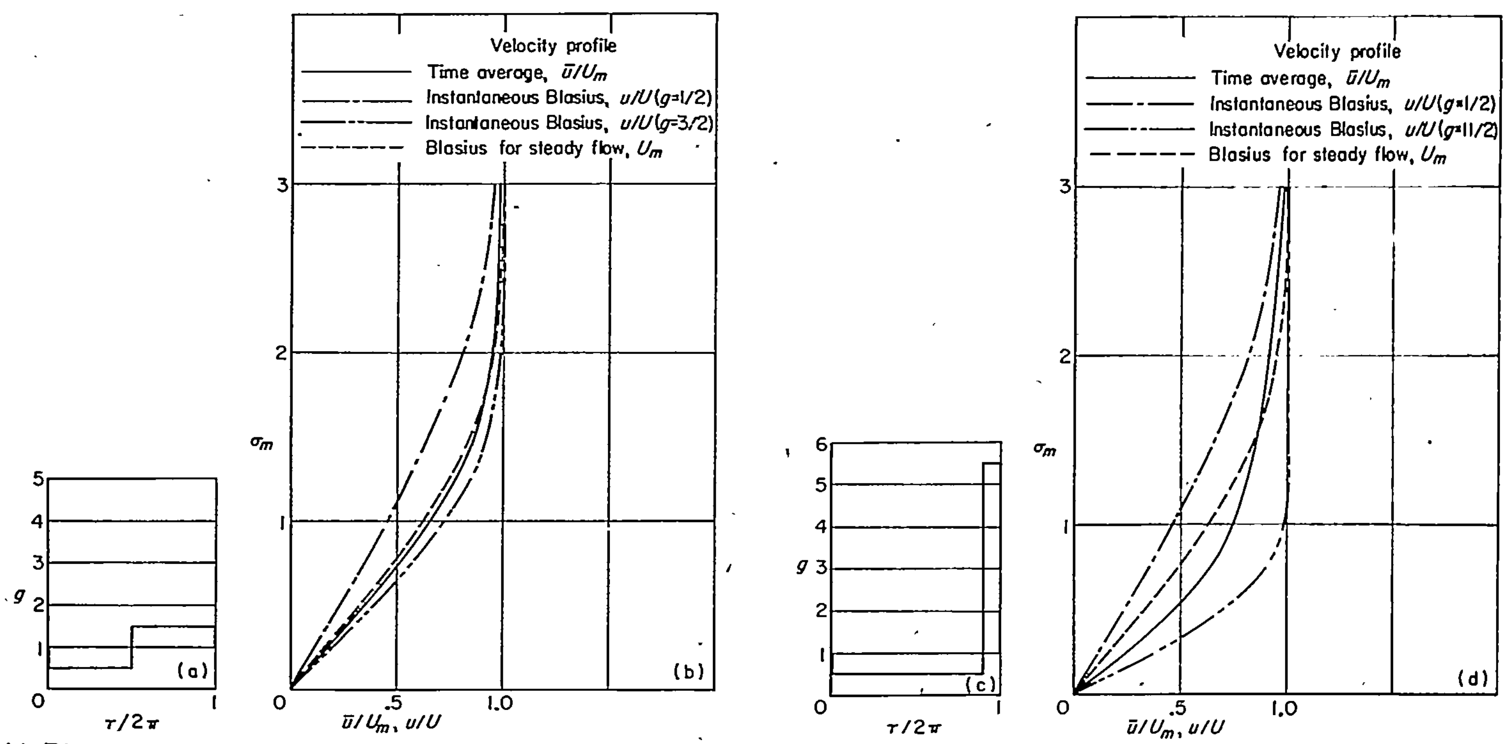
$$m^*\geq\frac{(-\Phi)}{0.424}\,\frac{1-\overline{g^{1/2}}}{\overline{g^{5/2}}-1} \tag{62}$$

in cases of negative Φ. If $\Phi>0$, the condition of zero average heat transfer is

$$m^*=\frac{\Phi}{0.424}\,\frac{\overline{g^{1/2}}}{\overline{g^{5/2}}} \tag{63}$$

DISCUSSION OF FREQUENCY-DEPENDENT TERMS

Equation (53) indicates that average skin friction is diminished in proportion to the square of the frequency. The dominant factor in this effect is the response of the boundary layer to rate of change of acceleration ($f_1''(0)$ in



(a) Distribution of plate velocity with time. Moderate velocity variation.

(b) Boundary-layer velocity profile. Moderate velocity variation.

