REPORT 1374

THE SIMILARITY RULES FOR SECOND-ORDER SUBSONIC AND SUPERSONIC FLOW [1]

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SUMMARY

The similarity rules for linearized compressible flow theory (Göthert's rule and its supersonic counterpart) are extended to second order. It is shown that any second-order subsonic flow can be related to "nearly incompressible" flow past the same body, which can be calculated by the Janzen-Rayleigh method.

INTRODUCTION

The linearized small-disturbance theory of steady compressible flow, based on the Prandtl-Glauert equation, yields a first approximation for thin objects moving at either subsonic or supersonic speeds. More precisely, it provides the first term in an asymptotic expansion of the solution for small disturbances, provided that the flight Mach number is not too close either to unity (transonic flow) or to infinity (hypersonic flow).

The similarity rule that governs linearized subsonic flow past general three-dimensional objects was first given correctly by Göthert (ref. 1). It has an obvious counterpart in supersonic flow, and the rules have rendered great service in both theoretical and experimental investigations.

Recently, various investigators have sought to improve on the linearized theory by finding higher approximations (see, e.g., refs. 2 to 5). The second step is commonly referred to as the second-order small-disturbance theory, or simply "second-order theory." It can be found in general by iterating upon the linearized solution, retaining all terms out of the full nonlinear equations of motion whose contribution is of the order of the square of the disturbances in linearized theory (ref. 3). In the simplest case of plane flow without stagnation points, the linearized disturbances are proportional to the thickness ratio τ of the airfoil, so that second-order theory adds terms in τ^2, and higher approximations extend the series in powers of τ^n. Stagnation points lead to the appearance of logarithmic terms, beginning with $\tau^4 ln\ \tau$ in the fourth approximation. The series diverges in the immediate vicinity of stagnation points, although it can be corrected there by simple techniques (ref. 6). Slender pointed objects, such as a smooth body of revolution, cause smaller disturbances than airfoils, but logarithmic terms always arise at the outset; hence the linearized solution contains terms of order $\tau^2 ln\ \tau$ and τ^2, and the second-order increment then consists of terms of order $\tau^4 ln^2 \tau$, $\tau^4 ln\ \tau$, and τ^4. Nothing is known of the convergence of these series; they are perhaps only asymptotic expansions for small thickness. Second-order theory, like linearized theory, breaks down in the transonic and hypersonic ranges, though it may penetrate somewhat farther into their fringes.

A similarity rule for second-order theory has recently been given in the special case of supersonic flow past thin flat wings by Fenain and Germain, who demonstrate its usefulness in theoretical studies (ref. 5). However, as in linearized theory, the rules for flat wings are only special cases of those for general three-dimensional shapes. The present paper is devoted to deducing the general rules for subsonic and supersonic flows, and examining their implications. In particular, it is shown how the rule for subsonic flow relates the second-order solution for any object to nearly incompressible flow past the same body, which can be calculated by the Janzen-Rayleigh method.

The author is indebted to Wallace D. Hayes for suggesting several improvements that have been incorporated in the present version of this paper. In particular, the procedure for recovering the second-order solution from the Janzen-Rayleigh solution (p. 930) is simpler and more logical than that originally given in NACA TN 3875.

DERIVATION OF RULES FOR BODIES OF REVOLUTION

A body of revolution is the simplest shape that is not a special case, but displays the full generality of the existing similarity rules for subsonic, supersonic, transonic, and hypersonic flows. The same can be shown to be true of the second-order rules to be discussed here. Hence for clarity of exposition, the second-order rules will be derived in detail only for an axisymmetric body at zero angle of attack. The rules for general three-dimensional thin or slender objects will thereafter be stated without proof. The subsonic and supersonic cases are so similar that they can be treated simultaneously.

Let the body be described by $r=\tau R(x)$, where τ is a thickness parameter or characteristic slope (say, the maximum slope, average slope, thickness ratio, or the like), and $R(x)$ is a function of order unity (fig. 1). As usual in similarity analysis, the characteristic slope τ is regarded as a parameter, so that different values of τ correspond to affinely related members of the same family of bodies.

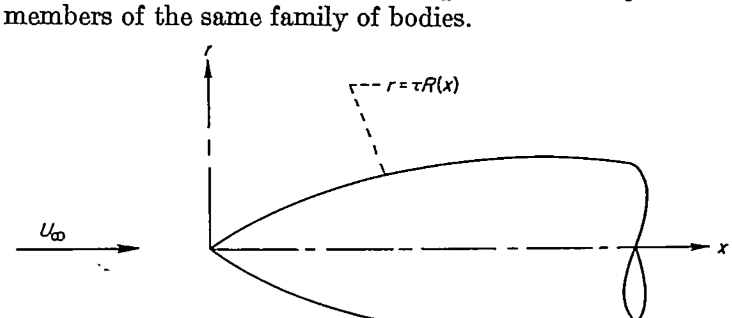

FIGURE 1.—Notation for body of revolution.

[1] Supersedes NACA TN 3875, by Milton D. Van Dyke, 1957.

To second order the flow is irrotational, so that there exists a velocity potential $\Phi(x, r; M, \gamma, \tau)$. This notation indicates that for each family of bodies (associated with a given function $R(x)$), the flow field is regarded as depending not only upon the two independent variables x and r but also upon the three parameters following the semicolon:

M free-stream Mach number
γ adiabatic exponent of gas[2]
τ characteristic slope of body

The aim of a similarity analysis is to transform the problem so as to reduce the number of parameters appearing in it. If that can be accomplished, flows having different values of the original parameters are related provided only that the reduced parameters are equal. The transformation to be used here consists in separating the dependent variable Φ into several components, and then stretching each component and the independent variables by factors that depend upon the original parameters. It is convenient, and involves no loss of generality, to leave streamwise coordinates unchanged, so that r is to be stretched but not x.

Perturbation potentials are first introduced by setting

$$\frac{\Phi}{U_\infty}=x+\phi+\varphi+\ldots \tag{1}$$

where ϕ is the potential of linearized theory, and φ the second-order increment.

