NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1522

.733

CALCULATION OF UNCOUPLED MODES AND FREQUENCIES IN BENDING OR TORSION OF NONUNIFORM BEAMS

By John C. Houbolt and Roger A. Anderson

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

NACA

Washington
February 1948

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1522

CALCULATION OF UNCOUPLED MODES AND FREQUENCIES IN BENDING OR TORSION OF NONUNIFORM BEAMS

By John C. Houbolt and Roger A. Anderson

SUMMARY

A procedure is presented for the calculation of frequencies and modes of nonuniform beams in uncoupled bending and torsional vibration. Based on the principle of the Stodola method, the procedure consists of solving the differential equation of equilibrium for vibration by a method of successive approximations. Basic principles of engineering beam theory are employed in the method, and the integrations involved are performed by improved numerical methods. An effort has been made to perform all calculations in a manner consistent with the accuracy to which physical constants in built-up beams are ordinarily known.

Higher modes are readily found by use of the orthogonality relation between normal modes. The frequency is found simply as the square root of the proportionality factor existing between modal deflection curves in successive approximations. All computations are tabular in form and are performed mentally or with the aid of a slide rule. Comparison made with available exact analytical solutions shows that the method gives for practical purposes the exact answer. Special consideration has been given to the treatment of various boundary conditions that are found in the vibration of aircraft structures. The cantilever beam, the free-free beam, beams with concentrated masses, beams mounted on springs, beams elastically coupled to masses, and so forth, are shown to be handled with practical simplicity. In order to serve as a guide in the solution of practical problems, the procedures for handling a number of different cases are illustrated by a liberal use of examples.

INTRODUCTION

In the dynamic analysis of aircraft structures, the determination of the natural modes and frequencies is of basic importance. A number of methods for calculating modes and frequencies have been developed; each method has certain desirable features. The objective of the present paper is to develop a procedure which is readily learned by anyone familiar with engineering beam theory, is easy and quick to apply without the use of complicated computing devices, and gives results within the range of accuracy with which the physical properties of the structure (mass, stiffness distribution, etc.) can ordinarily be determined. An adaptation of the successive-approximation procedure of Stodola (see reference 1) fulfills these requirements and has the additional advantage that data necessary for the analysis of stresses due to vibration are obtained during the computations for the modes. In the present paper, this successive-approximation or iteration method is employed to obtain solutions to the differential equations of equilibrium for bending and torsional vibrations.

The Stodola method, outlined in reference 1, was originally a graphical-integration procedure for determining fundamental modes and frequencies. Burgess presented in reference 2 a numerical procedure for finding the fundamental frequency of a cantilever, which was in essence a series of approximate numerical integrations to determine the modal deflection followed by an energy solution for the frequency. Boukidis and Ruggiero (reference 3) gave an application of the Stodola method, in the form of a numerical procedure, which permitted calculation of the higher as well as the fundamental modes and frequencies of a free-free beam in symmetrical vibration. Inherent disadvantages in the method of higher-mode determination, as presented in reference 3, however, have been pointed out by Beskin and Rosenberg in reference 4. These authors made use of the orthogonality relation between normal modes of vibration, in conjunction with Burgess's numerical procedure, to determine the higher modes of a cantilever.

Two improved methods of numerical integration are presented in the present paper for the solution of differential equations by iteration. One of the methods is a summation process, somewhat similar to the numerical procedure of reference 2; the other method is described by Newmark (reference 5), who used it in beam and column analysis and called attention to its applicability to vibration problems. Both methods are simple to apply and lead to relatively accurate modal deflections. The frequency is finally found to be the square root of the proportionality factor existing between modal-deflection curves in successive iterations.

In order to perform the integrations, the boundary conditions on the vibrating member must be taken into account. Consideration has been given herein to a number of different boundary conditions. Both the symmetrical and the antisymmetrical modes of a free-free beam are treated in detail. Boundary conditions for the vibration of beams supported on springs, beams elastically coupled to masses, and so forth will be shown to be handled with practical simplicity. Although the method can be applied to problems in which bending and torsion are coupled, the analysis will not be presented herein.

In order to determine modes higher than the fundamental by iteration, components of all modes lower than the one to be determined must be removed. This operation is performed by use of the orthogonality relation between normal modes. With this additional step accurate solutions for higher modes are readily obtained.

The basic steps of the iteration method and a condensed procedure for solving problems will be outlined in the following sections. Actual application of the procedure is illustrated by a number of examples. These examples are intended to serve as a guide in the solutions of practical problems and at the same time to give an indication of the simplicity attainable with this method of analysis. All computations are performed mentally or with the aid of a slide rule and comparisons made with the few available exact solutions show that the iteration method gives for practical purposes the exact answer. A theoretical verification of the analysis presented is given in the appendix.

SYMBOLS

L	length of beam; half span for symmetrical beams, full span for unsymmetrical beams
E	Young's modulus of elasticity
G	modulus of elasticity in shear
I	bending moment of inertia
I_p	mass polar moment of inertia per unit length of beam about axis of rotation
J	torsional stiffness constant
w	weight of beam per unit length
g	acceleration due to gravity
m	mass of beam per unit length $\left(\frac{w}{g}\right)$

S	shear
M	bending moment
T	torque
λ	distance between stations along beam
a_n	amplitude of nth mode $(n = 1,2,3, . . .)$
μ_n, ϵ_n and η_n	used in place of a_n to denote very small amplitudes
x	station coordinate
α_j	elastic spring constant at jth station
m_j	concentrated mass at jth station
f_n	frequency of nth natural mode (bending or torsional) vibration, cycles per second
ω_n	circular frequency of nth natural mode of (bending or torsional) vibration, radians per second $(\omega_n = 2\pi f_n)$
P_{eq}	equivalent loading
p_j	natural frequency of a spring-mass oscillator, radians per second
δ_j	deflection at station j
y or Y	general notation for deflection
$Y_n^{(0)}$ or $Y_n^{(0)}(x)$	assumed or reasonable approximate deflection of nth mode, usually written in terms of a unit tip deflection; $Y_n^{(0)}$ denotes deflection of a given point; $Y_n^{(0)}(x)$ denotes modal-deflection function
y_n or $y_n(x)$	exact deflection of nth mode written in terms of a unit tip deflection; y_n denotes deflection of a given point; $y_n(x)$ denotes modal-deflection function

$Y_n^{(i)}$ or $Y_n^{(i)}(x)$ — deflection of nth mode after i iterations for both derived values of deflection and values written in terms of unit tip deflection; $Y_n^{(i)}$ denotes deflection of a given point; $Y_n^{(i)}(x)$ denotes modal-deflection function

$Y_n'^{(i)}$ or $Y_n'^{(i)}(x)$ — deflection of nth mode after i iterations where deflection is given relative to an axis through center of beam and must be corrected to satisfy boundary conditions for both derived values of deflection and values written in terms of a unit tip deflection; $Y_n'^{(i)}$ denotes deflection of a given point; $Y_n'^{(i)}(x)$ denotes modal-deflection function

$Y_o^{(i)}$ — a constant correction to be applied to $Y_n'^{(i)}(x)$, after ith iteration, when a solution is sought for a symmetrical mode of a free-free beam

$K^{(i)}x$ — a linearly varying correction to be applied to $Y_n'^{(i)}(x)$, after the ith iteration, when a solution is sought for an antisymmetrical mode of a free-free beam

$y_n^{(i)}$ or $y_n^{(i)}(x)$ — deflection of nth mode after all modes lower than nth mode have been removed after ith iteration for both the derived values of deflection and values written in terms of unit tip deflection; $y_n^{(i)}$ denotes deflection of a given point; $y_n^{(i)}(x)$ denotes modal-deflection function

ϕ or Φ — general notation for rotation in torsional-vibration problems (with y and Y replaced by ϕ and Φ definition of symbols for rotation is similar to that for deflection)

BASIC STEPS OF ANALYSIS

Iteration Methods

Bending.- Examination of the differential equation of equilibrium for a beam in free harmonic bending vibration

$$\frac{d^2}{dx^2}\left[EI\ \frac{d^2y_n(x)}{dx^2}\right] = \omega_n{}^2m\ \ y_n(x) \qquad (1)$$

shows that a beam vibrating in one of its natural modes has an inertia loading at any point that is proportional to the product of the mass intensity and the deflection at that point. The left-hand side of the equation represents the elastic restoring force of the beam and, in the ordinary bending theory of beams, equals the external loading. The right-hand side represents the inertia loading which may be considered at any instant of time to be statically applied. Deflection functions $y_n(x)$ which satisfy both equation (1) and the boundary conditions of the beam are called the natural modes of the beam. The factor ω_n is the natural frequency in radians per second of vibration of the nth mode.

A curve approximating the deflection may be assumed and, as indicated by the right-hand side of the equation, values proportional to the loading at any point may be computed directly by multiplying the assumed deflection $Y_1{}^{(0)}$ by the mass intensity at the point. The factor $\omega_n{}^2$ can be ignored since it is a constant and since the amplitude of the deflection is purely arbitrary. The values of $mY_1{}^{(0)}$ then represent the inertia loading at each point. With this computed loading, a new deflection may be found by any of the known methods of engineering beam theory such as the direct-integration method; that is, with the loading on the beam as a starting point, successive integrations give in turn the shear, the moment, the slope, and the deflection.

In the present paper, the process of finding a new deflection from an assumed or given approximate deflection curve by four successive numerical integrations is considered one iteration. In the appendix the new deflection is shown to converge toward the lowest mode component present in the assumed deflection. By successive iteration, a natural-mode deflection curve may be found to an arbitrary degree of accuracy.

An iteration by two different methods for a beam in bending vibration is presented in table 1. The problem used for illustration is that of a nonuniform cantilever carrying a concentrated mass. The summation method is a simple numerical-integration procedure to determine the shear, the moment, the slope, and the new deflection. The second method presented is a form of numerical integration which makes use of the concept of equivalent concentrated loads. (See table 1.)

Most of the steps of an iteration will become self-evident on inspection of the tabulated computations and the graphical illustrations in figure 1.

For the summation method (table 1), the physical constants I and m for the cantilever beam are listed in columns 2 and 3, and the initially assumed values of deflection at each station are listed in column 4. The beam is divided into equally spaced stations with the root as station 0 and the tip as station 10. In the numerical integrations, the distance λ between stations may then be carried as a common factor as shown. The loading values $mY_1^{(0)}$ which are proportional to the inertia loading are given in columns 5 and 6. The shear (column 7) and moment (column 8) are then found by the summation process indicated in figure 2. Except for the initial value, the shear is found by successive addition of the loading ordinates. The initial value of shear is found by application of the following simple equation (from fig. 2):

$$\text{Area} = \lambda\left(\frac{3a + b}{8}\right) \qquad (2)$$

Addition starts from station 10 because the shear is known to be zero there. The moment is found by successive addition of the shear values; the addition starts at station 10 where the moment is zero. From the M/I values (column 9), the slope (column 10) is found in the same manner that was used to obtain the shear except that integration proceeds from station 0 where the slope is zero. Successive addition of the slope values, starting at station 0, gives finally the new deflection (column 11). In order to compare the new deflection with the assumed deflection, the new deflection is given in the next column in terms of a unit tip ordinate. Burgess's method (reference 2) consists of adding successive station ordinates but a clear physical interpretation of his procedure cannot readily be made.

Since the beam has been divided into equal intervals, mass concentrations on the beam will not, in general, be located at station points. The procedure, then, is to distribute a concentrated mass proportionally to two adjacent station points on the assumption that the mass is supported by a simple beam between the two points. When the station points are spaced reasonably close, this local redistribution of the loading produces no significant change in the derived deflection.

For curves that cannot be reasonably approximated by straight lines between stations without resorting to a large number of stations, another method of performing integrations numerically has been employed. In reference 5, the equivalent-load method is presented; this method makes use of the properties of a second-degree curve for computing the system of concentrated loads at the stations which produce the true moment at each station and for computing the concentrated values of M/EI which cause the deflection to be correct at each station. The method is easy to apply and gives results which are quite accurate even when curvatures are appreciable between stations.

For the equivalent-load method (table 1), the equivalent-loading diagram (column 4) is computed by using equations (a) and (b) given in figure 3. These equations were taken from reference 5 and are immediately applicable only to curves continuous over at least two station intervals. In order to keep $mY_1^{(0)}$ (column 3) from having a discontinuity at station 3, therefore, the concentrated mass (column 5) is handled separately. Summation of the equivalent loads gives the average shear between each station and summation of the shear gives the true-moment diagram. The equivalent values of M/I (E is carried as a common factor) are then found in the same manner as the equivalent loads with the exception of the value at station 3. The discontinuity in the shear diagram at station 3 causes an abrupt change in slope of the moment diagram. The equivalent M/I value at this station is computed by use of equation (c) in figure 3, which is derived by applying equation (a) to the ordinates of the smooth curve on either side of the abrupt change. The deflection (column 11) is found after two summations.

Either the summation method or the equivalent-load method may be used for an iteration. The summation method is simple and quick and gives good results for the lower modes and frequencies of vibration. The method of equivalent loads, on the other hand, is particularly suited for higher-mode determination. For the higher modes, curvatures are appreciable in the functions to be integrated, and the greater inherent accuracy of this method results in better approximations to the deflections.

Torsion.- The differential equation in torsional vibration is

$$-\frac{d}{dx}\left[GJ\,\frac{d\phi_n(x)}{dx}\right] = \omega_n^2\, I_p\phi_n(x) \qquad (3)$$

This equation is analogous to equation (1) for bending vibrations. The left-hand side of the equation represents the elastic restoring force, and the right-hand side represents the inertia loading. The iteration process in torsion is similar to that in bending; that is, a curve of angular displacement is assumed and on multiplication by the mass polar moment of inertia, a measure of the inertia torque loading is obtained. By direct integration a new angular-displacement curve is found. Only two integrations need be performed in an iteration for torsional problems as compared with the four integrations for bending problems because the differential equation of equilibrium in torsion is of the second order, whereas the equation in bending is of the fourth order.

A typical iteration for a cantilever in torsional vibration is given in tabular form in table 2 and presented graphically in figure 4. Integration of the torque-loading ordinates (column 5) by the summation method gives a value of torque (column 6) midway between each station. After dividing each torque ordinate by its corresponding value of J, another summation gives the rotation (column 8). The summation in each integration begins at the station where the function being determined is known to be zero. Only the summation method of integration is used in torsional problems because it is not convenient to determine equivalent concentrated inertia torque loads that will give the correct internal torque at specific points on the beam.

Treatment of Various Boundary Conditions

Bending.- The boundary conditions encountered in vibration problems are the same as those found in problems in statics; that is, the beam may have any combination of free, pinned, elastically restrained, or fixed ends. In addition, if the beam and its support conditions are symmetrical about a center line, the equilibrium conditions existing at the center line depend on whether the beam is vibrating in symmetrical or antisymmetrical modes. For such beams, the two types of modes are found separately by considering the beam to be cut at the center and by applying the proper boundary conditions for each type.

In the iteration process, the functions determined by numerical integration will be correct only if all the boundary conditions are satisfied. In the simplest cases, such as a cantilever or a symmetrical simply supported beam, where the shear, moment, slope, and deflection are each known to be zero at specific points on the beam, no difficulty is encountered in performing integrations. Summations simply proceed from the points where the functions are known to be zero. In other types of beams, there may be more than one known boundary condition on some of the functions and none at all on others. For those functions without boundary conditions, the point at which the function is zero is generally not known and a special treatment must therefore be made in the integration process. Although the treatment will vary with each problem, a general method of approach may be outlined. The method is to sketch the probable deflection, shear, moment, and slope diagrams in that order. In doing so, the conditions that must be satisfied to make each diagram follow from the preceding one will become evident. It will be seen that an arbitrary constant of integration is introduced when no boundary conditions are known for a particular function. The constant is then carried along and evaluated later when two boundary conditions are known for one of the other functions. In some problems, short-cut calculations may be made to evaluate the constant immediately.

As an illustration of the general approach, consider the case of a free-free beam symmetrical about its midpoint. In figure 5 the series of diagrams on the right represent the actual variation of the deflection, loading, shear, and so forth, along the half span of the beam for the first symmetrical bending mode. No boundary conditions are known for the deflection; the shear is zero at both the midpoint and the tip, the moment is zero at the tip, and the slope is zero at the midpoint. Since an arbitrarily assumed deflection will not, in general, produce a loading which will cause the shear to be zero at both the midpoint and the tip, the deflection has been assumed in two parts as shown in the two diagrams on the left. A variable part $Y_1'^{(0)}$ giving deflections relative to the center of the beam is assumed, together with a constant part $Y_o^{(0)}$, the magnitude of which is to be determined later by the shear boundary conditions. This separation of the deflection into two parts is permissible since the principle of superposition holds for problems of this type.

Loadings are computed for both deflections (each deflection is multiplied by the mass variation given in column 2 of table 3) and the shear diagram for each loading is found by integration from the tip inward. In order to make the shear zero at the center line, the ordinates of the S_o diagram must be adjusted to make the center-line ordinates of the S' and S_o diagrams equal in magnitude and opposite in sign; the two diagrams then added give the true shear diagram S (fig. 5). If desired, it is now possible at this point to correct the assumed deflection and to obtain the correct loading. Since the value of shear s_o is proportional to $Y_o^{(0)}$, the adjusted deflection $\frac{s'}{s_o} Y_o^{(0)}$ added to $Y_1'^{(0)}$ gives the deflection $Y_1^{(0)}$. When the true shear diagram is known, the moment and slope diagrams are found without any difficulty. For the new deflection integration proceeds from the center line which gives the deflections $Y_1'^{(1)}$ relative to the center of the beam. The correction $Y_o^{(1)}$ must now be determined. This correction is found from $Y_1'^{(1)}$ in the same manner that $Y_o^{(0)}$ was determined from $Y_1'^{(0)}$. After $Y_o^{(1)}$ is determined, $Y_1'^{(1)}$ is corrected to give the better modal approximation $Y_1^{(1)}$. Note that the process of determining the corrections $Y_o^{(i)}$ consists actually of performing a large part of another iteration. The method just described for integrating and satisfying boundary conditions is analogous to direct analytical integration of the differential equation, where the unknown constants of integration are carried along until enough boundary values are known to permit them to be evaluated.

A short-cut calculation is derived for the preceding case from the following consideration. It is recognized that in order to prevent translation of the beam the negative and positive loading areas must be equal. (See loading in fig. 5.) Expressed mathematically the condition is

$$\int_0^L mY_n \, dx = 0 \tag{4}$$

or if Y_n is broken into two parts, as in figure 5, equation 4 becomes

$$\int_0^L m(Y_n' - Y_o) \, dx = 0 \tag{5}$$

and Y_o is found to be

$$Y_o = \frac{\int_0^L mY_n' \, dx}{\int_0^L m \, dx} \tag{6}$$

In equations 4 to 6, and whenever convenient in the equations to follow, the superscripts indicating the iteration number have been purposely omitted for simplicity in presentation. The equations are of general form and apply regardless of the number of the iteration. The numerator and denominator of equation (6) can conveniently be evaluated in terms of the station ordinates by means of the following equation which is derived in the appendix:

$$A = \frac{125\lambda}{144}(0.38a + 1.50b + c + d + 1.50e + 0.76f + 1.50g + h + i + 1.50j + 0.38k) \qquad (7)$$

Equation (7) for computing the total area under a curve is used hereinafter because of its simplicity and accuracy. This equation is based on the properties of a fifth-degree curve and is therefore applicable to finite intervals in sets of five. After Y_n' has been assumed, Y_o can be computed directly by use of equation (6) and then the correct deflection $Y_n = Y_n' - Y_o$ can be found. This short-cut procedure is illustrated by the numerical example in table 3. A complete iteration is given in the section "EXAMPLES".

In table 4, the general approach is illustrated for the first antisymmetrical mode of a symmetrical free-free beam. The diagrammatic presentation of the data is shown in figure 6. As illustrated in the column of diagrams on the right-hand side the deflection and loading are known to be zero at the center of the beam and the shear is zero at the tip. The moment must be zero at both the center and tip, and no boundary values are known for the slope. The difficulty in this set of boundary conditions is that an arbitrarily assumed deflection will not produce a loading which will cause the moment to be zero at both the center and the tip. The moment boundary conditions can be satisfied, however, if the assumed curve is given a proper rotation about the midpoint of the beam. A linearly varying correction $K^{(0)}x$ is therefore integrated along with the assumed deflection $Y_1'^{(0)}$, as shown in the two left-hand columns in figure 6, until the moment diagrams are obtained. The two diagrams are added (fig. 6) to give the true moment diagram. When the correction to be applied is known, the corrected assumed deflection, the correct loading, and the shear can be obtained.

In the next step the interior point at which the slope is zero is unknown, so a zero point is assumed and integration for the slope proceeds from this point in both directions. An arbitrary constant correction is also assumed. Integration of the incorrect slope diagram and the constant correction gives the incorrect deflection $Y_1'^{(1)}$ and the linearly varying correction $K^{(1)}x$. The correction

$K^{(1)}x$ to the deflection $Y_1'^{(1)}$ is determined in the same manner that was used to find the correction $K^{(0)}x$ to the deflection $Y_1'^{(0)}$; that is, with $Y_1'^{(1)}$ and $K^{(1)}x$, integration proceeds in the next iteration (not shown in the table) until the moment diagrams are found, from which the true correction can be found by the condition that the center-line moment must be zero.

Again an equation can be derived for correcting the assumed deflection before an iteration and the number of integrations is thereby reduced. In order to prevent rotation of the beam about an axis at midspan perpendicular to the plane of vibration, the moment of the inertia loading about this axis must be zero. Written mathematically in terms of the two assumed loadings, the condition is

$$\int_0^L m(Y_n' - Kx)x\,dx = 0 \qquad (8)$$

where K is the slope of a linear correction. Solution for K gives

$$K = \frac{\int_0^L mxY_n'^{(i)}\,dx}{\int_0^L mx^2\,dx} \qquad (9)$$

This equation can be conveniently evaluated numerically by use of equation (7). The use of equation (9) is illustrated by the example in table 4. A complete iteration is given in a subsequent example.

As an indication of the flexibility of the general method of approach for handling boundary conditions, consider next a beam fixed at one end and simply supported at the other end. The logical procedure for performing an iteration on such a problem is given in figure 7. The process should become clear from a study of the sketches and the step-by-step outline of the iteration procedure.

