REPORT No. 731

DETERMINATION OF OPTIMUM PLAN FORMS FOR CONTROL SURFACES

By Robert T. Jones and Doris Cohen

SUMMARY

A theoretical analysis is made to determine the optimum chord distribution, location, and extent of control surfaces, with the ratio of hinge moment to effectiveness as the criterion. Expressions for the effectiveness —for ailerons, the rolling moment, and for tail surfaces, the change of lift on the tail due to deflection of the surface—were derived from lifting-line theory.

Solutions found for a range of airfoil plan forms indicate that, regardless of the characteristics of the tail surface, the chord of the rudder or of the elevator should be very nearly constant over its span. The optimum ailerons are also of a characteristic shape, varying little with the plan form of the wing.

INTRODUCTION

One of the primary difficulties in airplane design has been to keep the stick forces required to deflect the control surfaces at reasonably low values. This problem has inevitably increased in seriousness with the size and the speed of modern airplanes. There is as yet, however, no basic principle of control-surface design that engineers will agree minimizes the ratio of stick force to effectiveness. Examination of typical designs indicates that hinge-moment reductions as great as 40 percent may be achieved in some cases without lowering the effectiveness of the flap; that is, the efficiency may be increased by two-thirds.

The present study, which neglects structural and similar considerations, is a mathematical analysis leading to the plan forms for rudder, elevator, and ailerons that will be most effective in producing a given amount of control with the least operating force. The solutions are applicable to any airfoil of conventional plan form to the same extent as are the usual assumptions of the aerodynamic theory of airfoils, on which the analysis is based. Further discussion covers the extent, the location, and the shape of partial-span control surfaces to give the greatest efficiency.

THEORETICAL ANALYSIS

It is required to find the plan forms for rudder, elevator, and ailerons that will require the least stick force to produce a fixed amount of control per unit deflection of the surface. This is a problem in the calculus of variations, in which the expression for the effectiveness is the integral to be kept constant; the expression for the hinge moment is the integral to be minimized; and the hinge line, defined by a relation between the spanwise station and the ratio of flap chord to airfoil chord, may be considered the path of integration to be determined so as to satisfy the foregoing conditions. In the case of a rudder or an elevator, the effectiveness is measured by the change of lift on the tail surface produced by deflection of the flap; in the case of ailerons, it is measured by the rolling moment produced.

It will be seen in the course of the discussion that all constant factors may be combined in the final result into a single factor of proportionality. All the functions and relations discussed hereinafter will therefore be treated without regard to such factors.

The following symbols will be used in the development:

b	span of airfoil
$\frac{y}{b/2}$	spanwise station measured from plane of symmetry
$\theta=\cos^{-1}\frac{y}{b/2}$	parameter indicating spanwise station
Δl_1	lift per unit span at any section due to unit flap deflection
$\Delta\alpha_1$	change in effective angle of attack at any section due to unit flap deflection
c_l	section lift coefficient
c	chord of airfoil
c_f	chord of flap
r	ratio of flap chord to airfoil chord (c_f/c)
c_s	chord of airfoil at plane of symmetry
δ_f	angle of flap deflection
a_0	section lift-curve slope
$\mu=c_s a_0/4b$	aspect-ratio parameter
H	hinge moment
A_i,B_j,C_k	Fourier series coefficients
A_1	Fourier series coefficient proportional to lift
A_2	Fourier series coefficient proportional to moment
K_n	constants determined by C's and μ
λ	arbitrary proportionality factor

The expression for effectiveness is obtained with the aid of the Lotz method, an outline of which is found in reference 1. Results obtained by this method check reasonably well with experiment except for aspect ratios less than 2.

