REPORT 1335

MINIMUM WAVE DRAG FOR ARBITRARY ARRANGEMENTS OF WINGS AND BODIES [1]

By ROBERT T. JONES

SUMMARY

Studies of various arrangements of wings and bodies designed to provide favorable wave interference at supersonic speeds lead to the problem of determining the minimum possible value of the wave resistance obtainable by any disposition of the elements of an aircraft within a definitely prescribed region. Under the assumptions that the total lift and the total volume of the aircraft are given, conditions that must be satisfied if the drag is to be a minimum are found. The report concludes with a discussion of recent developments of the theory which lead to an improved understanding of the drag associated with the production of lift.

INTRODUCTION

The losses associated with the production of a given lift in frictionless flow are generally diminished by increasing the mass of air entrained or influenced by the wing system. At the same time, however, the loss due to friction becomes greater when the exposed surface area of the wing is increased. To minimize the resultant drag we thus require a lifting system which effects the largest entrainment and yet has the smallest exposed surface area.

At subsonic speeds the mass of air entrained depends only on the lateral dimensions of the wing and is not diminished by concentrating the lift within a narrow chordwise dimension. The fact that a lifting line perpendicular to the direction of flight has such an extensive lateral influence must be considered a peculiarity of subsonic flow; it depends of course on the unlimited propagation of the pressure field ahead of the wing. At supersonic speed the lateral entrainment begins only at the foremost points of the wing surface and is confined to the interior of the rearward-sloping Mach waves from this point. Finally, at extreme speeds for which Newtonian flow may be envisioned, the mass of air affected is limited to the mass coming directly into contact with the wing, so that the area of influence is simply the frontally projected area of the wing.

Another peculiarity of the subsonic inviscid flow is the complete lack of resistance associated with the thickness of the bodies or wings. At supersonic speeds, however, such a component of drag does arise and this drag appears in the energy required for the continual extension of the wave system.

Now the problem of minimizing drag at supersonic speeds may be treated mathematically in several ways, depending on the constraints adopted in the statement of the problem. If, following Munk's problem of the minimum induced drag at subsonic speeds, we impose a constraint merely on the lift L and the span b of the wing, then we obtain the same value for the drag at all Mach numbers, namely the induced drag associated with the vortex wake. However to achieve this value at supersonic speeds the wing would be required to have an infinitely great length in the flight direction so that the downward momentum associated with the lift could be introduced gradually along the flight path, without appreciable wave formation.

In order to put the problem of drag at supersonic speeds in a definite form the present writer proposed (ref. 1) that the outline or plan form S of the wing be adopted as a constraint rather than single lengthwise or spanwise dimensions. Thus for supersonic speeds we are led to consider the distribution of a given total lift L over a specified plan form S in such a way as to minimize the drag D.

In the latter problem it is presupposed that the lifting system is confined to a plane. However, the possibility of favorable interference with three-dimensional arrangements of airfoils and bodies should not be overlooked. Thus, Busemann has shown (ref. 2) that the wave drag can be completely canceled by reflection between the upper and lower wings of a biplane. Later Ferrari (ref. 3) showed that the drag of a body of revolution could be canceled by the addition of a ring airfoil to catch the wave from the nose and reflect it back to the tail.

The examples in which the wave cancellation is complete are, however, limited to systems in which the net lift and lateral force are zero. Nevertheless, examples cited by Ferri (ref. 4), Lomax and Heaslet (ref. 5), and Graham (ref. 6) indicate that the wave drag associated with the lift can be diminished by various three-dimensional arrangements of wings and bodies. These examples lead to a search for some general statements or criteria regarding the drag of such three-dimensional arrangements.

CONDITIONS FOR MINIMUM DRAG

To put the present question in a definite form it will be assumed the airfoils and bodies are disposed in the interior of a definite three-dimensional region R (see fig. 1). The region R thus represents a geometrical constraint on the dimensions of the aircraft. Three-dimensional problems of a similar type have been considered by E. W. Graham and his colleagues (ref. 6) who give, for example, the optimum distributions of lift in spherical and ellipsoidal regions. Here we assume the total lift L and the volume V to be given. In a typical situation the lift L will be produced

[1] Supersedes NACA TN 3530 by Robert T. Jones, 1956.

