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ON SIMILARITY RULES FOR TRANSONIC FLOWS

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SUMMARY

A method used by Tsien to derive similarity rules for hypersonic flows is utilized to derive Von Kármán's similarity rules for transonic flows. A slight generalization is introduced by the inclusion of γ, the ratio of specific heats, as a parameter. At the lower limit of the transonic region of flow the theory yields a formula for the critical stream Mach numbers of a given family of symmetrical profiles. It is further shown that this formula can also be obtained by means of the Prandtl-Glauert small-perturbation method. Investigation of the behavior of the similarity parameter in the region where the thickness coefficient approaches zero and the critical stream Mach number approaches unity shows that it possesses a limiting value characteristic of the prescribed family of shapes.

INTRODUCTION

The rigorous solution of the subsonic flow of a compressible fluid past a prescribed closed body thus far has proved to be of insurmountable difficulty. As a consequence of this difficulty the emphasis has been placed on the establishment of a correspondence between the flow past a given body in an incompressible fluid and the same body in a compressible fluid. Among the best known results of this mode of attack are the Prandtl-Glauert rule and the Von Kármán-Tsien velocity or pressure correction factor—both based on some form of linearization of the fundamental nonlinear flow equations. None of the methods based on the linearization of the flow equations, however, can yield correct results in the transonic range where the flow is partly subsonic and partly supersonic. For this case a certain amount of the feature of nonlinearity of the flow equations must be retained in order to obtain useful and nontrivial results. In the present paper a detailed derivation is given of the transonic similarity rules recently given by Von Kármán (reference 1).

FUNDAMENTAL EQUATIONS

In plane steady flow the equation governing the flow of a nonviscous compressible fluid can be written in the form

$$(c^2-u^2)\frac{\partial u}{\partial x}-uv\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+(c^2-v^2)\frac{\partial v}{\partial y}=0 \qquad (1)$$

In the derivation of this equation the pressure is assumed to be a function of the density only. If, further, the motion is irrotational, then

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0 \qquad (2)$$

Here u and v are, respectively, the component velocities along the Cartesian x- and y-axes and c is the local velocity of sound given by

$$c^2=c_0^2-\frac{\gamma-1}{2}(u^2+v^2) \qquad (3)$$

where c_0 is the velocity of sound at a stagnation point $u=0$, $v=0$ and γ is the ratio of specific heats at constant pressure and constant volume.

Equation (1) is far too complicated to afford an insight into the properties of potential flow in the neighborhood of Mach number unity. The discussion is, therefore, restricted to the flow past a thin profile. Thus, at first v is assumed to be small in comparison with the sound velocity c_0. Equation (1) is then simplified to

$$\frac{1-\frac{\gamma+1}{2}\frac{u^2}{c_0^2}}{1-\frac{\gamma-1}{2}\frac{u^2}{c_0^2}}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (4)$$

By the introduction of the sound velocity for which the local fluid and sound velocities are equal

$$c^*=\sqrt{\frac{2}{\gamma+1}}\,c_0 \qquad (5)$$

and of the maximum possible fluid velocity

$$q_{max}=\sqrt{\frac{2}{\gamma-1}}\,c_0 \qquad (6)$$

equation (4) becomes

$$\frac{1-\frac{u^2}{c^{*2}}}{1-\frac{u^2}{q_{max}^2}}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (7)$$

It is desirable to simplify this equation still further but yet to retain those features which yield nontrivial and useful results. Thus, if it is assumed that u is of the order of the critical velocity c^* and if only terms up to the order $1-\frac{u}{c^*}$ are retained, then equation (7) can be written as

$$-(\gamma+1)(u-c^*)\frac{\partial u}{\partial x}+c^*\frac{\partial v}{\partial y}=0 \qquad (8)$$

Thus far the irrotationality condition, equation (2), has not been used. If, now, the undisturbed stream past a slender body is of velocity U slightly different from the velocity of sound and in the direction of the positive x-axis; then, according to equation (2) and the assumptions leading to equation (8), a velocity potential Φ can be introduced with

$$\Phi=c^*x+(1-M^*)\varphi \tag{9}$$

where $M^*=\frac{U}{c^*}$ and $(1-M^*)\varphi$ is the disturbance-velocity potential.