If the lift distribution over the airfoil is given by the Fourier series

$$\Delta l_1 \propto \sum_{i=1}^{\infty} A_i \sin i\theta \tag{1}$$

the angle-of-attack distribution, by the series

$$\Delta \alpha_1 \sin \theta \propto \sum_{j=1}^{\infty} B_j \sin j\theta \tag{2}$$

and the chord distribution of the airfoil, by

$$\frac{\sin \theta}{c} = C_0 + \sum_{k=1}^{\infty} C_{2k} \cos 2k\theta \tag{3}$$

then the coefficients of the series are connected by the relation

$$\sum_k C_{2k} \cos 2k\theta \sum_i A_i \sin i\theta + \mu \sum_i iA_i \sin i\theta \equiv \sum_j B_j \sin j\theta \tag{4}$$

or

$$\sum_k \sum_i A_i C_{2k} [\sin (i+2k)\theta + \sin (i-2k)\theta] + 2\mu \sum_i iA_i \sin i\theta \equiv 2 \sum_j B_j \sin j\theta \tag{5}$$

This identity is equivalent to the set of simultaneous equations obtained by equating coefficients of terms in the same multiple of θ; thus, for each value of j,

$$(-1)^{(i\pm 2k)} \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} A_i C_{2k} + (C_0 + 2j\mu) A_j = 2B_j \tag{6}$$

where i and k are limited to values such that $i \pm 2k = \pm j$.

The plan form of the airfoil under consideration will determine the values for C_{2k} and will probably be approximated closely enough with a series of two or three terms. (It should be noted that the values of C_{2k} depend only on the distribution of the chord length rather than on the actual plan form.) The B's can be expressed as functions of the ratio r of flap chord to airfoil chord. This ratio will be considered the dependent variable, to be found as a function of the spanwise station $\frac{y}{b/2}$, or of θ. The set of simultaneous equations may now be solved for A_1, which is proportional to the total change in lift due to deflection of the flap, and for A_2, which is proportional to the rolling moment.

It should be possible to limit the number of equations to six or eight without introducing any noticeable inaccuracies. Only odd values of j, moreover, are involved in the solution for A_1 and only even values are involved in the solution for A_2. The value of A_1 will be found to be proportional to an expression of the form

$$K_1 B_1 + K_3 B_3 + \ldots + K_{2n-1} B_{2n-1} \ldots \tag{7}$$

and A_2 will be proportional to $\sum K_{2n} B_{2n}$, where K_n is a constant determined by the C's and μ. From equation (2)

$$B_n \propto \int_0^{\pi} \Delta \alpha_1(r) \sin \theta \sin n\theta \, d\theta \tag{8}$$

Substitute in expression (7); then

$$A_1 \propto \int_0^{\pi} \Delta \alpha_1(r) \sin \theta \, F_1(\theta) \, d\theta \tag{9}$$

where

$$F_1(\theta) = \sum_n K_{2n-1} \sin (2n-1)\theta \tag{10}$$

Similarly,

$$A_2 \propto \int_0^{\pi} \Delta \alpha_1(r) \sin \theta \, F_2(\theta) \, d\theta \tag{11}$$

where

$$F_2(\theta) = \sum_n K_{2n} \sin 2n\theta \tag{12}$$

A curve for $\Delta \alpha_1$ as a function of r, representing a summary of available experimental data on sealed-hinge flaps, is given in figure 1. (A theoretical curve

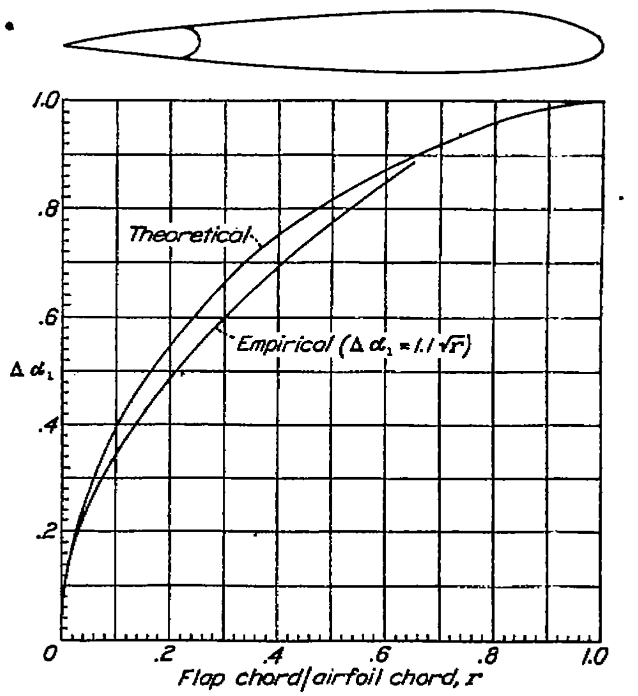

FIGURE 1.—Flap-effect curves for flaps with sealed hinges. (Theoretical curve from reference 2.) $\delta_f < \pm 20°$.