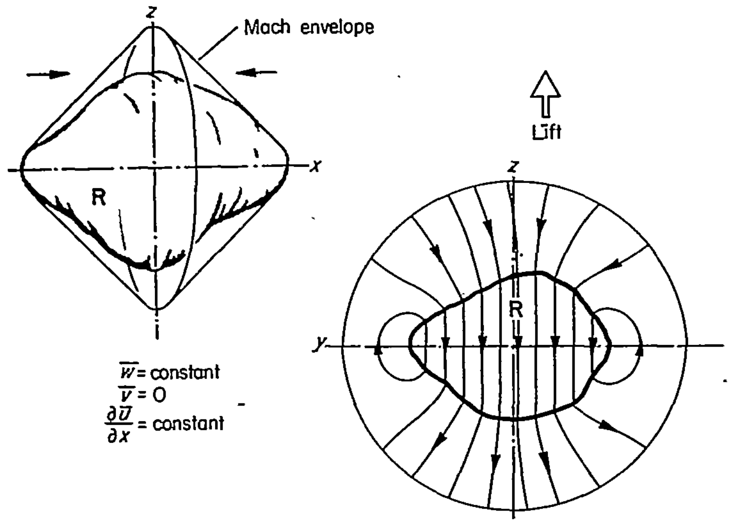

FIGURE 1.—Conditions for minimum drag distributions of lift and volume in region R.

by one or more airfoils while the volume V will represent the internal volume of one or more bodies of revolution plus the volume of the wing. The region R may then be thought of as the region within which the parts of the aircraft may be disposed so as to gain the maximum favorable interference. From a slightly different standpoint, R merely represents the maximum dimensions of the aircraft. We shall be especially interested in structures which minimize the drag for the largest possible region, but which in themselves occupy only a small part of this region.

Suppose a region R together with a distribution of singularities, such as sources or lifting vortices, is given (see fig. 1). Then by Kármán and Hayes' theorem (ref. 7) the drag will be unchanged by a reversal of the whole system. The geometry of the flow, including that of the airfoils and bodies, will be changed by the reversal but the total lift and the total volume will not. The drag for either direction of flow may then be computed by means of a fictitious "combined disturbance field" obtained by superimposing the disturbances in forward and reversed motion. The perturbation velocities in this combined field may be denoted by:

$$2\bar{u}=u_f+u_r$$

$$2\bar{v}=v_f+v_r$$

$$2\bar{w}=w_f+w_r$$

It may be shown that an arrangement of sources or lifting elements, or their combination, which yields the minimum drag is characterized by the following conditions

$$\left.\begin{aligned}\bar{w}&=\text{constant}\\ \bar{v}&=0\\ \frac{\partial\bar{u}}{\partial x}&=\text{constant}\end{aligned}\right\}\tag{1}$$

throughout R.

If conditions (1) are satisfied, then the integrated drag of the whole system will be given simply by

$$D_{min}=L\frac{\bar{w}}{U}+\rho UV\frac{\partial\bar{u}}{\partial x}\tag{2}$$

The first term on the right-hand side of this expression will be recognized as the drag arising from the rearward inclination of the lift vector, whereas the second term is simply the product of the volume and the constant gradient of pressure in the combined flow field.

These conditions may be verified by making use of a "mutual drag relation" (ref. 1), essentially similar to the well-known Ursell-Ward reciprocal relation, which connects the drag of any two interfering distributions of singularities in the combined flow field. According to this relation, the drag of distribution A caused by the interference of a second distribution B is equal to the drag added to B by the interference of A. Now let A be a distribution within R_A satisfying conditions (1). For B select a distribution having zero total lift and zero total volume. If R_B is contained within R_A, then the addition of B will amount simply to a redistribution, without changing the given lift L or volume V of A. The drag of $A+B$ may then be written in shorthand notation

$$D(A+B)=D_{AA}+D_{AB}+D_{BA}+D_{BB}\tag{3}$$

Then, since by the mutual drag relation $D_{AB}=D_{BA}$, this equation may be written as

$$D(A+B)=D_{AA}+2D_{BA}+D_{BB}\tag{4}$$

Here D_{BA} is the drag of B in the combined disturbance field of A. Since $\bar{w}_A=$constant, $\bar{v}_A=0$ and $\left(\frac{\partial\bar{u}}{\partial x}\right)_A=$constant in R_A, this interference drag may be written simply as

$$D_{BA}=L_B\frac{\bar{w}_A}{U}+\rho UV_B\left(\frac{\partial\bar{u}}{\partial x}\right)_A\tag{5}$$

However, since L_B and V_B are both zero D_{BA} vanishes and the added drag is that of distribution B alone, or D_{BB}. Now the drag of an isolated system can never be negative, hence $D(A+B)$ cannot be less than $D(A)$ under the conditions (1).