(c) Distribution of plate velocity with time. Abrupt velocity variation.

(d) Boundary-layer velocity profile. Abrupt velocity variation.

FIGURE 1.—Effect of plate velocity variation on profile of average velocity in boundary layer.

eq. (27)). With respect to skin friction, the quasi-steady term and the frequency term are in opposition. Subject to the requirement that $\Omega<<1$, a flat plate will experience a diminished average drag if

$$\Omega^2>\frac{\overline{g^{3/2}}-1}{(1.306)\,\overline{g^{-5/2}g'^2}} \tag{64}$$

Owing to the appearance of $\overline{g'^2}$ in inequality (64), abrupt velocity changes would favor drag reduction.

Insulated surface.—The average value of δ^*, from equation (54), will increase with Ω^2. The adiabatic wall temperature increases with Ω^2, which reinforces the effect of the quasi-steady term.

Heat transfer.—From equation (57), the terms proportional to Ω^2 all reinforce the effects of the quasi-steady terms with respect to the steady value. That is, the Ω^2 term in Φ reduced the magnitude of q as does the quasi-steady term since $\overline{g^{1/2}}\leq 1$, and the negative value of the Ω^2 term in m^* is consistent with the corresponding quasi-steady term since $\overline{g^{5/2}}\geq 1$.

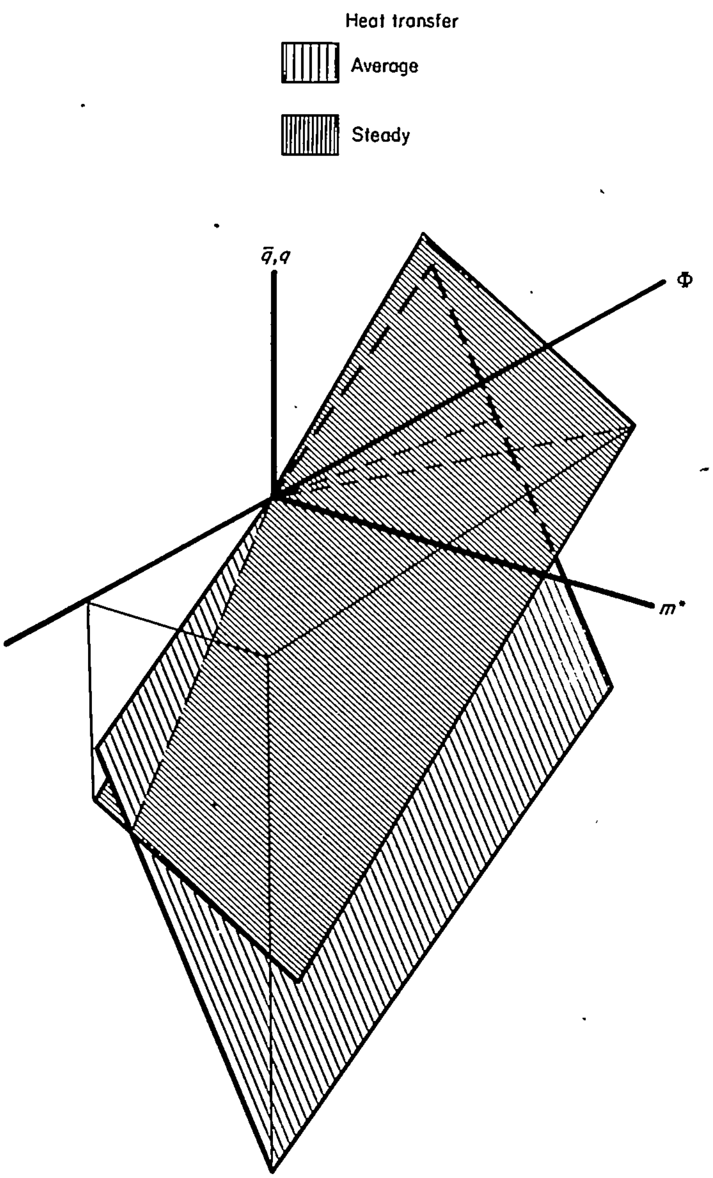


FIGURE 2.—Effect of unsteady velocity on relation among average heat transfer, temperature difference, and Mach number.

In the heat-transfer case, the Ω^2 terms of δ^* also reinforce the corresponding quasi-steady terms, when the groups depending on Φ and m^* are considered separately, just as in the foregoing discussion of q.