taken from reference 2 has been included in fig. 1 for comparison.) This empirical curve is closely approximated, for flaps up to 70-percent chord, by the equation

$$\Delta \alpha_1 = 1.1 \sqrt{r} \tag{13}$$

Then

$$A_1 \propto \int_0^{\pi} \sqrt{r} \sin \theta \, F_1(\theta) \, d\theta \tag{14}$$

and

$$A_2 \propto \int_0^{\pi} \sqrt{r} \sin \theta \, F_2(\theta) \, d\theta \tag{15}$$

The expressions for the lift and the rolling moment obtained by this method take into account the important effect of the aerodynamic induction associated with airfoils of finite span. In the expression for the hinge moment, this factor will be neglected; that is, the end loss in aerodynamic loading and a variation, caused by the floating tendency of the flap, in the hinge moment developed by the induced downwash will be omitted. Both of these effects are small for narrow-chord flaps. Without these effects the hinge moment is simply proportional to the square of the flap chord at each section. Thus,

$$H \propto \int_0^\pi c^2 r^2 \sin\theta \, d\theta \qquad (16)$$

This assumption checks reasonably well with experiments.

The problem may now be restated in more specific terms: to find r as a function of θ so that H is a minimum for a fixed value of A_1 (or A_2). Clearly, an equivalent condition would be that $H+\lambda A$ be rendered a minimum, where λ is a parameter associated with the value of A required. This condition is satisfied only if the variation of the integrand of the sum with the dependent variable is equal to zero.

Thus,

$$\frac{\partial}{\partial r}[r^2 c^2 \sin\theta + \lambda\sqrt{r} \sin\theta \, F(\theta) = 0] \qquad (17)$$

Then

$$2rc^2 \sin\theta + \frac{\lambda}{2\sqrt{r}} \sin\theta \, F(\theta) = 0 \qquad (18)$$

or

$$r \propto c^{-\frac{4}{3}}[F(\theta)]^{\frac{2}{3}} \qquad (19)$$

and

$$c_f \propto \sqrt[3]{\frac{[F(\theta)]^2}{c}} \qquad (20)$$

which is the most general form of the desired solution. In particular, if airfoils defined by two coefficients C_0 and C_2 are considered,

$$c = \frac{\sin\theta}{C_0 + C_2 \cos 2\theta} \qquad (21)$$

$$F_1 = \left[(C_0+7\mu)(C_0+5\mu)(C_0+3\mu) - \frac{C_2^2}{2}(C_0+5\mu)\right]\sin\theta - \frac{C_2}{2}\left[(C_0+7\mu)(C_0+5\mu) - \left(\frac{C_2}{2}\right)^2\right]\sin 3\theta + \left(\frac{C_2}{2}\right)^2 (C_0+7\mu)\sin 5\theta - \left(\frac{C_2}{2}\right)^3 \sin 7\theta \qquad (22)$$

and

$$F_2 = \left[(C_0+8\mu)(C_0+6\mu)(C_0+4\mu) - \frac{C_2^2}{2}(C_0+6\mu)\right]\sin 2\theta - \frac{C_2}{2}\left[(C_0+8\mu)(C_0+6\mu) - \left(\frac{C_2}{2}\right)^2\right]\sin 4\theta + \left(\frac{C_2}{2}\right)^2 (C_0+8\mu)\sin 6\theta - \left(\frac{C_2}{2}\right)^3 \sin 8\theta \qquad (23)$$

RESULTS AND DISCUSSION

The method just developed has been applied to airfoils with the following chord distributions:

$$\frac{\sin\theta}{c} = 2.071 - 0.6904\cos 2\theta \qquad \text{(blunt)}$$

$$\frac{\sin\theta}{c} = 2.356 \qquad \text{(elliptical)}$$

$$\frac{\sin\theta}{c} = 2.926 + 0.9755\cos 2\theta \qquad \text{(tapered)}$$

The elliptical distribution and the degrees of taper represented by the relations $C_2/C_0 = \pm 1/3$ were chosen to give an inclusive indication of the range and the manner of variation of the solutions. Particular values of C_0 were determined by the condition that the airfoils