On the other hand, suppose, for example that the sidewash $\bar{v}_A$ were not zero. A distribution of lateral forces could then be found which would result in a negative interference drag, dominating the quadratic term D_{BB}, so that the total drag could be reduced. Hence, if the drag of distribution A actually is a minimum value, then conditions (1) must be complied with.

The question of uniqueness depends on the existence of distributions of type B for which the drag is zero. As shown by Graham, such distributions exist in three dimensions and hence the minimum drag corresponding to a given region R may be achieved by a variety of arrangements. In the case of a planar region, such as the plan form S of a wing, distributions of lift or volume having zero drag do not exist, and hence in these cases the optimum distributions are unique.

Since $\bar{w}=\frac{\partial\bar{\varphi}}{\partial z}$, $\bar{v}=\frac{\partial\bar{\varphi}}{\partial y}$ and $\frac{\partial\bar{u}}{\partial x}=\frac{\partial^2\bar{\varphi}}{\partial x^2}$, it can be seen that conditions (1) do not agree with the linearized flow equation

$$(1-M^2)\bar{\varphi}_{xx}+\bar{\varphi}_{yy}+\bar{\varphi}_{zz}=0\tag{6}$$

in general, but only if

$$\frac{\partial\bar{u}}{\partial x}=0\tag{7}$$

Since $\frac{\partial \bar{u}}{\partial x}$ is proportional to the drag per unit volume, one concludes that the drag cannot be minimized in an absolute sense unless the drag associated with the volume of the system is zero, or unless the distribution of singularities is continuous throughout R. Examples such as the Busemann biplane satisfy the condition $\frac{\partial \bar{u}}{\partial x}=0$.

It is interesting to note that conditions analagous to $\bar{w}=$ constant and $\bar{v}=0$ were found by Munk in connection with the vortex drag of lifting systems at subsonic speeds. In that problem, the conditions apply to the two-dimensional motion associated with the trace of the wing system in the Trefftz plane. If the idea of superimposed disturbance fields is utilized in the subsonic problem, one finds that the cylindrical flow associated with the Trefftz plane extends along the whole flight path, including the region R. Conditions (1) thus apply at both subsonic and supersonic speeds.

Munk's condition of constant downwash and zero sidewash were used by Hemke (ref. 8) to calculate the effectiveness of end plates in reducing the vortex drag at low speeds. In such problems the condition is usually imposed by the statement that the trace of the airfoil system must move downward as a rigid body. It will be interesting to see how this condition might be used under more general circumstances. This application is illustrated in figure 2 for an end plate on the tip of a wing.

With the wing in forward motion, the lateral velocity v_f at the surface of the end plate is simply the lateral slope of the fin surface multiplied by the stream velocity. The condition $\bar{v}=0$ implies that $v_r=-v_f$ and this condition is obviously satisfied by keeping the geometry of the fin fixed when the flow is reversed. At the same time, however, recall that the distribution of lift and lateral force must be kept the same in forward and reversed flow. Hence, the problem of finding the optimum setting and camber for such a fin is solved by finding that particular shape for which the flow is exactly reversible, that is, the lateral pressure distribution remains unchanged by a reversal of flow direction. At first it seems impossible to satisfy such a requirement, since, for example, the direction of the force on an inclined surface is usually reversed by a reversal of the direction of flow. However, the form of the adjacent wing surface must, in general, change with the reversal, since $\bar{w}\neq 0$ and since the lift distribution on the wing must remain unchanged. Then it is evident that the conditions might be satisfied if the pressures on the fin surface were dominated by the wing pressures through interference.

Recently W. Wilmarth (ref. 9) has found several examples of wings with end plates which minimize the drag for certain prismatic regions.

The conditions for minimum drag are of course simply the result of the constraints adopted in the initial statement of the problem, and these are to a certain extent arbitrary. Nevertheless, experience shows that the study of such problems is likely to disclose essential relations in their clearest form.