RESULTS FOR HARMONIC $g(\tau)$

It may often be of interest to evaluate equations (53) to (57) when the function g is specialized to have harmonic form

$$g=1+\epsilon\sin\tau$$

where the restriction $0<\delta<1$ is imposed so that flight direction is not reversed.

Arbitrary ϵ.—The various integrals are evaluated for arbitrary ϵ in appendix B in the order of their appearance in equations (53) to (57) with the following results:

$$I_1\equiv\overline{g^{3/2}}=\frac{2}{3\pi}\sqrt{1+\epsilon}\,[4E(k)-(1-\epsilon)\,K(k)] \tag{B10}$$

$$I_2\equiv\overline{g^{-5/2}g'^2}=\frac{4}{3\pi\sqrt{1+\epsilon}}\left[\frac{E(k)}{1-\epsilon}-K(k)\right] \tag{B13}$$

$$I_3\equiv\overline{g^{-1/2}}=\frac{2K(k)}{\pi\sqrt{1+\epsilon}} \tag{B1}$$

$$I_4\equiv\overline{g^{-9/2}g'^2}=\frac{2}{105\pi\sqrt{1+\epsilon}(1-\epsilon^2)^2}\times[(3+16\epsilon-6\epsilon^2)\,E(k)-2(3+4\epsilon-3\epsilon^2)\,K(k)] \tag{B15}$$

$$I_5\equiv\overline{g^2}=1+\frac{\epsilon^2}{2} \tag{B4}$$

$$I_6\equiv\overline{g^{-2}g'^2}=-1+\frac{1}{\sqrt{1-\epsilon^2}} \tag{B5}$$

$$I_7\equiv\overline{g^{1/2}}=\frac{2\sqrt{1+\epsilon}}{\pi}E(k) \tag{B2}$$

$$I_8\equiv\overline{g^{5/2}}=\frac{2\sqrt{1+\epsilon}}{15\pi}[(23+9\epsilon^2)\,E(k)-8(1-\epsilon)\,K(k)] \tag{B11}$$

$$I_9\equiv\overline{g^{-7/2}g'^2}=\frac{4}{15\pi\sqrt{1+\epsilon}(1-\epsilon^2)}\left[\frac{1+2\epsilon+6\epsilon^2}{1+\epsilon}E(k)-K(k)\right] \tag{B14}$$

$$I_{10}\equiv\overline{g^{-3/2}g'^2}=-\frac{4}{\pi}\sqrt{1+\epsilon}\left[E(k)-\frac{K(k)}{1+\epsilon}\right] \tag{B12}$$

(65)

where

$$k=\sqrt{2\epsilon/(1+\epsilon)} \tag{B3}$$

and K and E are the complete elliptic integrals of the first and second kind, respectively. The results are presented in table II.

Small ϵ.—In the event that ϵ is very small, to a first approximation, each of the foregoing equations for arbitrary ϵ may be replaced by making the appropriate substitution for N in the formulas

$$\left.\begin{aligned} \overline{g^N} &\cong 1+\frac{1}{4}N(N-1)\,\epsilon^2 \\ \overline{g^N g'^2} &\cong \frac{1}{2}\,\epsilon^2 \end{aligned}\right\} \tag{66}$$

ϵ Near unity.—If ϵ is near unity, then k (eq. (B3)) is also near unity, or $k'\equiv\sqrt{1-k^2}\approx\sqrt{1-\epsilon}$ is nearly zero; and the asymptotic relations

$$K(k)\approx\ln\frac{4}{\sqrt{1-\epsilon}}; \qquad E(k)\approx 1$$

apply (ref. 7). Equations (65) become

$$\left.\begin{aligned} &\overline{g^{3/2}}\rightarrow\frac{8\sqrt{2}}{3\pi}; \qquad \overline{g^{-5/2}g'^2}\rightarrow\frac{2\sqrt{2}}{3\pi}\,\frac{1}{1-\epsilon} \\ &\overline{g^{-1/2}}\rightarrow\frac{\sqrt{2}}{\pi}\ln\frac{4}{\sqrt{1-\epsilon}} \\ &\overline{g^{-9/2}g'^2}\rightarrow-\frac{2\sqrt{2}}{105\pi}\,\frac{1}{(1-\epsilon)^2}\left(\ln\frac{4}{\sqrt{1-\epsilon}}-\frac{13}{8}\right) \\ &\overline{g^2}\rightarrow\frac{3}{2}; \qquad \overline{g^{-2}g'^2}\rightarrow\frac{\sqrt{2}}{2\sqrt{1-\epsilon}} \\ &\overline{g^{1/2}}\rightarrow\frac{2\sqrt{2}}{\pi}; \qquad \overline{g^{5/2}}\rightarrow\frac{32\sqrt{2}}{3\pi} \\ &\overline{g^{-7/2}g'^2}\rightarrow-\frac{\sqrt{2}}{15\pi}\,\frac{1}{1-\epsilon}\left(\ln\frac{4}{\sqrt{1-\epsilon}}-\frac{9}{2}\right) \\ &\overline{g^{-3/2}g'^2}\rightarrow\frac{2\sqrt{2}}{\pi}\left(\ln\frac{4}{\sqrt{1-\epsilon}}-2\right) \end{aligned}\right\} \tag{67}$$

