NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2002

APPLICATION OF THE LAPLACE TRANSFORMATION TO THE SOLUTION OF THE LATERAL AND LONGITUDINAL STABILITY EQUATIONS

By G. A. Mokrzycki

Ryan Aeronautical Company
Lindbergh Field, San Diego, Calif.

Washington
January 1950

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2002

APPLICATION OF THE LAPLACE TRANSFORMATION TO THE SOLUTION OF THE LATERAL AND LONGITUDINAL STABILITY EQUATIONS

By G. A. Mokrzycki

SUMMARY

The application of the Laplace transformation to the solution of the lateral and longitudinal stability equations is presented. The expressions for the time history of the motion in response to a sinusoidal control motion are derived for the general case in which the initial conditions, initial displacements and initial velocities, are assumed different from zero. Some illustrative examples of the application of the Laplace transform to ordinary linear differential equations with constant coefficients and a numerical example of a specific problem are presented in appendixes.

INTRODUCTION

Recent developments in piloted and pilotless aircraft, equipped with automatic devices, have directed the attention of engineers to the theoretical investigation of dynamic longitudinal and lateral stability problems of aircraft designed for high-speed and high-altitude flight. In the past, the dynamic stability investigations were usually limited to the determination of Routh's condition for stability and for the calculation of the roots of the characteristic stability equation to determine the damping of the modes of motion and the period of the oscillation. A more complete analysis of the problem requires the calculation of a time history of the airplane motion in response to a gust disturbance or in response to the application of the control surfaces. As the methods of classical analysis (references 1 and 2) proved to be inadequate for this purpose, new methods of operational mathematics, representing a more powerful tool, were used. These methods are known today as the Heaviside operational calculus and the Laplace transformation. The application of the Heaviside operational calculus to the calculation of airplane motions is discussed in references 3, 4, and 5. However, the Laplace transformation is considered a more powerful method than the Heaviside operational calculus because the initial conditions of the problem, initial displacements and initial velocities, are inherently taken into account by the Laplace transformation, whereas in the Heaviside operational calculus, all initial conditions are zero.

In this paper, the Laplace transformation is applied to both the longitudinal and lateral stability equations for the general case where the initial displacements and initial velocities were assumed different from zero. The operational equations obtained for this general case were then solved and the time history of the motion was obtained by the Heaviside expansion theorem and by the inversion theorem for Laplace transformation. The Laplace transformation is simple and effective. Its principles are easily understood and its technique quickly learned. It represents a further development in operational mathematics because it is a more powerful mathematical tool and because the difficulties and obscurities of the work of Heaviside are avoided.

A short historical sketch tracing the development of operational mathematics and its application to airplane dynamics is presented in appendix A.

The author is indebted to Mr. Leonard Sternfield of the Langley Stability Research Division, NACA, for information and collaboration he has contributed in connection with this paper.

SYMBOLS

c	chord, feet
b	span, feet
S	wing area, square feet
W	weight, pounds
m	mass, slugs $\left(\frac{W}{g}\right)$
ρ	density, slugs per cubic foot
V	airspeed, feet per second
t	time, seconds
s_c	nondimensional time parameter based on chord $\left(t \frac{V}{c}\right)$

s_b	nondimensional time parameter based on span $\left(t \frac{V}{b}\right)$
μ_c	relative density coefficient based on chord $\left(\frac{m}{\rho Sc}\right)$
μ_b	relative density coefficient based on span $\left(\frac{m}{\rho Sb}\right)$
D_c	differential operator with respect to s_c $\left(\frac{d}{ds_c}\right)$
D_b	differential operator with respect to s_b $\left(\frac{d}{ds_b}\right)$
P	operator in Laplace transformation
λ	root of stability equation
k_{X_o}	radius of gyration about principal longitudinal axis, feet
k_{Y_o}	radius of gyration about principal lateral axis, feet
k_{Z_o}	radius of gyration about principal vertical axis, feet
K_{X_o}	nondimensional radius of gyration about principal longitudinal axis $\left(k_{X_o}/b\right)$
K_{Y_o}	nondimensional radius of gyration about principal lateral axis $\left(k_{Y_o}/c\right)$
K_{Z_o}	nondimensional radius of gyration about principal vertical axis $\left(k_{Z_o}/b\right)$
η	angle between principal longitudinal axis of inertia and flight path (fig. 1), degrees
ϵ	angle between reference axis and principal longitudinal axis (fig. 1), degrees
α	angle of attack (fig. 1), degrees