RULES FOR LINEARIZED THEORY

The linearized problem is

$$\left.\begin{array}{ll}\Box\phi\equiv(1-M^2)\phi_{xx}+\phi_{rr}+\dfrac{\phi_r}{r}=0 & \\ \phi\rightarrow 0 & \text{at infinity} \\ \phi_r=\tau R'(x) & \text{at } r=\tau R(x)\end{array}\right\} \tag{2}$$

The first relation is the linearized Prandtl-Glauert equation. The second is a statement, sufficiently definite for present purposes, of the requirement that the flow approach a uniform stream far from the body in almost all directions. The third is the linearized condition of tangent flow at the body surface.[3] The linearized problem is seen not to involve γ, so that the solution depends upon only the two parameters M and τ.

The similarity rules can be obtained by considering arbitrary scale transformations of ϕ and r. It is readily found that the only choice that reduces the number of parameters from two to one is (temporarily suppressing the dependence on parameters)

$$\phi(x,r)=\frac{1}{\beta^2}F(x,\rho) \tag{3a}$$

and

$$\rho=\beta r \tag{3b}$$

where

$$\beta=\begin{cases}\sqrt{1-M^2} & \text{for subsonic flow}\\ \sqrt{M^2-1} & \text{for supersonic flow}\end{cases} \tag{3c}$$

Then the problem becomes

$$\left.\begin{array}{ll}\triangle F\equiv F_{\rho\rho}+\dfrac{F_\rho}{\rho}\pm F_{xx}=0 & \\ F\rightarrow 0 & \text{at infinity} \\ F_\rho=\beta\tau R'(x) & \text{at } \rho=\beta\tau R(x)\end{array}\right\} \tag{4}$$

where here and later the upper and lower signs apply, respectively, to the subsonic and supersonic problems.

The transformations of ϕ and r have been so chosen that the problem is reduced to one involving the two parameters M and τ not separately, but only in the combination $\beta\tau$. This is the similarity parameter. Two subsonic or supersonic flows past bodies of the same family are related if the corresponding Mach numbers are such that the parameter $\beta\tau$ is the same for both. The nature of the relationship is found by reintroducing the dependence on parameters into equation (3a), which gives the similarity rules

$$\phi(x,r;M,\tau)=\frac{1}{\beta^2}F(x,\beta r;\beta\tau) \tag{5}$$

SECOND-ORDER RULES

The second-order problem is found to be (ref. 3)

$$\left.\begin{array}{ll}\Box\varphi=M^2\{[(\gamma+1)M^2+2(1-M^2)]\phi_x\phi_{xx}+2\phi_r\phi_{xr}+\phi_r^2\phi_{rr}\} & \\ \varphi\rightarrow 0 & \text{at infinity} \\ \varphi_r=\tau\phi_x R'(x) & \text{at } r=\tau R(x)\end{array}\right\} \tag{6}$$

Note that the first equation contains not only quadratic terms on the right-hand side, but also the triple product $\phi_r^2\phi_{rr}$ whose contribution is of second order in some cases.

The parameter γ appears only linearly in the combination $(\gamma+1)$ and can accordingly be separated out. Thus the appropriate transformation is found to be

$$\varphi(x,r)=\frac{1}{\beta^4}\left[f_1(x,\rho)+M^2f_2(x,\rho)+(\gamma+1)\frac{M^4}{\beta^2}f_3(x,\rho)\right] \tag{7}$$

Then equating like powers of M^2 yields the following set of three problems for f_1,f_2,f_3 in which the parameters M,γ,τ appear again only in the form of the single similarity parameter $\beta\tau$:

$$\left.\begin{array}{ll}\triangle f_1=0 & \\ f_1\rightarrow 0 & \text{at infinity} \\ f_{1\rho}=\beta\tau F_x R'(x) & \text{at } \rho=\beta\tau R(x)\end{array}\right\} \tag{8a}$$

[2] Hayes has pointed out (ref. 7) that to second order an imperfect gas corresponds to a polytropic gas having a γ equal to the free-stream value of

$$\frac{2}{c}\left(\frac{\partial\rho c}{\partial\rho}\right)_s-1=1+\frac{\rho}{c}\left(\frac{\partial c^2}{\partial\rho}\right)_s$$

where c is the speed of sound and ρ the density, the partial derivative being taken at constant entropy s.

[3] In what is generally called the slender-body approximation, the body is assumed to be so smooth and slender that the tangency condition can be imposed on the axis rather than on the actual surface, for bodies of revolution in the form $\lim_{r\to 0} r\phi_r=\tau^2R(x)R'(x)$. Thus slender-body theory is a further approximation within linearized theory (being, in fact, the leading term in the asymptotic expansion of the linearized solution for small thickness τ). Consequently, the slender-body solution obeys the similarity rules of linearized theory, and the second-order slender-body solution likewise obeys the second-order similarity rules.

$$\left.\begin{aligned}&\Delta f_2=\pm 2F_xF_{xx}+2F_\rho F_{x\rho}+F_\rho^2F_{\rho\rho}\\&f_2\to 0 \quad \text{at infinity}\\&f_{2\rho}=0 \quad \text{at } \rho=\beta\tau R(x)\end{aligned}\right\} \qquad (8b)$$

$$\left.\begin{aligned}&\Delta f_3=F_xF_{xx}\\&f_3\to 0 \quad \text{at infinity}\\&f_{3\rho}=0 \quad \text{at } \rho=\beta\tau R(x)\end{aligned}\right\} \qquad (8c)$$

Then reintroducing the explicit dependence on parameters into the functions f_1, f_2, f_3 of equation (7) gives the similarity rules for the second-order increment in perturbation potential:

$$\varphi(x,r;M,\gamma,\tau)=\frac{1}{\beta^4}\left[f_1(x,\beta r;\beta\tau)+M^2f_2(\;)+(\gamma+1)\frac{M^4}{\beta^2}f_3(\;)\right] \qquad (9)$$

where the arguments of f_2 and f_3 are the same as those of f_1. Hereafter the arguments of later terms will be omitted in this fashion when they are the same as for the leading term. It should be emphasized that, just as in linearized theory, the subsonic and supersonic rules are quite distinct, although they have the same form (9). Because of the different definitions of β, and the resulting $\pm$ signs in equations (4) and (8b), a supersonic flow is not related to a subsonic flow. Discussion of these results is deferred to the general case.