Then

$$\left.\begin{aligned} u&=\frac{\partial\Phi}{\partial x}=c^*+(1-M^*)\frac{\partial\varphi}{\partial x}\\ v&=\frac{\partial\Phi}{\partial y}=(1-M^*)\frac{\partial\varphi}{\partial y}\end{aligned}\right\} \tag{10}$$

and equation (8) becomes

$$-(\gamma+1)(1-M^*)\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}+c^*\frac{\partial^2\varphi}{\partial y^2}=0 \tag{11}$$

Equation (11) is a nonlinear simplified form of the fundamental differential equation (1) and has been treated recently by Von Kármán in connection with similarity rules in two-dimensional transonic flow (reference 1). Equation (11), when expressed in hodograph variables, is of the type treated by the Italian mathematician F. Tricomi some years ago (reference 2).

DERIVATION OF SIMILARITY CONDITIONS

Recently, Tsien (reference 3) derived similarity rules for hypersonic flows where the fluid velocity is much larger than the velocity of sound. In the present paper the same procedure is employed to derive similarity rules for transonic flows where the fluid velocity is very nearly that of sound.

According to the assumptions leading to equation (11), it is implied that the solid body is thin and possesses no stagnation points and that the velocity of the fluid is everywhere in the neighborhood of the local velocity of sound. Now, suppose the profile of the obstacle to be symmetrical with respect to both the x- and y-axes and to possess cusps at both the leading and the trailing edges. Such profiles with uniform flow in the direction of the long axis-of-symmetry x fulfill the assumptions leading to equation (11). The flows past these profiles are said to be similar if the equation of motion (11) and the boundary conditions can be expressed in nondimensional variables in such a way that only a single constant factor is involved. Thus, if $2a$ is the chord and $2b$ is the maximum thickness of the body, then the following nondimensional variables are introduced:

$$\left.\begin{aligned} x&=a\xi\\ y&=a(\gamma+1)^m t^n\eta\end{aligned}\right\} \tag{12}$$

where $t=\frac{b}{a}$ and m and n are exponents yet to be determined. It is clear that the nondimensional quantity involved is the thickness coefficient t since this quantity determines the magnitude of the disturbance velocities. The exponents m and n are to be determined in such a way that the same constant factor appears in both the equation of motion and the boundary conditions.

The appropriate nondimensional form for the velocity potential φ is

$$\varphi=ac^*f(\xi,\eta) \tag{13}$$

By substitution from equations (12) and (13), the equation of motion (11) becomes

$$-(\gamma+1)^{2m+1}(1-M^*)t^{2n}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}+\frac{\partial^2 f}{\partial\eta^2}=0 \tag{14}$$

The boundary conditions at infinity require that the flow velocity be U. Hence, from the first of equations (10),

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x}&=-c^*\\ \frac{\partial\varphi}{\partial y}&=0\end{aligned}\right\}\quad(\text{at }\infty)$$

The boundary condition at the surface of the slender body is

$$(1-M^*)\frac{\partial\varphi}{\partial y}=c^*g\left(\frac{x}{a}\right)t \qquad (\text{at } y=0,\ -a\leqq x\leqq a)$$

where $g\left(\frac{x}{a}\right)$ describes the distribution of slope along the surface of the body.