The foregoing illustrations indicate that rather complicated and even statically indeterminate structures can be handled if they are broken up into their basic component parts. In the examples that are presented subsequently, the method of adding elastic restraints, such as springs, to the beams is shown. The solutions given for these problems make use of short-cut equations, similar to equation (6), which are derived from basic sketches such as those shown in figures 5 and 6.

Torsion.- As was true in bending-vibration problems, no difficulties arise in the handling of the ordinary boundary conditions for problems in torsion with the exception of the free-free and certain fixed-end beams. For these cases a procedure analogous to that in bending must be used.

Removal of Lower-Mode Components

Bending.- In the determination of higher modes by the iteration process any components of modes lower than the one to be determined must be removed from the assumed or given deflection. Unless this procedure is followed before each iteration, convergence will tend toward the lowest-mode component present in the assumed deflection. Assuming that the shapes of the lower modes are known, the components of each mode present in the assumed higher-mode shape can be found by use of the orthogonality relation between the normal modes of vibration. The equation for computing the amplitude a_n of any mode y_n present in a given deflection curve Y_m is

$$a_n = \frac{\int_0^L m y_n Y_m \, dx}{\int_0^L m {y_n}^2 \, dx} \tag{10}$$

This equation is given in reference 1 and, for completeness, a derivation is included in the present appendix.

The procedure for removing lower-mode components is illustrated in table 5 where the fundamental-mode component is subtracted from an assumed second-mode shape for a cantilever. A graphical representation of this procedure is given in figure 8. The integrals in equation (10) are evaluated in columns 5 and 8, and the fundamental-mode amplitude is computed at the bottom of the table. The assumed second-mode shape minus the component of the fundamental mode is given in column 11 in terms of a unit tip deflection. The same procedure applies when n-1 lower modes are being removed from an assumption for the deflected shape of the nth mode.

Torsion.- The method for removing lower-mode components in torsional problems is the same as that for bending.

Frequency Determination

Bending.- Once a close approximation for the deflection of a given mode has been established, the frequency of vibration can be determined. For bending, the frequency may be found from the following equation

$$\omega_n^2 = \frac{y_n^{(i)}}{y_n^{(i+1)}} \tag{11}$$

where

$y_n^{(i)}$ value of deflection of a point on beam before iteration

$y_n^{(i+1)}$ value of deflection of point after iteration when fundamental mode is being determined; whereas, it is deflection of point after iteration and removal of lower-mode components for higher-mode determination

(The deflection $y_n^{(i)}$ is usually written in terms of a unit tip deflection and, when equation (11) is used, $y_n^{(i+1)}$ is the absolute value of the deflection found in an iteration. Before the next iteration is begun, however, $y_n^{(i+1)}$ is for convenience written in terms of a unit tip deflection.) The use of equation (11) in the frequency determination of a beam is illustrated in a subsequent section "EXAMPLES."

The frequency may also be determined from the following equations which are derived in the appendix:

$$\omega_n^2 = \frac{\int m\left[y_n^{(i)}\right]^2 dx}{\int \frac{M^2}{EI} dx} \tag{12}$$

$$\omega^2 = \frac{\int m y^{(i)} y^{(i+1)} dx}{\int m\left[y^{(i+1)}\right]^2 dx} \tag{13}$$

The moment M in equation (12) is the moment which results from the loading $my_n^{(i)}$. Both equations are derived from the energy relations of a beam in vibration and have been used by other investigators. Since the frequency, as given by these equations, is evaluated from the entire deflection curve, values of deflection which are as accurate as those required by equation (11), where the frequency is evaluated from the deflections at a point, are not necessary. Thus, if only the frequency is desired in a given example, a good value may be found from either of these equations by use of a deflection which is found from one to two less iterations than those that would normally be required if both mode and frequency are to be determined.

Torsion.- Frequency determination in torsion is analogous to that in bending. The frequency may be found from the following equation

$$\omega_n^2 = \frac{\phi_n^{(i)}}{\phi_n^{(i+1)}} \tag{14}$$

where

$\phi_n^{(i)}$ value of rotation of a station on the beam before iteration

$\phi_n^{(i+1)}$ value of rotation of station after iteration for fundamental-mode determination; value of rotation of station after iteration and removal of lower-mode components for higher-mode determination.

SUMMARY OF PROCEDURE

Thus far, the more important steps in the analysis have been explained in some detail. In order to facilitate the actual working of problems, the various steps will be summarized in their logical order. The procedure will be given for computing the modes and frequencies of a beam in bending vibration. An analogous procedure is used for torsional problems.

Fundamental Mode.- The steps for computing the fundamental mode are as follows:

(1) Divide the beam into a convenient number of equal stations. If only the fundamental mode is to be computed, six to eight stations will be sufficient. If higher modes are to be determined later, ten

stations should be used because any errors present in the fundamental mode will have an effect on the accuracy of the higher modes. With the beam divided into ten stations, a very accurate fundamental mode can be obtained. From mass and moment-of-inertia distribution plots, read off the values of these variables at each station. Any large concentrated masses should be distributed to the nearest station points.

(2) Assume a reasonable approximation to the fundamental mode. A convenient way is to sketch the deflected shape of the beam and read off the deflection ordinates at each station.

(3) Perform an iteration with proper regard to boundary conditions.

(4) Use the deflected shape from the previous iteration and successively repeat step 3 until the desired degree of convergence is obtained.

(5) Compute the frequency by equation (11).

Higher Modes.- The steps for computing higher modes are as follows:

(1) Divide the beam into equal stations. For convenience in applying the integration formula (equation (7)), which is based on a fifth-degree curve, the number of stations should be a multiple of five. In order to retain the same degree of accuracy for each higher mode, the number of stations must be increased when successively higher modes are being determined. For the first four modes, however, ten stations are sufficient for engineering accuracy if the iterations are performed by use of the method of equivalent loads. Determine the mass and moment-of-inertia values at each station.

(2) Sketch the most probable shape of the higher mode to be determined and read off the values of deflection at each station. (Each successive higher mode has one additional nodal point.)

(3) Remove all lower-mode components from the assumed higher-mode deflection.

(4) Perform an iteration with proper regard to boundary conditions and remove lower-mode components from the derived deflection.

(5) Successively repeat step 4 until the desired degree of convergence is obtained. It may be found that all the lower-mode components need not be removed after each iteration. (A more complete explanation as to when lower modes ought to be removed is presented in the example on the determination of the third mode of a cantilever.) Before and after the final iteration all the lower modes should be removed.

(6) Compute frequency by equation (11).

EXAMPLES

The solution to a number of typical problems is illustrated by examples. These examples are presented to serve as a guide when actual problems of the type they represent are being solved and each one need not be studied to understand the basic procedure given in this paper. The examples are:

Example 1 – Nonuniform cantilever, first bending mode

Example 2 – Nonuniform cantilever, first torsional mode

Example 3 – Uniform cantilever, second bending mode

Example 4 – Uniform cantilever, third bending mode

Example 5 – Free-free beam with concentrated masses, first symmetrical bending mode

Example 6 – Free-free beam coupled to masses through springs, first symmetrical bending mode

Example 7 – Free-free beam with concentrated masses, first antisymmetrical bending mode

Example 8 – Beam with concentrated mass and mounted on spring, first bending mode

Example 9 – Beam with concentrated mass and mounted on spring, second bending mode

Examples 1 and 2 are simple illustrations of the iteration process for bending and for torsion. Examples 3 and 4 show the removal of lower-mode components in the determination of higher modes. A comparison is given with the exact solution for modes and frequencies. Examples 5 to 7 illustrate the manner in which boundary conditions are satisfied for the vibration of free-free beams. Boundary conditions for beams coupled to springs are illustrated in examples 8 and 9 and the results are compared with exact solutions. In every example one or two complete iterations are shown in detail, and results of any other iterations necessary for convergence are indicated.

Example 1: Nonuniform cantilever, first bending mode.- The physical constants and first iteration for example 1 are given in table 1 and the solution is completed in table 6. Each step follows very closely the procedure outlined for fundamental-mode determination. It should be noted that convergence to the fundamental mode in cantilever beams is very rapid. After only two iterations the ratio was taken at each station of two successive deflections, column 12, and it is seen to be fairly constant. A comparison is given in figure 9 between the assumed deflection and the calculated fundamental mode.

Example 2: Nonuniform cantilever, first torsional mode.- An example of beam torsional vibration is presented in table 7. The first iteration is the one used to illustrate the iteration process for torsion (table 2). It is seen in columns 2, 7, 9, 11, and 14 in table 7 that convergence to the fundamental torsional mode for a cantilever is slower than for the corresponding mode in bending. The rate of convergence in the iteration method is a function of the separation of the frequencies of the natural modes and, for a cantilever, the frequencies of the torsional modes are not as widely separated as the frequencies of the bending modes. The greater number of iterations needed in this particular example can also be attributed in part to the large difference in the shapes of the assumed mode and derived mode. (See fig. 10.) The labor involved, however, in finding a torsional mode is comparable to that in bending since only two integrations are needed for an iteration.

Example 3: Uniform cantilever, second bending mode.- A uniform-beam case is presented in table 8 so that a comparison can be made between the exact solution and the results obtained by the iteration method. The procedure used is that outlined for higher modes. Previously computed ordinates for the fundamental mode are listed in column 2 and the assumed second-mode ordinates are given in column 5. Because the beam is uniform, the solution is somewhat simplified as the mass m and moment of inertia I can be ignored during the iterations and are taken into account only in the frequency computation. By comparison of columns 35 and 38 it can be seen that the mode obtained after two iterations is good enough for most practical purposes. If modes higher than the second are to be calculated, however, more accurate values should be obtained for the deflection of the second mode than those given in column 35. The results of the third iteration are shown in the sketch of figure 11(a), and they present a close check on the mode and frequency obtained by an exact analysis. The denominator of equation (10) is evaluated in column 40 for use in the third-mode determination, presented in the next section. The fact that column 42 sums to zero means that the derived second mode is orthogonal to the fundamental mode.

Example 4: Uniform cantilever, third bending mode.- The third mode and frequency computations for the same uniform beam are presented in table 9. The assumed third mode is given in column 2 and the fundamental mode and derived second mode from table 8 are used in computing columns 3 and 5. After each iteration a large fundamental-mode component must be removed from the derived deflection. It is difficult to remove completely all traces of lower modes, and during each iteration the residual lower-mode components are amplified; in this case in the ratios $(\omega_3/\omega_1)^2$ for the first mode and $(\omega_3/\omega_2)^2$ for the second mode. For a cantilever the ratio $(\omega_3/\omega_1)^2$ is very large and after one iteration the first-mode impurity has been greatly amplified (column 23). The actual amount of amplification of the second-mode impurity (column 24) is small enough to be negligible. Some labor could have been saved in this example by postponing the removal of the second-mode component until after the second iteration. After three iterations and removal of the lower modes, the derived third mode (column 46) and frequency as computed by equation (11) give an indication of the high accuracy attainable by the procedure described herein when the beam is divided into ten stations. Figure 11(b) shows the assumed third mode, the computed mode, the exact mode, and a comparison of the exact and computed frequency.

Example 5: Free-free beam with concentrated masses, first symmetrical bending mode.- In table 10, the first symmetrical bending mode and frequency of a uniform free-free beam carrying concentrated masses are computed. A convenient assumption for the deflection is one with a zero center-line ordinate (column 5). In order to insure that the shear be zero at the center line, the assumed deflection is then corrected according to equation 6, the use of which has been illustrated in table 3. With the computation starting at the corrected deflection, given in terms of a unit tip deflection and multiplied by the constant mass of unity (column 8), iteration is performed in a straightforward manner. The new deflection (column 15) is again given with a zero center-line ordinate and corrected (column 19) by use of equation 6. Two iterations are sufficient to determine the mode and frequency. The computed deflected shape is shown in figure 12.

Example 6: Free-free beam coupled to masses through springs, first symmetrical bending mode.- Example 6 (table 11) is the same as example 5 except that the concentrated masses are elastically connected to the beam by springs. The equation used to make the assumed loading (column 5) satisfy the shear boundary conditions is given in figure 13 and is developed in the appendix. This equation contains terms involving the unknown coupled frequency, which necessitates an assumption for this frequency before each iteration. Unless the natural frequency p_j^2 of the spring-mass system nearly coincides with the unknown frequency ω^2 of the coupled system, the equation for Y_o is relatively insensitive to incorrect assumptions for ω^2. In the general case, a more accurate assumption is made for ω^2 before each succeeding iteration on the basis of a frequency computed from the preceding iteration.

The system under consideration in this example involves a relatively stiff restraint between the masses and the beam. The unknown frequency will be only slightly less than the frequency obtained in example 5 which considers the mass rigidly connected to the beam. In that example (see table 10) the square of the frequency was found to be 11.6 (radians/sec)^2; consequently, an assumption of 11 (radians/sec)^2 was made for this case. The initial assumption for the deflection is the deflected shape obtained in the last iteration in table 10. Once the correct inertia loading is computed (table 11, column 8) iteration proceeds as before. Since the assumed deflected shape and frequency were very close to their actual values, one iteration results in a uniform ratio at each station (column 20) between the deflection and the assumed deflected shape. The computed deflection is shown as the dashed curve in figure 12. Elastic coupling between the beam and masses is seen to reduce the frequency of the entire system and to cut down the deflection of the center of the beam relative to the ends. In table 11 a computation is also presented for the force in the spring for the given beam-tip deflection.

Example 7: Free-free beam with concentrated masses, first antisymmetrical bending mode.- Table 12 illustrates the solution for the antisymmetrical modes of the same free-free beam analyzed in example 5. A deflection is assumed (column 7) and by means of equation (9) this deflection is corrected to satisfy the moment boundary conditions. This procedure has been illustrated in table 4. With the correct loading (column 12) the integrations are straightforward until the slope (column 19) is to be determined. The point of zero slope is unknown and is assumed to be between stations 3 and 4. Integration of the assumed slope diagram results in incorrect deflections (column 20). A true deflection is computed as before, however, which satisfies the moment boundary conditions and automatically has the correct slope at each point. The frequency was computed after the second iteration. The computed deflected shape is shown in figure 14.

Example 8: Beam mounted on spring, first bending mode.- A first mode solution for the problem of a beam vibrating on a spring is given in table 13. The boundary conditions are the same as for the symmetrical free-free beam modes except that the unbalanced inertia loading on the beam must equal the spring reaction. This condition is satisfied when the distance Y_o in figure 15 is computed by the equation given in the figure which is developed in the appendix. In order to solve this equation an estimation of the probable value of ω^2 is first necessary. The following considerations will aid in making this estimate. If the spring is infinitely stiff, the lowest mode and frequency is the fundamental for the half beam vibrating as a cantilever. If the spring has a finite stiffness, however, one natural frequency of the system will be lower than the cantilever frequency. The more flexible the spring is relative to the beam stiffness, the lower the frequency of the first bending mode will be relative to the cantilever frequency. The limiting case is that in which the beam is considered to have infinite stiffness and corresponds to an inelastic mass vibrating on a spring. Call this frequency of vibration p_j^2 which is given as $p_j^2 = \frac{\alpha_j}{mL}$. The lowest natural bending frequency ω of the system will always be lower than p_j.

The example worked in table 13 is for the case of a flexible beam carrying a heavy concentrated mass and mounted on a relatively stiff spring. The cantilever and spring-mass frequencies are first calculated in order to form a basis for estimating the probable natural frequency. Since p_j^2 is several times larger than ${\omega_1}^2_{cant}$, a natural frequency only slightly lower than the cantilever frequency was assumed. After $Y^{(0)}$ is calculated by use of the formula given, the succeeding steps of an iteration are the same as those for the symmetrical modes of a free-free beam. The trial frequencies calculated in column 17 show that the estimated frequency of 5.9 used in calculating $Y_1^{(1)}$ is as accurate an assumption as could have been made. After the second iteration a few trial calculations for ${\omega_n}^2$ (column 28) showed that an estimated frequency of 5.93 gave the proper values of $Y_1^{(2)}$. A third iteration is indicated with the resulting frequency and mode given in columns 31 and 32. In figure 16 the deflection is presented with a comparison between the exact and computed frequencies.

<u>Example 9: Beam mounted on spring, second bending mode</u>.- A second-mode solution for the same beam-mass-spring system is presented in table 14. The procedure used is that for higher-mode determination. The most reasonable assumption for the deflection (column 5) is a curve having a unit deflection at the tip and center. The first-mode component present in the assumed curve is calculated and removed and one iteration performed. The deflection (column 16) is given in terms of a zero center-line ordinate. This ordinate must be computed by use of the equation which was used in the first-mode determination.

In order to solve this equation, the frequency must first be estimated. A reasonably acceptable value may be obtained by equating the total inertia force $\left(\omega^2 \int m y_2^{(0)}\, dx\right)$ of the assumed loading to the spring force (center deflection times spring constant) and solving for ω^2. This calculation is shown in table 14. The accuracy of ω^2 depends, of course, on the closeness of the assumed curve to the actual deflection. By use of this value of ω^2, the equation for Y_o in figure 15 is solved for $Y_o^{(1)}$. The value of $Y_o^{(1)}$ is subtracted from the value of $Y_2{'}^{(1)}$ (column 19) and the first-mode component is again removed and another iteration performed. In each succeeding iteration a better value of ω^2 is found by considering the center-line shear boundary condition and finally ω^2 converges to the frequency as computed by equation (11). In this example, three iterations are sufficient for convergence. The determined mode and frequency are presented in figure 16. A comparison is given between the exact and computed frequency.

CONCLUDING REMARKS

The examples presented have shown the manner in which the determination of vibration modes and frequencies is related to beam-deflection theory. By successive approximation, solutions are readily found to an accuracy consistent with that to which the physical constants in built-up beams are ordinarily known. Although exact solutions, for practical purposes, are obtained to the differential equations, the equations themselves are based on limiting assumptions. Since the equations do not include the effects of structural damping, rotary inertia, and deflection due to shear, which are known to cause a change in the shapes and frequencies of the higher modes, engineering judgement must be used to interpret solutions obtained from these equations.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., July 23, 1947

APPENDIX

THEORETICAL DERIVATIONS

Determination of the Deflection by the Iteration Method

Differential equation of equilibrium.- The differential equation of equilibrium of a beam in free harmonic bending vibration is

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = \omega^2 my \tag{A1}$$

The left-hand side of the equation represents the elastic restoring forces of the beam and in the ordinary bending theory of beams equals the external loading. For beams in vibration the loading is composed of inertia forces which at any instant of time may be considered as static forces given by the expression on the right-hand side of the equation. Equation (A1) has solutions $y = y_n$ and associated characteristic values $\omega = \omega_n$, where y_n describes the deflection of the nth mode of vibration and ω_n is the frequency of vibration of that mode. Except for some simple cases these solutions cannot be obtained by exact analysis. It is possible, however, to solve equation (A1) by an iteration process in which successive approximations are found, each approximation more closely representing the true solution.

Integration of differential equation.- Suppose that some arbitrary function Y is substituted for y on the right-hand side of equation (A1). Except for the factor ω^2, which may be ignored because Y is purely arbitrary, the equation would be

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = mY \tag{A2}$$

This equation can be solved by direct integration. Assume for the moment that the solution is $y = F(x)$. From the static loading concept indicated in the previous section it follows that a beam with a loading $\omega_n{}^2my_n$ would have a deflection y_n. A beam loaded with forces my_n would therefore have a deflection $y_n/\omega_n{}^2$. Thus, if Y were an exact solution - say y_n - of equation (A1), the solution of equation (A2) would be simply

$$Y = F(x) = \frac{y_n}{\omega_n^2} \tag{A3}$$

In general, Y is not the exact solution y_n, and the proportionality given by equation (A3) would not be constant along the beam. Because the process of finding a new curve from a given curve is a converging process, however, curves giving a constant proportionality may be obtained even though the starting curve is approximate. The newly found curve is used to determine the loading, and another deflection is computed. This operation is repeated until two successively determined deflection curves are of constant ratio to each other. The final curve found is the fundamental or first mode $(n = 1)$ of vibration of the system. This process of finding the curve is commonly known as the Stodola method. The fact that the process is converging and converges to the lowest mode will be shown in the next section. In the section "Frequency Determination" the way in which the method is made to converge to a given higher mode of vibration is shown.

<u>Proof of convergence of the iteration process</u>.— The arbitrarily chosen curve Y may be expressed in terms of a series involving the normal modes of vibration of the system, thus

$$Y^{(0)}(x) = a_1y_1(x) + a_2y_2(x) + a_3y_3(x) + \ldots \tag{A4}$$

If this series is substituted for Y on the right-hand side of equation (A2) and integration is performed, then, by the principle of superposition, the solution will be

$$Y^{(1)}(x) = \frac{a_1}{\omega_1^2}y_1(x) + \frac{a_2}{\omega_2^2}y_2(x) + \frac{a_3}{\omega_3^2}y_3(x) + \ldots \tag{A5}$$

If the process is repeated successively with the newly found curve as the approximation to the deflection, the curve found after i iterations will be

$$Y^{(i)}(x) = \frac{a_1}{\omega_1^{2i}}y_1(x) + \frac{a_2}{\omega_2^{2i}}y_2(x) + \frac{a_3}{\omega_3^{2i}}y_3(x) + \ldots \tag{A6}$$

This equation may be written in the following form

$$Y^{(i)}(x) = \frac{1}{\omega_1^{2i}}\left[a_1y_1(x) + \left(\frac{\omega_1}{\omega_2}\right)^{2i}a_2y_2(x) + \left(\frac{\omega_1}{\omega_3}\right)^{2i}a_3y_3(x) + \ldots\right] \quad \text{(A7)}$$

Since $\omega_1 < \omega_2$, $\omega_1 < \omega_3$, . . ., increasing i causes the higher-mode components to decrease, and the first mode y_1 appears more and more pure. The greater the separation of ω_1, ω_2, ω_3, . . ., the stronger will be the convergence to y_1.