γ	flight-path angle between path and horizontal (fig. 1), degrees
θ	attitude angle between reference line and horizontal line, degrees $(\alpha + \gamma)$
δ_a, δ_e, δ_r	deflection angles of aileron, elevator, and rudder, degrees
q	pitching angular velocity, radians per second $(\dot{\theta})$
β	angle of sideslip, radians
ψ	azimuth angle, radians
r	yawing angular velocity, radians per second $(\dot{\psi})$
ϕ	angle of bank, radians
p	rolling angular velocity, radians per second $(\dot{\phi})$
u	increment of forward velocity, feet per second
u'	nondimensional increment of forward velocity $\left(\frac{u}{V}\right)$
K_X	nondimensional radius of gyration about longitudinal stability axis $\left(\sqrt{{K_{X_o}}^2\cos^2\eta + {K_{Z_o}}^2\sin^2\eta}\right)$
K_Z	nondimensional radius of gyration about vertical stability axis $\left(\sqrt{{K_{Z_o}}^2\cos^2\eta + {K_{X_o}}^2\sin^2\eta}\right)$
K_{XZ}	nondimensional product-of-inertia parameter $\left(\left({K_{Z_o}}^2 - {K_{X_o}}^2\right)\sin\eta\cos\eta\right)$
$\frac{pb}{2V}$	rolling-velocity parameter (helix angle generated by wing tip in roll), radians
$\frac{qc}{2V}$	pitching-velocity parameter, radians

$\frac{rb}{2V}$	yawing-velocity parameter, radians
q	dynamic pressure
x, y, z	rectangular coordinates (fig. 1)
X	longitudinal force, pounds (fig. 1)
Y	lateral force, pounds (fig. 1)
Z	normal force, pounds (fig. 1)
L	rolling moment, foot-pounds (fig. 1)
M	pitching moment, foot-pounds (fig. 1)
N	yawing moment, foot-pounds (fig. 1)
$C_D = C_X$	drag coefficient $\left(\frac{\text{Drag}}{qS}\right)$
$C_L = C_Z$	lift coefficient $\left(\frac{\text{Lift}}{qS}\right)$
C_X	longitudinal-force coefficient $\left(\frac{X}{qS}\right)$
C_Y	lateral-force coefficient $\left(\frac{Y}{qS}\right)$
C_Z	normal-force coefficient $\left(\frac{Z}{qS}\right)$
C_m	pitching-moment coefficient $\left(\frac{M}{qSc}\right)$
C_l	rolling-moment coefficient $\left(\frac{L}{qSb}\right)$
C_n	yawing-moment coefficient $\left(\frac{N}{qSb}\right)$
$C_{X_{u'}} = \frac{\partial C_X}{\partial u'}$	
$C_{X_\alpha} = \frac{\partial C_X}{\partial \alpha}$	

$$C_{X_{D\alpha}} = \frac{\partial C_X}{\partial\left(\frac{\dot{\alpha}c}{2V}\right)}$$

$$C_{X_\theta} = \frac{\partial C_X}{\partial \theta}$$

$$C_{X_q} = \frac{\partial C_X}{\partial\left(\frac{qc}{2V}\right)}$$

$$C_{X_{\delta_e}} = \frac{\partial C_X}{\partial \delta_e}$$

$$C_{Z_{u'}} = \frac{\partial C_Z}{\partial u'}$$

$$C_{Z_\alpha} = \frac{\partial C_Z}{\partial \alpha}$$

$$C_{Z_{D\alpha}} = \frac{\partial C_Z}{\partial\left(\frac{\dot{\alpha}c}{2V}\right)}$$