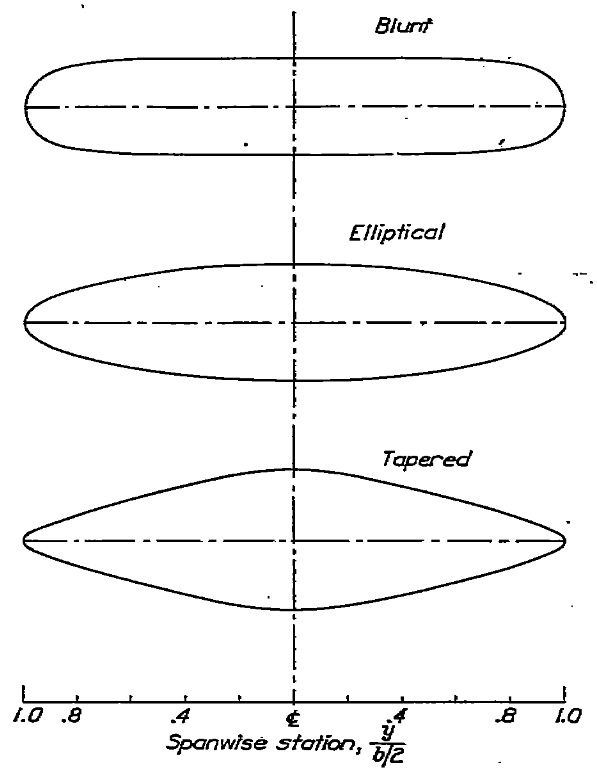

FIGURE 2.—Airfoil plan forms defined by the two-term series: $\frac{\sin\theta}{c} = C_0 + C_2\cos 2\theta$. Aspect ratio, 6.

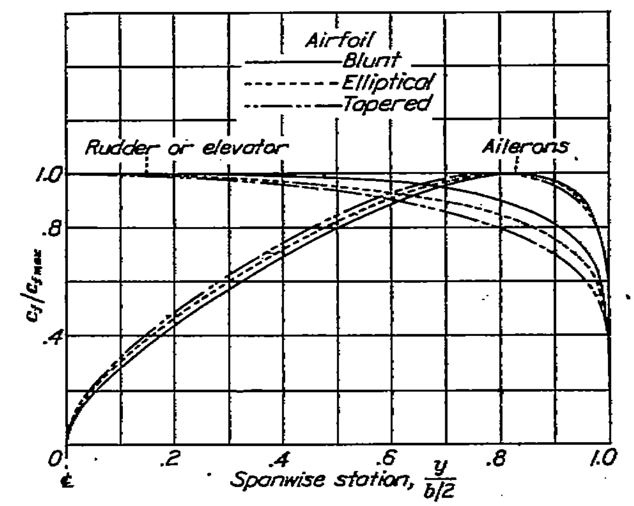

FIGURE 3.—Optimum chord distribution for control surfaces.

be of unit semispan and aspect ratio 6. The resulting chord distributions, plotted about a straight 50-percent chord line, are shown in figure 2.

The solutions were found to be strikingly independent of the form of the airfoil. In figure 3, the shapes for the movable surfaces are shown for the three airfoils just described. The optimum ailerons are seen to vary hardly at all from airfoil to airfoil. The outlines for maximum lift efficiency, although differing slightly, nevertheless suggest the general conclusion that the most desirable control surfaces in this respect are of nearly constant chord.