Thus, as $\epsilon\rightarrow 1$, equations (53) to (57) show that the effect of the Ω^2 terms is greatly accentuated because their coefficients approach infinity from equations (67). For example, equation (53) becomes

$$\overline{C_f}\rightarrow(0.6640)\sqrt{\frac{C\nu_\infty}{XU_m}}\left[(1.20)-(0.3918)\,\frac{\Omega^2}{1-\epsilon}+\ .\ .\ .\right] \tag{68}$$

It should be noted that under these circumstances ($\epsilon\rightarrow 1$) the higher order terms in Ω^2 become significant.

REMARKS ON HEAT-TRANSFER PROBLEM

In a previous section it was shown that the heat-transfer rate with an oscillating surface differs from that with a surface in steady motion (see fig. 2). The relative merits of the oscillations, however, depend upon the particular configuration or application. For example, it has been determined that a greater heat-transfer rate to a cooled oscillating wall can be obtained for $\Phi<0$ provided that inequality (62) applies. This result suggests that, if one wishes to increase the rate of heat abstraction from a gas flow over a cooled surface (as, e. g., in a heat exchanger), it may be advantageous to oscillate the surface mechanically in its own plane so that the relative stream velocity oscillates. The power required to oscillate the plate should properly be assessed to the system. Unless this power is less than the additional heat transferred, no heat is abstracted from the hot gas; also in this case that power could be used directly to increase the energy obtained with the surface fixed or in steady motion. Of course, if the oscillations were inherent to the system, the assessment would not be required.

If the power required to oscillate the plate is taken into account, the comparison is as follows: The excess of power required beyond that corresponding to steady flow is given by [4]

$$\Delta P=\int_0^X[\overline{U\tau_w}-U_m\tau_w(U_m)]\,\mathrm{d}X=\frac{1}{2}\,\rho_\infty U_m^3\int_0^X[\overline{gC_f}-C_f(U_m)]\,\mathrm{d}X \tag{69}$$

and the increment of heat transfer into the wall owing to the oscillation is

$$\Delta Q=-\int_0^X[\overline{q}-q(U_m)]\,\mathrm{d}X \tag{70}$$

where X is the length of the plate.

Equation (69) can be evaluated from equation (53), because $\overline{gC_f}$ is given by the same formula as $\overline{C_f}$ with the exponent of g increased by 1 wherever it appears in equation (53):

$$\Delta P=(0.6640)\,\rho_\infty U_m^3\sqrt{\frac{C\nu_\infty X}{U_m}}\,[\overline{g^{5/2}}-1+\mathcal{O}(\Omega^2)] \tag{71}$$

Equation (70) can be evaluated directly from equation (57), setting $g\equiv 1$ to define $q(U_m)$:

$$\Delta Q=(0.8211)c_p\theta_\infty\rho_\infty U_m\sqrt{\frac{C\nu_\infty X}{U_m}}[(1-\overline{g^{1/2}})\Phi+ (0.424)m^*(\overline{g^{5/2}}-1)+\mathcal{O}(\Omega^2)] \tag{72}$$

The question is whether $\Delta Q>\Delta P$. Therefore, equation (71) is subtracted from equation (72) to yield

$$\Delta Q-\Delta P=(0.8211)c_p\theta_\infty\rho_\infty U_m\sqrt{\frac{C\nu_\infty X}{U_m}}[(1-\overline{g^{1/2}})\Phi- (0.385)(\overline{g^{5/2}}-1)m^*+\mathcal{O}(\Omega^2)] \tag{73}$$

Thus, if $\Delta Q-\Delta P>0$,

$$\frac{\theta_w}{\theta_\infty}-1>(0.385)m^*\frac{\overline{g^{5/2}}-1}{1-\overline{g^{1/2}}} \tag{74}$$