$$C_{Z_\theta} = \frac{\partial C_Z}{\partial \theta}$$

$$C_{Z_q} = \frac{\partial C_Z}{\partial\left(\frac{qc}{2V}\right)}$$

$$C_{Z_{\delta_e}} = \frac{\partial C_Z}{\partial \delta_e}$$

$$C_{m_{u'}} = \frac{\partial C_m}{\partial u'}$$

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}$$

$$C_{m_{D\alpha}} = \frac{\partial C_m}{\partial\left(\frac{\dot{\alpha}c}{2V}\right)}$$

$$C_{m_\theta} = \frac{\partial C_m}{\partial \theta}$$

$$C_{m_q} = \frac{\partial C_m}{\partial\left(\frac{qc}{2V}\right)}$$

$$C_{m_{\delta_e}} = \frac{\partial C_m}{\partial \delta_e}$$

$$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{l_p} = \frac{\partial C_l}{\partial\left(\frac{pb}{2V}\right)}$$

$$C_{Y_p} = \frac{\partial C_Y}{\partial\left(\frac{pb}{2V}\right)}$$

$$C_{n_p} = \frac{\partial C_n}{\partial\left(\frac{pb}{2V}\right)}$$

$$C_{Y_r} = \frac{\partial C_Y}{\partial\left(\frac{rb}{2V}\right)}$$

$$C_{l_r} = \frac{\partial C_l}{\partial\left(\frac{rb}{2V}\right)}$$

$$C_{n_r} = \frac{\partial C_n}{\partial\left(\frac{rb}{2V}\right)}$$

$$C_{Y_\delta}\delta = \frac{\partial C_Y}{\partial \delta_a}\delta_a + \frac{\partial C_Y}{\partial \delta_r}\delta_r$$

$$C_{l_\delta}\delta = \frac{\partial C_l}{\partial \delta_a}\delta_a + \frac{\partial C_l}{\partial \delta_r}\delta_r$$

$$C_{n_\delta}\delta = \frac{\partial C_n}{\partial \delta_a}\delta_a + \frac{\partial C_n}{\partial \delta_r}\delta_r$$

$G_1, G_2, G_3, H_1, H_2, H_3$ functions of P on right side of operational equations

$\bar{\Delta}, \bar{\Delta}_1, \bar{\Delta}_2, \bar{\Delta}_3$; $\Delta, \Delta_1, \Delta_2, \Delta_3$ determinants

A, B, C, D, E coefficients in fourth-degree characteristic equations

$a_{11} \ldots a_{33}$; $b_{11} \ldots b_{33}$ abbreviated coefficients in operational equation

$g_{11}(P) \ldots g_{33}(P)$; $h_{11}(P) \ldots h_{33}(P)$ abbreviated functions of P in operational equation

R residue

The subscript o is used to indicate initial conditions, a bar is used to denote variables in the operational equations, and a dot is used to denote differentiation with respect to time.

ANALYSIS

The purpose of this paper is to show how the longitudinal and lateral stability equations can be solved by the Laplace transformation. Thus no attempt is made to present a detailed discussion on the theory of Laplace transform, which can be found in references 6 and 7 and in the bibliography presented in appendix I of reference 6, but rather to present sufficient background of the theory to permit a clear understanding of its application to this particular problem.