By means of equations (12) and (13), the boundary conditions can be written as:

$$\left.\begin{aligned} &\left.\begin{aligned}\frac{\partial f}{\partial\xi}&=-1\\ \frac{\partial f}{\partial\eta}&=0\end{aligned}\right\}(\text{at }\infty)\\ &(\gamma+1)^{-m}(1-M^*)t^{-n-1}\frac{\partial f}{\partial\eta}=g(\xi)\qquad(\text{at }\eta=0,\ -1\leqq\xi\leqq 1)\end{aligned}\right\} \tag{15}$$

A comparison of the differential equation (14) and the boundary conditions, equations (15), shows that a single parameter is involved if

$$2n=-(n+1)\qquad\text{or}\qquad n=-\frac{1}{3}$$

$$2m+1=-m\qquad\text{or}\qquad m=-\frac{1}{3}$$

that is,

$$(\gamma+1)^{1/3}(1-M^*)t^{-2/3}=2K$$

The undisturbed-stream Mach number $M_\infty=\frac{U}{c_\infty}$ can be introduced in the following way:

The general relation between M^* and M_∞ is

$$M^*=\left(\frac{\frac{\gamma+1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right)^{1/2} \tag{16}$$

or

$$M_\infty=\left\{\frac{1-(1-M^*)[2-(1-M^*)]}{1+\frac{\gamma-1}{2}(1-M^*)[2-(1-M^*)]}\right\}^{1/2}$$

Then, if powers of $1-M^*$ higher than the first are neglected,

$$1-M^*=\frac{2}{\gamma+1}(1-M_\infty)+\ldots$$

Therefore

$$(1-M_\infty)[(\gamma+1)t]^{-2/3}=K \tag{17}$$

The results obtained thus far are such that by means of the substitution equations

$$\left.\begin{aligned} x&=a\xi \\ y&=a[(\gamma+1)t]^{-1/3}\eta \\ \varphi&=ac^*f(\xi,\eta) \\ K&=(1-M_\infty)[(\gamma+1)t]^{-2/3} \end{aligned}\right\} \tag{18}$$

the differential equation for $f(\xi, \eta)$ and the boundary conditions become:

$$\frac{\partial^2 f}{\partial\eta^2}=2K\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2} \tag{19}$$

and

$$\left.\begin{aligned} &\left.\begin{aligned}\frac{\partial f}{\partial\xi}&=-1\\ \frac{\partial f}{\partial\eta}&=0\end{aligned}\right\}\quad (\text{at } \infty) \\ &2K\frac{\partial f}{\partial\eta}=g(\xi)\qquad (\text{at } \eta=0, -1\leq\xi\leq1) \end{aligned}\right\} \tag{20}$$

The meaning of the similarity rule implied in the definition of the parameter K is the following:

If a series of bodies having the same distribution function $g(\xi)$ for the slope but different thickness ratios t are placed in flows of different undisturbed-stream Mach numbers M_∞ and different values of γ, such that the parameter $K=\frac{1-M_\infty}{[(\gamma+1)t]^{2/3}}$ remains constant, then the flow patterns are similar in the sense that the same function $f(\xi, \eta)$ describes the flows.

RESULTS DERIVED FROM THE SIMILARITY RULE

PRESSURE COEFFICIENT

In the case of a uniform flow past a fixed boundary, the pressure coefficient is defined as

$$C_{p,M_\infty}=\frac{p-p_\infty}{\frac{1}{2}\rho_\infty U^2}$$

where p_∞ and ρ_∞ are, respectively, the pressure and density in the undisturbed stream and the static pressure p in the fluid is given by

$$p=p_\infty\left[1-\frac{\gamma-1}{2}M_\infty^2\left(\frac{u^2+v^2}{U^2}-1\right)\right]^{\frac{\gamma}{\gamma-1}} \tag{21}$$

Then

$$C_{p,M_\infty}=\frac{2}{\gamma M_\infty^2}\left\{-1+\left[1-\frac{\gamma-1}{2}M_\infty^2\left(\frac{u^2+v^2}{U^2}-1\right)\right]^{\frac{\gamma}{\gamma-1}}\right\} \tag{22}$$

By means of equations (10) and (18), if powers of $1-M_\infty$ higher than the first are neglected, the following result is obtained:

$$C_{p,M_\infty}=-\frac{4}{\gamma+1}(1-M_\infty)\left(1+\frac{\partial f}{\partial\xi}\right)$$

or

$$C_{p,M_\infty}=\frac{t^{2/3}}{(\gamma+1)^{1/3}}P(\xi,\eta;K) \tag{23}$$

where $P(\xi, \eta; K)$ depends on the form of the solution $f(\xi, \eta)$ for the particular family of profiles treated.