From this result it follows that the most efficient airfoil plan form for longitudinal control is also one of nearly constant chord, because such a combination of airfoil and flap will have a nearly uniform distribution

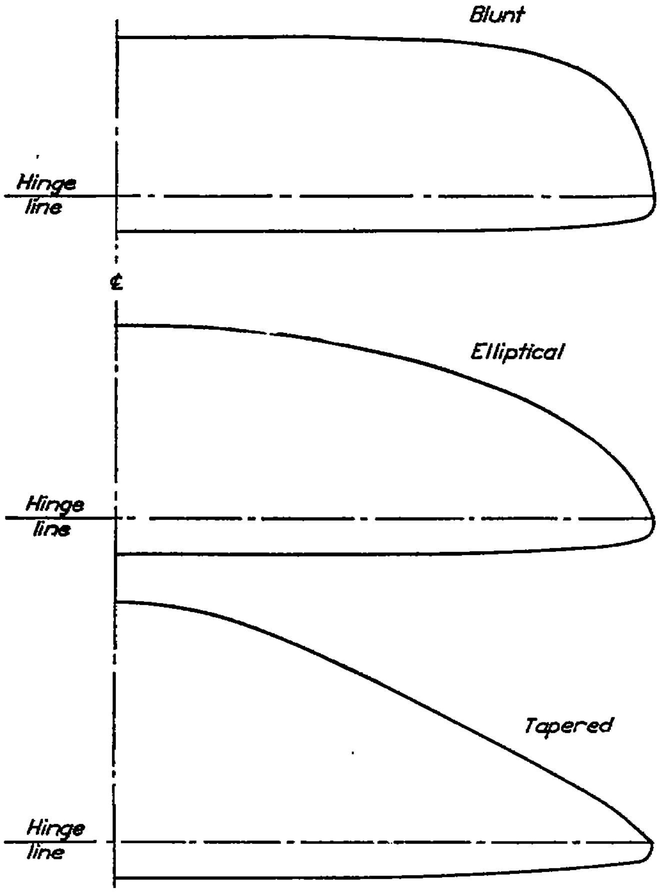

FIGURE 4.—Airfoils of figure 3 with flaps of optimum plan form for lift. Average chord ratio, 20 percent.

of effective angle of attack. Calculations also show that, if each of the airfoils given by the foregoing distributions were provided with a flap of optimum shape, a somewhat smaller movable surface, and hence a smaller hinge moment, would be required to increase the lift on the blunt wing a given amount per unit deflection than on either of the other two. On the other hand, since aileron effectiveness depends on the ratio of rolling moment to damping moment, a tapered wing is seen to be most efficient for lateral control.

In figure 4 the flaps are plotted in relation to the airfoils. The flaps as drawn are 20 percent of the mean chord of the airfoil. It will be noted that, for the tapered airfoil, this is the maximum width at which the shape of the flap can be maintained. Ordinarily as high a taper as shown would not be used and a 20-percent or wider flap would be possible. It is not important, in any case, to hold rigidly to the optimum shape at the extreme tip if such a design introduces a cusp in the fixed portion of the surface.

The corresponding presentation of the solution for the ailerons is given in figure 5. The ailerons shown are approximately 15 percent of the mean chord of the airfoil. Seen in this aspect ratio, the ailerons appear not to vary greatly in width over their span. As is to be expected, however, they do taper off somewhat toward the center where the small moment arm would obviously make a larger area inefficient.

The choice of a straight hinge line has led to the introduction of sweepback in these plan forms. It should be remembered, however, that the solutions as expressed by figure 3 are mathematically very general ones and cover, within the limits of accuracy of the lifting-line theory, any width of flap, any shape of hinge line or quarter-chord line, and any normal aspect ratio. The