If a function $x(t)$, defined for all positive values of the variable t, is multiplied by e^{-Pt} and integrated with respect to t from zero to infinity, a new function $\bar{x}(P)$ of the variable P is obtained; that is

$$\bar{x}(P) = \int_0^\infty e^{-Pt}x(t)dt$$

This operation on a function $x(t)$ is called the Laplace transformation of $x(t)$. The necessary and sufficient conditions for the existence of the Laplace transform of a function $x(t)$ are discussed in reference 6. Let

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \ldots a_{n-1}\frac{dx}{dt} + a_n x = x(t) \tag{1}$$

represent an ordinary linear differential equation with constant coefficients $a_1, a_2, a_3, \ldots a_{n-1}, a_n$. If D is substituted for $\frac{d}{dt}$, D^2 for $\frac{d^2}{dt^2}$, and so forth, equation (1) can be written in operational form

$$\left(D^n + a_1 D^{n-1} + \ldots a_{n-1}D^{n-1} + a_n\right)x = x(t) \tag{1a}$$

When $t = 0$, the following initial conditions are assumed:

$$x = x_0$$

$$x_1 = \frac{dx}{dt}$$

$$\ldots\ldots$$

$$x_{n-1} = \frac{d^{n-1}x}{dt}$$

The Laplace transformation of equation (1a), with the use of the letter P for the operator, is

	Term corresponding to
$\left(P^n + a_1P^{n-1} + \dots a_{n-1}P + a_n\right)\bar{x} = \bar{x}(P)$	
$+ \left(P^{n-1}x_o + P^{n-2}x_1 + \dots Px_{n-2} + x_{n-1}\right)$	D^nx
$+ a_1\left(P^{n-2}x_o + P^{n-3}x_1 + \dots Px_{n-3} + x_{n-2}\right)$	$D^{n-1}x$
$\dots\dots\dots\dots\dots\dots\dots$	$\dots$
$+ a_{n-3}\left(P^2x_o + Px_1 + x_2\right)$	D^3x
$+ a_{n-2}\left(Px_o + x_1\right)$	D^2x
$+ a_{n-1}x_o$	Dx (2)

The transform $\bar{x}(P)$ for $x(t)$ is taken from table I which presents some simple Laplace transforms. A more complete table of Laplace transforms is given in appendix III of reference 6 and in appendix A of reference 7. Appendix B shows two illustrative examples of the application of Laplace transform to ordinary linear differential equations with constant coefficients.

Longitudinal Motion

The nondimensional linearized stability equations for longitudinal motion are given by NACA in the form:

$$\left.\begin{aligned}
2\mu_cD_cu' &= C_{X_{\delta_e}}\delta_e + C_{X_{u'}}u' + \left(C_{X_\alpha} + \frac{1}{2}C_{X_{D\alpha}}D_c\right)\alpha + \left(C_{X_\theta} + \frac{1}{2}C_{X_q}D_c\right)\theta \\
2\mu_cD_c(\alpha - \theta) &= C_{Z_{\delta_e}}\delta_e + C_{Z_{u'}}u' + \left(C_{Z_\alpha} + \frac{1}{2}C_{Z_{D\alpha}}D_c\right)\alpha + \left(C_{Z_\theta} + \frac{1}{2}C_{Z_q}D_c\right)\theta \\
2\mu_cK_Y{}^2D_c{}^2\theta &= \left(C_{m_{\delta_e}} + \frac{1}{2}C_{m_{D\delta_e}}D_c\right)\delta_e + C_{m_{u'}}u' + \left(C_{m_\alpha} + \frac{1}{2}C_{m_{D\alpha}}D_c\right)\alpha \\
&\quad + \left(C_{m_\theta} + \frac{1}{2}C_{m_q}D_c\right)\theta
\end{aligned}\right\} \quad (3)$$