Removal of Lower-Mode Components

The iterative process will always cause convergence to the lowest mode present in the originally assumed deflection. In equation (A7) $\left(\frac{\omega_1}{\omega_{n+1}}\right)^{2i} < \left(\frac{\omega_1}{\omega_n}\right)^{2i} < \left(\frac{\omega_1}{\omega_{n-1}}\right)^{2i}$. Not only, therefore, is the component of each higher mode reduced during an iteration but also the amount of reduction becomes greater with each successive higher mode. Thus, if the components of all the modes lower than the nth mode were to be removed completely from a given deflection curve, the solution would necessarily have to converge to the nth mode (the lowest mode remaining). Provided that the shape of a given mode is known, the component of that mode in a given curve may be found conveniently by use of the orthogonality of the normal modes of vibration. Suppose that the given curve is expressed by equation (A4) and that y_n is the mode for which the component is being determined. Multiplication through by $my_n(x)$ gives

$$my_n(x)Y^{(0)}(x) = a_1my_n(x)y_1(x) + a_2my_n(x)y_2(x) + \ldots a_nmy_n^2(x) + \ldots \quad \text{(A8)}$$

Integration over the length of the beam gives the relation

$$\int_0^L my_n(x)Y^{(0)}(x)\,dx = \int_0^L a_nmy_n^2(x)\,dx \quad \text{(A9)}$$

since the y_n-terms are orthogonal functions defined by the orthogonality condition

$$\int_0^L m y_m(x) y_n(x)\, dx = 0 \quad \text{when} \quad m \neq n \tag{A10}$$

In the integrals to follow, the limits of integrations have been omitted but should be interpreted to be over the half span for all beams symmetrical about a center line and over the full span in unsymmetrical beams. Solution of equation (A9) for a_n gives

$$a_n = \frac{\int m y_n(x) Y^{(0)}(x)\, dx}{\int m y_n^2(x)\, dx} \tag{A11}$$

With a_n known, the nth mode component $a_n y_n(x)$ may be readily removed from the given curve by subtraction, thus

$$Y^{(0)}(x) - a_n y_n(x) = a_1 y_1(x) + a_2 y_2(x) + \ldots$$

$$+ a_{n-1}\, y_{n-1}(x) + a_{n+1}\, y_{n+1}(x) + \ldots \tag{A12}$$

Treatment of Free-Free Beams

In the case of symmetrical vibrations of free-free beams the first normal mode (pure translation, frequency equal to zero) is not an elastic mode but must be included when an arbitrary curve is developed in a series involving the normal modes. Thus, the series for an arbitrarily chosen deflection $Y'(x)$ is

$$Y'(x) = Y_o + a_1 y_1(x) + a_2 y_2(x) + a_3 y_3(x) + \ldots \tag{A13}$$

where Y_o is simply a constant. Since it is desirable to have the chosen curve in terms of the elastic modes alone, the value of Y_o must be determined. In order to prevent translation of the beam, the inertia loading in each symmetrical mode of vibration must sum to zero over the span. The following relation must therefore be true:

$$\int m y_n(x)\, dx = 0 \tag{A14}$$

Multiplication of equation (A13) through by m and integrating over the length of the beam by use of equation (A14) yields

$$\int mY'(x)\, dx = \int mY_o\, dx \tag{A15}$$

which when solved for Y_o gives

$$Y_o = \frac{\int mY'(x)\, dx}{\int m\, dx} \tag{A16}$$

Y_o may now be subtracted from the assumed deflection, thus

$$Y(x) = Y'(x) - Y_o = a_1y_1(x) + a_2y_2(x) + a_3y_3(x) + \ldots \tag{A17}$$

The resulting deflection is now given in terms of the normal elastic modes of a free-free beam, satisfies the equilibrium condition (equation (A15)), and may be substituted in the right-hand side of equation (A2). The expression may then be integrated without difficulty.

For convenience in the present solution for symmetrical modes of free-free beams, the assumed deflection $Y'(x)$ is always given in terms of a zero center-line ordinate. With this assumption, Y_o then represents the center-line deflection.

In the case of antisymmetrical vibrations of free-free beams, the first normal mode is a pure rotation with frequency equal to zero. An arbitrarily assumed curve, representing the deflection of an antisymmetrical mode and developed in a series involving the normal modes, must include a term corresponding to deflection due to rotation; thus,

$$Y'(x) = Kx + a_1y_1(x) + a_2y_2(x) + a_3y_3(x) + \ldots \tag{A18}$$

where K is a slope and Kx therefore is a linearly varying deflection. In order to prevent rotation, the moment of the inertia loading about the midpoint of the beam must equal zero. Thus,

$$\int xmy_n(x)\ dx = 0 \tag{A19}$$

Multiplication of equation (A18) through by xm and integrating over half the length of the beam by use of equation (A19) yields

$$\int xmY'(x)\ dx = \int Kx^2m\ dx \tag{A20}$$

which, solved for K, gives

$$K = \frac{\int xmY'(x)\ dx}{\int x^2m\ dx} \tag{A21}$$

After this value of K is multiplied by x, Kx may be subtracted from the assumed deflection; thus,

$$Y(x) = Y'(x) - Kx = a_1y_1(x) + a_2y_2(x) + a_3y_3(x) + \ldots \tag{A22}$$

This expression may now be substituted in the right-hand side of equation (A2) and integrated without difficulty.

Treatment for Beams Coupled to Springs

In order to analyze structures which contain an elastic restraint, such as a spring, the equilibrium of forces existing between the structure and the restraint must first be considered. For the case shown in figure 15, a deflection cannot be assumed directly because the equilibrium condition (spring force equals unbalanced shear at center line) may not be satisfied. On assuming the variable part $Y'(x)$ of the deflection, however, the constant part Y_o may be determined by equating the total inertia load to the force in the spring; thus,

$$\omega^2 \int mY(x)\ dx = -\alpha Y_o \tag{A23}$$

Since $Y(x)$ has been broken up into the variable part $Y'(x)$ and the constant Y_o, equation (A23) may be written

$$\omega^2 \int m\left[Y'(x) - Y_o\right] dx = -\alpha Y_o \tag{A24}$$

Solution for Y_o gives

$$Y_o = \frac{\int mY'(x)\, dx}{-\frac{\alpha}{\omega^2} + \int m\, dx} \tag{A25}$$

The generalization of equation (A25) for a spring placed at the jth station along the beam is

$$Y_o = \frac{\int mY'(x)\, dx - \frac{\delta_j \alpha_j}{\omega^2}}{-\frac{\alpha_j}{\omega^2} + \int m\, dx} \tag{A26}$$

where δ_j is the deflection of the beam at the jth station relative to the center of the beam and α_j is the elastic constant of the spring at the jth station. Subtraction of Y_o from $Y'(x)$ then gives

$$Y(x) = Y'(x) - Y_o = a_1 y_1(x) + a_2 y_2(x) + a_3 y_3(x) \tag{A27}$$

which is the arbitrarily assumed curve corrected to satisfy the shear boundary conditions.

For the case of a free-free beam coupled to spring-mass oscillators (fig. 13), the total inertia load must again equal the force in the spring. At any instant of time

$$\omega^2 \int mY(x)\, dx + F = 0 \tag{A28}$$

where F is the spring force. For convenience, the equilibrium of forces at maximum displacement will be considered. If equation (30) in reference 1 (p. 59) is multiplied through by $\alpha_j a_o$ and then $\alpha_j y_o$ is replaced by F, a_o by $(-Y_o + \delta_j)$, and ω_n by p_j, the following expression is obtained for the spring force F:

$$F = \frac{\left(\frac{\omega}{p_j}\right)^2 \alpha_j \left(-Y_o + \delta_j\right)}{1 - \left(\frac{\omega}{p_j}\right)^2} \tag{A29}$$

Since the natural frequency p_j of the simple oscillator is given by

$$p_j^2 = \frac{\alpha_j}{m_j} \tag{A30}$$

equation (A29) may be written

$$F = \frac{m_j \left(-Y_o + \delta_j\right)}{1 - \left(\frac{\omega}{p_j}\right)^2} \omega^2 \tag{A31}$$

Substituting this value of F in equation (A28) and writing Y(x) in terms of Y'(x) and Y_o gives the expression

$$\omega^2 \int m\left[Y'(x) - Y_o\right] dx + \frac{m_j\left(-Y_o + \delta_j\right)}{1 - \left(\frac{\omega}{p_j}\right)^2} \omega^2 = 0 \tag{A32}$$

which can be solved for Y_o

$$Y_o = \frac{\int mY' \, dx + \dfrac{m_j \delta_j}{1 - \left(\frac{\omega}{p_j}\right)^2}}{\int m \, dx + \dfrac{m_j}{1 - \left(\frac{\omega}{p_j}\right)^2}} \tag{A33}$$

The generalization of equation (A33) for any number of elastically mounted masses symmetrically placed at j points along the beam is

$$Y_o = \frac{\int mY'(x)\,dx + \sum_j \frac{m_j\delta_j}{1-\left(\frac{\omega}{p_j}\right)^2}}{\int m\,dx + \sum_j \frac{m_j}{1-\left(\frac{\omega}{p_j}\right)^2}} \tag{A34}$$

Subtracting the value Y_o from the variable deflection $Y'(x)$ gives

$$Y(x) = Y'(x) - Y_o = a_1y_1(x) + a_2y_2(x) + a_3y_3(x) + \ldots \tag{A35}$$

which represents an arbitrarily assumed deflection of the beam corrected to satisfy the equilibrium condition between beam and spring.

Frequency Determination

In equation (A7) all terms except the first become negligible when i is large enough, and the equation reduces simply to

$$Y^{(i)}(x) = \frac{1}{\omega_1^{2i}}\,a_1y_1(x) \tag{A36}$$

If one more iteration had been performed the equation would have reduced to

$$Y^{(i+1)}(x) = \frac{1}{\omega_1^{2(i+1)}}\,a_1y_1(x) \tag{A37}$$

Division of equation (A36) by equation (A37) gives for the square of the frequency

$$\omega_1{}^2 = \frac{Y^{(i)}}{Y^{(i+1)}} \tag{A38}$$

Thus, after enough iterations have been performed to cause the higher-mode components to become small, the ratio of two successively found deflections is the square of the natural frequency of the fundamental mode of vibration.

Although equation (A38) denotes the frequency of the fundamental mode, a similar equation can be derived for the higher modes. Suppose that the nth mode is being determined and that i iterations with the necessary removal of lower-mode components have been performed; then, with the removal of all modes lower than the nth mode after the ith iteration, the resulting curves may be expressed by the equation

$$y_n^{(i)}(x) = \epsilon_1 y_1(x) + \epsilon_2 y_2(x) + \ldots a_n y_n(x)$$

$$+ \ldots \mu_p y_p(x) + \ldots \tag{A39}$$

The small coefficients ϵ_1, ϵ_2, up to a_n remain because in the numerical procedure each mode lower than the nth mode cannot be removed precisely. These values, however, are very small in comparison with values of a_n. Iterating once again, beginning with equation (A39), gives the curve

$$Y_n^{(i+1)}(x) = \frac{\epsilon_1}{\omega_1{}^2} y_1(x) + \frac{\epsilon_2}{\omega_2{}^2} y_2(x) + \ldots \frac{a_n}{\omega_n{}^2} y_n(x)$$

$$+ \ldots \frac{\mu_p}{\omega_p{}^2} y_p(x) + \ldots \tag{A40}$$

If $\frac{1}{\omega_n{}^2}$ is factored out of the right-hand side of equation (A40), it is seen that, in comparison with a_n, the first-mode component has been amplified by the factor $\left(\frac{\omega_n}{\omega_1}\right)^2$, the second mode by the factor

$\left(\frac{\omega_n}{\omega_2}\right)^2$, and so forth. Removal again of all modes lower than the nth mode gives then the deflections

$$y_n^{(i+1)}(x) = \eta_1 y_1(x) + \eta_2 y_2(x) + \ldots \frac{a_n}{\omega_n^2} y_n(x)$$

$$+ \ldots \frac{\mu_p}{\omega_p^2} y_p(x) + \ldots \tag{A41}$$

where η_1, η_2 are extremely small in comparison with $\frac{a_n}{\omega_n^2}$. Division of equation (A39) by equation (A41) and neglecting all terms in ϵ, μ, and η gives, as a close approximation to the square of the frequency, the relation

$$\omega_n^2 = \frac{y_n^{(i)}}{y_n^{(i+1)}} \tag{A42}$$

Alternate Method of Frequency Determination

Another method for determining the frequency can be found from the energy expressions for a beam in vibration. It can be shown that the potential energy of bending U for a beam in its maximum displaced position is given by the expression

$$U = \frac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx \tag{A43}$$

and the kinetic energy of the beam V as it passes through the equilibrium position is

$$V = \frac{1}{2}\int \omega^2 m y^2 \, dx \tag{A44}$$

These energies must be equal, hence

$$\frac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{1}{2}\int \omega^2 m y^2 \, dx \tag{A45}$$

Solution of this equation for ω^2 gives

$$\omega^2 = \frac{\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int m y^2 \, dx} \tag{A46}$$

With the use of the relation $M = EI\frac{d^2y}{dx^2}$, equation (A46) may be written

$$\omega^2 = \frac{\int \frac{M^2}{EI} dx}{\int my^2 \, dx} \tag{A47}$$

As applied to the iteration process, the moment M in equation (A47) is associated with the deflection that is derived. That is, y in equation (A47) corresponds to $y^{(i+1)}$, the deflection derived in an iteration, where $y^{(i)}$ is the given deflection or the deflection used at the start of the iteration. By use of equation (A42), the frequency may be written in terms of $y^{(i)}$ and the moment rather than $y^{(i+1)}$ and the moment.

Substitution of the expression for $y^{(i+1)}$ from equation (A42) for y in equation (A47) and solving for ω_n^2 gives

$$\omega_n^2 = \frac{\int m\left[y_n^{(i)}\right]^2 dx}{\int \frac{M^2}{EI} dx} \tag{A48}$$

Thus, after a reasonably good approximation to the deflection of a mode has been established, the moment which results from a loading computed from this deflection is found, and by use of equation (A48) a good value of the frequency of that mode can be determined.

Equation (A42) expresses the frequency in terms of the successively found ordinates at any station along the beam. Equation (A46) can be transformed so that the frequency is expressed in terms of all the ordinates along the beam. The transformation is given herein as a matter of interest. On integration of the numerator by parts, equation (A46) would appear

$$\omega^2 = \frac{\int Ey\frac{d^2}{dx^2}\left[I\frac{d^2y}{dx^2}\right] dx}{\int my^2 \, dx} \tag{A49}$$

For the (i + 1) deflection this equation would be

$$\omega^2 = \frac{\int Ey^{(i+1)} \frac{d^2}{dx^2}\left[I \frac{d^2y^{(i+1)}}{dx^2}\right] dx}{\int m\left[y^{(i+1)}\right]^2 dx} \tag{A50}$$

If $y^{(i+1)}$ were the exact deflection, it would necessarily have to satisfy equation (A1); thus,

$$\frac{d^2}{dx^2}\left[EI(x) \frac{d^2y^{(i+1)}}{dx^2}\right] = \omega^2 m(x)y^{(i+1)} \tag{A51}$$

With the use of equation (A42), equation (A51) may be written

$$\frac{d^2}{dx^2}\left[EI(x) \frac{d^2y^{(i+1)}}{dx^2}\right] = m(x)y^{(i)} \tag{A52}$$

Substitution of equation (A52) in equation (A50) then gives for the frequency the relation

$$\omega^2 = \frac{\int my^{(i)}y^{(i+1)} dx}{\int m\left[y^{(i+1)}\right]^2 dx} \tag{A53}$$

This equation gives, in effect, a weighted average of the frequencies given by equation (A42) for all the points on the beam. The accuracy of the frequency obtained from equation (A53) will in general be greater than that obtained from equation (A42). Any local error in deflection will cause no appreciable error in the frequency as given by equation (A53); whereas an error in the deflection of a point will cause a like error in the frequency if determined by equation (A42) with the deflections at that point.

Iteration as Applied to Torsional Problems

For torsional problems the differential equation of equilibrium is

$$-\frac{d}{dx}\left[GJ\frac{d\phi}{dx}\right] = \omega^2 I_p \phi \qquad \text{(A54)}$$

This equation is analogous to equation (A1) for bending vibrations. The left-hand side of the equation represents the elastic restoring forces and the right-hand side represents the inertia loading. A procedure similar to that used in the case of bending vibrations is used to solve equation (A54); that is, a curve representing the mode to be determined is assumed and is substituted on the right-hand side and, by direct integration, a new curve is found; and so on. Only two integrations need be performed in an iteration for torsional problems as compared with the four for bending problems because the differential equation of equilibrium in torsion is of the second order, whereas the equation in bending is of the fourth order. The proof of the process for obtaining a solution by iteration follows closely that given for bending vibrations and, therefore, no further discussion will be given.

Derivation of Equation Used in Numerical Evaluation of Areas

Suppose the area under a given curve is to be determined. In general, any part of the curve over a finite interval may be approximated by a second-, third-, or higher-degree curve; the accuracy of approximation, of course, increases with the degree of the curve. In the present analysis a fifth-degree curve has been used for convenience.

Consider the plot of a curve $y = f(x)$. Let $z = \frac{x}{\lambda}$ in order to obtain a dimensionless coordinate, and let the ordinates y at $z = 0, 1, 2, 3, 4, 5$ be a, b, c, d, e, f, respectively. It can be readily verified that the following equation represents a factored general fifth-degree equation having the required values $y = a, b, c, \ldots$ at $z = 0, 1, 2, \ldots$

$$y = -\frac{1}{120}a(z - 1)(z - 2)(z - 3)(z - 4)(z - 5) + \frac{1}{24}bz(z - 2)(z - 3)(z - 4)(z - 5)$$

$$- \frac{1}{12}cz(z - 1)(z - 3)(z - 4)(z - 5) + \frac{1}{12}dz(z - 1)(z - 2)(z - 4)(z - 5)$$

$$- \frac{1}{24}ez(z - 1)(z - 2)(z - 3)(z - 5) + \frac{1}{120}fz(z - 1)(z - 2)(z - 3)(z - 4) \qquad \text{(A55)}$$

The area under the curve in the interval $z = 0$ to $z = 5$ is found simply by integration of this equation; thus,

$$A = \int_0^{5\lambda} y\,dx = \lambda \int_0^5 y\,dz = \frac{125\lambda}{144}(0.38a + 1.50b + c + d + 1.50e + 0.38d) \qquad \text{(A56)}$$

By application of equation (A56) to adjacent sets of five intervals the complete area under any curve may be found. Usually ten intervals are adequate for most purposes; the area in terms of the eleven coordinates would then be

$$A = \frac{125\lambda}{144}(0.38a + 1.50b + c + d + 1.50e + 0.76f + 1.50g + h + i + 1.50j + 0.38k) \qquad \text{(A57)}$$

For most purposes an area may be evaluated most conveniently in tabular form. Columns 3 and 6 in table 3 are illustrations of such an evaluation.

REFERENCES

1. Den Hartog, J. P.: Mechanical Vibrations. Second ed., McGraw-Hill Book Co., Inc., 1940, p. 188.

2. Burgess, C. P.: The Frequencies of Cantilever Wings in Beam and Torsional Vibrations. NACA TN No. 746, 1940.

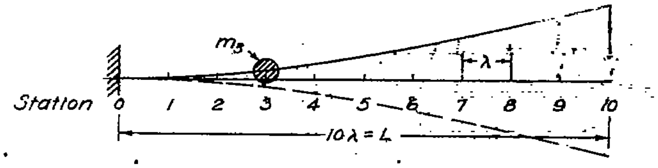
3. Boukidis, N. A., and Ruggiero, R. J.: An Iterative Method for Determining Dynamic Deflections and Frequencies. Jour. Aero. Sci., vol. 11, no. 4, Oct. 1944, pp. 319-328.

4. Beskin, L., and Rosenberg, R. M.: Higher Modes of Vibration by a Method of Sweeping. Jour. Aero. Sci., vol. 13, no. 11, Nov. 1946, pp. 597-604.

5. Newmark, N. M.: Numerical Procedure for Computing Deflections, Moments, and Buckling Loads. Paper No. 2202, Trans. A.S.C.E., vol. 108, 1943, pp. 1161-1188.