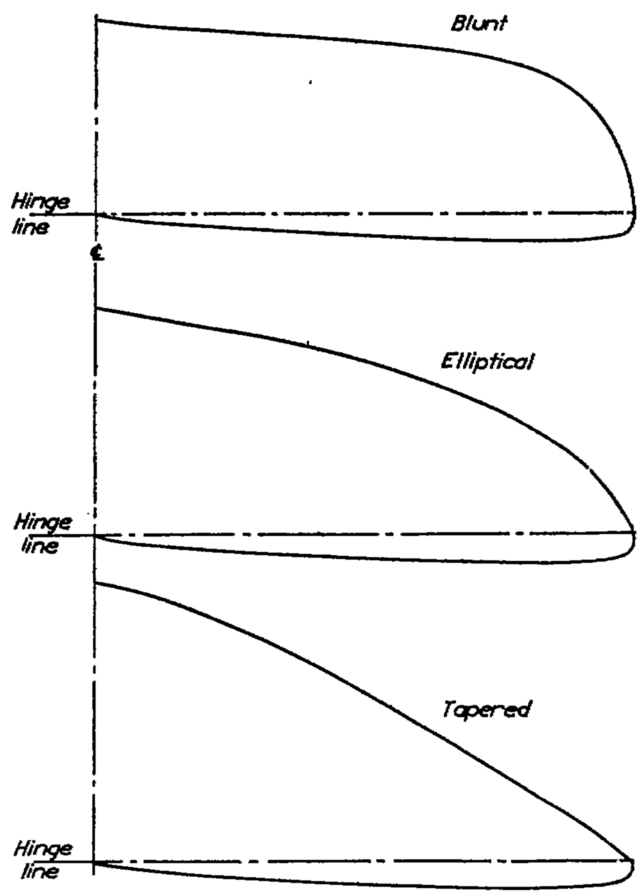

FIGURE 5.—Airfoils of figure 3 with ailerons of optimum plan form. Average chord ratio, 15 percent.

aspect ratio, in μ, affects the expressions for $F(\theta)$ in equations (22) and (23), but calculations made for an aspect ratio of 2 resulted in outlines that could not be differentiated in plotting from those given in figure 3. Thus, the aspect ratio enters the solutions only in so far as it limits the applicability of the lifting-line theory. The solutions are also expected to apply in a general way to unsealed flaps which, although less effective than sealed flaps, may be supposed to show similar variations of effectiveness with chord.

A separate investigation was made to determine on which of the airfoils considered the derived control surfaces would produce the most lift for a given hinge moment, with a view to estimating the loss resulting from the use of a tapered stabilizer. The calculated difference of 8 percent between the tapered and the blunt airfoils was not so great as might have been expected. It is supplemented, however, by other aerodynamic effects such as fuselage interference, which may make the use of area near the fuselage inefficient.

Ailerons on a blunt wing would similarly give more rolling moment for a certain hinge moment than on a more tapered wing. Since the tapered wing requires less powerful ailerons because of the small damping moment, the actual rate of rolling would be definitely greater for a given hinge moment; thus in the last analysis, the tapered wing must be considered the most efficient from considerations of aileron control.

PARTIAL-SPAN FLAPS

At this point a question arises as to the design of partial-span flaps, their shape, optimum length, and location. The most efficient shape for such flaps can be deduced from a review of the preceding development. If a portion of the airfoil span has no movable area, the value of $\Delta\alpha_1$ will be zero over that region and the expressions for A_1 and A_2 given in equations (14) and (15) will reduce to integrals covering only flapped portions of the span. The limits for H given in equation (16) may be similarly changed. Then the reasoning remains the same; only the limits of integration are changed to agree with the extent of the flap and, these limits being identical for the functions involved, the relations between the integrands may be expressed as before and the same solutions will be obtained. It follows that flaps extending over a part of the span of an airfoil should have the same shape as the portion of a flap of optimum shape covering that same part of the span.

Another interesting characteristic of these shapes, one from which further deductions concerning partial-span flaps may be made, appears when the solution given in equation (19) is substituted in expressions (14), (15), and (16) for the effectiveness and the hinge moment. It is seen that the integrands of these expressions are identical except for the discarded factors of proportionality. This fact may be interpreted as meaning that the surfaces found have, for any particular solution, a constant ratio of effectiveness to hinge moment all along the span, or that any portion of a given flap of optimum shape is as efficient as any other portion.