The Laplace transformation is demonstrated for the case in which the elevator motion can be simulated by the sine function

$$\delta_e = \delta_m \sin as_c \tag{4}$$

where δ_m is the amplitude. (In most cases δ_m is assumed to be 1.) Rearranging and substituting equation (4) into equation (3) give:

$$\left.\begin{aligned}
&\left(2\mu_c D_c - C_{X_{u'}}\right)u' - \left(C_{X_\alpha} + \frac{1}{2} C_{X_{D\alpha}} D_c\right)\alpha - \left(C_{X_\theta} + \frac{1}{2} C_{X_q} D_c\right)\theta \\
&\qquad = C_{X_{\delta_e}} \delta_m \sin as_c \\
&-C_{Z_{u'}}u' + \left[\left(2\mu_c - \frac{1}{2} C_{Z_{D\alpha}}\right)D_c - C_{Z_\alpha}\right]\alpha - \left[\left(2\mu_c + \frac{1}{2} C_{Z_q}\right)D_c + C_{Z_\theta}\right]\theta \\
&\qquad = C_{Z_{\delta_e}} \delta_m \sin as_c \\
&-C_{m_{u'}}u' - \left(C_{m_\alpha} + \frac{1}{2} C_{m_{D\alpha}} D_c\right)\alpha + \left(2\mu_c K_Y{}^2 D_c{}^2 - C_{m_\theta} - \frac{1}{2} C_{m_q} D_c\right)\theta \\
&\qquad = \left(C_{m_{\delta_e}} + \frac{1}{2} C_{m_{D\delta_e}} D_c\right)\delta_m \sin as_c
\end{aligned}\right\} \tag{3a}$$

In order to illustrate the use of the Laplace transformation for a very general case, the only initial condition assumed to be zero is $\delta_{e_o} = 0$; that is, the deflection is measured from its trim position before the maneuver begins. For all other parameters the initial conditions are assumed to be different from zero; thus the values are u_o', α_o, θ_o, and q_o at $s_c = 0$. The equations can then be written in general form, in which the four initial disturbances are combined with elevator motion. In a specific problem some of the initial conditions would probably be zero. For practical engineering purposes, in fact, the most interesting cases are

(1) Disturbance only in angle of attack α_o (due, for example, to a gust); elevator fixed; all other disturbances zero $(u_o' = \theta_o = q_o = 0)$

(2) Change in thrust, thus $u_o' \neq 0$; elevator fixed; other disturbances zero $(\alpha_o = \theta_o = q_o = 0)$

(3) Disturbance caused by elevator motion; other disturbances zero $(u_o' = \alpha_o = \theta_o = q_o = 0)$

Each of these assumptions greatly simplifies the equations and shortens the computations, because many terms in equations developed for a general case will vanish.

The Laplace transformation of equation (3a) can be written as follows:

$$\left.\begin{aligned}
&\left(2\mu_c P - C_{X_{u'}}\right)\bar{u}' - \left(C_{X_\alpha} + \frac{1}{2}C_{X_{D\alpha}}P\right)\bar{\alpha} - \left(C_{X_\theta} + \frac{1}{2}C_{X_q}P\right)\bar{\theta} \\
&\qquad = C_{X_{\delta_e}}\delta_m\frac{a}{P^2 + a^2} + 2\mu_c u_o' - \frac{1}{2}C_{X_{D\alpha}}\alpha_o - \frac{1}{2}C_{X_q}\theta_o \\
&-C_{Z_{u'}}\bar{u}' + \left[\left(2\mu_c - \frac{1}{2}C_{Z_{D\alpha}}\right)P - C_{Z_\alpha}\right]\bar{\alpha} - \left[\left(2\mu_c + \frac{1}{2}C_{Z_q}\right)P + C_{Z_\theta}\right]\bar{\theta} \\
&\qquad = C_{Z_{\delta_e}}\delta_m\frac{a}{P^2 + a^2} + \left(2\mu_c - \frac{1}{2}C_{Z_{D\alpha}}\right)\alpha_o - \left(2\mu_c + \frac{1}{2}C_{Z_q}\right)\theta_o \\
&-C_{m_{u'}}\bar{u}' - \left(C_{m_\alpha} + \frac{1}{2}C_{m_{D\alpha}}P\right)\bar{\alpha} + \left(2\mu_c K_Y^2 P^2 - \frac{1}{2}C_{m_q}P - C_{m_\theta}\right)\bar{\theta} \\
&\qquad = \left(C_{m_{\delta_e}} + \frac{1}{2}C_{m_{D\delta_e}}P\right)\delta_m\frac{a}{P^2 + a^2} - \frac{1}{2}C_{m_{D\alpha}}\alpha_o + 2\mu_c K_Y^2\left(P\theta_o + q_o\right) \\
&\qquad\quad - \frac{1}{2}C_{m_q}\theta_o
\end{aligned}\right\} \quad (5)$$