If inequality (74) is to hold, $\dfrac{\theta_w}{\theta_\infty}-1$ must be positive. Now, the greatest value which θ_w/θ_∞ can approach consistent with

[4] The formula used for ΔP accounts for all the power required to maintain the plate in flight. If only that power required to oscillate the plate in a steady wind-tunnel flow or in a stationary heat exchanger were desired, the formula $\Delta P=\int_0^X\overline{(U-U_m)\tau_w}\,\mathrm{d}X$ would be used, which is less than that given in equation (69) in the amount $\int_0^X\overline{U}[\overline{\tau_w}-\tau_w(U_m)]\,\mathrm{d}X$. Equation (69) is selected for discussion on the grounds that the difference cited would usually be charged as an energy loss of a practical airborne system.

heat flow into the surface is the insulated surface value from equation (55). Thus, inequality (74) becomes

$$0.424m^*\overline{g^2} > \frac{\theta_w}{\theta_\infty} - 1 > 0.385m^* \frac{\overline{g^{5/2}} - 1}{1 - \overline{g^{1/2}}} \qquad (75)$$

or,

$$1.101\overline{g^2} > \frac{\overline{g^{5/2}} - 1}{1 - \overline{g^{1/2}}} \qquad (76)$$

It is doubtful that any reasonable function of g can be advised to satisfy inequality (76). Inspection of table II shows that inequality (76) cannot be satisfied by a harmonic function of any amplitude.

For a stationary configuration, inequality (74) is computed using the ΔP expression in footnote 4 and again the inequality cannot be satisfied. Therefore, for air at normal conditions, the power required to oscillate the plate would exceed the extra heat transfer obtained by oscillation.

The results of the foregoing discussion may be compared with the result of reference 3, wherein it was concluded that oscillation of a doubly infinite plate in a fluid otherwise at rest would also result in an increased heat transfer to the plate greater than the extra power required only if $\frac{\theta_w}{\theta_\infty} > 1$.

CONCLUSIONS

The unsteady laminar boundary layer on a flat plate in compressible flow has been analyzed for the case of time-variable velocity of flight with a view to describing the time-average characteristics of such a boundary layer. Flight velocity is assumed to vary slowly enough so that the resulting boundary-layer flow is nearly, but not quite, quasi-steady. The wall temperature has been assumed constant both along the plate and in time; further analysis would be required for the case of fluctuating surface temperature.

In order to obtain time averages, the expansions of flow quantities must include terms in $\zeta_0 \equiv \frac{XU'(t)}{U^2}$, $\zeta_1 \equiv \frac{X^2U''(t)}{U^3}$, and ζ_0^2. The terms for ζ_0 and ζ_1 were taken from previous work, while the ζ_0^2 terms were obtained by numerical integration, and are presented in tabular form herein.

The time-averages of skin friction and "displacement thickness" are presented, as well as heat-transfer rate at the surface, and for the special case of an adiabatic wall, the surface temperature.

A significant amount of information may be obtained without specifying $U(t)$ beyond the requirements that $U(t)$ remain positive and have an average value U_m. Each time average involves two groups of terms to the order contemplated in the present report: A time average of quasi-steady terms, and terms proportional to the inverse square of the characteristic time of the velocity fluctuation (i. e., the square of reduced frequency).

The quasi-steady terms differ from the values for steady flow at the corresponding average velocity, owing to the nonlinear dependence of the physical quantities on $U(t)$. In fact, especially for extreme variations of $U(t)$ about the mean value, the average velocity profile is steeper near the wall and more gradual in its outer portion than the Blasius profile which applies at each instant. Thus, in the quasi-steady approximation, skin friction, "displacement thickness," and adiabatic wall temperatures are greater on the average than for the case of constant velocity. The magnitude of the part of the heat-transfer rate that is independent of Mach number is less in the quasi-steady approximation, whereas the Mach number dependent part differs in the direction of less heat out of the surface into the gas.

The differences cited are reinforced by the frequency dependent averages for adiabatic wall temperature and heat-transfer rate. The effects oppose one another in the case of skin friction.

The various time averages are derived for the special case of harmonic velocity variation. Large amplitude affects chiefly the frequency-dependent terms, greatly accentuating their importance.