TABLE 1.— ILLUSTRATIVE EXAMPLES OF ITERATION PROCESS FOR A BEAM IN BENDING VIBRATION

[L = 100 in.; λ = 10 in.; E = 10,000,000 psi; m_3 = 3000 $\frac{\text{lb-sec}^2}{\text{in.}}$]



Summation method											
1	2	3	4	5	6	7	8	9	10	11	12
Station	I	m	$Y_1^{(0)}$	$mY_1^{(0)}$	$m_3\delta_3$	S	M	M/I	Slope	$Y_1^{(1)}$	$Y_1^{(1)}$
	Common factors →				λ	λ	$λ^2$	$λ^2$	$λ^3/E$	$λ^4/E$	
10	20	10	1.00	10.00		[b]5.01	0	0	25.50	173.9	1.000
9	22	12	.84	10.08		15.09	5.0	.227	25.27	148.4	.854
8	25	15	.68	10.20		25.29	20.1	.805	24.46	123.1	.708
7	31	18	.53	9.54		34.83	45.4	1.465	23.00	98.60	.567
6	38	22	.40	8.80		43.63	80.2	2.112	20.89	75.60	.435
5	46	26	.29	7.54		51.17	123.8	2.692	13.20	54.71	.315
4	56	30	.19	5.70		56.87	175.0	3.120	15.08	36.51	.210
3	66	35	.11	3.85	[a]33.0	93.72	231.9	3.515	11.56	21.43	.123
2	77	40	.05	2.00		95.72	325.6	4.230	7.33	9.87	.0568
1	88	45	.01	.45		96.17	421.3	4.790	[c]2.54	2.54	.0146
0	100	50	0	0			517.5	5.175		0	0

[a] $m_3\delta_3 = \frac{3000 \times 0.11}{\lambda} = 33.0$

[b] $5.01 = \frac{(3 \times 10.00) + 10.08}{8}$

[c] $2.54 = \frac{(3 \times 5.175) + 4.790}{8}$

Equivalent-load method											
1	2	3	4	5	6	7	8	9	10	11	12
Station	$Y_1^{(0)}$	$mY_1^{(0)}$	P_{eq}	P_{eq}	S	M	M/I	$(M/I)_{eq}$	Slope	$Y_1^{(1)}$	$Y_1^{(1)}$
Common factors →			λ/12	λ/12	λ/12	$λ^2/12$	$λ^2/12$	$λ^3/144$	$λ^3/144E$	$λ^4/144E$	
10	1.00	10.00	[a]60.1		60.1	0	0		3668	24978	1.000
9	.84	10.08	[b]121.0		181.1	60	2.73	37	3631	21310	.853
8	.68	10.20	121.6		302.7	241	9.65	116	3515	17679	.706
7	.53	9.54	114.4		417.1	544	17.52	210	3305	14164	.566
6	.40	8.80	105.1		522.2	961	25.30	303	3002	10859	.434
5	.29	7.54	89.9		612.1	1483	32.25	385	2617	7857	.314
4	.19	5.70	68.4		680.5	2095	37.42	448	2169	5240	.210
3	.11	3.85	46.2	[c]396	1122.7	2776	42.10	[d]513	1656	3071	.123
2	.05	2.00	24.3		1147.0	3899	50.60	605	1051	1415	.0566
1	.01	.45	6.5		1153.5	5046	57.40	687	364	364	.0146
0	0	0				6199	61.99	364		0	0

[a] $60.1 = \frac{(7 \times 10.00) + (6 \times 10.08) - 10.20}{2}$

[b] $121.0 = 10.00 + (10 \times 10.08) + 10.20$

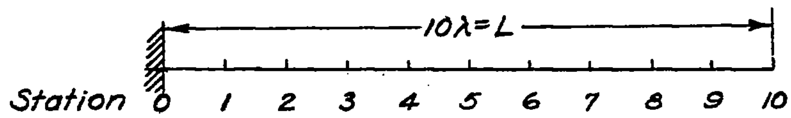
[c] $396 = \frac{12}{\lambda} \times 3000 \times 0.11$

[d] $513 = -\frac{57.40}{2} + (3 \times 50.60) + (7 \times 42.10) + (3 \times 37.42) - \frac{32.25}{2}$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 2.– ILLUSTRATIVE EXAMPLE OF ITERATION PROCESS FOR A CANTILEVER IN TORSIONAL VIBRATION

[L = 100 in.; λ = 10 in.; G = 4,000,000 psi]



1	2	3	4	5	6	7	8	9
Station	J	I_p	$\phi_1^{(0)}$	$I_p\phi_1^{(0)}$	T	T/J	$\phi_1^{(1)}$	$\phi_{1_m}^{(1)}$
Common factors →					λ	λ	λ^2/G	
10		120	1.00	120.0			8.905	1.000
	90				[a] 60.2	0.669		
9		135	.90	121.5			8.236	.925
	145				181.7	1.252		
8		163	.80	130.3			6.984	.785
	235				312.0	1.329		
7		202	.70	141.3			5.655	.635
	380				453.3	1.192		
6		259	.60	155.2			4.463	.501
	595				608.5	1.023		
5		334	.50	167.0			3.440	.386
	875				775.5	.886		
4		471	.40	188.2			2.554	.286
	1255				963.7	.768		
3		726	.30	218.0			1.786	.200
	1750				1181.7	.676		
2		1100	.20	220.0			1.110	.125
	2360				1401.7	.594		
1		1480	.10	148.0			.516	.058
	3000				1549.7	.516		
0		1850	0	0			0	0

[a] $60.2 = \frac{(3 \times 120) + 121.5}{8}$

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TABLE 3.– SATISFACTION OF BOUNDARY CONDITION FOR THE FIRST SYMMETRICAL MODE OF A FREE-FREE BEAM BY A SHORT-CUT CALCULATION

1	2	3	4	5	6	7	8
Station	m	Σ	$Y_1'(0)$	$mY_1'(0)$	Σ	$Y_1(0)$ (a)	$Y_1(0)$
Common factors →		$\frac{125\lambda}{144}$			$\frac{125\lambda}{144}$		
10	10	3.8	1.00	10.0	3.8	0.806	1.000
9	13	19.5	.84	10.9	16.4	.646	.801
8	16	16.0	.68	10.9	10.9	.486	.603
7	20	20.0	.53	10.6	10.6	.336	.416
6	25	37.5	.40	10.0	15.0	.206	.256
5	31	22.6	.29	9.0	6.8	.096	.119
4	38	57.0	.19	7.2	10.8	-.004	-.005
3	47	47.0	.11	5.2	5.2	-.084	-.104
2	59	59.0	.05	3.0	3.0	-.144	-.179
1	75	112.5	.01	.8	1.2	-.184	-.228
0	100	38.0	0	0	0	-.194	-.240
		432.9			83.7		

[a] $Y_1(0) = Y_1'(0) - Y_o(0)$ where $Y_o(0) = \frac{\Sigma mY_1'(0)}{\Sigma m}$

$$= \frac{83.7}{432.9} = 0.194$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 4.– SATISFACTION OF BOUNDARY CONDITIONS FOR THE FIRST ANTISYMMETRICAL MODE OF A FREE-FREE BEAM BY A SHORT-CUT CALCULATION

1	2	3	4	5	6	7	8	9	10	11
Station	m	x	mx^2	Σ	$Y_1'(0)$	$mxY_1'(0)$	Σ	$K^{(0)}x$	$Y_1^{(0)}$ (a)	$Y_1^{(0)}$
Common factors →		L	L^2	$\frac{125\lambda L^2}{144}$		L	$\frac{125\lambda L}{144}$			
10	10	1.0	10.00	3.80	1.00	10.00	3.80	-0.418	1.418	1.000
9	13	.9	10.52	15.80	.60	7.01	10.51	-.376	.976	.689
8	16	.8	10.23	10.23	.24	3.07	3.07	-.334	.574	.405
7	20	.7	9.80	9.80	-.10	-1.40	-1.40	-.293	.193	.136
6	25	.6	9.00	13.50	-.38	-5.70	-8.55	-.251	-.129	-.091
5	31	.5	7.75	5.89	-.62	-9.60	-7.29	-.209	-.411	-.290
4	38	.4	6.08	9.11	-.70	-10.62	-15.92	-.167	-.533	-.376
3	47	.3	4.23	4.23	-.60	-8.46	-8.46	-.125	-.475	-.335
2	59	.2	2.36	2.36	-.42	-4.96	-4.96	-.084	-.336	-.237
1	75	.1	.75	1.12	-.22	-1.65	-2.47	-.042	-.178	-.126
0	100	0	0	0	0	0	0	0	0	0
				75.84			-31.67			

[a] $Y_1^{(0)} = Y_1'^{(0)} - K^{(0)}x$ where $K^{(0)} = \frac{\Sigma mxY_1'^{(0)}}{L\ \Sigma mx^2} = \frac{-31.67}{75.84 \times 100} = -0.00418$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 5.– REMOVAL OF THE FUNDAMENTAL-MODE COMPONENT IN AN ASSUMED SECOND MODE FOR A CANTILEVER IN BENDING VIBRATION

1	2	3	4	5	6	7	8	9	10	11
Station	m	y_1	$m(y_1)^2$	Σ	$Y_2(0)$	$my_1Y_2(0)$	Σ	$a_1(0)y_1$	$y_2(0)$ (a)	$y_2(0)$
Common factors →				$\frac{125\lambda}{144}$			$\frac{125\lambda}{144}$			
10	1	1.000	1.000	0.380	1.00	1.000	0.380	−0.514	1.514	1.000
9	1	.847	.718	1.077	.50	.423	.634	−.435	.935	.617
8	2	.698	.974	.974	.06	.084	.084	−.358	.418	.276
7	3	.545	.891	.891	−.32	−.524	−.524	−.280	−.040	−.026
6	4	.409	.670	1.005	−.54	−.884	−1.325	−.210	−.330	−.218
5	5	.287	.412	.313	−.60	−.861	−.655	−.147	−.453	−.299
4	6	.180	.194	.291	−.52	−.561	−.841	−.092	−.428	−.282
3	7	.105	.077	.077	−.36	−.264	−.264	−.054	−.306	−.202
2	8	.044	.015	.015	−.18	−.063	−.063	−.023	−.157	−.104
1	9	.011	.001	.002	−.04	−.004	−.006	−.006	−.034	−.022
0	10	0	0	0	0	0	0	0	0	0
				5.025			−2.580			

[a] $y_2(0) = Y_2(0) - a_1(0)y_1$ where $a_1(0) = \frac{\Sigma m y_1 Y_2(0)}{\Sigma m(y_1)^2} = \frac{-2.580}{5.025} = -0.514$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 6.– NONUNIFORM CANTILEVER, FIRST BENDING MODE

1	2	3	4	5	6	7	8	9	10	11	12	13
Station	$Y_1^{(1)}$	$mY_1^{(1)}$	P_{eq}	P_{eq}	S	M	M/I	$(M/I)_{eq}$	Slope	$Y_1^{(2)}$	$Y_1^{(1)}/Y_1^{(2)}$	$Y_1^{(2)}$
Common factors →			$\lambda/12$	$\lambda/12$	$\lambda/12$	$\lambda^2/12$	$\lambda^2/12$	$\lambda^3/144$	$\lambda^3/144E$	$\lambda^4/144E$		
10	1.000	10.00	60.4			0	0			26221	5.49	1.000
					60.4				3833			
9	.853	10.23	122.9			60.4	2.74	37		22388	5.49	.853
					183.3				3796			
8	.706	10.59	126.3			243.7	9.74	118		18592	5.48	.709
					309.6				3678			
7	.566	10.19	123.0			553.3	17.83	214		14914	5.47	.569
					432.6				3464			
6	.434	9.55	113.8			985.9	25.95	311		11450	5.46	.436
					564.4				3153			
5	.314	8.16	97.5			1532.3	33.30	398		8297	5.45	.316
					643.9				2755			
4	.210	6.30	75.5			2176.2	38.85	466		5542	5.45	.211
					719.4				2289			
3	.123	4.31	51.7	443		2895.6	43.90	253 283		3253	5.45	.124
					1214.1				1753			
2	.0566	2.26	27.6			4109.7	53.40	639		1500	5.44	.0572
					1241.7				1114			
1	.0146	.66	8.9			5351.4	60.85	728		386	5.44	.0147
					1250.6				386			
0	0	0	0			6602.0	66.02	386		0		0

$\omega^2 = 5.45$ (radians/sec)^2

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 7.– NONUNIFORM CANTILEVER, FIRST TORSIONAL MODE

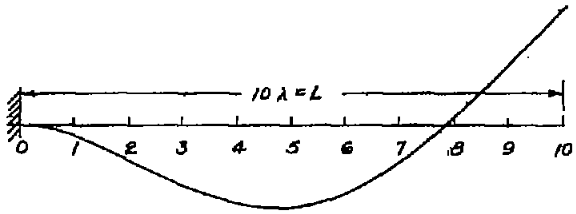
1	2	3	4	5	6	7	8	9	10	11	12	13	14
Station	$\phi_1(1)$	$I_p\phi_1(1)$	T	T/J	$\phi_1(2)$	$\phi_1(2)$	$\phi_1(3)$	$\phi_1(3)$	$\phi_1(4)$	$\phi_1(4)$	$\phi_1(5)$	$\phi_1(4)/\phi_1(5)$	$\phi_1(5)$
Common factors ⟶			λ	λ	λ^2/G		λ^2/G		λ^2/G		λ^2/G		
10	1.000	120.0	60.6	0.674	8.314	1.000	8.068	1.000	7.947	1.000	7.919	5050	1.000
9	.925	124.9	185.5	1.280	7.640	.919	7.395	.916	7.276	.915	7.248	5050	.915
8	.785	128.0	313.5	1.333	6.360	.765	6.123	.759	6.007	.755	5.979	5050	.755
7	.635	128.0	441.5	1.163	5.027	.605	4.805	.595	4.697	.591	4.671	5060	.590
6	.501	129.8	571.3	.962	3.864	.465	3.644	.451	3.570	.449	3.549	5060	.449
5	.386	129.0	700.3	.800	2.902	.349	2.715	.336	2.654	.334	2.638	5060	.333
4	.286	134.8	835.1	.666	2.102	.252	1.950	.242	1.903	.240	1.891	5070	.239
3	.200	145.2	980.3	.560	1.436	.173	1.323	.164	1.288	.162	1.280	5060	.161
2	.125	137.3	1117.6	.474	.876	.105	.802	.0995	.779	.0980	.774	5060	.0976
1	.058	85.9	1203.5	.402	.402	.048	.366	.0454	.355	.0446	.353	5050	.0445
0	0	0			0	0	0	0	0	0	0	--	0

$\omega^2 = 5050$ (radians/sec)^2

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 8.– SECOND BENDING MODE OF A UNIFORM CANTILEVER

$$\left[L = 100 \text{ in.};\quad \lambda = 10 \text{ in.};\quad m = 5\,\frac{\text{lb (sec)}^2}{\text{in.}^2};\quad I = 50 \text{ in.}^4;\quad E = 10{,}000{,}000 \text{ psi}\right]$$



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Station	y_1	$(y_1)^2$	Σ	$Y_2(0)$	$y_1Y_2(0)$	Σ	$a_1(0)y_1$	$y_2(0)$ (a)	$y_2(0)$	P_{eq}	S	M	M_{eq}	Slope
Common factors →		m	$\frac{125\lambda m}{144}$		m	$\frac{125\lambda m}{144}$				$\frac{\lambda m}{12}$	$\frac{\lambda m}{12}$	$\frac{\lambda^2 m}{12}$	$\frac{\lambda^3 m}{144EI}$	$\frac{\lambda^3 m}{144EI}$
10	1.000	1.000	0.380	1.00	1.000	0.3800	0.038	0.962	1.000	4.943		0		
											4.94			1296
9	.863	.744	1.116	.50	.431	.6460	.033	.467	.486	5.893		4.94	65.2	
											10.84			1231
8	.726	.526	.526	.06	.0436	.0436	.028	.032	.033	.460		15.78	189.8	
											11.30			1041
7	.591	.349	.349	–.32	–.189	–.1890	.023	–.343	–.356	–4.107		27.08	320.8	
											7.19			720
6	.461	.212	.318	–.54	–.249	–.3740	.018	–.558	–.580	–6.792		34.27	404.5	
											.40			316
5	.340	.116	.088	–.60	–.204	–.1550	.013	–.613	–.636	–7.490		34.67	408.5	
											–7.09			– 93
4	.230	.053	.079	–.52	–.1198	–.1798	.009	–.529	–.550	–6.515		27.58	324.4	
											–13.61			–417
3	.1365	.0186	.019	–.36	–.0491	–.0491	.005	–.365	–.379	–4.529		13.97	163.1	
											–18.14			–580
2	.0638	.0041	.004	–.18	–.0115	–.0115	.002	–.182	–.189	–2.311		–4.17	–52.3	
											–20.45			–528
1	.0168	.0003	0	–.04	–.0007	–.0010	.001	–.041	–.042	–.609		–24.62	–296.0	
											–21.06			–232
0	0	0	0	0	0	0	0	0	0	0		–45.68	–232.0	
			2.879			0.1102								

[a] $y_2(0) = Y_2(0) - a_1(0)y_1$ where $a_1(0) = \frac{0.1102}{2.879} = 0.0383$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 8.— SECOND BENDING MODE OF A UNIFORM CANTILEVER — Continued

$\left[L = 100 \text{ in}; \quad \lambda = 10 \text{ in.}; \quad m = 5 \frac{\text{lb (sec)}^2}{\text{in.}^2}; \quad I = 50 \text{ in.}^4; \quad E = 10{,}000{,}000 \text{ psi}\right]$

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Station	$Y_2(1)$	$y_1Y_2(1)$	Σ	$a_1(1)y_1$	$y_2(1)$	$y_2(1)$	F_{eq}	S	M	M_{eq}	Slope	$Y_2(2)$	$y_1Y_2(2)$	Σ	$a_1(2)y_1$
Common factors →	$\frac{\lambda^4 m}{144EI}$	$\frac{\lambda^4 m^2}{144EI}$	$\frac{125\lambda^5 m^2}{144^2EI}$	$\frac{\lambda^4 m}{144EI}$	$\frac{\lambda^4 m}{144EI}$		$\frac{\lambda m}{12}$	$\frac{\lambda m}{12}$	$\frac{\lambda^2 m}{12}$	$\frac{m\lambda^3}{144EI}$	$\frac{m\lambda^3}{144EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{\lambda^4 m^2}{144EI}$	$\frac{125\lambda^5 m^2}{144^2EI}$	$\frac{\lambda^4 m}{144EI}$
10	2754	2754	1047	82	2672	1.000	5.026		0			2816	2816	1070	-111
								5.03			1381				
9	1458	1258	1887	71	1387	.519	6.252		5.03	66.6		1435	1239	1859	-96
								11.28			1314				
8	227	165	165	60	167	.0625	.821		16.31	196.5		121	88	88	-80
								12.10			1117				
7	-814	-481	-481	49	-863	-.323	-3.757		28.41	337.2		-997	-589	-589	-66
								8.34			780				
6	-1534	-708	-1061	38	-1572	-.589	-6.916		36.75	434.1		-1777	-818	-1228	-51
								1.42			346				
5	-1850	-629	-478	28	-1878	-.703	-8.284		38.18	449.9		-2123	-722	-549	-38
								-6.86			-104				
4	-1757	-404	-605	19	-1776	-.665	-7.859		31.32	368.0		-2019	-464	-696	-25
								-14.72			-472				
3	-1340	-183	-183	11	-1351	-.506	-6.011		16.60	193.2		-1548	-211	-211	-15
								-20.73			-665				
2	-760	-48	-48	5	-765	-.286	-3.453		-4.13	-53.0		-883	-56	-56	-7
								-24.18			-612				
1	-232	-4	-6	1	-233	-.0872	-1.158		-28.31	-340.9		-271	-5	-7	-2
								-25.34			-271				
0	0	0	0	0	0	0	0		-53.65	-270.9		0	0	0	0
			237											-319	

$a_1(1) = \frac{237}{2.879} = 82.3$ $\qquad a_1(2) = \frac{-319}{2.879} = -111$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 8.– SECOND BENDING MODE OF A UNIFORM CANTILEVER – Concluded

$\left[L = 100 \text{ in.}; \quad \lambda = 10 \text{ in.}; \quad m = 5\ \frac{\text{lb (sec)}^2}{\text{in.}^2}; \quad I = 50 \text{ in.}^4; \quad E = 10{,}000{,}000 \text{ psi}\right]$

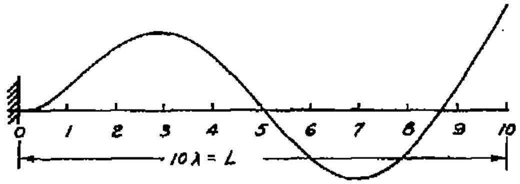
32	33	34	35	36	37	38	39	40	41	42
Station	$y_2(2)$	$y_2(1)/y_2(2)$	$y_2(2)$	$y_2(3)$	$y_2(2)/y_2(3)$ (b)	$y_2(3)$	$(y_2)^2$	Σ	y_1y_2	Σ
Common factors	$\frac{m\lambda^4}{144EI}$			$\frac{m\lambda^4}{144EI}$			m	$\frac{125\lambda m}{144}$	m	$\frac{125\lambda m}{144}$
10	2927	491	1.000	2959	486	1.000	1.0000	0.3800	1.000	0.380
9	1531	488	.523	1550	486	.523	.2740	.4110	.451	.677
8	201	448	.0686	207	478	.070	.0049	.0049	.051	.051
7	–931	500	–.318	–938	488	–.317	.1005	.1005	–.187	–.187
6	–1726	491	–.589	–1744	486	–.589	.3470	.5210	–.272	–.408
5	–2085	485	–.713	–2112	486	–.714	.5090	.3870	–.243	–.185
4	–1994	480	–.681	–2022	485	–.684	.4675	.7010	–.157	–.235
3	–1533	475	–.524	–1557	485	–.526	.2765	.2765	–.072	–.072
2	–876	470	–.299	–891	484	–.301	.0906	.0906	–.019	–.019
1	–269	467	–.0919	–274	484	–.0926	.0086	.0129	–.0016	–.002
0	0	– –	0	0	– –	0	0	0	0	0
								2.8854		0

b $\omega_2^2 = \frac{y_2(2)}{y_2(3)} \approx 486$ (radians/sec^2)

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 9.- THIRD BENDING MODE OF A UNIFORM CANTILEVER

$\left[L = 100 \text{ in};\quad \lambda = 10 \text{ in.};\quad m = 5\ \frac{\text{lb (sec)}^2}{\text{in.}^2};\quad I = 50 \text{ in.}^4;\quad E = 10{,}000{,}000 \text{ psi}\right]$



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Station	$Y_3^{(0)}$	$y_1Y_3^{(0)}$	Σ	$y_2Y_3^{(0)}$	Σ	$a_1^{(0)}y_1$	$a_2^{(0)}y_2$	column 7 + column 8	$y_3^{(0)}$ (a)	$y_3^{(0)}$	P_{eq}	S	M	M_{eq}	Slope
Common factors →		m	$\frac{125\lambda m}{144}$	m	$\frac{125\lambda m}{144}$						$\frac{\lambda m}{12}$	$\frac{\lambda m}{12}$	$\frac{\lambda^2 m}{12}$	$\frac{m\lambda^3}{144EI}$	$\frac{m\lambda^3}{144EI}$
10	1.00	1.0000	0.380	1.0000	0.3800	-0.078	.034	-0.044	1.044	1.000	4.123	4.123	0		284.4
9	.10	.0863	.129	.0523	.0784	-.068	.018	-.050	.150	.143	2.042	6.165	4.12	51.5	232.9
8	-.46	-.3440	-.344	-.0322	-.0322	-.057	.002	-.055	-.405	-.388	-4.257	1.908	10.29	119.2	113.7
7	-.60	-.3550	-.355	.1902	.1902	-.046	-.011	-.057	-.543	-.520	-5.936	-4.028	12.20	140.5	-26.8
6	-.42	-.1935	-.290	.2475	.3710	-.036	-.020	-.056	-.364	-.348	-3.951	-7.979	8.17	94.1	-120.9
5	0	0	0	0	0	-.027	-.024	-.051	0.051	.049	.564	-7.415	.19	2.8	-123.7
4	.40	.0920	.138	-.2735	-.4100	-.018	-.023	-.041	.441	.422	4.871	-2.544	-7.23	-81.9	-41.8
3	.60	.0819	.082	-.3160	-.3160	-.011	-.018	-.029	.629	.602	6.916	4.372	-9.77	-110.3	68.5
2	.48	.0306	.031	-.1445	-.1445	-.005	-.010	-.015	.495	.474	5.480	9.852	-5.40	-59.3	127.8
1	.14	.0023	.003	-.0130	-.0195	-.001	-.003	-.004	.144	.138	1.854	11.706	4.45	55.3	72.5
0	0	0	0	0	0	0	0	0	0	0	0		16.16	72.5	
			-0.226		0.0974										