LIFT COEFFICIENT

The lift l of the body is given by

$$l=\oint (p)_{\eta=0}dx$$

By a similar procedure, as in the derivation of equation (23), the lift coefficient is given by

$$c_l=\frac{l}{\frac{1}{2}\rho_\infty U^2(2a)}=\frac{\alpha^{2/3}}{(\gamma+1)^{1/3}}L(K) \tag{24}$$

where

$$L(K)=\int_{-1}^{1}P(\xi,0;K)d\xi$$

and where for an extremely thin straight-line profile the thickness coefficient t has been replaced by the angle of attack α.

DRAG COEFFICIENT

The pressure drag d of the body is given by the following expression:

$$d=2\int_{-a}^{a}(p)_{\eta=0}\,g\left(\frac{x}{a}\right)t\,dx$$

Hence the drag coefficient is given by

$$c_d=\frac{d}{\frac{1}{2}\rho_\infty U^2(2a)}=\frac{t^{5/3}}{(\gamma+1)^{1/3}}D(K) \tag{25}$$

where

$$D(K)=2\int_{-1}^{1} g(\xi)P(\xi,0;K)\,d\xi$$

ADDITIONAL CONSIDERATIONS

The results derived in the present paper apply to two-dimensional near-sonic flows past thin shapes. Such flows have been calculated for a family of symmetrical shapes with cusped leading and trailing edges (reference 4) and for a family of elliptic cylinders (reference 5). These calculations are valid at least up to the critical stream Mach number M_{cr}. The critical Mach number may be considered from two points of view. First, it may be considered to denote the lower limit of a mixed subsonic-supersonic flow, that is, where the imbedded supersonic region is simply the point of maximum fluid velocity at the surface of the solid. From this point of view, according to equation (17), a critical value of the similarity parameter K can be defined. Thus

$$K_{cr}=\frac{1-M_{cr}}{[(\gamma+1)t]^{2/3}} \tag{26}$$

This equation can also be written in a form that yields the critical stream Mach number for a given family of shapes; that is,

$$M_{cr}=1-K_{cr}\,[(\gamma+1)t]^{2/3} \tag{27}$$

Second, the critical-stream Mach number may be considered to denote the upper limit of the purely subsonic range of speeds. This point of view suggests a derivation of equation (27) by means of the Prandtl-Glauert small-perturbation method (reference 6). The procedure is as follows:

The relation between the local and the undisturbed-stream Mach number, within the approximation of the small-disturbance theory, is given by

$$M^2=M_\infty^2\left[1+2\frac{u'}{U}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\right] \tag{28}$$

where u' is the disturbance velocity and U is the undisturbed-stream velocity.

By definition, $M=1$ for $M_\infty=M_{cr}$ and, since $C_{p,M_\infty}=-2\frac{u'}{U}$ approximately, equation (28) becomes

$$1=M_{cr}^2\left[1-C_{p,M_{cr}}\left(1+\frac{\gamma-1}{2}M_{cr}^2\right)\right] \tag{29}$$

The relation between $C_{p,M_{cr}}$ and the pressure coefficient $C_{p,0}$ of the incompressible fluid is given by the Prandtl-Glauert rule

$$C_{p,M_{cr}}=\frac{C_{p,0}}{\sqrt{1-M_{cr}^2}}$$

Hence equation (29) becomes

$$\frac{(1-M_{cr}^2)^{3/2}}{M_{cr}^2\left(1+\frac{\gamma-1}{2}M_{cr}^2\right)}=-C_{p,0} \tag{30}$$

As seen from equation (29), $1-M_{cr}^2$ is of the first order in the small perturbation u'/U and, accordingly, equation (30) includes terms of higher order. This fact can be seen by rewriting equation (30) in the following form:

$$\frac{(1-M_{cr}^2)^{3/2}}{\frac{\gamma+1}{2}[1-(1-M_{cr}^2)]\left[1-\frac{\gamma-1}{\gamma+1}(1-M_{cr}^2)\right]}=-C_{p,0}$$