RULES FOR GENERAL BODIES

Consider a family of general three-dimensional bodies, whose members are derived from one another by a uniform magnification or reduction of all dimensions normal to the free stream (fig. 2). Each member of such a family can be characterized, as before, by some characteristic slope τ. It may be emphasized that τ can be identified with thickness, camber, or angle of attack, all of which vary together for related bodies.

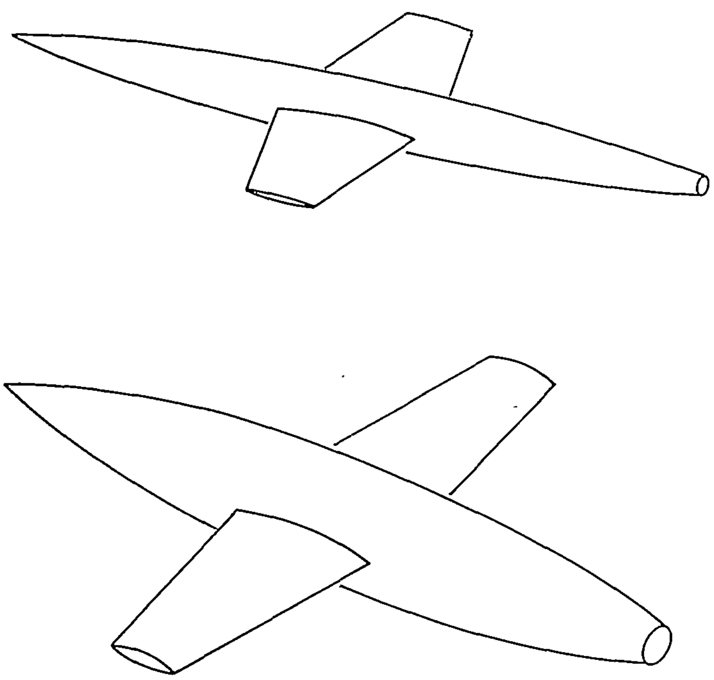

FIGURE 2.—Example of two related bodies.

The preceding analysis can be extended in a straightforward way to such general bodies, at the expense only of typographic complexity. Both cross-stream dimensions behave in the way that r did before. Hence the subsonic and supersonic second-order rules for the velocity potential, corresponding to equations (5) and (9), are, in Cartesian coordinates:

$$\frac{1}{U_\infty}\Phi(x,y,z;M,\gamma,\tau)=x+\frac{1}{\beta^2}F(x,\beta y,\beta z;\beta\tau)+\frac{1}{\beta^4}\left[f_1(x,\beta y,\beta z;\beta\tau)+M^2f_2(\;)+(\gamma+1)\frac{M^4}{\beta^2}f_3(\;)\right] \qquad (10a)$$

Differentiation yields the corresponding rules for velocity components (those for w having the same form as for v):

$$\frac{u}{U_\infty}=1+\frac{1}{\beta^2}U(x,\beta y,\beta z;\beta\tau)+\frac{1}{\beta^4}\left[u_1(x,\beta y,\beta z;\beta\tau)+M^2u_2(\;)+(\gamma+1)\frac{M^4}{\beta^2}u_3(\;)\right] \qquad (10b)$$

$$\frac{v}{U_\infty}=\frac{1}{\beta}V(x,\beta y,\beta z;\beta\tau)+\frac{1}{\beta^3}\left[v_1(x,\beta y,\beta z;\beta\tau)+M^2v_2(\;)+(\gamma+1)\frac{M^4}{\beta^2}v_3(\;)\right] \qquad (10c)$$

(The functions appearing here are actually related to derivatives of the functions in equation (10a), but the connection is of little interest.) To second order the pressure coefficient is given by

$$C_p=-2\phi_x-(\phi_y^2+\phi_z^2)-2\varphi_x-2(\phi_y\varphi_y+\phi_z\varphi_z)+(M^2-1)\phi_x^2+M^2\phi_x(\phi_y^2+\phi_z^2)+\tfrac{1}{4}M^2(\phi_y^2+\phi_z^2)^2$$

where the terms in the second line may be significant for slender shapes. Substituting the expressions (10) for velocity components and simplifying shows that the similitude for pressure coefficient has the same form as that for the streamwise velocity increment $\Delta u/U_\infty$:

$$C_p(x,y,z;M,\gamma,\tau)=\frac{1}{\beta^2}P(x,\beta y,\beta z;\beta\tau)+\frac{1}{\beta^4}\left[p_1(x,\beta y,\beta z;\beta\tau)+M^2p_2(\;)+(\gamma+1)\frac{M^4}{\beta^2}p_3(\;)\right] \qquad (10d)$$

The similarity rules for the perturbation stream function in plane flow are the same as those for v/U_∞ (eq. (10c)).