(Values in columns 13 and 16 are printed between stations, each between the station shown and the next lower station.)

a $y_3^{(0)} = Y_3^{(0)} - a_1^{(0)}y_1 - a_2^{(0)}y_2$

where $a_1^{(0)} = \frac{-0.226}{2.879} = -0.0785$

and $a_2^{(0)} = \frac{0.0974}{2.885} = 0.0338$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 9.– THIRD BENDING MODE OF A UNIFORM CANTILEVER – Continued

$\left[L = 100 \text{ in.}; \quad \lambda = 10 \text{ in.}; \quad m = 5 \frac{\text{lb (sec)}^2}{\text{in.}^2}; \quad I = 50 \text{ in.}^4; \quad E = 10{,}000{,}000 \text{ psi}\right]$

17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Station	$Y_3^{(1)}$	$y_1Y_3^{(1)}$	Σ	$y_2Y_3^{(1)}$	Σ	$a_1^{(1)}y_1$	$a_2^{(1)}y_2$	column 23 + column 24	$Y_3^{(1)}$	$y_3^{(1)}$	P_{eq}	S	M	M_{eq}	Slope
Common factors →	$\frac{m\lambda^4}{144EI}$	$\frac{\lambda^4 m^2}{144EI}$	$\frac{125\lambda^5 m^2}{144^2 EI}$	$\frac{\lambda^4 m^2}{144EI}$	$\frac{125\lambda^5 m^2}{144^2 EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{m\lambda^4}{144EI}$		$\frac{\lambda m}{12}$	$\frac{\lambda m}{12}$	$\frac{\lambda^2 m}{12}$	$\frac{\lambda^3 m}{144EI}$	$\frac{\lambda^3 m}{144EI}$
10	586.6	586.6	223.0	586.6	223.0	267	-2	265	322	1.000	4.371		0		
												4.371			422.3
9	302.2	260.5	376.0	158.0	237.0	230	-1	229	73	.226	2.872		4.37	55.3	
												7.243			367.0
8	69.3	50.3	50.3	4.8	4.8	194	0	194	-125	-.388	-4.284		11.61	135.0	
												2.959			232.0
7	-44.4	-26.2	-26.2	14.1	14.1	158	1	159	-203	-.630	-7.129		14.57	167.7	
												-4.170			64.3
6	-17.6	-8.1	-12.2	10.4	15.6	123	1	124	-142	-.441	-5.006		10.40	119.8	
												-9.176			-55.5
5	103.0	35.2	26.7	-73.8	-56.0	91	1	92	11	.034	.411		1.22	15.1	
												-8.765			-70.6
4	227.0	52.2	78.4	-155.2	-232.8	61	1	62	165	.512	5.874		-7.54	-84.6	
												-2.891			14.0
3	268.8	36.7	36.7	-141.7	-141.7	36	1	37	232	.720	8.277		-10.43	-116.9	
												5.386			130.9
2	200.3	12.8	12.8	-60.2	-60.2	17	1	18	182	.565	6.584		-5.04	-53.9	
												11.970			184.8
1	72.5	1.2	1.8	-6.7	-10.0	4	0	4	69	.214	2.705		6.93	85.9	
												14.675			98.9
0	0	0	0	0	0	0	0	0	0	0			21.60	98.9	
			769.3		-6.2										

$a_1^{(1)} = \frac{769.3}{2.879} = 267 \qquad a_2^{(1)} = \frac{-6.2}{2.885} = -2.15$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 9.– THIRD BENDING MODE OF A UNIFORM CANTILEVER – Concluded

[L = 100 in.; λ = 10 in.; $m = 5\ \frac{\text{lb (sec)}^2}{\text{in.}^2}$; I = 50 in.^{4}; E = 10,000,000 psi]

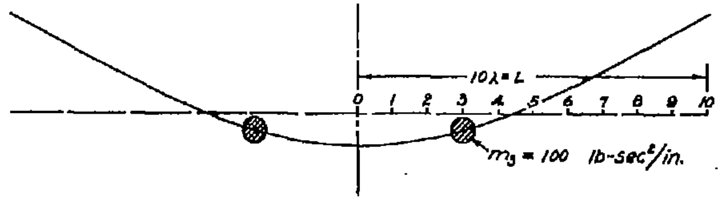
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
Station	$Y_3^{(2)}$	$y_1Y_3^{(2)}$	Σ	$y_2Y_3^{(2)}$	Σ	$a_1^{(2)}y_1$	$a_2^{(2)}y_2$	column 39 + column 40	$y_3^{(2)}$	$Y_3^{(1)}/y_3^{(2)}$	$y_3^{(2)}$	$Y_3^{(3)}$	$y_3^{(3)}$	$Y_3^{(2)}/y_3^{(3)}$	$y_3^{(3)}$
Common factors →	$\frac{\lambda^4 m}{144EI}$	$\frac{\lambda^4 m^2}{144EI}$	$\frac{125\lambda^5 m^2}{144^2EI}$	$\frac{\lambda^4 m^2}{144EI}$	$\frac{125\lambda^5 m^2}{144^2EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{\lambda^4 m}{144EI}$			$\frac{\lambda^4 m}{144EI}$	$-\frac{\lambda^4 m}{144EI}$		
10	1388.1	1388	527	1388	527	1027	−2	1025	363	3960	1.000	−295.6	377	3820	1.000
9	965.8	834	1251	505	757	885	−1	884	82	3970	.226	−493.7	86	3780	.228
8	598.8	435	435	42	42	745	0	745	−146	3830	−.402	−636.4	−150	3860	−.398
7	366.8	217	217	−116	−116	606	1	607	−240	3780	−.661	−644.1	−249	3820	−.661
6	302.5	139	208	−178	−267	473	1	474	−171	3720	−.471	−486.5	−180	3770	−.478
5	358.0	122	93	−256	−194	349	2	351	7	———	.019	−217.9	8	———	.021
4	428.6	98.6	148	−293	−440	236	2	238	191	3860	.526	46.7	199	3810	.528
3	414.6	56	56	−218	−218	140	1	141	274	3780	.755	195.4	284	3830	.753
2	283.7	18	18	−85	−85	66	1	67	217	3760	.598	185.6	228	3780	.605
1	98.9	1.7	2	−9	−13	17	0	17	82	3760	.226	74.4	85	3830	.225
0	0	0	0	0	0	0	0	0	0	———	0	0	0	———	0
			2955		−7										

$a_1^{(2)} = \frac{2955}{2.879} = 1027$ $\qquad a_2^{(2)} = \frac{-7}{2.835} = -2.42$ $\qquad \omega_3^2 \approx 3820\ (\text{radians/sec})^2$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 10.— FIRST SYMMETRICAL BENDING MODE OF A FREE-FREE BEAM CARRYING CONCENTRATED MASSES

[L = 100 in.; λ = 10 in.; I = 5 in.[4]; E = 10,000,000 psi]



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Station	I	m	Σ	$Y_1'(0)$	Σ	$Y_1(0)$ (a)	$mY_1(0)$	P_{eq}	P_{eq}	S	M	M_{eq}	Slope	$Y_1'(1)$	$Y_1'(1)$	Σ
Common factors →			$\frac{125\lambda}{144}$	m	$\frac{125\lambda m}{144}$			$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{125\lambda^5 m}{144^2 EI}$
10	5	1	0.38	1.00	0.38	0.766	1.000	5.58		5.58	0		11399	78669	78669	29900
9	5	1	1.50	.84	1.26	.606	.791	9.49		15.07	5.6	77	11303	67290	67290	100900
8	5	1	1.00	.68	.68	.446	.582	7.00		22.07	20.7	255	11048	55987	55987	55987
7	5	1	1.00	.53	.53	.296	.386	4.66		26.73	42.7	517	10531	44939	44939	44939
6	5	1	1.50	.40	.60	.166	.217	2.63		29.36	69.4	836	9695	34408	34408	51550
5	5	1	.76	.29	.22	.056	.073	.89		30.25	98.8	1186	8509	24713	24713	18780
4	5	1	1.50	.19	.28	−.044	−.057	−.66		29.59	129.0	1548	6961	16204	16204	24300
3	5	1	1.00	.11	.11	−.1243	−.1625	−1.92	[b]−19.50	8.17	158.6	892 969	5100	9243	9243	9243
2	5	1	1.00	.05	.05	−.184	−.240	−2.86		5.31	166.8	1998	3102	4143	4143	4143
1	5	1	1.50	.01	.02	−.224	−.293	−3.48		1.83	172.1	2061	1041	1041	1041	1561
0	5	1	.38	0	0	−.234	−.306	−1.83			173.9	1041		0	0	0
			11.52		4.13											341,303

[a] $Y_1(0) = Y_1'(0) - Y_0(0)$

where $Y_0(0) = \frac{\left(\frac{125\lambda m}{144} \times 4.13\right) + (100 \times 0.11)}{\left(\frac{125\lambda}{144} \times 11.52\right) + 100} = \frac{46.85}{200} = 0.2343$

[b] $P_{eq} = \frac{12}{\lambda} \times 100 \times (-0.1625) = -19.50$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 10.– FIRST SYMMETRICAL BENDING MODE OF A FREE-FREE BEAM CARRYING CONCENTRATED MASSES – Concluded

[L = 100 in.; λ = 10 in.; I = 5 in.^{4}; E = 10,000,000 psi]

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
Station	$Y_1(1)$	$Y_1(0)/Y_1(1)$	$mY_1(1)$	P_{eq}	P_{eq}	S	M	M_{eq}	Slope	$Y_1'(2)$	$Y_1'(2)$	Σ	$Y_1(2)$	$Y_1(1)/Y_1(2)$	$Y_1(2)$
Common factors →	$\frac{\lambda^4}{144EI}$			$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$	$\frac{m\lambda^4}{144EI}$	$\frac{125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$		
10	59240	12.1	1.000	5.61			0			82148	82148	31200	61833	11.63	1.000
9	47860	11.9	.808	9.70		5.61	5.6	77	11856	70292	70292	105300	49977	11.63	.808
8	36560	11.5	.616	7.40		15.31	20.9	258	11779	58513	58513	58513	38198	11.62	.617
7	25510	10.9	.431	5.18		22.71	43.6	529	11521	46992	46992	46992	26677	11.61	.431
6	14980	10.4	.253	3.05		27.89	71.5	861	10992	36000	36000	54000	15685	11.60	.254
5	5280	10.0	.0892	1.09		30.94	102.5	1231	10131	25869	25869	19670	5554	11.58	.090
4	–3230	12.7	–.0545	–.62		32.03	134.5	1613	8900	16969	16969	25450	–3346	11.72	–.054
3	–10190	11.5	–.1720	–2.03	–20.65	31.41	165.9	932 1013	7287	9682	9682	9682	–10633	11.63	–.172
2	–15290	11.3	–.258	–3.06		8.73	174.6	2092	5342	4340	4340	4340	–15975	11.62	–.258
1	–18390	11.5	–.310	–3.68		5.67	180.3	2160	3250	1090	1090	1630	–19225	11.62	–.311
0	–19430	11.3	–.328	–1.95		1.99	182.3	1090	1090	0	0	0	–20315	11.62	–.328
												356,777			

$$Y_0(1) = \frac{\frac{125\lambda m}{144} \times 341303 + 100 \times 9234}{200} = \frac{3886000}{200} = 19{,}430$$

$$Y_0(2) = \frac{\frac{125\lambda m}{144} \times 356777 + 100 \times 9682}{200} = \frac{4063000}{200} = 20{,}315$$

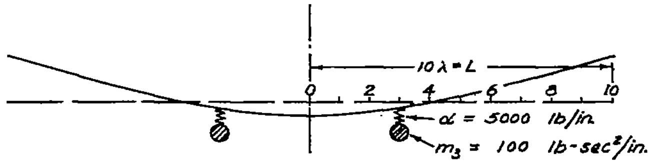
$\omega^2 \approx 11.62$ (radians/sec)^2

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 11.– FIRST SYMMETRICAL BENDING MODE OF A FREE-FREE BEAM

CARRYING MASSES MOUNTED THROUGH SPRINGS

[L = 100 in.; λ = 10 in.; I = 5 in.^{4}; E = 10,000,000 psi]



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Station	I	m	Σ	$Y_1'(0)$	Σ	$Y_1(0)$ (a)	$mY_1(0)$	P_{eq}	P_{eq}		M	M_{eq}	Slope	$Y_1'(1)$
Common factors →			$\frac{125\lambda}{144}$	m	$\frac{125\lambda m}{144}$			$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$
10	5	1	0.38	1.000	0.380	0.769	1.000	5.62			0			82982
										5.62			11974	
9	5	1	1.50	.855	1.282	.624	.811	9.73			5.6	77		71008
										15.35			11897	
8	5	1	1.00	.712	.712	.481	.625	7.50			21.0	259		59111
										22.85			11638	
7	5	1	1.00	.571	.571	.340	.442	5.31			43.8	531		47473
										28.16			11107	
6	5	1	1.50	.438	.656	.207	.269	3.24			72.0	867		36366
										31.40			10240	
5	5	1	.76	.315	.239	.084	.109	1.33			103.4	1242		26126
										32.73			8998	
4	5	1	1.50	.206	.309	–.025	–.032	–.36			136.1	1633		17128
										32.37			7365	
3	5	1	1.00	.118	.118	–.113	–.147	–1.73	[b]–22.65		168.5	946 1029		9763
										7.99			5390	
2	5	1	1.00	.053	.053	–.178	–.232	–2.75			176.5	2115		4373
										5.24			3275	
1	5	1	1.50	.013	.019	–.218	–.283	–3.36			181.7	2177		1098
										1.88			1098	
0	5	1	.38	0	0	–.231	–.300	–1.78			183.6	1098		0
			11.52		4.339									

[a] $Y_1(0) = Y_1'(0) - Y_0(0)$

$$\text{where } Y_0(0) = \frac{\Sigma mY_1'(0) + \frac{\left(m_3 Y_1'(0)\right)_{\text{Station 3}}}{1 - \left(\frac{\omega}{p}\right)^2}}{\Sigma m + \frac{m_3}{1 - \left(\frac{\omega}{p}\right)^2}} = \frac{\left(\frac{125\lambda m}{144} \times 4.339\right) + \left(\frac{100 \times 0.118}{1 - \frac{11}{50}}\right)}{\left(\frac{125\lambda}{144} \times 11.52\right) + \left(\frac{100}{1 - \frac{11}{50}}\right)} = \frac{52.73}{228.2} = 0.231$$

and $p^2 = \frac{\alpha}{m_3} = \frac{5000}{100} = 50(\text{radians/sec})^2$ and $\omega^2 = 11$ (assumed)

16	17	18	19	20
Station		$Y_1(1)$	$Y_1(0)/Y_1(1)$	$Y_1(1)$
Common factors →	$\frac{125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$		
10	31500	63780	11.28	1.000
9	106600	51810	11.28	.811
8	59111	39910	11.28	.625
7	47473	28270	11.28	.443
6	54500	17170	11.28	.269
5	19830	6926	11.32	.1083
4	25700	–2072	11.12	–.0324
3	9763	–9437	11.22	–.1478
2	4373	–14830	11.25	–.232
1	1647	–18100	11.26	–.284
0	0	[a]–19200	11.25	–.301
	360,497			

$$Y_0(1) = \frac{\frac{125\lambda m}{144} \times 360497 + \frac{100 \times 9763}{0.78}}{228.2} = \frac{4382000}{228.2} = 19200$$

$\omega^2 \approx 11.27$ (radians/sec)^2

For 1-inch beam tip deflection,

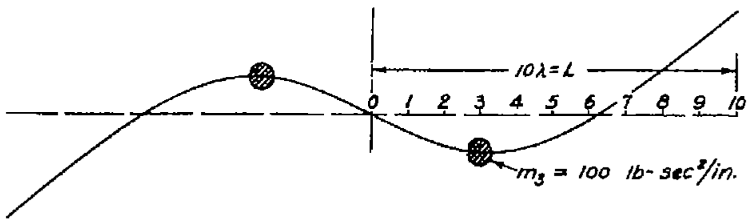
$$\text{Maximum spring force} = \frac{m_3\omega^2\left(Y_1(1)\right)_{\text{Station 3}}}{1 - \left(\frac{\omega}{p}\right)^2}$$

$$= \frac{100 \times 11.28 \times 0.1478}{1 - \frac{11.28}{50}} = 216 \text{ lb}$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 12.– FIRST ANTISYMMETRICAL BENDING MODE OF A FREE-FREE BEAM CARRYING CONCENTRATED MASSES

[L = 100 in.; λ = 10 in.; I = 5 in.⁴; E = 10,000,000 psi]



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Station	I	m	x	mx^2	Σ	$Y_1'(0)$	$xmY_1'(0)$	Σ	$K^{(0)}x$	$Y_1^{(0)}$ (a)	$mY_1^{(0)}$	P_{eq}	P_{eq}	S	M	M_{eq}
Common factors →			L	L^2	$\frac{125\lambda L^2}{144}$		L	$\frac{125\lambda L}{144}$				$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$
10	5	1	1.0	1.00	0.38	1.00	1.000	0.380	–.421	1.421	1.000	5.36			0	
9	5	1	.9	.81	1.22	.60	.540	.810	–.379	.979	.689	8.30		5.36	5.36	73
8	5	1	.8	.64	.64	.24	.192	.192	–.337	.577	.406	4.89		13.66	19.02	233
7	5	1	.7	.49	.49	–.10	–.070	–.070	–.295	.195	.137	1.68		18.55	37.57	452
6	5	1	.6	.36	.54	–.38	–.228	–.342	–.253	–.127	–.089	–1.05		20.23	57.80	693
5	5	1	.5	.25	.19	–.62	–.310	–.236	–.210	–.410	–.288	–3.34		19.18	76.98	920
4	5	1	.4	.16	.24	–.70	–.280	–.420	–.169	–.531	–.374	–4.36		15.84	92.82	1109
3	5	1	.3	.09	.09	–.60	–.180	–.180	–.126	–.474	–.333	–3.94	[b]–40.00	11.48	104.30	605 563
2	5	1	.2	.04	.04	–.42	–.084	–.084	–.084	–.336	–.236	–2.82		–32.46	71.84	859
1	5	1	.1	.01	.01	–.22	–.022	–.033	–.042	–.178	–.125	–1.49		–35.28	36.56	439
0	5	1	0	0	0	0	0	0	0	0	0			–36.77	–.21	
					3.84			.017								

[a] $Y_1^{(0)} = Y_1'(0) - K^{(0)}x$

$$\text{where } K^{(0)} = \frac{\left(\frac{125\lambda L}{144} \times 0.017\right) + \left(10.0 \times 30 \times -0.60\right)}{\left(\frac{125\lambda L^2}{144} \times 3.84\right) + \left[100 \times (30)^2\right]} = \frac{-1785.2}{423000} = -0.00421$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 12.– FIRST ANTISYMMETRICAL BENDING MODE OF A FREE-FREE BEAM CARRYING CONCENTRATED MASSES – Concluded

[L = 100 in.; λ = 10 in.; I = 5 in.^{4}; E = 10,000,000 psi]

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Station	Slope	$Y_1'^{(1)}$	$mxY_1'^{(1)}$	Σ	$K^{(1)}x$	$Y_1^{(1)}$	$mY_1^{(1)}$	P_{eq}	P_{eq}	S	M	M_{eq}	Slope	$Y_1'^{(2)}$
Common factors →	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$	L	$\frac{125\lambda L}{144}$		$\frac{\lambda^4}{144EI}$		$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$
10		10260	10260	3895	–2880	13140	1.000	5.42			0			10643
	3480									5.42			3630	
9		6780	6100	9150	–2590	9370	.713	8.56			5.42	74		7013
	3407									13.98			3556	
8		3373	2700	2700	–2300	5670	.431	5.19			19.40	238		3457
	3174									19.17			3318	
7		199	139	139	–2020	2220	.169	2.06			38.57	465		139
	2722									21.23			2853	
6		–2523	–1515	–2270	–1730	–793	–.060	–.67			59.80	717		–2714
	2029									20.56			2136	
5		–4552	–2275	–1730	–1440	–3110	–.236	–2.76			80.36	962		–4850
	1109									17.80			1174	
4		–5661	–2265	–3395	–1150	–4510	–.343	–4.03			98.16	1174		–6024
	0									13.77			0	
3		–5661	–1700	–1700	–860	–4800	–.365	–4.29	–43.80		111.93	646 585		–6024
	–1168									–34.32			–1231	
2		–4493	–900	–900	–570	–3920	–.298	–3.51			77.61	928		–4793
	–2027									–37.83			–2159	
1		–2466	–247	–370	–290	–2170	–.165	–1.95			39.78	475		–2634
	–2466									–39.78			–2634	
0		0	0	0	0	0	0	0			0			0
				5510										

$$K^{(1)} = \frac{5510\ \frac{125\lambda L}{144} + 100 \times 30 \times (-5661)}{423000} = \frac{12200000}{423000} = -28.8$$

33	34	35	36	37	38	39
Station	$mxY_1'^{(2)}$	Σ	$K^{(2)}x$	$Y_1^{(2)}$	$Y_1^{(1)}/Y_1^{(2)}$	$Y_1^{(2)}$
Common factors →	L	$\frac{125\lambda L}{144}$		$\frac{\lambda^4}{144EI}$		
10	10643	4045	–3180	13820	52.1	1.000
9	6310	9460	–2860	9870	51.9	.714
8	2765	2765	–2540	6000	51.7	.434
7	97	97	–2220	2360	51.5	.171
6	–1629	–2443	–1910	–804	– –	–.058
5	–2425	–1843	–1590	–3260	52.1	–.236
4	–2410	–3615	–1270	–4750	52.0	–.344
3	–1807	–1807	–950	–5070	51.9	–.367
2	–959	–959	–640	–4150	51.7	–.300
1	–263	–394	–320	–2310	51.4	–.167
0	0	0	0	0	– –	0
		5306				

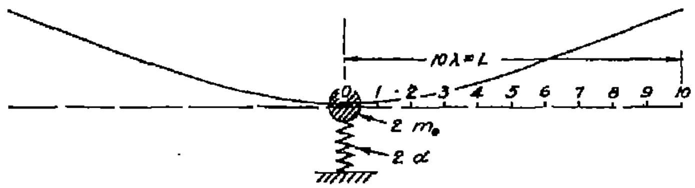
$$K^{(2)} = \frac{5306\ \frac{125\lambda L}{144} + 100 \times 30 \times (-6024)}{423000} = \frac{-13470000}{423000} = -31.8$$