Then, to within the lowest order in the small perturbation,

$$1-M_{cr}^2=\left(-\frac{\gamma+1}{2}C_{p,0}\right)^{2/3}$$

or, approximately,

$$M_{cr}=1-\frac{1}{2}\left(-\frac{\gamma+1}{2}C_{p,0}\right)^{2/3} \tag{31}$$

It is quite easy to show the connection between equation (31) and equation (27). Thus, for an incompressible fluid, the pressure coefficient is given by

$$C_{p,0}=1-\frac{q^2}{U^2} \tag{32}$$

where q is the magnitude of the fluid velocity at any point in the field of flow.

In the case of the family of shapes of reference 4, the maximum velocity at the surface is given by

$$\frac{q}{U}=1+\frac{3}{2}t+\ .\ .\ .$$

Hence

$$C_{p,0}=-3t+\ .\ .\ .$$

and equation (31) becomes

$$M_{cr}=1-\left(\frac{9}{32}\right)^{1/3}[(\gamma+1)t]^{2/3} \tag{33}$$

In the case of the family of elliptic cylinders of reference 5, the maximum velocity at the surface is given by

$$\frac{q}{U}=1+t$$

Hence

$$C_{p,0}=-2t+\ .\ .\ .$$

and equation (31) becomes

$$M_{cr}=1-\frac{1}{2}[(\gamma+1)t]^{2/3} \tag{34}$$

An examination of equations (33) and (34) suggests that K_{cr} possesses a limiting value; that is,

$$(K_{cr})_{lim}=\lim_{\substack{t\to 0\\ M_{cr}\to 1}}\frac{1-M_{cr}}{[(\gamma+1)t]^{2/3}} \tag{35}$$

It is noteworthy that the value of $(K_{cr})_{lim}$ depends on the family of shapes, although in the limit $t\rightarrow 0$ the profile in every case is a straight-line segment. The numerical values of $(K_{cr})_{lim}$ shown in equations (33) and (34) represent, however, only the effect of the Prandtl-Glauert or first-order term in the power-series development in t of the maximum velocity at the solid surface. It is rather surprising that the higher-order terms involving the higher powers of t also contribute to the value of $(K_{cr})_{lim}$. The procedure is simply to replace the maximum velocity by the critical speed; that is,

$$\left(\frac{q}{U}\right)_{max}=\left(\frac{q}{U}\right)_{cr}=\left(\frac{1+\frac{\gamma-1}{2}M_{cr}^2}{\frac{\gamma+1}{2}M_{cr}^2}\right)^{1/2}$$

or

$$\left(\frac{1+\frac{\gamma-1}{2}M_{cr}^2}{\frac{\gamma+1}{2}M_{cr}^2}\right)^{1/2}=1+a_1t+a_2t^2+a_3t^3+\ldots \qquad (36)$$

where a_1, a_2, a_3, . . . depend on the given family of shapes and involve only γ and M_{cr}.

Then as suggested by equation (26), when $1-M_{cr}^2$ is replaced by $2K_{cr}[(\gamma+1)t]^{2/3}$, equation (36) is identically satisfied for $t\rightarrow 0$, with each additional higher-order term contributing to the value of K_{cr}. In this fashion there are obtained, successively, linear, quadratic, cubic, and higher-order equations for the determination of $(K_{cr})_{lim}$.