Equation (5) can be expressed in a shorter form as

$$\left.\begin{aligned} a_{11}\overline{u}' + a_{12}\overline{\alpha} + a_{13}\overline{\theta} &= G_1 \\ a_{21}\overline{u}' + a_{22}\overline{\alpha} + a_{23}\overline{\theta} &= G_2 \\ a_{31}\overline{u}' + a_{32}\overline{\alpha} + a_{33}\overline{\theta} &= G_3 \end{aligned}\right\} \qquad (5a)$$

where

$$\left.\begin{aligned} a_{11} &= 2\mu_c P - C_{X_{u'}} \\ a_{12} &= -\left(C_{X_\alpha} + \frac{1}{2} C_{X_{D\alpha}} P\right) \\ a_{13} &= -\left(C_{X_\theta} + \frac{1}{2} C_{X_q} P\right) \\ a_{21} &= -C_{Z_{u'}} \\ a_{22} &= \left(2\mu_c - \frac{1}{2} C_{Z_{D\alpha}}\right)P - C_{Z_\alpha} \\ a_{23} &= -\left[\left(2\mu_c + \frac{1}{2} C_{Z_q}\right)P + C_{Z_\theta}\right] \\ a_{31} &= -C_{m_{u'}} \\ a_{32} &= -\left(C_{m_\alpha} + \frac{1}{2} C_{m_{D\alpha}} P\right) \\ a_{33} &= 2\mu_c {K_Y}^2 P^2 - \frac{1}{2} C_{m_q} P - C_{m_\theta} \end{aligned}\right\} \qquad (6)$$

$$G_1 = \frac{aC_{X_{\delta_e}}\delta_m + \left(2\mu_c u_o' - \frac{1}{2}C_{X_{D\alpha}}\alpha_o - \frac{1}{2}C_{X_q}\theta_o\right)\left(P^2 + a^2\right)}{\left(P^2 + a^2\right)}$$

$$= \frac{g_1(P)}{P^2 + a^2}$$

$$= \frac{g_1(P)}{(P + ia)(P - ia)} \tag{7a}$$

$$G_2 = \frac{aC_{Z_{\delta_e}}\delta_m + \left[\left(2\mu_c - \frac{1}{2}C_{Z_{D\alpha}}\right)\alpha_o - \left(2\mu_c + \frac{1}{2}C_{Z_q}\right)\theta_o\right]\left(P^2 + a^2\right)}{\left(P^2 + a^2\right)}$$

$$= \frac{g_2(P)}{(P + ia)(P - ia)} \tag{7b}$$

$$G_3 = \frac{aC_{m_{\delta_e}}\delta_m + \frac{a}{2}C_{m_{D\delta_e}}\delta_m P + \left[2\mu_c K_Y^2\left(P\theta_o + q_o\right) - \frac{1}{2}C_{m_{D\alpha}}\alpha_o - \frac{1}{2}C_{m_q}\theta_o\right]\left(P^2 + \alpha^2\right)}{\left(P^2 + a^2\right)}$$

$$= \frac{g_3(P)}{(P + ia)(P - ia)} \tag{7c}$$

Now the system (5) or (5a) represents three simultaneous algebraic equations which can be solved for $\overline{u}'$, $\overline{\alpha}$, and $\overline{\theta}$ by the method of determinants. Thus