$\omega_1^2 \approx 51.8$ (radians/sec)^2

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 13.– BEAM MOUNTED ON SPRING, FIRST BENDING MODE

$$\left[L = 100 \text{ in.}; \quad \lambda = 10 \text{ in.}; \quad I = 5 \text{ in.}^4; \quad m = 1 \frac{\text{lb(sec)}^2}{\text{in.}^2}; \quad \alpha = 10{,}000 \frac{\text{lb}}{\text{in.}}; \quad m_o = mL = 100 \frac{\text{lb(sec)}^2}{\text{in.}}; \quad E = 10{,}000{,}000 \text{ psi} \right]$$



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Station	I	m	Σ	$Y_1'^{(0)}$	Σ	$Y_1^{(0)}$ (a)	$mY_1^{(0)}$	P_{eq}	S	M	M_{eq}	Slope	$Y_1'^{(1)}$	Σ	$Y_1^{(1)}$	ω_1^2
Common factors →			$\frac{125\lambda}{144}$		$\frac{125\lambda m}{144}$			$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$	$\frac{125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$	
10	5	1	0.38	1.00	0.38	1.024	1.000	5.68	5.68	0		15322	111119	42200	114000	6.31
9	5	1	1.50	.84	1.26	.864	.842	10.11	15.79	5.7	78	15244	95797	143700	98700	
8	5	1	1.00	.68	.68	.704	.686	8.24	24.03	21.5	266	14978	80553	80550	83500	5.92
7	5	1	1.00	.53	.53	.554	.540	6.50	30.53	45.5	552	14426	65575	65580	68500	
6	5	1	1.50	.40	.60	.424	.414	4.99	35.52	76.0	917	13509	51149	76800	54000	5.52
5	5	1	.76	.29	.22	.314	.306	3.68	39.20	111.5	1342	12167	37640	28600	40500	
4	5	1	1.50	.19	.28	.214	.209	2.53	41.73	150.7	1812	10355	25473	38200	28370	5.31
3	5	1	1.00	.11	.11	.134	.131	1.59	43.32	192.4	2310	8045	15118	15120	18020	
2	5	1	1.00	.05	.05	.074	.072	.88	44.20	235.7	2829	5216	7073	7070	9970	5.20
1	5	1	1.50	.01	.02	.034	.033	.42	44.62	279.9	3359	1857	1857	2780	4760	
0	5	1	.38	0	0	.024	.023	.14		324.5	1857		0	0	2900	5.71
			11.52		4.14									500600		

$$^{a}\; Y_1^{(0)} = Y_1'^{(0)} - Y_o^{(0)} \quad \text{where} \quad Y_o^{(0)} = \frac{\Sigma m Y_1'^{(0)}}{-\frac{\alpha}{\omega_1^2} + \Sigma m + m_o} = \frac{\frac{125\lambda m}{144} \times 4.10}{-\frac{10000}{5.9} + \left(\frac{125\lambda}{144} \times 11.52\right) + 100} = \frac{35.90}{-1495} = -0.024$$

and ω_1^2 is assumed to be 5.9 based on the relations

$$\omega_1^2 \text{ (cant.)} = \frac{12.4EI}{mL^4} = 6.20; \qquad p^2 = \frac{\alpha}{m_o} = \frac{10000}{200} = 50$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 13.– BEAM MOUNTED ON SPRING, FIRST BENDING MODE – Concluded

[$L = 100$ in.; $\lambda = 10$ in.; $I = 5$ in.^{4}; $m = 1\ \frac{\text{lb(sec)}^2}{\text{in.}^2}$; $\alpha = 10{,}000\ \frac{\text{lb}}{\text{in.}}$; $m_o = mL = 100\ \frac{\text{lb(sec)}^2}{\text{in.}}$; $E = 10{,}000{,}000$ psi]

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Station	$mY_1^{(1)}$	P_{eq}	S	M	M_{eq}	Slope	$Y_1'^{(2)}$	Σ	$Y_1^{(2)}$	ω_1^2	$mY_1^{(2)}$	$Y_1^{(3)}$	ω_1^2	y_1
Common factors →		$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$	$\frac{.125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$			$\frac{\lambda^4}{144EI}$		
10	1.000	5.73		0			117900	44800	121000	5.95	1.000	121200	5.94	1.000
			5.73			16209								
9	.866	10.39		5.7	79		101700	152500	104800		.865	105000	5.93	.866
			16.12			16130								
8	.732	8.79		21.8	270		85600	85600	88700	5.94	.734	88800	5.95	.732
			24.91			15860								
7	.601	7.22		46.7	567		69740	69740	72800		.601	73000	5.93	.602
			32.13			15293								
6	.474	5.70		78.8	951		54450	81900	57600	5.94	.476	57600	5.94	.475
			37.83			14342								
5	.355	4.27		116.6	1404		40110	30500	43200		.357	43300	5.94	.357
			42.10			12938								
4	.248	2.99		158.7	1907		27170	40700	30280	5.90	.250	30300	5.94	.250
			45.09			11031								
3	.158	1.92		203.8	2447		16140	16140	19250		.159	19290	5.94	.159
			47.01			8584								
2	.088	1.08		250.8	3011		7559	7560	10670	5.95	.088	10690	5.93	.0881
			48.09			5573								
1	.042	.53		298.9	3587		1986	2980	5096		.042	5110	5.93	.0421
			48.62			1986								
0	.025	.17		347.5	1986		0	0	3110	5.83	.0257	3120	5.94	.0257
								532420						

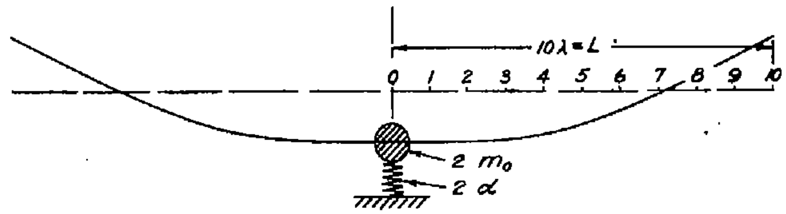
$$Y_o^{(2)} = \frac{\frac{125\lambda m}{144} \times 532400}{-\frac{10000}{5.93} + \frac{125\lambda}{144} \times 11.52 + 100} = \frac{4620000}{-1487} = -3110$$

$\omega_1^2 \approx 5.94$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 14.– BEAM MOUNTED ON A SPRING, SECOND BENDING MODE

$$\left[L = 100 \text{ in.}; \quad \lambda = 10 \text{ in.}; \quad I = 5 \text{ in.}^4; \quad m = 1 \frac{\text{lb(sec)}^2}{\text{in.}^2}; \quad \alpha = 10{,}000 \frac{\text{lb}}{\text{in.}}; \quad m_o = mL = 100 \frac{\text{lb(sec)}^2}{\text{in.}}; \quad E = 10{,}000{,}000 \text{ psi}\right]$$



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Station	y_1	y_1^2	Σ	$Y_2(0)$	$y_1Y_2(0)$	Σ	$a_1^{(0)}y_1$ (a)	$y_2(0)$	$my_2(0)$	P_{eq}	S	M	M_{eq}	Slope	$Y_2'(1)$
Common factors →		m	$\frac{125\lambda m}{144}$		m	$\frac{125\lambda m}{144}$				$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$
10	1.000	1.000	0.380	1.00	1.000	0.380	−0.185	1.185	1.000	5.12	5.12	0		3379	17393
9	.866	.750	1.125	.52	.450	.675	−.160	.680	.574	6.94	12.06	5.1	68	3311	14014
8	.732	.536	.536	.10	.073	.073	−.135	.235	.198	2.43	14.49	17.2	209	3102	10703
7	.602	.362	.362	−.26	−.157	−.157	−.111	−.149	−.126	−1.44	13.05	31.7	379	2723	7601
6	.475	.226	.339	−.54	−.256	−.384	−.088	−.452	−.382	−4.53	8.52	44.7	532	2191	4878
5	.357	.127	.096	−.76	−.271	−.206	−.066	−.694	−.585	−6.94	1.58	53.2	631	1560	2687
4	.250	.063	.094	−.88	−.220	−.330	−.046	−.834	−.704	−8.39	−6.81	54.8	649	911	1127
3	.159	.025	.025	−.94	−.149	−.149	−.029	−.911	−.769	−9.19	−16.00	48.0	567	344	216
2	.0881	.008	.008	−.96	−.084	−.084	−.016	−.944	−.796	−9.55	−25.55	32.0	374	−30	−128
1	.0421	.001	.003	−.98	−.041	−.061	−.008	−.972	−.820	−9.84	−35.39	6.5	68	−98	−98
0	.0257	0	0	−1.00	−.026	−.010	−.005	−.995	−.840	−5.00		−28.9	−98		0
			2.968			−0.253									

$$a \quad a_1(0) = \frac{\Sigma m y_1 Y_2(0) + m_o y_1 Y_2(0) \text{ station } 0}{\Sigma m (y_1)^2 + m_o (y_1)^2 \text{ station } 0} = \frac{\frac{125\lambda m}{144}(-0.253) - 2.57}{\frac{125\lambda m}{144}(2.968) + 0.066} = \frac{-4.765}{25.82} = -0.185$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 14.– BEAM MOUNTED ON A SPRING, SECOND BENDING MODE – Continued

$$\left[L = 100 \text{ in.}; \quad \lambda = 5 \text{ in.}; \quad I = 5 \text{ in.}^4; \quad m = 1 \frac{\text{lb(sec)}^2}{\text{in.}^2}; \quad \alpha = 10{,}000 \frac{\text{lb}}{\text{in.}}; \quad m_0 = mL = 100 \frac{\text{lb(sec)}^2}{\text{in.}}; \quad E = 10{,}000{,}000 \text{ psi}\right]$$

17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Station	Σ	$Y_2^{(1)}$ (b)	$y_1Y_2^{(1)}$	Σ	a_1y_1	$y_2^{(1)}$	$my_2^{(1)}$	P_{eq}	S	M	M_{eq}	Slope	$Y_2'^{(2)}$	Σ	$Y_2^{(2)}$
Common factors ⟶	$\frac{125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$	$\frac{\lambda^4 m}{144EI}$	$\frac{125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$	$\frac{\lambda^4}{144EI}$		$\frac{\lambda}{12}$	$\frac{\lambda}{12}$	$\frac{\lambda^2}{12}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^3}{144EI}$	$\frac{\lambda^4}{144EI}$	$\frac{125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$
10	6610	9153	9153	3480	260	8893	1.000	5.24		0			19350	7360	9960
									5.24			3725			
9	21000	5774	5000	7500	226	5548	.624	7.50		5.2	70		15625	23450	6235
									12.74			3655			
8	10703	2463	1803	1803	191	2272	.256	3.09		17.9	218		11970	11970	2580
									15.83			3437			
7	7601	–639	–384	–384	157	–796	–.089	–1.03		33.7	403		8533	8533	–857
									14.80			3034			
6	7310	–3362	–1598	–2395	124	–3486	–.393	–4.65		48.5	577		5499	8850	–3891
									10.15			2457			
5	2045	–5553	–1980	–1505	93	–5646	–.635	–7.55		58.7	697		3042	2310	–6348
									2.60			1760			
4	1690	–7113	–1778	–2665	65	–7178	–.807	– 9.61		61.3	726		1282	1922	–8108
									–7.01			1034			
3	216	–8024	–1275	–1275	41	–8065	–.907	–10.82		54.3	641		248	248	–9142
									–17.83			393			
2	–128	–8368	–737	–737	23	–8391	–.944	–11.29		36.5	427		–145	–145	–9535
									–29.12			–34			
1	–147	–8338	–351	–526	11	–8349	–.939	–11.86		7.4	77		–111	–166	–9501
									–40.38			–111			
0	0	–8240	–211.8	–80	7	–8247	–.928	–5.59		–33.0	–111		0	0	–9390
	56900			3216										63732	

b $Y_2^{(1)} = Y_2'^{(1)} - Y_0^{(1)}$ where $Y_0^{(1)} = \dfrac{\Sigma m Y_2'^{(1)}}{-\dfrac{\alpha}{\omega_2^2} + \Sigma m + m_0} = \dfrac{\left(\dfrac{125\lambda m}{144} \times 56900\right)}{-\dfrac{10000}{71.5} + \dfrac{125\lambda}{144} \times 11.52 + 100} = \dfrac{494000}{60} = 8240$

and assumption for ω_2^2 is found from the relation

$$\omega_2^2 \frac{\lambda}{12} S \text{ (Station 0)} + \omega_2^2 \frac{\lambda}{12} P_{eq} \text{ (Station 0)} + \omega_2^2 m_0 y_2^{(0)} \text{ (Station 0)} = \alpha y_2^{(0)} \text{ (Station 0)}$$

$$\omega_2^2 = \frac{10000\ (-0.840)}{-\frac{\lambda}{12} \times 35.39 - \frac{\lambda}{12} \times 5.00 - 100 \times 0.840} = \frac{-10090}{-141.3} = 71.5$$

$$a_1^{(1)} = \frac{\frac{125\lambda m}{144}(3216) - 21180}{25.82} = 260$$

$$\omega_2^2 = \frac{10000\ (-0.928)}{-\frac{\lambda}{12} \times 40.38 - \frac{\lambda}{12} \times 5.59 - 100 \times 0.928} = 70.9$$

$$Y_0^{(1)} = \frac{\frac{125\lambda m}{144} \times 63732}{-\frac{10000}{70.9} + 200} = \frac{554000}{59} = 9390$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 14.— BEAM MOUNTED ON A SPRING, SECOND BENDING MODE – Concluded

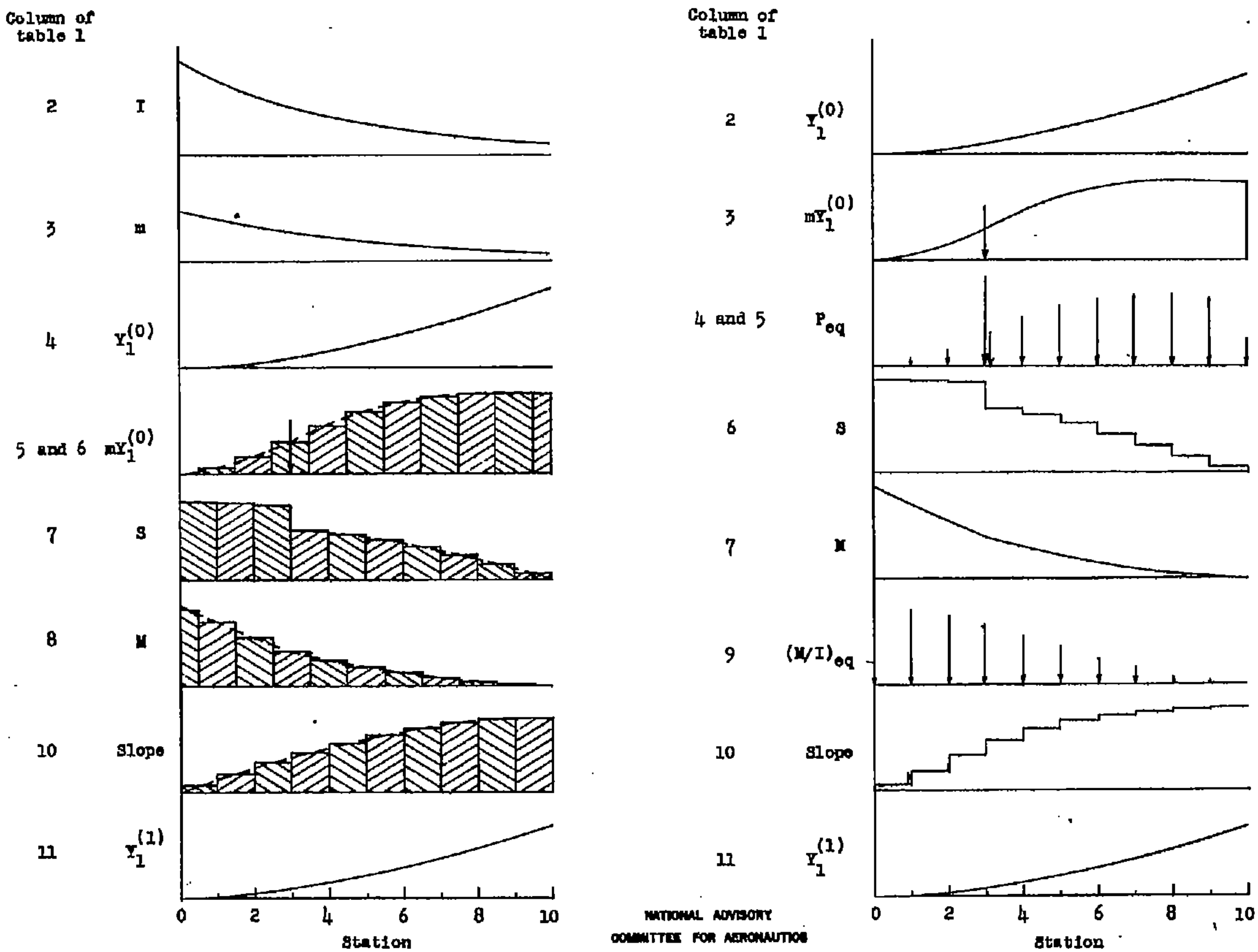
$\left[L = 100 \text{ in.}; \quad \lambda = 10 \text{ in.}; \quad I = 5 \text{ in.}^4; \quad m = 1 \frac{\text{lb(sec)}^2}{\text{in.}^2}; \quad \alpha = 10{,}000 \frac{\text{lb}}{\text{in.}}; \quad m_0 = mL = 100 \frac{\text{lb(sec)}^2}{\text{in.}}; \quad E = 10{,}000{,}000 \text{ psi}\right]$

33	34	35	36	37	38	39	40	41	42
Station	$y_1Y_2^{(2)}$	Σ	$a_1^{(2)}y_1$	$y_2^{(2)}$	ω_2^2	my_2	$y_2^{(3)}$	ω_2^2	y_2
Common factors →	$\frac{\lambda^4 m}{144EI}$	$\frac{125\lambda^5 m}{144^2 EI}$	$\frac{\lambda^4}{144EI}$	$\frac{\lambda^4}{144EI}$			$\frac{\lambda^4}{144EI}$		
10	9960	3785	−13	9973	72.1	1.000	10125	71.0	1.000
9	5400	8100	−11	6246	72.0	.625	6348	70.9	.626
8	1890	1890	−9	2589	71.1	.259	2640	70.6	.260
7	−516	−516	−8	−849	75.5	−.085	−849	72.1	−.084
6	−1849	−2770	−6	−3885	72.8	−.389	−3931	71.3	−.388
5	−2265	−1722	−5	−6343	72.1	−.635	−6423	71.2	−.634
4	−2025	−3040	−3	−8105	71.7	−.812	−8212	71.1	−.810
3	−1452	−1452	−2	−9140	71.5	−.915	−9268	71.1	−.915
2	−840	−840	−1	−9534	71.4	−.955	−9676	71.1	−.955
1	−400	−600	0	−9501	71.2	−.951	−9650	71.0	−.951
0	−241.3	−92	0	−9390	71.2	−.940	−9540	71.0	−.941
		2743							

$$a_1^{(2)} = \frac{\frac{125\lambda m}{144} \times 2743 - 24130}{25.82} = -12.8$$

$$\omega_2^2 \approx 71.1$$

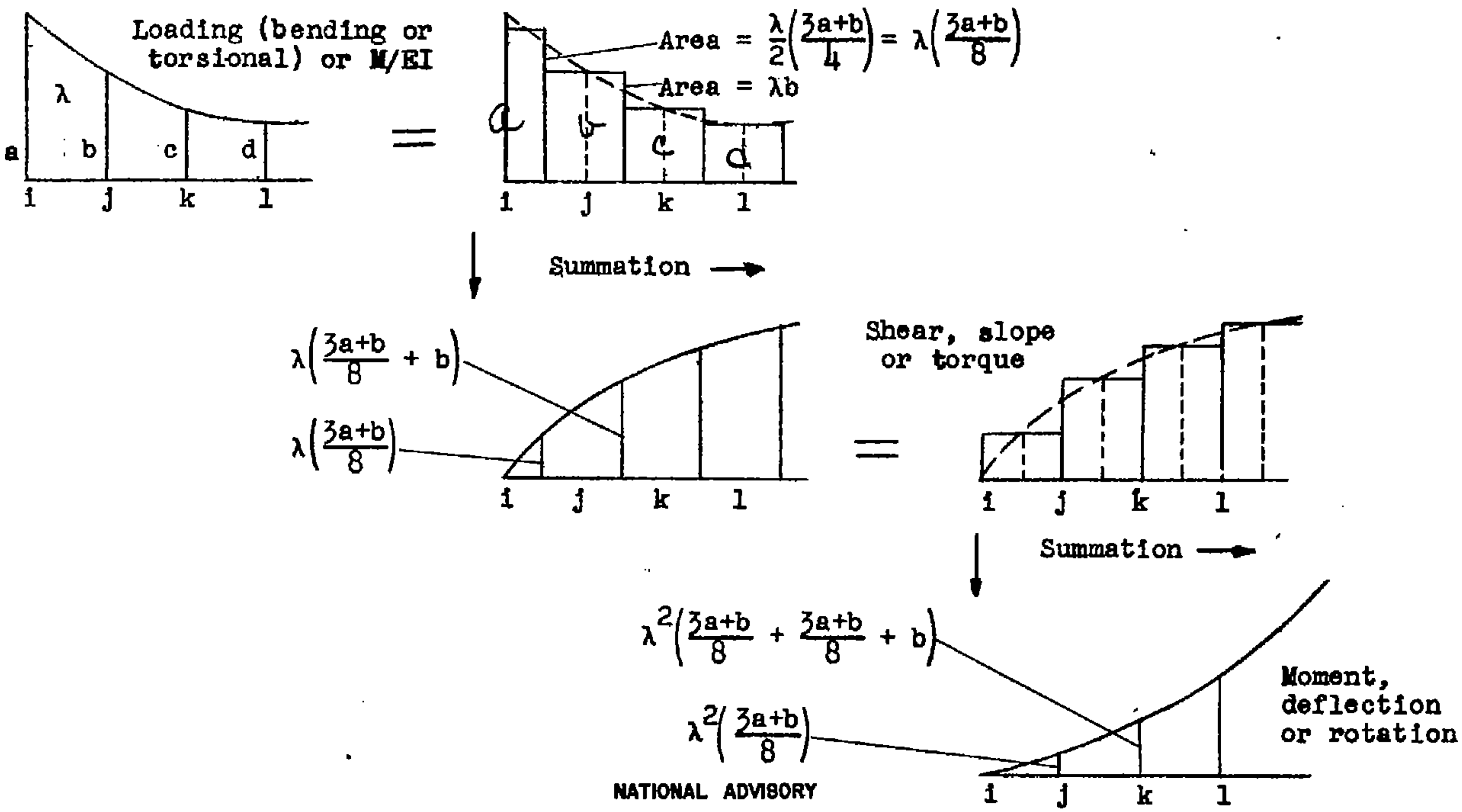
NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS



(a) Summation method.