This characteristic leads again to the conclusion that partial-span flaps should be segments of the optimum full-span shapes. The extent and the location of the flaps for greatest efficiency are also indicated by these considerations. If the ratio for a given shape is everywhere the same, the greatest lift or rolling moment must be contributed by the element of the flap that has the maximum chord (and therefore the maximum hinge moment). If any other element of equal span is to be used to develop the same lift, that element must either be deflected through a greater angle or increased in width. It is assumed that the maximum degree of control possible within the efficient range of flap deflection, which is about ±20°, is desired, and the control should therefore not be increased at the expense of

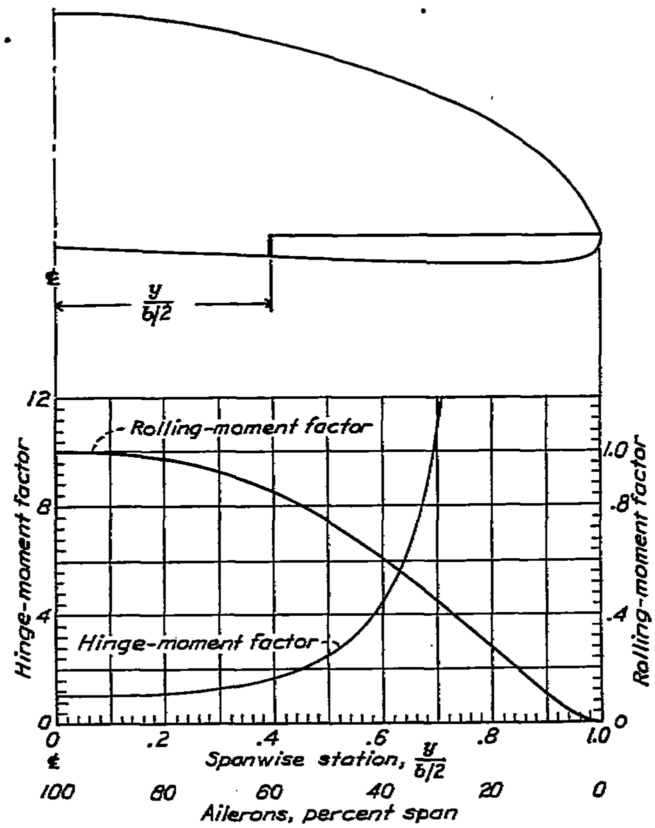

FIGURE 6.—Rolling-moment and hinge-moment curves for partial-span ailerons. Elliptical wing; aspect ratio, 6.

flap deflection. If the chord is increased, it will be with the square of the lift (equation (13)) and the hinge moment will then be increased with the square of the chord (equation (16)) or the fourth power of the lift. The conclusion is obvious: The most efficient flap of a given span will cover the portion over which the ordinates of figure 3 are the greatest. This result is of particular significance as applied to ailerons, which should therefore extend inward from the tips.

It also follows that the greatest efficiency is obtained by using the longest possible control surfaces. Shorter flaps must of necessity be wider to produce the same effect, and the increase in chord causes a sharp drop in efficiency. This consideration should influence not only the design of the flaps but also the design of the tail surfaces themselves.

Figures 6 and 7 are a quantitative representation of the situation. In these figures, the "hinge-moment factor" is the number by which the hinge moment of the partial-span control surfaces would be multiplied if their effectiveness were increased (by increasing the chord) to equal that of the full-span surfaces. The "rolling-moment factor" is the rolling moment developed by partial-span ailerons (to the tip) of optimum shape, expressed as a fraction of the moment developed by the full-span ailerons of which they are a part.

In the case of ailerons, the loss of efficiency is not very great if the extreme inboard portion is dispensed with, but the rate at which the efficiency drops increases rapidly as the span of the ailerons is lessened. For example, if on a particular wing only the outer 60 percent of the optimum ailerons of a certain percentage chord were used, approximately 16 percent of the rolling moment would be sacrificed. If it were desired, however, to retain the full power of the control, a 60-percent increase in hinge moment would be incurred; or, from