$$\overline{u}' = \frac{\overline{\Delta}_1}{\overline{\Delta}} \tag{8a}$$

$$\overline{\alpha} = \frac{\overline{\Delta}_2}{\overline{\Delta}} \tag{8b}$$

$$\overline{\theta} = \frac{\overline{\Delta}_3}{\overline{\Delta}} \tag{8c}$$

where the determinant

$$\bar{\Delta} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{9}$$

The expansion of the determinant $\bar{\Delta}$ results in a quartic equation in P

$$\bar{\Delta} = AP^4 + BP^3 + CP^2 + DP + E \tag{10}$$

which generally has two pairs of complex conjugate roots, namely

$$P_{1,2} = -a \pm ib$$

$$P_{3,4} = -c \pm id$$

Thus

$$\bar{\Delta} = (P + a - ib)(P + a + ib)(P + c - id)(P + c + id) \tag{10a}$$

The coefficients of the quartic (equation (10)) are

$$A = 2\mu_c{}^2 K_Y{}^2 \left(4\mu_c - C_{Z_{D\alpha}}\right)$$

$$B = \mu_c\Big[K_Y^2\left(C_{X_{u'}}C_{Z_{D\alpha}} - 4C_{X_{u'}}\mu_c - 4C_{Z_\alpha}\mu_c - C_{Z_{u'}}C_{X_{D\alpha}}\right) + \frac{1}{2}\left(C_{Z_{D\alpha}}C_{m_q} - C_{m_{D\alpha}}C_{Z_q}\right) - 2\mu_c\left(C_{m_q} + C_{m_{D\alpha}}\right)\Big]$$

$$\begin{aligned} C = \Big[&\mu_c\Big(C_{X_{u'}}C_{m_q} + C_{m_q}C_{Z_\alpha} - 4C_{m_\theta}\mu_c + C_{Z_{D\alpha}}C_{m_\theta} - 4C_{m_\alpha}\mu_c + C_{m_{D\alpha}}C_{X_{u'}} \\ &- C_{m_\alpha}C_{Z_q} - C_{m_{D\alpha}}C_{Z_\theta} - C_{m_{u'}}C_{X_{D\alpha}} - C_{m_{u'}}C_{X_q}\Big) + \frac{1}{4}\Big(C_{m_{D\alpha}}C_{X_{u'}}C_{Z_q} \\ &- C_{m_q}C_{X_{u'}}C_{Z_{D\alpha}} + C_{m_q}C_{Z_{u'}}C_{X_{D\alpha}} - C_{X_q}C_{Z_{u'}}C_{m_{D\alpha}} - C_{Z_q}C_{m_{u'}}C_{X_{D\alpha}} \\ &+ C_{Z_{D\alpha}}C_{X_q}C_{m_{u'}}\Big) + 2\mu_c K_Y^2\left(C_{X_{u'}}C_{Z_\alpha} - C_{Z_{u'}}C_{X_\alpha}\right)\Big] \end{aligned}$$

$$\begin{aligned} D = \Big[&\frac{1}{2}\Big(C_{m_\alpha}C_{X_{u'}}C_{Z_q} - C_{m_q}C_{X_{u'}}C_{Z_\alpha} - C_{m_\theta}C_{X_{u'}}C_{Z_{D\alpha}} + C_{m_{D\alpha}}C_{X_{u'}}C_{Z_\theta} + C_{m_q}C_{Z_{u'}}C_{X_\alpha} \\ &+ C_{m_\theta}C_{Z_{u'}}C_{X_{D\alpha}} - C_{X_q}C_{Z_{u'}}C_{m_\alpha} - C_{X_\theta}C_{Z_{u'}}C_{m_{D\alpha}} - C_{Z_q}C_{m_{u'}}C_{X_\alpha} - C_{Z_\theta}C_{m_{u'}}C_{X_{D\alpha}} \\ &+ C_{Z_{D\alpha}}C_{m_{u'}}C_{X_\theta} + C_{m_{u'}}C_{Z_\alpha}C_{X_q}\Big) + 2\mu_c\Big(C_{m_\theta}C_{X_{u'}} + C_{m_\theta}C_{Z_\alpha} \\ &+ C_{m_\alpha}C_{X_{u'}} - C_{m_\alpha}C_{Z_\theta} - C_{m_{u'}}C_{X_\alpha} - C_{m_{u'}}C_{X_\theta}\Big)\Big] \end{aligned}$$