ALTERNATIVE FORMS

As with other similarity rules, an unlimited number of alternative forms can be produced by multiplying by powers of the similarity parameter. Thus, of the many possible alternatives to the second-order rules (10d) for pressure coefficient, two of the most useful are:

$$C_p=\frac{\tau}{\beta}\bar{P}(x,\beta y,\beta z;\beta\tau)+\frac{\tau^2}{\beta^2}\left[\bar{p}_1(\;)+M^2\bar{p}_2(\;)+(\gamma+1)\frac{M^4}{\beta^2}\bar{p}_3(\;)\right] \qquad (10e)$$

$$C_p=\tau^2\tilde{P}(x,\beta y,\beta z;\beta\tau)+\tau^4\left[\tilde{p}_1(\;)+M^2\tilde{p}_2(\;)+(\gamma+1)\frac{M^4}{\beta^2}\tilde{p}_3(\;)\right] \qquad (10f)$$

In addition, the first two second-order terms can be manipulated, using the connection between M^2 and β^2, to yield additional alternative forms such as the following, which correspond to the three forms above:

$$C_p=\frac{1}{\beta^2}P(x,\beta y,\beta z;\beta\tau)+\frac{1}{\beta^2}\left[p_1()+\frac{M^2}{\beta^2}p_2'()+(\gamma+1)\frac{M^4}{\beta^4}p_3()\right] \tag{10g}$$

$$C_p=\frac{\tau}{\beta}\bar{P}(x,\beta y,\beta z;\beta\tau)+\tau^2\left[\bar{p}_1()+\frac{M^2}{\beta^2}\bar{p}_2'()+(\gamma+1)\frac{M^4}{\beta^4}\bar{p}_3()\right] \tag{10h}$$

$$C_p=\tau^2\tilde{P}(x,\beta y,\beta z;\beta\tau)+\tau^4\left[\beta^2\tilde{p}_1()+\tilde{p}_2'()+(\gamma+1)\frac{M^4}{\beta^2}\tilde{p}_3()\right] \tag{10i}$$

FORCE COEFFICIENTS

The rules for pressure imply rules for the lift and drag coefficients. For example, equation (10e) leads to

$$C_L(M,\gamma,\tau)=\frac{\tau}{\beta}L(\beta\tau)+\frac{\tau^2}{\beta^2}\left[l_1(\beta\tau)+M^2l_2(\beta\tau)+(\gamma+1)\frac{M^4}{\beta^2}l_3(\beta\tau)\right] \tag{10j}$$

$$C_D(M,\gamma,\tau)=\frac{\tau^2}{\beta}D(\beta\tau)+\frac{\tau^3}{\beta^2}\left[d_1(\beta\tau)+M^2d_2(\beta\tau)+(\gamma+1)\frac{M^4}{\beta^2}d_3(\beta\tau)\right] \tag{10k}$$

if the coefficients are referred to some plan-form area. If some cross-sectional area is used, each term is reduced by one power of τ. Various alternative forms are again useful. In the case of lift coefficient, one will ordinarily choose to identify τ with the angle of attack.

RULES FOR QUASI-CYLINDRICAL BODIES

A special class of objects must be distinguished, which will be termed quasi-cylindrical bodies. These are shapes that lie everywhere so close to some cylinder (not necessarily circular) parallel to the free stream that to a first approximation the condition of tangent flow can be imposed at the cylinder rather than on the actual body surface. Likewise, in second-order theory the tangency condition can be transferred to the cylinder by Taylor series expansion. The simplest example is an airfoil whose thickness, camber, and angle of attack are so small that the tangency condition can be transferred from the airfoil surface to a mean plane parallel to the stream (fig. 3). Another example is an open-nosed body of revolution whose radius varies only slightly. Others are biplanes, cruciform wings, any of these in an open or closed wind tunnel, in combination with one another, etc.

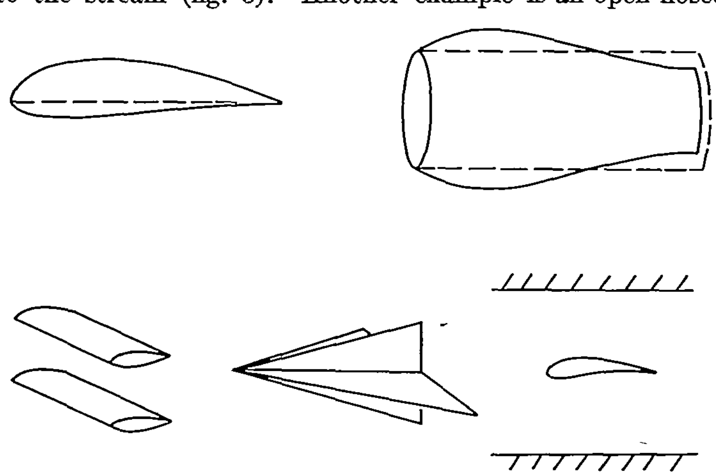

FIGURE 3.—Examples of quasi-cylindrical bodies.

A quasi-cylindrical body can be regarded as consisting of a skeleton upon which is superimposed a small slope distribution. The skeleton is simply the projection of the body onto the basic cylinder. For example, the skeleton of the quasi-cylindrical body of revolution is the circular tube shown dashed in figure 3.

The special place of quasi-cylindrical bodies in similarity theory arises from the fact that the skeleton and the slope distribution can be varied independently. This extra freedom is important. For example, it leads to a useful transonic similarity rule for quasi-cylindrical bodies whereas none exists for general shapes. It is convenient always to leave streamwise dimensions unaltered. Hence, we consider families of quasi-cylindrical bodies that are derived from one another by a lateral compression or expansion of the skeleton, and a quite independent magnification or reduction of all surface slopes. Two members of such a family are shown in figure 4.

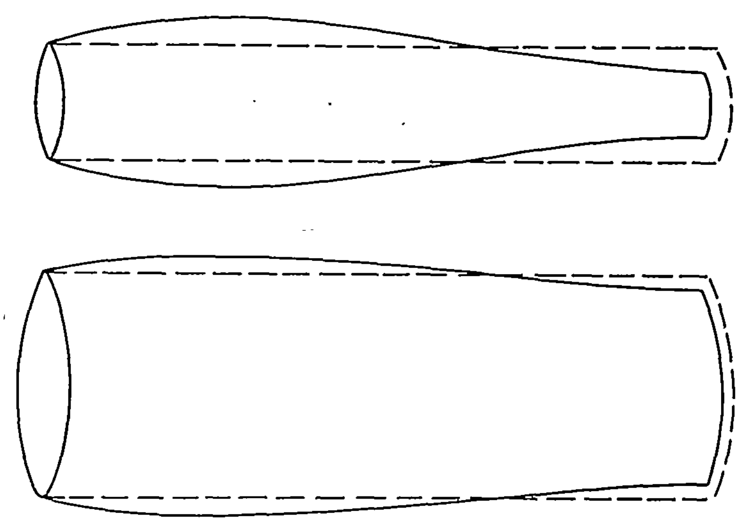

FIGURE 4.—Example of two related quasi-cylindrical bodies.