The question discussed is whether it would be advantageous to oscillate the surface of a heat exchanger in order to take advantage of the increased rate of heat transfer to the wall, and it is concluded that the heat-transfer advantage would generally be vitiated by the power requirement for oscillating the surface against the action of skin friction.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, *September 5, 1956*

APPENDIX A

SYMBOLS

C	constant defined by eq. (11)
C_f	local skin-friction coefficient
c_p	specific heat at constant pressure
F	related to stream function for flat plate in steady flow
f,f_i	functions related to stream function for unsteady flat-plate flow, i=0, 1, 2, 00, . . .
g	function related to plate velocity, defined by eq. (20)
H	temperature function related to steady isothermal flat-plate flow
h,h_i	functions related to temperature for unsteady isothermal flat-plate flow, i=0, 1, 2, 00, . . .
k	thermal conductivity coefficient
M	Mach number
m	function related to Mach number, defined by eq. (15)
m^*	constant defined by eq. (31)
N	general exponent
P	power
Pr	Prandtl number
Q	total heat-transfer along plate
q	local heat-transfer rate
R	function related to temperature for steady insulated flat-plate flow
r,r_i	functions related to temperature for unsteady insulated flat-plate flow, i=0, 1, 2, 00, . . .
S	function related to temperature for steady isothermal flat-plate flow
s,s_i	functions related to temperature for unsteady isothermal flat-plate flow, i=0, 1, 2, 00, . . .
T,t	time
U	stream or plate velocity in X-direction (see sketch (b))
U_m	mean velocity
u	relative velocity in X-direction
u_a	absolute velocity in x-direction
X	coordinate along surface measured from leading edge
x	coordinate along surface in system fixed in fluid (see sketch (a))
Y	coordinate defined by eq. (8)
y	coordinate normal to surface
γ	ratio of specific heats
δ^*	displacement thickness
ϵ	amplitude of velocity fluctuations
ζ_n	dimensionless parameter, n=0, 1, 2, . . . (eq. (2))
Θ	dimensionless temperature difference
θ	temperature
μ	absolute viscosity coefficient
ν	kinematic viscosity coefficient
ρ	density
σ	dimensionless coordinate defined by eq. (6)
σ_m	dimensionless coordinate, $\frac{Y}{2}\sqrt{\frac{U_m}{C\nu_\infty X}}$
τ	function related to time by eq. (22)
τ_w	local wall shear stress
Φ	constant defined by eq. (17)
ψ	stream function
Ω	frequency parameter defined by eq. (24)
ω	frequency of velocity fluctuations

Subscripts:

w	evaluation at wall (Y=0)
∞	evaluation in stream ($Y\rightarrow\infty$)

Subscript notation for partial differentiation is used when convenient. Primes denote ordinary differentiation.

Superscripts:

——	time average as defined in eq. (21)

APPENDIX B

DETERMINATION OF INTEGRALS FOR HARMONIC OSCILLATIONS

Given that $g(\tau)\equiv 1+\epsilon\sin\tau$, the various averages appearing in the equations are to be found. The averages $\overline{g^{-1/2}}$ and $\overline{g^{1/2}}$ can quickly be expressed as complete elliptic integals by replacing τ by $2z$ and taking account of the symmetry of the integrands:

$$\overline{g^{-1/2}}\equiv\frac{1}{2\pi}\int_0^{2\pi}\frac{d\epsilon}{\sqrt{1+\epsilon\sin\tau}}=\frac{2}{\pi\sqrt{1+\epsilon}}\int_0^{\pi/2}\frac{dz}{\sqrt{1-k^2\sin^2 z}}\equiv\frac{2K(k)}{\pi\sqrt{1+\epsilon}} \tag{B1}$$

$$\overline{g^{1/2}}\equiv\frac{1}{2\pi}\int_0^{2\pi}\sqrt{1+\epsilon\sin\tau}\,d\tau=\frac{2\sqrt{1+\epsilon}}{\pi}\int_0^{\pi/2}\sqrt{1-k^2\sin^2 z}\,dz \equiv\frac{2\sqrt{1+\epsilon}}{\pi}E(k) \tag{B2}$$

where

$$k\equiv\sqrt{2\epsilon/(1+\epsilon)} \tag{B3}$$

and $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind, respectively (eqs. (773.1) and (773.3), and tables (1040) and (1041) of ref. 7).