The foregoing considerations have been applied to the family of symmetrical shapes of reference 4 and to the family of elliptic cylinders of reference 5. The values of K_{cr} for values of t different from zero were obtained from equation (26) with the required values of M_{cr} determined by means of the theoretical results of references 4 and 5. The limiting values of K_{cr} were obtained from equations corresponding to equation (36). Tables I and II show the results of these calculations for air and for Freon-12. Note the approach to the limiting values of K_{cr} as t approaches zero and M_{cr} approaches unity. The first column of values of K_{cr} shows the effect of the first-order or Prandtl-Glauert term; the second and third columns show, respectively, the effects of the second-order and third-order terms in the thickness coefficient t. The successive values of K_{cr} for the various values of the thickness coefficient t indicate good convergence.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
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REFERENCES

1. Von Kármán, Theodore: Supersonic Aerodynamics—Principles and Applications. Jour. Aero. Sci., vol. 14, no. 7, July 1947, p. 396.
2. Tricomi, F.: Sulle equazioni lineari alle derivate parziali di 2° ordine di tipo misto. Atti della R. Accad. Naz. dei Lincei, Mem. della Cl. Sci. Fis., Mat. e Nat., ser. 5, vol. 14, 1923, p. 134.
3. Tsien, Hsue-shen: Similarity Laws of Hypersonic Flows. Jour. Math. and Phys., vol. XXV, no. 3, Oct. 1946, pp. 247–251.
4. Kaplan, Carl: The Flow of a Compressible Fluid past a Curved Surface. NACA Rep. No. 768, 1943.
5. Kaplan, Carl: Effect of Compressibility at High Subsonic Velocities on the Moment Acting on an Elliptic Cylinder. NACA TN No. 1218, 1947.
6. Liepmann, Hans Wolfgang, and Puckett, Allen E.: Introduction to Aerodynamics of a Compressible Fluid. John Wiley & Sons, Inc., 1947, pp. 142–143.

TABLE I

VALUES OF K_{cr} FOR THE FAMILY OF SYMMETRICAL SHAPES OF REFERENCE 4

[For air, γ=1.4; for Freon-12, γ=1.136]

t	M_{cr}			K_{cr}		
	Approximation			Approximation		
	First	Second	Third	First	Second	Third
Air						
0	1.000	1.000	1.000	0.655	0.717	0.738
.002	.982	.980	.979	.647	.710	.732
.005	.967	.963	.962	.638	.705	.726
.010	.947	.942	.940	.632	.700	.719
.020	.918	.909	.906	.624	.689	.709
.040	.872	.859	.854	.609	.675	.696
.060	.837	.818	.812	.595	.664	.685
.080	.807	.784	.774	.581	.650	.678
.100	.781	.753	.743	.568	.639	.665
Freon-12						
0	1.000	1.000	1.000	0.655	0.717	0.738
.002	.983	.981	.980	.647	.713	.735
.005	.969	.966	.964	.642	.709	.730
.010	.951	.946	.944	.636	.704	.725
.020	.924	.915	.912	.627	.694	.716
.040	.881	.867	.863	.612	.682	.706
.060	.848	.830	.823	.600	.670	.698
.080	.819	.797	.787	.588	.658	.691
.100	.794	.769	.756	.576	.647	.682

TABLE II

VALUES OF K_{cr} FOR THE FAMILY OF ELLIPTIC CYLINDERS OF REFERENCE 5

[For air, γ=1.4; for Freon-12, γ=1.136]

t	M_{cr}			K_{cr}		
	Approximation			Approximation		
	First	Second	Third	First	Second	Third
Air						
0	1.000	1.000	1.000	0.500	0.537	0.556
.002	.986	.985	.984	.494	.530	.550
.005	.974	.972	.971	.490	.526	.544
.010	.960	.957	.955	.487	.522	.537
.020	.936	.932	.930	.481	.515	.526
.040	.901	.894	.892	.470	.505	.515
.060	.873	.863	.861	.462	.497	.507
.080	.848	.837	.833	.455	.490	.500
.100	.827	.813	.809	.447	.484	.494
Freon-12						
0	1.000	1.000	1.000	0.500	0.537	0.556
.002	.987	.986	.985	.497	.534	.553
.005	.976	.974	.973	.495	.532	.550
.010	.962	.959	.958	.491	.528	.543
.020	.941	.936	.935	.484	.520	.534
.040	.908	.900	.899	.474	.509	.522
.060	.881	.872	.871	.469	.504	.513
.080	.858	.847	.845	.462	.496	.506
.100	.838	.826	.822	.455	.488	.498