Distortions of the skeleton will be measured by some characteristic "aspect ratio" A. It is important to note that the term "aspect ratio" is used here in a very general sense to mean any typical ratio of gross cross stream to streamwise dimensions. For example, in the last shape in figure 3, the ratio of wind-tunnel height to airfoil chord is an appropriate characteristic aspect ratio. Changes of slope are measured, as before, by some characteristic slope τ.

The preceding similarity rules can be simplified for quasi-cylindrical bodies by using the facts that first-order perturbation quantities are directly proportional[4] to τ, and second-order terms to τ^2. The simplification can be carried out by first imagining the quasi-cylindrical body to be restricted to be a general body, which means that both βA and $\beta\tau$ must be the same for similarity. Then consider the preceding rules in the particular alternative forms in which τ appears explicitly outside the first-order term and

[4] This is by no means true for general bodies; as noted previously, the first-order pressure coefficient on a smooth slender pointed body of revolution varies as $\tau^2 \ln \tau$ for small τ.

τ^2 outside the second-order terms. For the pressure coefficient, this form is that of equation (10e):

$$C_p=\frac{\tau}{\beta}\overline{P}(x,\beta y,\beta z;\beta\tau,\beta A)+\frac{\tau^2}{\beta^2}\left[\overline{p}_1(\ )+M^2\overline{p}_2(\ )+(\gamma+1)\frac{M^4}{\beta^2}\overline{p}_3(\ )\right]$$

So far the functions $\overline{P},\overline{p}_1,\overline{p}_2,\overline{p}_3$ have been supposed to depend parametrically upon both βA and $\beta\tau$. However, the first- and second-order terms can be proportional to τ and τ^2, respectively, only if the supposed dependence upon $\beta\tau$ is nonexistent. Hence, the similarity parameter is βA alone, and the rules for pressure are (dropping bars from the functional symbols):

$$C_p(x,y,z;M,\gamma,\tau,A)=\frac{\tau}{\beta}P(x,\beta y,\beta z;\beta A)+\frac{\tau^2}{\beta^2}\left[p_1(x,\beta y,\beta z;\beta A)+M^2p_2(\ )+(\gamma+1)\frac{M^4}{\beta^2}p_3(\ )\right] \qquad (11a)$$

The corresponding rules for the potential and velocity components can, if desired, be written down by inspection from equations (10).

With $\hat{p}_2=p_2\pm p_1$ (where, as before, the upper sign applies to subsonic and the lower to supersonic flow), these rules can be rewritten as

$$C_p=\frac{\tau}{\beta}P(\ )+\tau^2\left[p_1(\ )+\frac{M^2}{\beta^2}\hat{p}_2(\ )+(\gamma+1)\frac{M^4}{\beta^4}p_3(\ )\right] \qquad (11b)$$

and this is the result that Fenain and Germain found in their treatment of the flat diamond cone in supersonic flow (ref. 5).

CONNECTION WITH HAYES' RULE

For plane flow past a single body, Hayes has discovered a remarkable rule for the second-order surface pressure (ref. 7). It implies that, on the surface, the functions in equation (11a) are such that $p_2=0$ and $p_1=4p_3$. Hence,

$$C_{p_s}=\frac{\tau}{\beta}P(x)+\tau^2\frac{(\gamma+1)M^4+4(1-M^2)}{4(1-M^2)^2}p_1(x) \qquad (12)$$

In supersonic flow this is simply Busemann's well-known second-order solution, P being twice the local slope of the surface and p_1 twice its square; in subsonic flow P and p_1 are more complicated (ref. 8). This rule implies a corresponding, but more complicated, rule for surface velocity (ref. 8).

In addition to the restriction to single bodies and plane flow, these rules are not similarity rules in the sense of the preceding results, because they apply only at the surface rather than throughout the field.

EXAMPLES

The rules will be illustrated by two simple examples, attention being confined to the surface pressure coefficient.

SLENDER CIRCULAR CONE IN SUPERSONIC FLOW

Broderick has derived the second-order slender-body solution for a circular cone at zero angle in a supersonic stream (ref. 2). The surface pressure coefficient on a cone of slope τ is

$$C_{p_s}=\tau^2\left(2\ln\frac{2}{\beta\tau}-1\right)+\tau^4\left[3\beta^2\left(\ln\frac{2}{\beta\tau}\right)^2-(5M^2-1)\ln\frac{2}{\beta\tau}+\frac{13}{4}M^2+\frac{1}{2}+(\gamma+1)\frac{M^4}{\beta^2}\right] \qquad (13)$$

This has the form of equation (10i) with

$$\tilde{P}=2\ln\frac{2}{\beta\tau}-1 \qquad \tilde{p}_2'=-4\ln\frac{2}{\beta\tau}+\frac{15}{4}$$

$$\tilde{p}_1=3\ln^2\frac{2}{\beta\tau}-5\ln\frac{2}{\beta\tau}+\frac{13}{4} \qquad \tilde{p}_3=1$$

WAVY WALL IN CLOSED SUBSONIC WIND TUNNEL

Consider the sinusoidal wall $y=\tau\sin x$ at a distance h from a flat wall (or a distance $2h$ from its mirror image) as indicated in figure 5. Subsonic flow between the walls at a mean

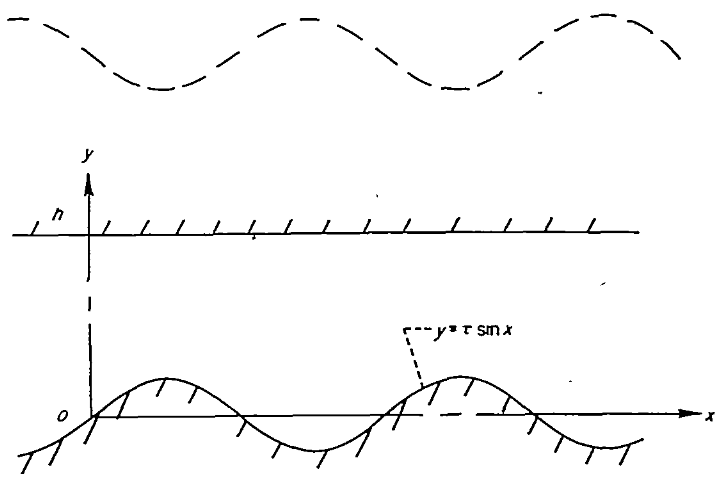

FIGURE 5.—Wavy wall in wind tunnel.