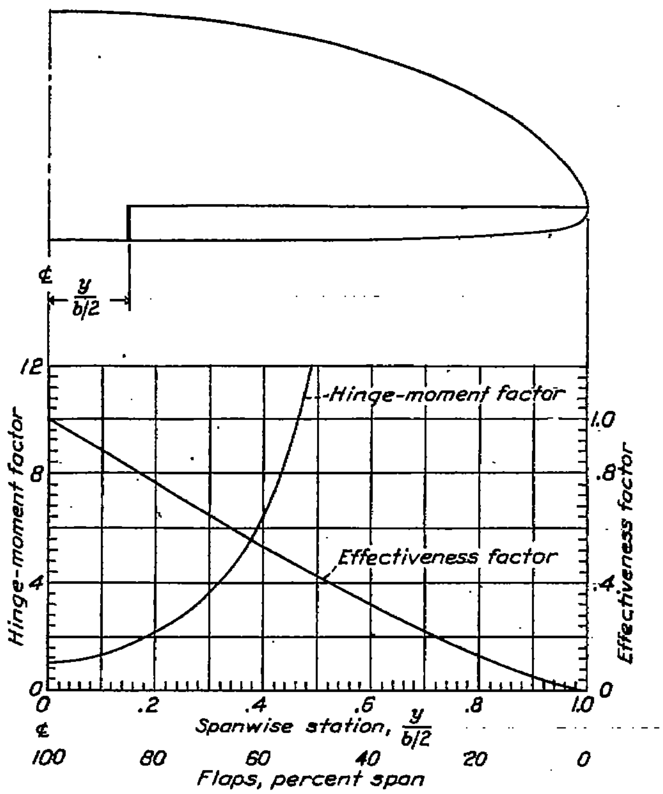

FIGURE 7.—Hinge-moment and effectiveness curves for partial-span flaps. Elliptical tail surface; aspect ratio, 6.

another point of view, a 60-percent aileron will always require an operating force 60 percent greater than is entirely necessary for the same effectiveness. If only 40 percent of the span is used for control, the hinge moment required will be almost three times as great as for equally effective 60-percent-span ailerons and 4.4 times as great as for full-span ailerons.

The corresponding curves for the elevator (fig. 7) show a much sharper decline in efficiency as the flaps are shortened along the span. Because the fixed surface between flaps (or the cut-out) is seldom more than 15 percent, however, the resultant increase of control force is not so great as for partial-span ailerons.

Application of the principles outlined has been made to a modern airplane, with ailerons as shown in figure 8. Calculations indicate a 30-percent reduction in hinge moment (with no loss of rolling moment) due to improvement of the plan form of the ailerons alone. If an additional 10 percent of the semispan were allotted to each aileron, the required operating force would be reduced to 45 percent of that for the original ailerons, and the efficiency would be more than doubled.

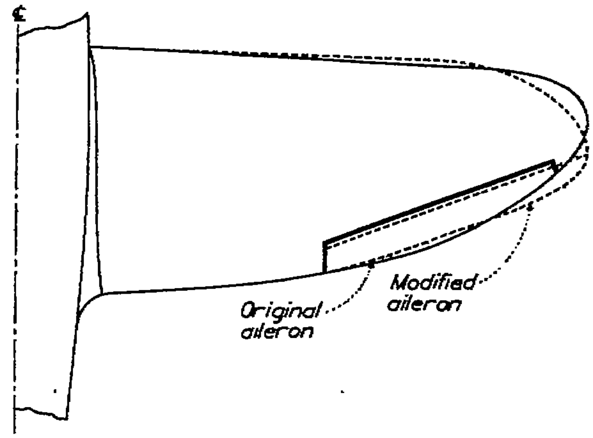

FIGURE 8.—Modification of a typical aileron to reduce the hinge moment.

CONCLUSIONS

Control surfaces of maximum efficiency (requiring a minimum operating force to achieve a given amount of control) may be designed almost without regard to the characteristics of the wings or tail surfaces to which they are to be attached. Except, perhaps, on very low-aspect-ratio tail surfaces (aspect ratio less than 2), flaps should be of almost constant chord over the span. The optimum shape for ailerons is of maximum width near the tip of the wing and has a slightly convex curvature as it tapers toward the center. Partial-span control surfaces should be sections of these optimum shapes and should include the regions of maximum chord. For maximum efficiency, however, because the hinge moment increases as the fourth power of the lift when the gain in lift must be achieved by increasing the chord, flaps and ailerons should be as long and narrow as is compatible with structural and other design considerations.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *January 30, 1941*.

REFERENCES

1. Pearson, H. A.: Span Load Distribution for Tapered Wings with Partial-Span Flaps. Rep. No. 585, NACA, 1937.
2. Munk, Max M.: Fundamentals of Fluid Dynamics for Aircraft Designers. The Ronald Press Co., 1929, p. 75.