With help from equations (858.3), (436.00), and (436.03) of reference 7, these results are obtained:

$$\overline{g^2}=\frac{1}{2\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^2\,d\tau=1+\frac{\epsilon^2}{2} \tag{B4}$$

$$\overline{g^{-2}g'^2}=\frac{\epsilon^2}{2\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^{-2}\cos^2\tau\,d\tau=-1+\frac{1}{\sqrt{1-\epsilon^2}} \tag{B5}$$

The remaining averages are of the forms

$$\overline{g^{N+\frac{1}{2}}}\equiv\frac{1}{2\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{1}{2}}d\tau \tag{B6}$$

and also

$$\begin{aligned}\overline{g^{N+\frac{1}{2}}g'^2}&\equiv\frac{\epsilon^2}{2\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{1}{2}}\cos^2\tau\,d\tau\\&=-(1-\epsilon^2)\frac{1}{2\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{1}{2}}d\tau\\&+\frac{1}{\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{3}{2}}d\tau\\&-\frac{1}{2\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{5}{2}}d\tau\end{aligned} \tag{B7}$$

Thus, equation (B7) can be evaluated knowing equation (B6), which may be obtained from a recursion formula, as follows: From integrating by parts,

$$\begin{aligned}\frac{\epsilon^2}{2\pi}\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{1}{2}}\cos^2\tau\,d\tau&=\frac{\epsilon}{\pi(2N+3)}\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{3}{2}}\\ \times\sin\tau\,d\tau&=\frac{1}{\pi(2N+3)}\left[\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{5}{2}}d\tau-\right.\\&\left.\int_0^{2\pi}(1+\epsilon\sin\tau)^{N+\frac{3}{2}}d\tau\right]\end{aligned} \tag{B8}$$

Combining equations (B7) and (B8) yields

$$\left(N+\frac{3}{2}\right)(1-\epsilon^2)\overline{g^{N+\frac{1}{2}}}-2(N+2)\overline{g^{N+\frac{3}{2}}}+\left(N+\frac{5}{2}\right)\overline{g^{N+\frac{5}{2}}}=0 \tag{B9}$$

Substituting equations (B1) and (B2) into (B9) yields all necessary relations for evaluating equations (B6) and (B7):

$$\overline{g^{3/2}}=\frac{2}{3\pi}\sqrt{1+\epsilon}\,[4E-(1-\epsilon)K] \tag{B10}$$

$$\overline{g^{5/2}}=\frac{2}{15\pi}\sqrt{1+\epsilon}\,[(23+9\epsilon^2)E-8(1-\epsilon)K] \tag{B11}$$

$$\overline{g^{-3/2}g'^2}=-\frac{4}{\pi}\sqrt{1+\epsilon}\left(E-\frac{K}{1+\epsilon}\right) \tag{B12}$$

$$\overline{g^{-5/2}g'^2}=\frac{4}{3\pi\sqrt{1+\epsilon}}\left(\frac{E}{1-\epsilon}-K\right) \tag{B13}$$

$$\overline{g^{-7/2}g'^2}=\frac{4}{15\pi\sqrt{1+\epsilon}\,(1-\epsilon^2)}\left(\frac{1+2\epsilon+6\epsilon^2}{1+\epsilon}E-K\right) \tag{B14}$$

$$\overline{g^{-9/2}g'^2}=\frac{2}{105\pi\sqrt{1+\epsilon}\,(1-\epsilon^2)^2}[(3+16\epsilon-6\epsilon^2)E-2(3+4\epsilon-3\epsilon^2)K] \tag{B15}$$

The foregoing averages are presented in table II as functions of ϵ.

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TABLE I.—SECOND-ORDER SOLUTIONS