Mach number M can be readily calculated to second order by separation of variables. The resulting pressure coefficient on the surface of the wavy wall is

$$C_{p_s}=-\frac{2\tau}{\beta}\frac{1+e^{-2\beta h}}{1-e^{-2\beta h}}\sin x+\tau^2\frac{(\gamma+1)M^4+4(1-M^2)}{4(1-M^2)^2}\times\frac{1+4e^{-2\beta h}+e^{-4\beta h}}{(1-e^{-2\beta h})^2}\cos 2x-2\tau^2\frac{e^{-2\beta h}}{(1-e^{-2\beta h})^2}+2(\gamma+1)\frac{M^4}{\beta^4}\tau^2\frac{\beta he^{-4\beta h}}{(1+e^{-2\beta h})(1-e^{-2\beta h})^3}\cos 2x \qquad (14)$$

The relevant aspect ratio is the height h (which is really a multiple of the height-chord ratio, because of the choice of scale for the wavy wall). Thus the result is seen to have the similitude of equation (11a). As the tunnel height increases indefinitely, the last two terms disappear, and the remainder follows the similitude of equation (12) for the surface of a single plane body.

REDUCTION OF SUBSONIC PROBLEM TO NEARLY INCOMPRESSIBLE FLOW

In linearized theory, an important application of the similitude is Göthert's rule, which reduces any subsonic flow problem to a related incompressible flow (ref. 1). As the rule is usually stated, the incompressible flow is that past a thinner affinely related body. However, the incompressible solution for one member of an affinely related family of bodies determines that for all other members, so that the subsonic flow may, if desired, be related to the same body rather than a thinner one, and that viewpoint will be adopted here as being the simplest.

In second-order theory, the explicit appearance of terms in M^2 and $(\gamma+1)M^4$ in equations (10) means that reduction to an incompressible problem is impossible (except for the special case of the surface of a single plane body, where eq. (12) applies). The second-order problem can, however, be reduced to a nearly incompressible flow.

Flows at low Mach numbers can be calculated by the Janzen-Rayleigh method, which involves iterating on the incompressible solution to obtain a power series in M^2. Thus the velocity potential is approximated by

$$\frac{1}{U_\infty}\Phi(x,y,z;M,\gamma,\tau)=\Phi_0(x,y,z;\tau)+M^2\Phi_1(x,y,z;\tau)+(\gamma+1)M^4\Phi_2(\;)+M^4\Phi_3(\;)+\ldots \tag{15a}$$

The two terms in M^4 are ordinarily considered together, but for present purposes it is essential to separate them because only Φ_2 is required. This is fortunate because Φ_2 can be calculated almost as easily as Φ_1, whereas the determination of Φ_3 is much more difficult.

The small-disturbance and Janzen-Rayleigh series represent two different asymptotic expansions of the actual solution. They are believed to complement each other, so that an expansion of the Janzen-Rayleigh solution for small thickness must be identical with the expansion of the small-disturbance solution in powers of M^2, as has been verified in all worked examples. This fact permits the small-disturbance solution to be recovered from the Janzen-Rayleigh series. The converse is not true, however, except for bodies without stagnation points, because the small-disburbance expansion is not uniformly valid near a stagnation point.

PROCEDURE FOR RECOVERING SECOND-ORDER SOLUTION

Another alternative form for the second-order velocity potential of equation (10a), which is useful here, is

$$\frac{1}{U_\infty}\Phi(x,y,z;M,\gamma,\tau)=x+\tau^2F(x,\beta y,\beta z;\beta\tau)+\tau^2\left[f_1(\;)+\frac{M^2}{\beta^2}f_2(\;)+(\gamma+1)\frac{M^4}{\beta^4}f_3(\;)\right] \tag{10l}$$

(Here F and f_1, f_2, f_3 are not the same functions as in equation (10a) but related ones; in the notation of equations (10h) and (10i) they are actually $\widetilde{F}$ and $\overline{f}_1$, $\overline{f}_2$, $\overline{f}_3$.)

For present purposes it is unnecessary to distinguish between the first-order term F and the second-order increment f_1; combining them as $F_1=F+f_1$ gives

$$\frac{\Phi-U_\infty x}{U_\infty\tau^2}=F_1(x,\beta y,\beta z;\beta\tau)+\frac{M^2}{\beta^2}f_2(\;)+(\gamma+1)\frac{M^4}{\beta^4}f_3(\;) \tag{10m}$$

The Janzen-Rayleigh solution is now to be manipulated into this same form. The first three terms as given in equation (15a) are equivalent to

$$\frac{1}{U_\infty}\Phi(x,y,z;M,\gamma,\tau)=\Phi_0(x,y,z;\tau)+\frac{M^2}{\beta^2}\Phi_1(\;)+(\gamma+1)\frac{M^4}{\beta^4}\Phi_2(\;)+0\left(\frac{M^4}{\beta^4}\right) \tag{15b}$$

which may be rewritten as

$$\frac{\Phi-U_\infty x}{U_\infty\tau^2}=\varphi_0(x,y,z;\tau)+\frac{M^2}{\beta^2}\varphi_1(\;)+(\gamma+1)\frac{M^4}{\beta^4}\varphi_2(\;)+0\left(\frac{M^4}{\beta^4}\right) \tag{15c}$$

with

$$\left.\begin{aligned}\varphi_0&=\frac{\Phi_0-x}{\tau^2}\\ \varphi_1&=\frac{\Phi_1}{\tau^2}\\ \varphi_2&=\frac{\Phi_2}{\tau^2}\end{aligned}\right\} \tag{15d}$$