With the aid of the combined flow field and the mutual drag theorem, it is a relatively simple matter to extend the

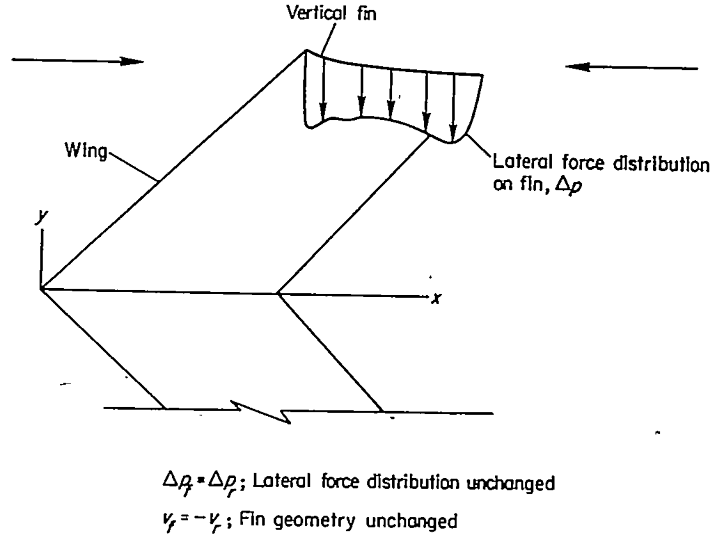

FIGURE 2.—Use of condition $\bar{v}=0$ to determine optimum setting of vertical fin on wing tip.

constraints in various ways. Thus in the case of a planar wing if both the total lift L and the spanwise loading are specified, minimum drag requires that $\bar{w}$ be constant along chordwise strips, but may vary laterally. Here we have

$$\bar{w}=f(y) \tag{8}$$

In case the lengthwise loading is specified we have

$$\bar{w}=f(x) \tag{9}$$

Again, if the first moment of the load distribution about the y axis is specified

$$\bar{w}=a_o+a_1x \tag{10}$$

and so on.

If the conditions on the combined disturbance velocities hold beyond the boundaries of the region R, then the drag cannot be changed by extending the distribution of lift or volume into the new region. In general, this will not be the case, however, and the drag can be continually diminished by increasing the dimensions of R. Thus in the case of a monoplane wing a strong upwash appears beyond the wing tips, indicating that the drag could be diminished by increasing the span. Similarly, sidewash velocities appear just above and below the planar region, and the drag could be reduced by extending vertical fins, or "fences" into this region.

It must be admitted that the considerations have thus far been rather abstract. A more concrete result would yield the actual magnitudes of the drag associated with various regions, as well as the shapes of the bodies or wings. Although no direct method of calculation has been proposed, numerous examples have been found. Thus reference 6 gives the optimum distributions of lift in spherical and ellipsoidal regions.

A rough lower bound for the minimum wave drag associated with any region may be obtained from Hayes' formula (ref. 7) or the formula of Lomax (ref. 5). With these formulas a spatial distribution of lift or volume may be resolved into a number of equivalent linear distributions, the latter obtained from the intersections of the region R by plane waves lying at various angles θ around the x axis. The wave drag of the system is then the sum of values for the linear distributions integrated from $\theta=0$ to $\theta=2\pi$. The expression for the wave drag of a single linear distribution is the same (except for a constant factor) as the expression for the vortex drag of a lifting line in subsonic flow. Thus, for a single elliptically loaded lifting line of length l parallel to the flight direction the wave drag is:

$$D_{wave}=\frac{M^2-1}{2}\frac{L^2}{\pi q l^2} \tag{11}$$

This value may be used as an approximation for the wave drag of any narrow wing lying near the center of the Mach cone. Deviations are to be expected for wider wings; however, these deviations are not very pronounced, as figure 3 shows. In this figure values of the wave drag obtained from exact theoretical formulas are compared with the values given by the approximate expression (11). The "exact" values were obtained by superimposing uniformly loaded wings of elliptical plan form and are not the minimum values for the resulting plan forms.

A sufficient condition for the wave drag of a lifting system to have a minimum value is that all the projected loadings, in addition to the lengthwise loading, be elliptical. In this case we obtain the formula

$$D_{wave} \geq \frac{M^2-1}{2}\frac{L^2}{\pi q \bar{l}^2} \tag{12}$$

where

$$\frac{1}{\bar{l}^2}=\frac{1}{\pi}\int_0^{2\pi}\frac{\sin^2\theta \, d\theta}{[l(\theta)]^2} \tag{13}$$

and $l(\theta)$ is the projected length of the region R as defined in figure 4 with $\beta=\sqrt{M^2-1}$.