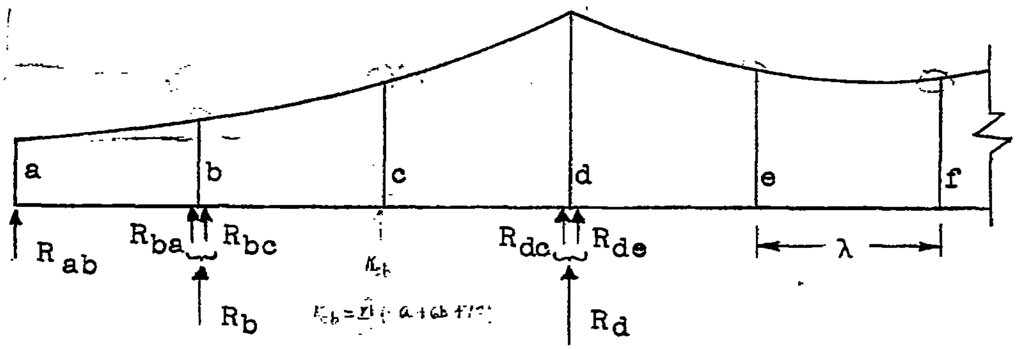
(b) Equivalent-load method.

Figure 1.- Graphical illustration of the summation and equivalent-load methods presented in table 1.



NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

Figure 2.- Two successive numerical integrations of a curve by the summation method.



$$R_{ab} = \frac{\lambda}{24}(7a+6b-c) = \frac{\lambda}{12}\left(\frac{7a+6b-c}{2}\right) \qquad (a)$$

$$R_{ba} = \frac{\lambda}{24}(3a+10b-c)$$

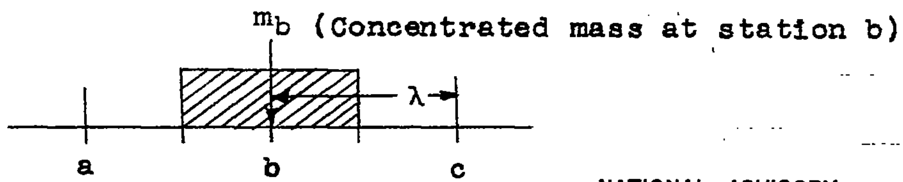
$$R_{bc} = \frac{\lambda}{24}(-a+10b+3c)$$

$$R_b = \frac{\lambda}{12}(a+10b+c) \qquad (b)$$

$$R_{dc} = \frac{\lambda}{24}(7d+6c-b)$$

$$R_{de} = \frac{\lambda}{24}(7d+6e-f)$$

$$R_d = \frac{\lambda}{24}(-b+6c+14d+6e-f) = \frac{\lambda}{12}\left(-\frac{b}{2}+3c+7d+3e-\frac{f}{2}\right) \qquad (c)$$



NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

$$R_{m_b} = \frac{\lambda}{12}\left(\frac{12}{\lambda}m_b\right) \qquad (d)$$

Figure 3.- Formulas for equivalent concentrated loads. (Equations (a) and (b) from reference 5.)

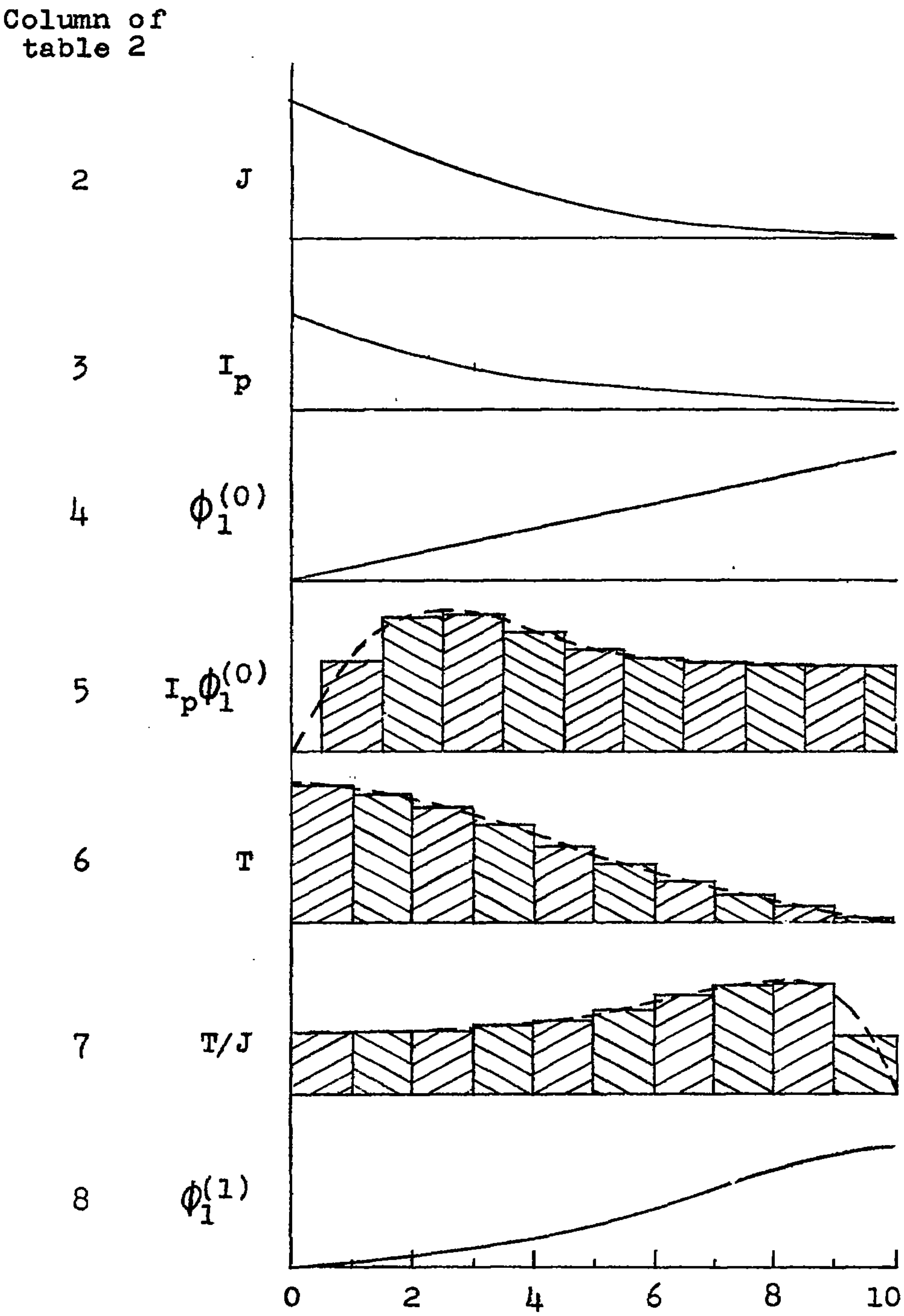


Figure 4.- Graphical illustration of the torsional iteration presented in table 2.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

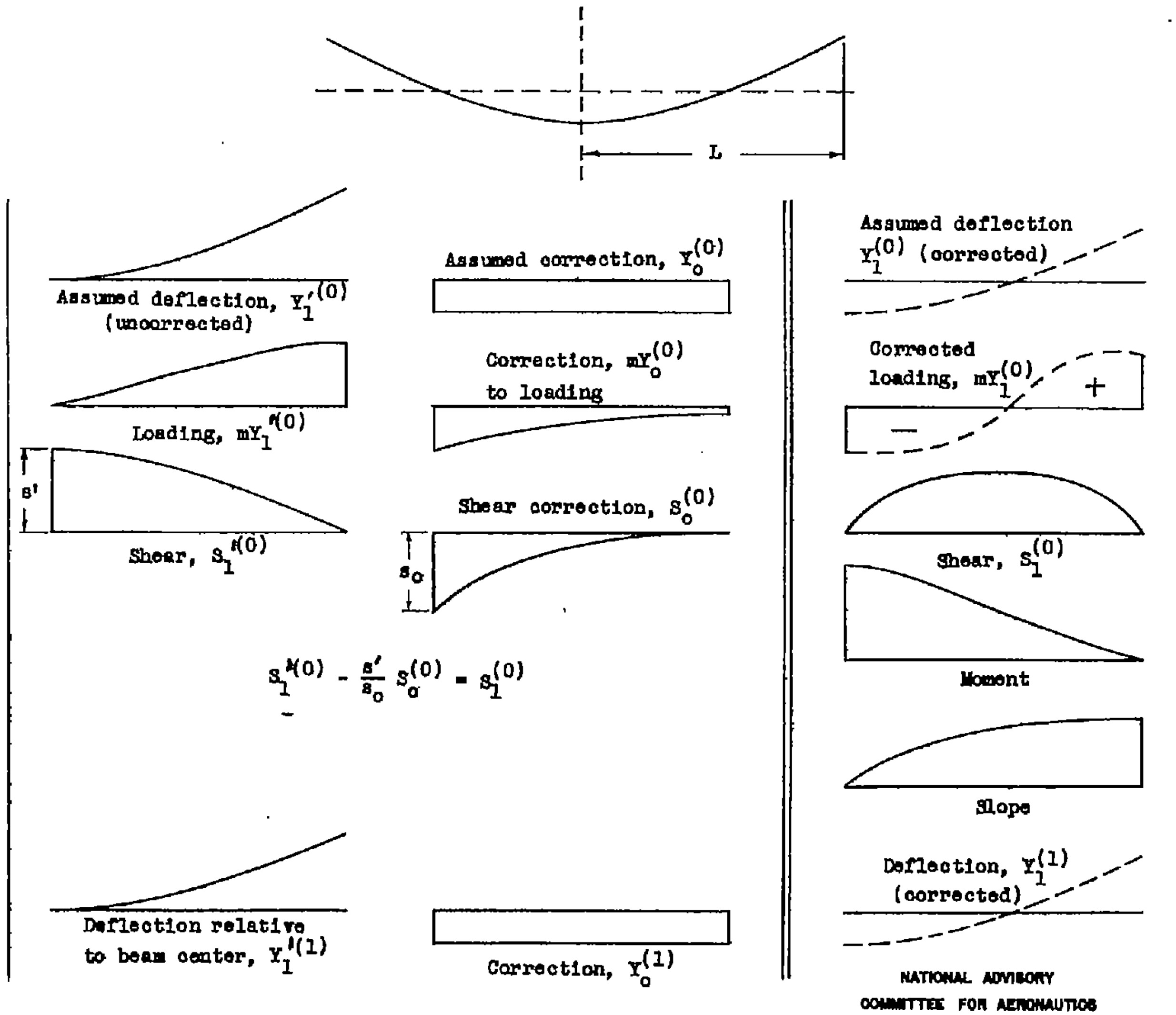


Figure 5.- Satisfaction of boundary conditions for the first symmetrical mode of a free-free beam by the general method of approach.

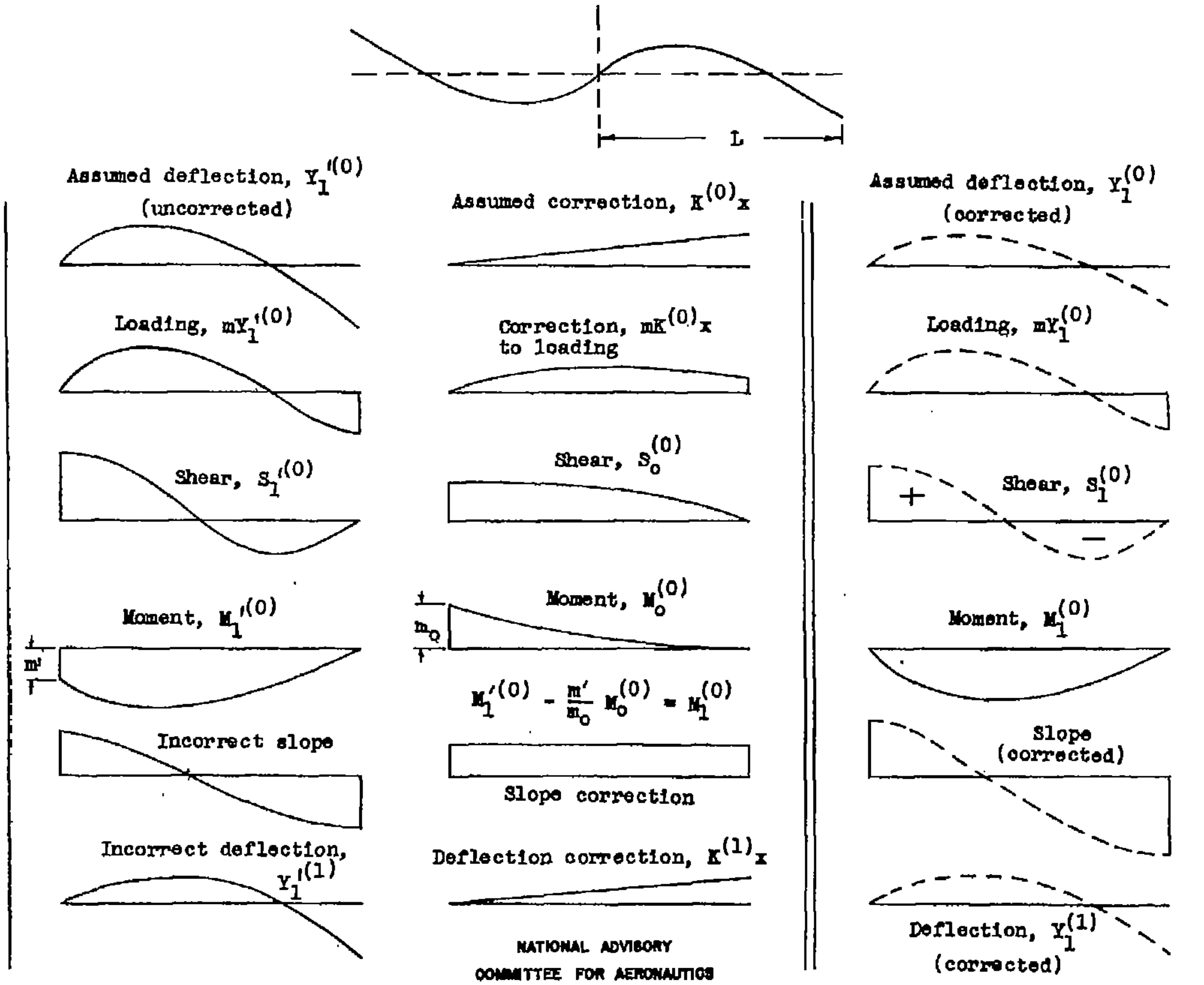
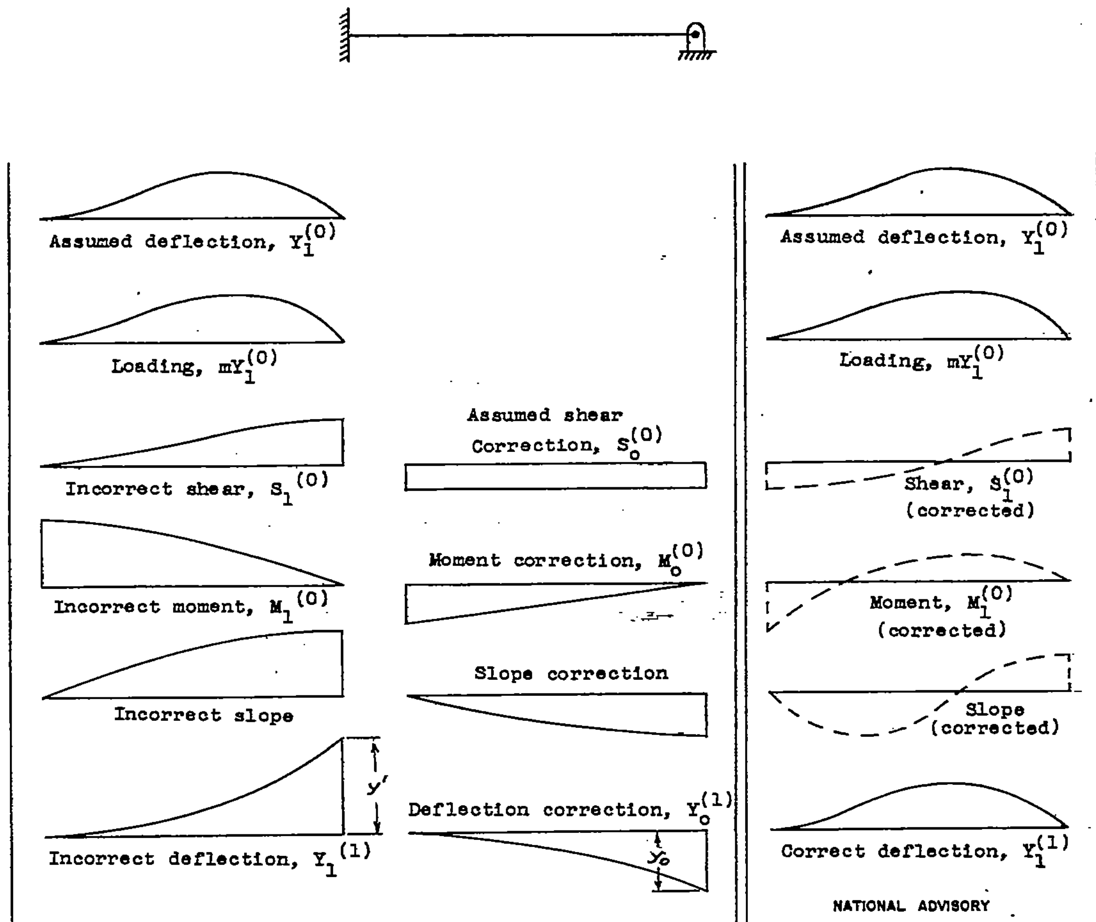


Figure 6.- Satisfaction of boundary conditions for first antisymmetrical mode of a free-free beam by general method of approach.



$$Y_1'^{(1)} - \frac{y'}{y_0} Y_0^{(1)} = Y_1^{(1)}$$

Outline of Iteration Procedure:

(1) Deflection $Y_1^{(0)}$ is assumed and loading $mY_1^{(0)}$ is computed.

(2) Shear is determined in two parts; known shear variation relative to left end, and an assumed correction for the left-end reaction.

(3) By successive integrations, the moment, slope, and deflection diagrams are found from each of the two shear diagrams.

(4) Deflection y_0 must equal y' ; therefore, the deflection arising from the shear correction is adjusted. The two diagrams then added give the final deflection $Y_1^{(1)}$, which satisfies the deflection boundary conditions.

(5) Process is repeated until $Y_1^{(i)} \propto Y_1^{(i+1)}$.

Figure 7.- Boundary conditions and iteration procedure for a beam fixed at one end and simply supported at the other end.

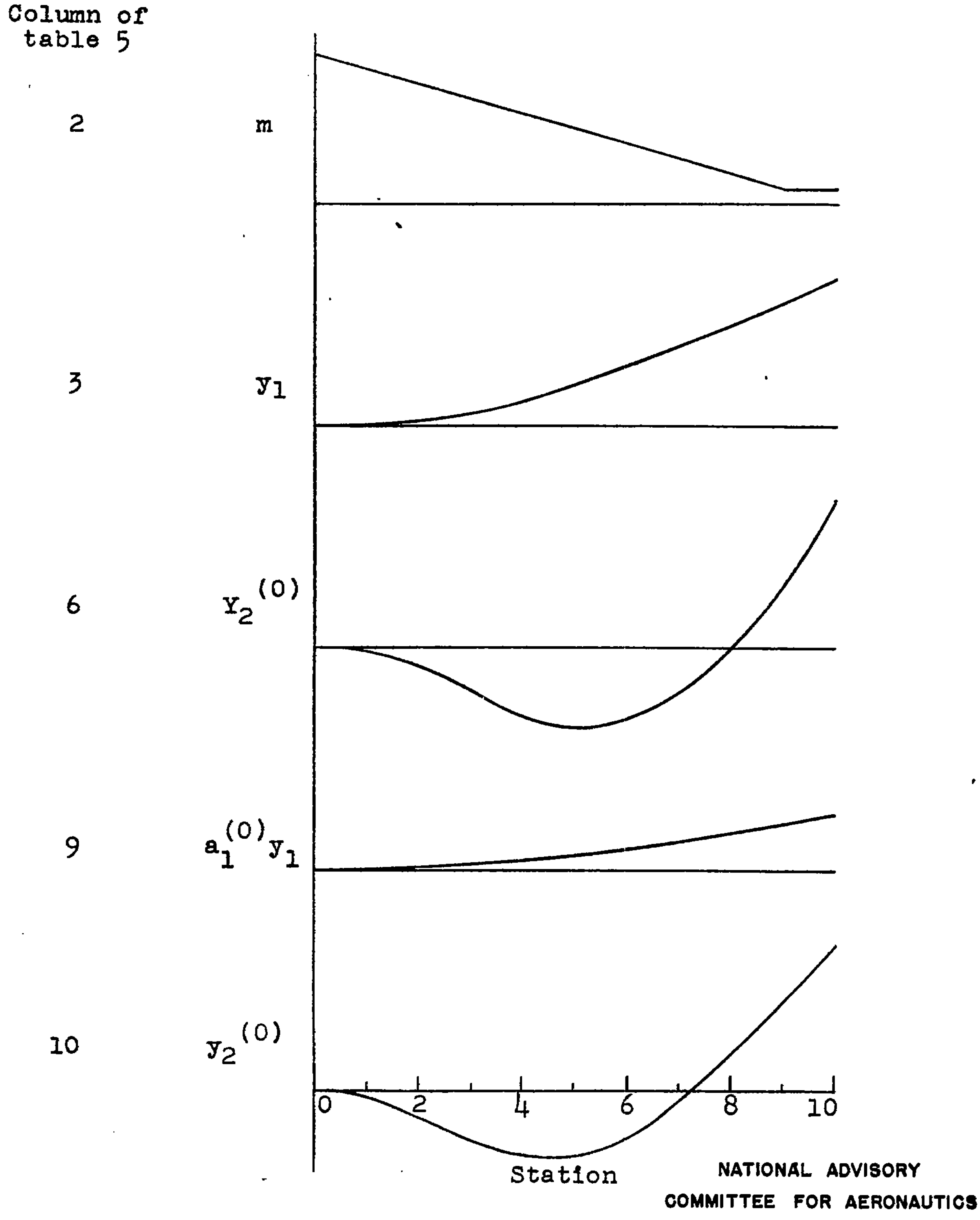


Figure 8.- Graphical representation of table 5.