Finally, with the aid of

$$y=\beta y\sqrt{1+\frac{M^2}{\beta^2}}=\beta y\left[1+\frac{1}{2}\frac{M^2}{\beta^2}+0\left(\frac{M^4}{\beta^4}\right)\right] \tag{16}$$

and corresponding expansions for z and τ, this may be re-expressed as

$$\frac{\Phi-U_\infty x}{U_\infty\tau^2}=\varphi_0(x,\beta y,\beta z;\beta\tau)+\frac{M^2}{\beta^2}\left[\varphi_1(\;)+\frac{\beta y}{2}\varphi_{0_{\beta y}}(\;)+\frac{\beta z}{2}\varphi_{0_{\beta z}}(\;)+\frac{\beta\tau}{2}\varphi_{0_{\beta\tau}}(\;)\right]+(\gamma+1)\frac{M^4}{\beta^4}\varphi_2(\;) \tag{15e}$$

which is the desired form. Here, for example, $\varphi_{0_{\beta y}}(x,\beta y,\beta z;\beta\tau)$ means $(\partial/\partial y)\varphi_0(x,y,z;\tau)$ evaluated at $x=x$, $y=\beta y$, $z=\beta z$, and $\tau=\beta\tau$.

The second-order solution is thus recovered from the Janzen-Rayleigh approximation simply by calculating in turn the expressions in equations (15d) and (15e). The procedure can actually be expressed by a single equation as follows. From the Janzen-Rayleigh approximation in the form of equation (15a), the second-order small-disturbance solution is recovered according to

$$\frac{\Phi}{U_\infty}=x+\tau^2\left\{\frac{\Phi_0(x,y,z;\tau)-x}{\tau^2}+\frac{M^2}{\beta^2}\left[\frac{\Phi_1}{\tau^2}+\frac{y}{2}\left(\frac{\Phi_0-x}{\tau^2}\right)_y+\frac{z}{2}\left(\frac{\Phi_0-x}{\tau^2}\right)_z+\frac{\tau}{2}\left(\frac{\Phi_0-x}{\tau^2}\right)_\tau\right]+(\gamma+1)\frac{M^4}{\beta^4}\frac{\Phi_2}{\tau^2}\right\}_{\substack{y\to\beta y\\ z\to\beta z\\ \tau\to\beta\tau}} \tag{17}$$

where Φ_n means $\Phi_n(x,y,z;\tau)$ throughout, and subscripts indicate differentiation.

APPLICATION TO PARABOLA

As an example, consider plane subsonic flow at zero angle of attack past the parabola described by $y=\tau\sqrt{x}$ (fig. 6).

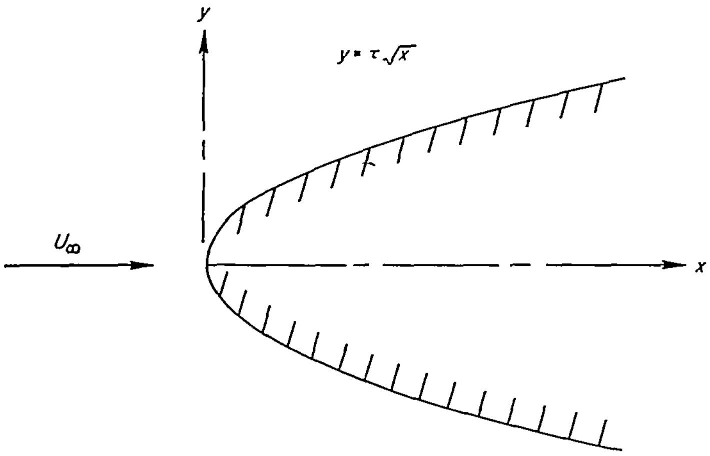

FIGURE 6.—Parabola at zero angle of attack.

The Janzen-Rayleigh solution including terms in $(\gamma+1)M^4$ has been calculated by Imai (ref. 9). Although the velocity potential is complicated, it simplifies when only second-order terms in τ are retained to

$$\frac{\Phi}{U_\infty}=x+\tau\eta+\frac{1}{2}M^2\tau\eta^2\frac{\eta-\tau}{\xi^2+\eta^2}-\frac{1}{8}M^2\tau^2\,ln\frac{\xi^2+\eta^2}{\tau^2}-\frac{\gamma+1}{32}M^4\tau^2\left[4\frac{\eta^4}{(\xi^2+\eta^2)^2}+ln\frac{\xi^2+\eta^2}{\tau^2}\right]+\cdots \quad (18a)$$

Here ξ,η are parabolic coordinates related to the Cartesian coordinates by

$$\left[\left(x-\frac{1}{4}\tau^2\right)+iy\right]=(\xi+i\eta)^2$$

so that to second order

$$\left.\begin{aligned}2\xi^2&=\sqrt{x^2+y^2}+x\\2\eta^2&=\sqrt{x^2+y^2}-x\end{aligned}\right\} \quad (18b)$$

In this case the expressions given by equation (15d) are

$$\left.\begin{aligned}\varphi_0&=\frac{\eta}{\tau}\\\varphi_1&=\frac{1}{8}\left[4\frac{\eta^2(\eta/\tau-1)}{\xi^2+\eta^2}+ln\frac{\xi^2+\eta^2}{\tau^2}\right]\\\varphi_2&=\frac{1}{32}\left[4\frac{\eta^4}{(\xi^2+\eta^2)^2}+ln\frac{\xi^2+\eta^2}{\tau^2}\right]\end{aligned}\right\} \quad (18c)$$