The value given by equations (12) and (13) is actually attained by elliptic wings and by distributions of lift in spherical or ellipsoidal regions (ref. 6). However, for

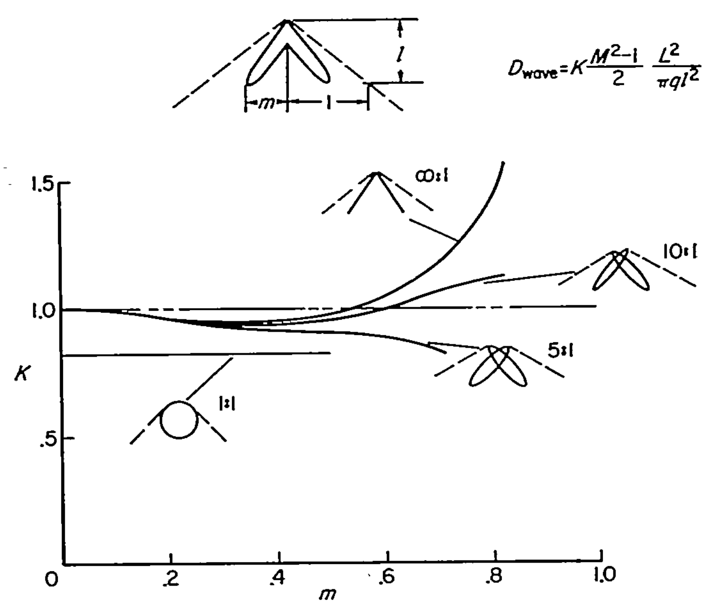

FIGURE 3.—Approximate expression for wave drag of lifting surface.

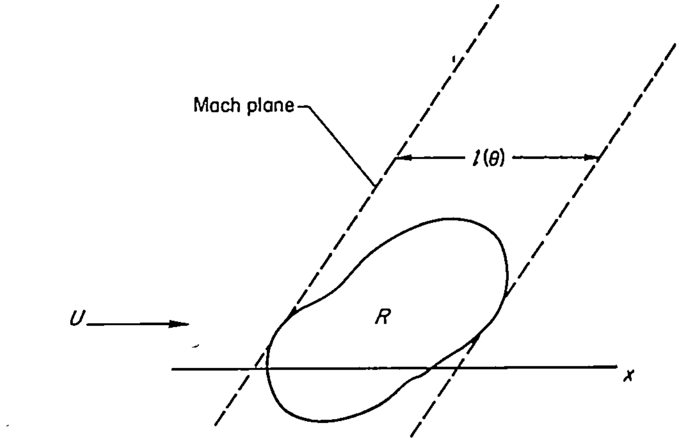

FIGURE 4.—Lower bound for wave drag associated with the region R and the lift L.

triangular or swept wings of the type depicted in figure 3, the values given by the simpler expression equation (11) are more accurate.

In a recent paper (ref. 10) M. I. Kogan has pointed out that determination of the minimum drag of a lifting surface having no subsonic edges can be reduced to the solution of Laplace's equation in the two-dimensional region bounded externally by the trace of the characteristic envelope, and bounded internally by the vortex trace of the wing. In addition to $\varphi_{yy}+\varphi_{zz}=0$, the boundary condition that no disturbance extend beyond the Mach cone corresponds to the condition $\varphi=0$ on the outline of the characteristic trace S_j (i. e., the outer rim of the Mach envelope in fig. 1) while the condition of constant downwash corresponds to φ_z=constant on the vortex trace.

The result given by Kogan has been derived independently by E. W. Graham (ref. 11) and by G. N. Ward (ref. 12). Graham makes use of the combined flow field, and shows that fields which are two-dimensional throughout the interior of any given characteristic envelope, and which satisfy the condition $\bar{\varphi}_z=\bar{w}$=constant on a vortex trace passing through the region, can be constructed.

Such solutions correspond to our previous conditions (6) and (7) and are not restricted to wings having supersonic edges.

In Ward's analysis the physical flow is used, but the drag is calculated by using the forward-going surface of the characteristic envelope as a control surface. Since $\varphi=0$ there in the reversed flow, it can be seen that the values of φ in the real flow coincide with those in the combined flow on this surface. By a projection of the disturbance velocities on this surface, Ward reduces the integral for drag to Dirichlet's integral, which is a minimum when the derived velocity field satisfies Laplace's equation.