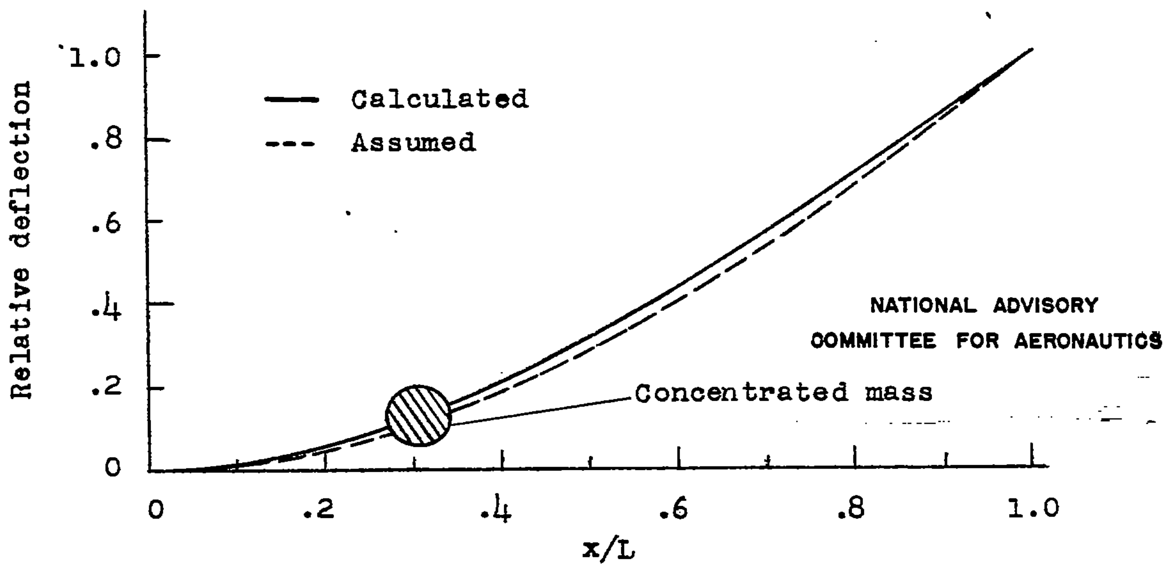


Figure 9.- Fundamental bending mode of a nonuniform cantilever beam.

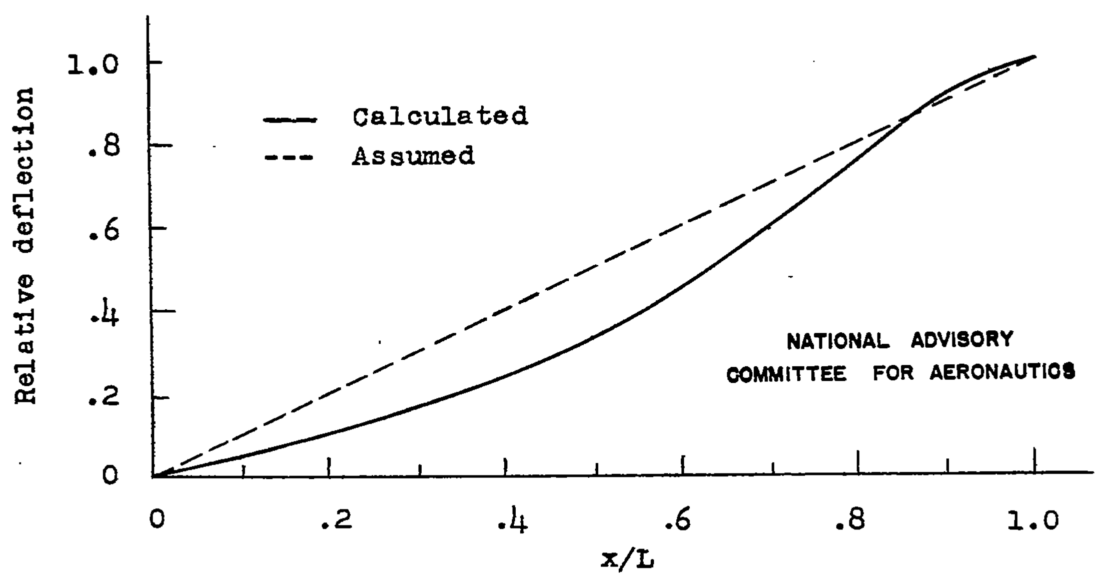


Figure 10.- Fundamental torsional mode of a nonuniform cantilever beam.

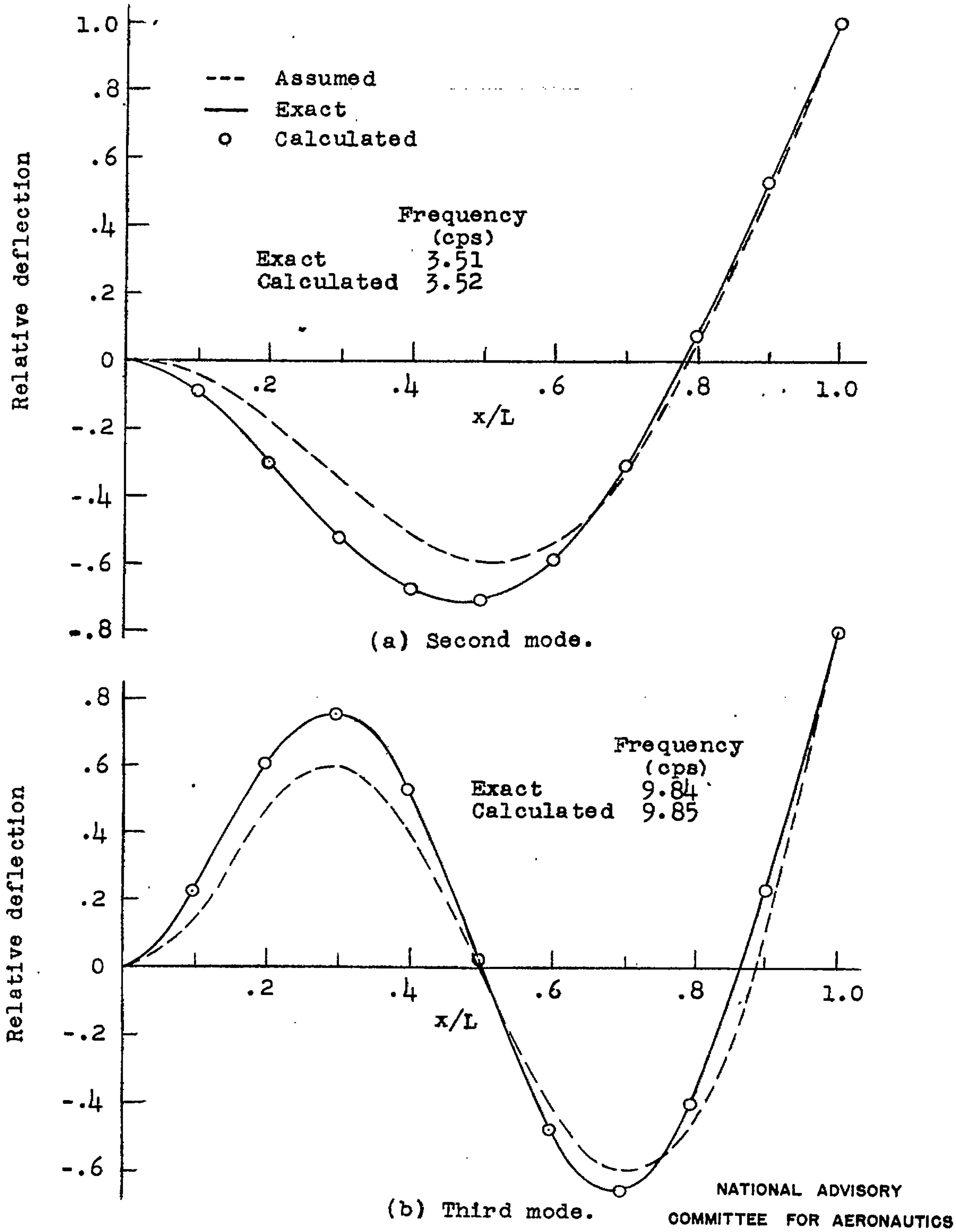


Figure 11.- Second and third bending modes of a uniform cantilever beam.

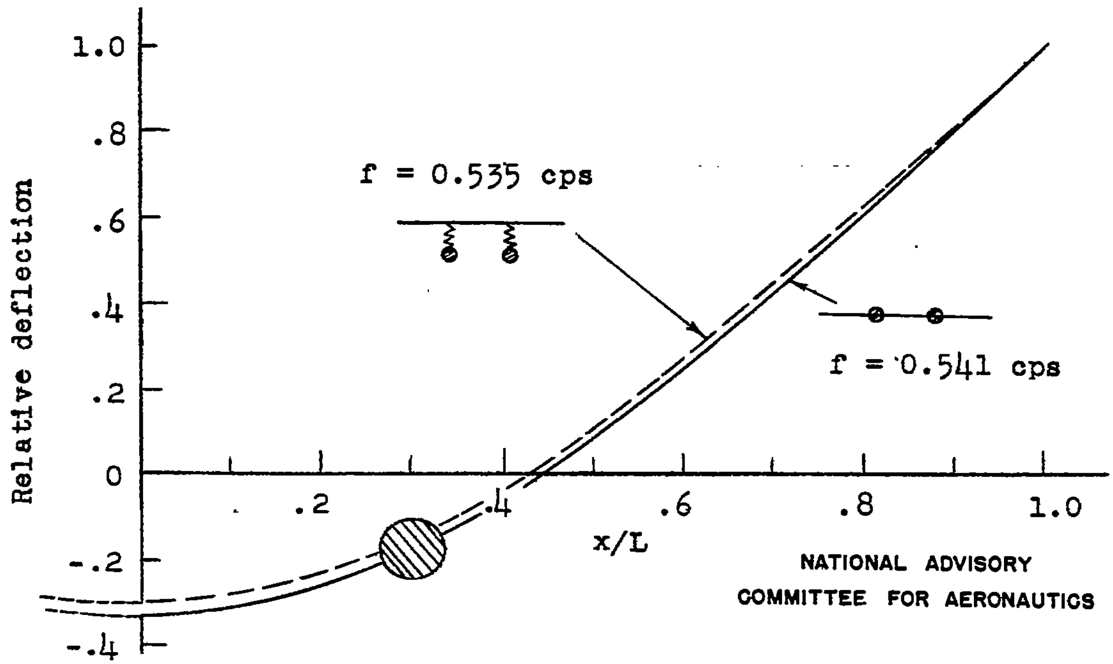
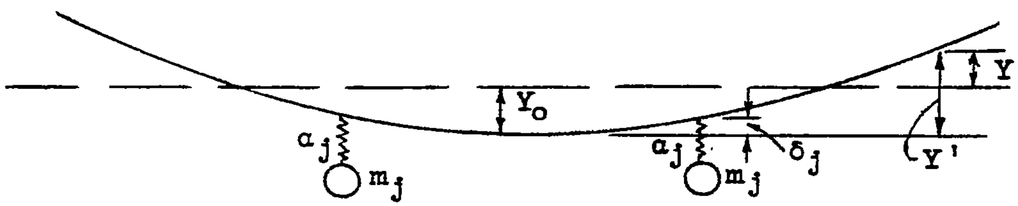


Figure 12.- First symmetrical bending mode of a free-free beam carrying concentrated masses.



$$Y_o = \frac{\int mY'dx + \dfrac{m_j\delta_j}{1-\left(\dfrac{\omega}{p_j}\right)^2}}{\int m\,dx + \dfrac{m_j}{1-\left(\dfrac{\omega}{p_j}\right)^2}}$$

$$p_j = \sqrt{\frac{a_j}{m_j}}$$

$$Y = Y' - Y_o$$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Figure 13.- Equations for correcting assumed deflection curve for a beam carrying spring-mounted masses.

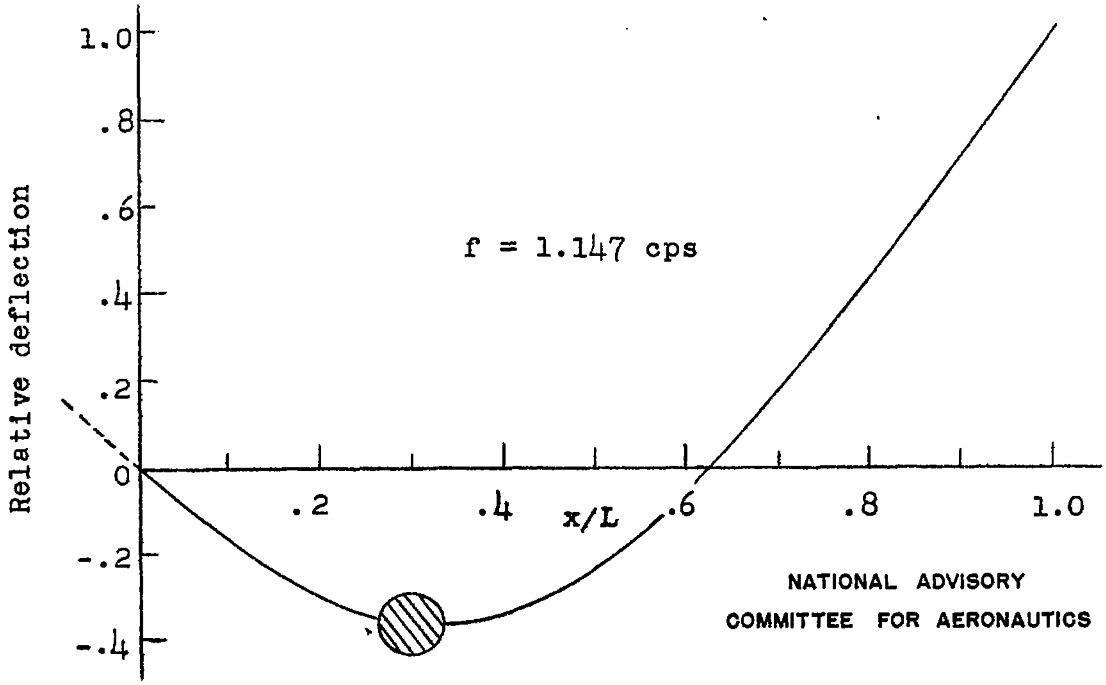
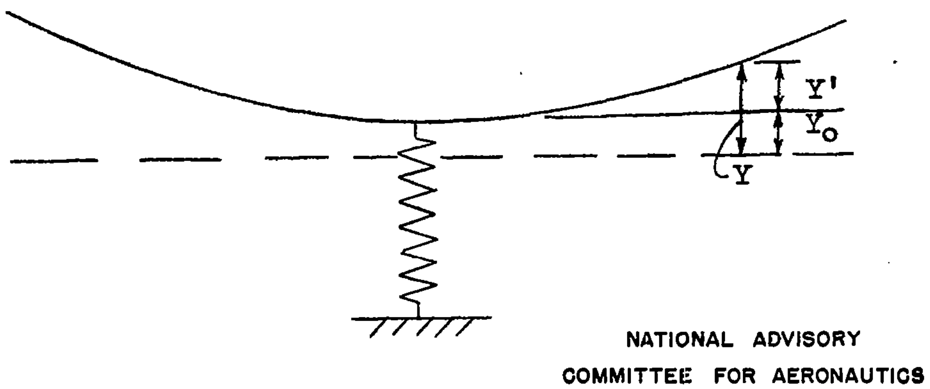


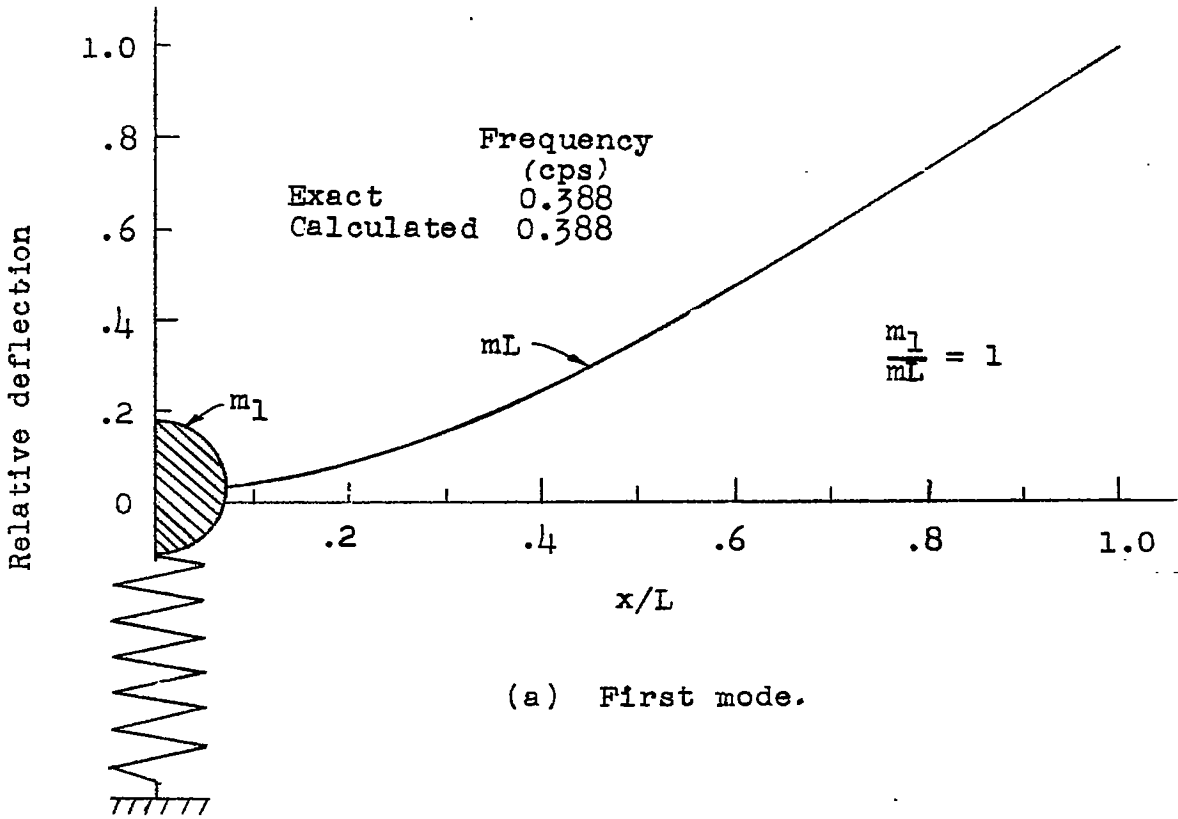
Figure 14.- First antisymmetrical bending mode of a free-free beam carrying concentrated masses.



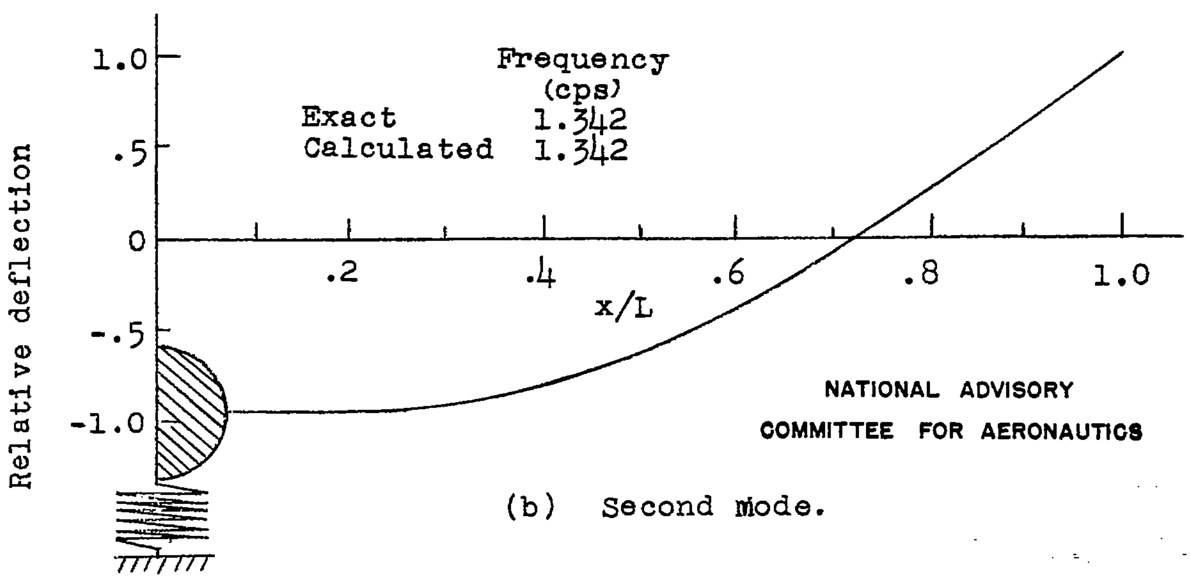
$$Y_o = \frac{\int mY' dx}{-\frac{a}{\omega^2} + \int m \, dx}$$

$$Y = Y' - Y_o$$

Figure 15.- Equation for correcting assumed deflection curve for a beam mounted on a spring.



(a) First mode.



NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

(b) Second mode.

Figure 16.- First and second bending modes of a beam-mass-spring system.