This example illustrates the fact that for planar systems these terms are not of order unity in τ. Then according to equation (15e) the second-order small-disturbance solution is

$$\frac{\Phi}{U_\infty}=x+\frac{\tau}{\beta}\bar{\eta}-\frac{1}{8}\frac{M^2}{\beta^2}\tau^2\left(4\frac{\bar{\eta}^2}{\bar{\xi}^2+\bar{\eta}^2}+ln\frac{\bar{\xi}^2+\bar{\eta}^2}{\beta^2\tau^2}\right)-\frac{\gamma+1}{32}\frac{M^4}{\beta^4}\tau^2\left[4\frac{\bar{\eta}^4}{(\bar{\xi}^2+\bar{\eta}^2)^2}+ln\frac{\bar{\xi}^2+\bar{\eta}^2}{\beta^2\tau^2}\right] \quad (19a)$$

where

$$\left.\begin{aligned}2\bar{\xi}^2&=\sqrt{x^2+\beta^2y^2}+x\\2\bar{\eta}^2&=\sqrt{x^2+\beta^2y^2}-x\end{aligned}\right\} \quad (19b)$$

This result is of interest because it apparently cannot be found directly. Plane small-disturbance flows can be calculated easily if one adopts the thin-airfoil approximation of transferring the boundary conditions to the line $y=0$ by Taylor series expansion, but that approximation fails near round noses in the second approximation and, as a consequence, divergent integrals arise (ref. 8). Instead, one can try to treat the round nose more carefully using conformal mapping (cf. ref. 11, pp. 361–367), but the result is found to be indeterminate to the extent of a multiple of $\ln(\bar{\xi}^2+\bar{\eta}^2)$. This is the potential of a point source at the origin, which is an eigensolution, the proper multiple of which (appearing in eq. (19a)) is not determined by the suggested method.

The second-order increment in equation (19a) is seen to include terms in $\tau^2 ln\,\tau$, whose function is to render the argument of the logarithm dimensionless. However, these terms are simply constants, so that no logarithms of thickness appear in the actual flow quantities such as velocity and pressure. As remarked in the introduction, logarithmic terms in thickness arise in the actual flow disturbances only in the fourth approximation.

The second-order small-disturbance solution for the stream function can in the same way be extracted from Imai's Janzen-Rayleigh solution, and the result is found to agree with that calculated directly by Kaplan (ref. 10) using conformal mapping. It contains no terms in $ln\,\tau$. (The direct approach succeeds for the stream function, although it fails for the velocity potential, because the tangency condition is imposed on the mass flux, which is affected by the above eigensolution.) Then using the connections between the stream function and velocity potential, one can verify the correctness of equations (19).

CONCLUDING REMARKS

UTILITY OF THE RULES

The second-order rules are scarcely suited for correlating experimental data, since tests on four related bodies would be needed in order to isolate the four functions involved. That they are, however, useful in theoretical analyses has already been pointed out by Fenain and Germain in the special case of supersonic flow past flat wings (ref. 5). Previous investigators had calculated (erroneously, as it turns out) the second-order solution for the flat diamond cone shown in figure 7, and carried out numerical computations

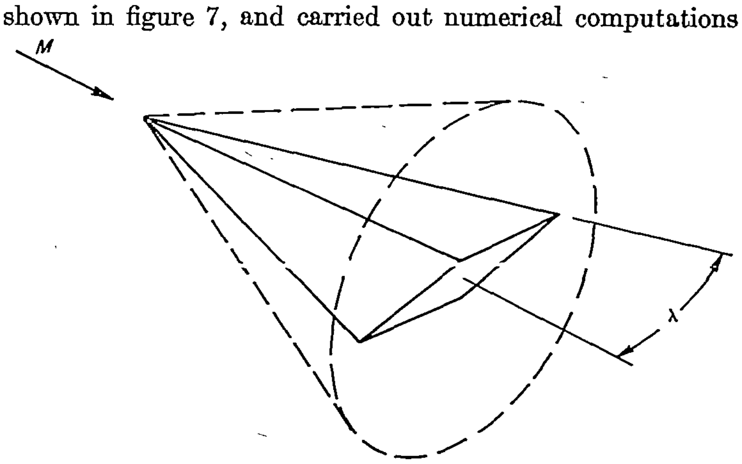

FIGURE 7.—Flat diamond cone.

for three different Mach numbers and four values of the parameter β tan λ (reported in ref. 11). Because the latter is the similarity parameter βA of equations (11), failure to take advantage of the similitude resulted in three fold unnecessary duplication of computing labor.

The reduction to nearly incompressible flow assumes importance for bodies with stagnation points. The small-disturbance assumption is violated, and, as was noted in the example of the parabola, the second-order solution consequently cannot be found directly. For bodies of revolution the difficulties appear to be even more severe. In such cases it is convenient to calculate the Janzen-Rayleigh solution, and from it extract the true second-order solution by the procedure outlined above. This process has been carried out for the paraboloid of revolution in reference 12.

NONUNIFORMITY IN SUPERSONIC FLOW

It should be noted that in supersonic flow the similarity rules may fail in localized regions. Shock waves or expansion fans spring from corners and edges, and in their vicinity the formal interation procedure is not uniformly valid. The similarity rules for surface pressure fail locally when such waves intersect other parts of the body, as in the case of a triangular wing with leading edges ahead of the Mach cone. The rules for integrated lift and drag are correct to first order, but may be in error in second-order terms. These difficulties can in principle be eliminated by straining the coordinates according to Lighthill's technique (ref. 13).

FURTHER EXTENSIONS

The similarity rules can readily be extended to third and higher order in the same fashion (except for additional complications in supersonic flow because of the ultimate appearance of significant vorticity engendered by curved shock waves). The similarity parameter remains unchanged; the complexity arising in a proliferation of functions multiplied by powers of $(\gamma+1)^m M^{2n}\beta^{-2p}$. Likewise, the small-disturbance solution to any order can be recovered from the nearly incompressible solution provided by an appropriate number of terms of the Janzen-Rayleigh solution.

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