σ	f''_{00}	f'_{00}	f_{00}	r'_{00}	r_{00}	h'_{00}	h_{00}	s''_{00}	s_{00}
0	1.08350	0	0	0	1.51298	−0.13182	0	1.03328	0
.1	1.05201	.10728	.00539	−.45493	1.48921	−.13245	−.01321	.60292	.08043
.2	.96756	.20864	.02126	−.79778	1.42573	−.13329	−.02649	.31413	.12530
.3	.84501	.29952	.04677	−1.04778	1.33275	−.13283	−.03981	.12345	.14650
.4	.69821	.37683	.08071	−1.22022	1.21876	−.12966	−.05296	−.00100	.15217
.5	.53920	.43876	.12162	−1.32738	1.09088	−.12273	−.06562	−.08207	.14773
.6	.37789	.48460	.16792	−1.37922	.95513	−.11146	−.07737	−.13533	.13668
.7	.22196	.51452	.21801	−1.38395	.81660	−.09577	−.08776	−.17062	.12125
.8	.07702	.52936	.27032	−1.34855	.67967	−.07608	−.09638	−.19353	.10297
.9	−.05306	.53042	.32342	−1.27929	.54802	−.05328	−.10286	−.20673	.08287
1.0	−.16583	.51932	.37600	−1.18215	.42474	−.02857	−.10696	−.21111	.06190
1.1	−.25986	.49788	.42694	−1.06312	.31232	−.00329	−.10855	−.20687	.04093
1.2	−.33452	.46800	.47530	−.92844	.21264	.02116	−.10764	−.19421	.02081
1.3	−.38974	.43162	.52032	−.78455	.12694	.04352	−.10439	−.17382	.00237
1.4	−.42593	.39069	.56147	−.63797	.05582	.06272	−.09905	−.14709	−.01373
1.5	−.44399	.34704	.59837	−.49498	−.00077	.07802	−.09199	−.11611	−.02692
1.6	−.44530	.30245	.63085	−.36120	−.04348	.08895	−.08360	−.08334	−.03689
1.7	−.43189	.25847	.65888	−.24104	−.07347	.09541	−.07435	−.05113	−.04361
1.8	−.40828	.21648	.68261	−.13766	−.09225	.0753	−.06466	−.02166	−.04721
1.9	−.37091	.17753	.70228	−.05322	−.10164	.09577	−.05496	.00333	−.04809
2.0	−.32905	.14254	.71825	.01202	−.10354	.09073	−.04562	.02295	−.04674
2.1	−.28380	.11182	.73092	.05906	−.09984	.08314	−.03690	.03695	−.04369
2.2	−.23782	.08580	.74078	.08980	−.09227	.07381	−.02905	.04565	−.03953
2.3	−.19386	.06419	.74823	.10679	−.08234	.06352	−.02216	.04973	−.03472
2.4	−.15345	.04688	.75374	.11281	−.07128	.05298	−.01635	.05009	−.02969
2.5	−.11804	.03339	.75776	.11065	−.06005	.04278	−.01156	.04770	−.02479
2.6	−.08819	.02304	.76051	.10287	−.04934	.03338	−.00776	.04347	−.02022
2.7	−.06387	.01559	.76247	.09172	−.03959	.02507	−.00486	−.03828	−.01613
2.8	−.04494	.01008	.76369	.07900	−.03105	.01799	−.00271	.03263	−.01258
2.9	−.03052	.00642	.76452	.06587	−.02380	.01225	−.00121	.02698	−.00961
3.0	−.02006	.00392	.76507	.05335	−.01785	.00776	−.00022	.02171	−.00717
3.1	−.01271	.00223	.76528	.04207	−.01309	.00440	.00038	.01704	−.00523
3.2	−.00773	.00134	.76557	.03233	−.00939	.00200	.00069	.01304	−.00374
3.3	−.00448	.00060	.76555	.02424	−.00657	.00040	.00081	.00976	−.00260
3.4	−.00238	.00038	.76566	.01774	−.00449	−.00059	.00080	.00714	−.00177
3.5	−.00116	.00018	.76573	.01268	−.00297	−.00111	.00071	.00509	−.00116
3.6	−.00046	.00006	.76562	.00886	−.00191	−.00133	.00059	.00355	−.00073
3.7	−.00015	.00013	.76579	.00604	−.00117	−.00133	.00046	.00243	−.00044
3.8	.00009	−.00003	.76561	.00403	−.00067	−.00124	.00033	.00164	−.00022
3.9	.00010	.00014	.76575	.00263	−.00034	−.00106	.00022	.00108	−.00009
4.0	.00019	.00006	.76576	.00169	−.00013	−.00087	.00012	.00070	−.00001
4.1	.00021	.00013	.76565	.00106	.00000	−.00070	.00003	.00045	.00004
		$\int_0^\infty f_{00}\,d\sigma=2.5490$		$\int_0^\infty r_{00}\,d\sigma=1.0422$		$\int_0^\infty h_{00}\,d\sigma=-0.1033$		$\int_0^\infty s_{00}\,d\sigma=0.0713$	

TABLE II.—EVALUATION OF INTEGRALS

ϵ	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}
0.2	1.00748	0.02087	1.00769	0.02269	1.020	0.02059	0.99747	1.03747	0.02168	0.02050
.4	1.03019	.09673	1.03299	.13704	1.080	.09107	.98960	1.14963	.11282	.08680
.6	1.06872	.29227	1.08539	.71028	1.180	.24999	.97522	1.33554	.43430	.22032
.8	1.12432	.96153	1.20016	7.28120	1.320	.66663	.95129	1.59356	2.41368	.49776
.9	1.15938	2.38511	1.33236	63.68454	1.405	1.29413	.93287	1.74880	10.88546	.79898
.95	1.17904	5.30960	1.47463	529.649	1.45125	2.20252	.92025	1.83273	45.84304	1.10872