Applications of this method to problems involving thickness and volume have been given by M. A. Heaslet (ref. 13). Problems in which both the lift and the center of pressure are given have been treated by P. Germain (ref. 14).

These theoretical developments provide an interesting intuitive picture of the drag associated with the production of lift at supersonic speeds. At subsonic speeds the lifting wing leaves in its wake a two-dimensional, essentially incompressible downwash flow bounded internally by the vortex wake, but unbounded externally. According to Kelvin's theorem such an incompressible downwash flow satisfying $\varphi_{yy}+\varphi_{zz}=0$, minimizes the kinetic energy relative to all other streamline motions satisfying the same boundary conditions. For a given lift, (or downward momentum) the kinetic energy, and hence the drag, is minimized when the wake moves with constant downwash. At supersonic speeds we are led to consider not the flow in the Trefftz plane at infinity, but the flow in the last characteristic surface where the zone of influence lies entirely behind the wing. The two-dimensional flow obtained by projection on this surface will be limited laterally by its intersection with the real Mach wave, where φ must vanish, and will be bounded internally by the vortex wake on the the trailing edge of the wing. This flow is certainly not incompressible in general. However, if the wing is to have the minimum drag consistent with the given span and with the given limitations of the lateral zone of influence, then by Kelvin's theorem the flow must imitate the streamlines of an incompressible lateral flow in this intervening limited region. For a given total lift the vortex wake should again move with constant downwash.

The condition $\varphi=0$ on the rim of the characteristic envelope is exactly the same as that imposed at the boundary of an open-jet wind tunnel. Hence, we are led to compare the action of the wing in supersonic flow with that of a wing in a finite jet (fig. 5). Wings having small fore and aft dimensions have a limited lateral entrainment, as shown by the small cross sections of their equivalent incompressible jets (see fig. 6).

In Munk's theory of the minimum induced drag the "area of the additional apparent mass" associated with the vortex trace of the wing plays an important role. Denoting this area by S_w', we have for the drag due to lift

$$D=\frac{1}{2\rho U^2}\frac{L^2}{S_w'} \qquad (14)$$

This formula actually applies in perfect fluid flow at all speeds if S_w' is replaced by S_{w_j}', the additional apparent

FIGURE 5.—Equivalent incompressible jet for wing at supersonic speed.

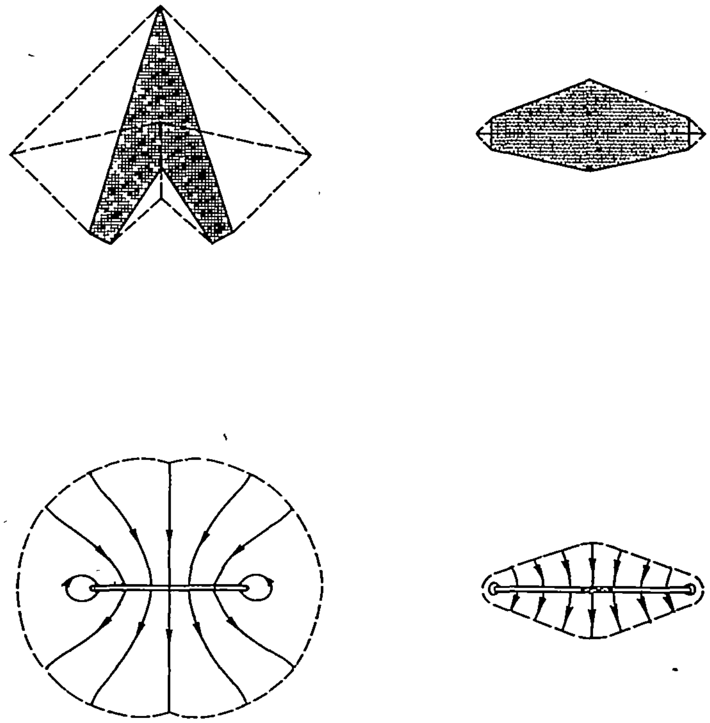

FIGURE 6.—Effect of fore and aft dimension of wing on area of lateral entrainment.

mass of the wing in the limited jet determined by the Mach waves. If the Mach number of the stream is reduced, the waves become more nearly vertical and the equivalent jet expands laterally, reaching an infinite cross section at $M=1.0$. Below $M=1.0$ the wing is operating in unlimited flow and we then have:

$$S_w'=\pi\frac{b^2}{4} \qquad (15)$$

which leads to

$$D=\frac{2L^2}{\pi\rho U^2b^2} \qquad (16)$$

On the other hand, at extremely high supersonic speeds, the equivalent jet contracts into a narrow space around the frontal projection of the wing. In this case the streamlines of the downflow in the jet will be nearly straight and parallel, as illustrated in figure 6, and the area S_{w_j}' will be substantially equal to the area of the jet S_j.

In special cases the two-dimensional downflow in the characteristic trace or jet S_j can be readily calculated. Thus in the case of the elliptic wing the envelope of characteristics has an elliptic cross section, with the vortex trace of the wing extending between the foci. Now if a flat plate moves downward (along z) in unlimited flow, the potential at the surface of any confocal elliptic cylinder will be of the form $\varphi_s=kz_s$. Hence the boundary condition $\phi=0$ may be satisfied on any such confocal ellipse by adding a uniform downwash throughout its interior so that $w=-k$ or $\varphi=-kz$. When the downward momentum of the resultant flow is computed, it is found to correspond to a virtual mass with area S_{w_j}' given by

$$\frac{1}{S_{w_j}'}=\frac{1}{S_j}-\frac{1}{S_j+S_j'}+\frac{1}{S_w'} \qquad (17)$$

where

$$S_w'=\frac{\pi b^2}{4} \qquad (18)$$

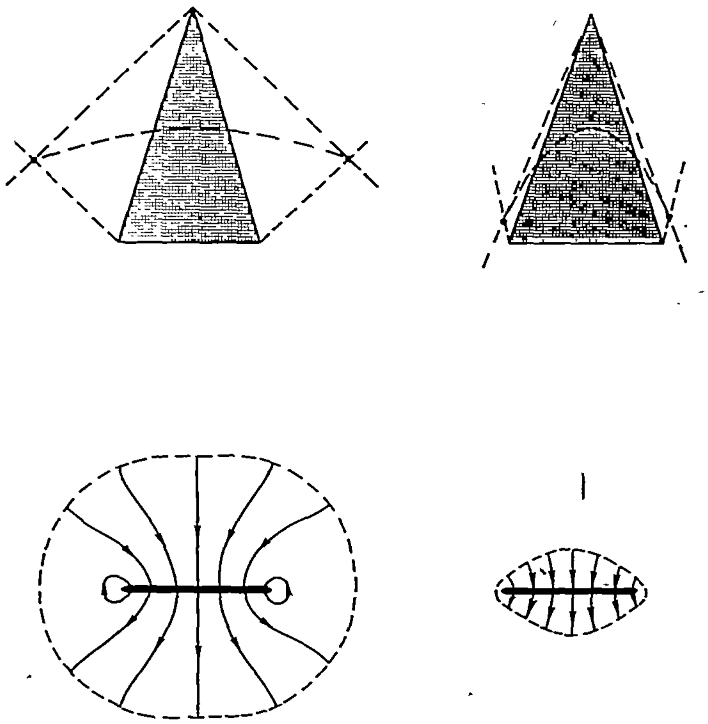

FIGURE 7.—Effect of Mach number on lateral entrainment.

In the case of a long slender wing lying near the center of the Mach cone (slender wing theory) the characteristic trace will be circular. An elliptical lengthwise distribution of lift then produces an incompressible downwash flow resembling that of a dipole at the center of the circle. The added downwash required to make $\varphi=0$ on this circular boundary then yields our formula (11) for the wave drag.

If we try to find the surface loading or shape that corresponds to the drag given by equation (14), we discover that Kogan's analysis has in fact carried us away from our original problem in which the plan form of the wing (or the region R occupied by the lifting system) was given. The information given now concerns only the trace of the wing and its characteristic envelope. Now, the relation between the plan form of a wing and its characteristic trace is certainly not unique. On the other hand the particular form of the two-dimensional flow on the reversed characteristic surface must require a unique distribution of lift in the plane of the wing. Otherwise one could show by superposition that planar distributions of lift having no drag would exist. It must be concluded therefore that of all the plan forms having a given characteristic envelope, only those whose surface area is extensive enough to enclose the required surface distribution of lift can achieve the minimum dràg given by equation (14).

AMES AERONAUTICAL LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
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