$$E = C_{X_{u'}}\left(C_{m_\alpha}C_{Z_\theta} - C_{m_\theta}C_{Z_\alpha}\right) + C_{Z_{u'}}\left(C_{m_\theta}C_{X_\alpha} - C_{m_\alpha}C_{X_\theta}\right) + C_{m_{u'}}\left(C_{Z_\alpha}C_{X_\theta} - C_{Z_\theta}C_{X_\alpha}\right)$$

The other determinants are

$$\bar{\Delta}_1 = \begin{vmatrix} G_1 & a_{12} & a_{13} \\ G_2 & a_{22} & a_{23} \\ G_3 & a_{32} & a_{33} \end{vmatrix} \qquad (11)$$

$$\bar{\Delta}_2 = \begin{vmatrix} a_{11} & G_1 & a_{13} \\ a_{21} & G_2 & a_{23} \\ a_{31} & G_3 & a_{33} \end{vmatrix} \qquad (12)$$

$$\bar{\Delta}_3 = \begin{vmatrix} a_{11} & a_{12} & G_1 \\ a_{21} & a_{22} & G_2 \\ a_{31} & a_{32} & G_3 \end{vmatrix} \qquad (13)$$

When expanded the determinants can be written

$$\bar{\Delta}_1 = G_1 f_{11}(P) - G_2 f_{12}(P) + G_3 f_{13}(P) \qquad (11a)$$

$$\bar{\Delta}_2 = -G_1 f_{21}(P) + G_2 f_{22}(P) - G_3 f_{23}(P) \qquad (12a)$$

$$\bar{\Delta}_3 = G_1 f_{31}(P) - G_2 f_{32}(P) + G_3 f_{33}(P) \qquad (13a)$$

where the minors $f_{ij}(P)$ are

$$f_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad f_{12} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad f_{13} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$f_{21} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad f_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \qquad f_{23} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$f_{31} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \qquad f_{32} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad f_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

After expanding, there result

$$\begin{aligned} f_{11}(P) = \Bigg\{ & \left[K_Y{}^2\mu_c\left(4\mu_c - C_{Z_{D\alpha}}\right)\right] P^3 \\ & + \left[C_{m_q}\left(\frac{1}{4} C_{Z_{D\alpha}} - \mu_c\right) - C_{m_{D\alpha}}\left(\frac{1}{4} C_{Z_q} + \mu_c\right) - 2C_{Z_\alpha}\mu_c K_Y{}^2\right] P^2 \\ & + \left[C_{m_\theta}\left(\frac{1}{2} C_{Z_{D\alpha}} - 2\mu_c\right) - C_{m_\alpha}\left(\frac{1}{2} C_{Z_q} + 2\mu_c\right) + \frac{1}{2}\left(C_{Z_\alpha}C_{m_q} - C_{Z_\theta}C_{m_{D\alpha}}\right)\right] P \\ & + \left(C_{Z_\alpha}C_{m_\theta} - C_{Z_\theta}C_{m_\alpha}\right) \Bigg\} \end{aligned} \tag{14}$$

The other determinants are

$$\bar{\Delta}_1 = \begin{vmatrix} G_1 & a_{12} & a_{13} \\ G_2 & a_{22} & a_{23} \\ G_3 & a_{32} & a_{33} \end{vmatrix} \tag{11}$$

$$\bar{\Delta}_2 = \begin{vmatrix} a_{11} & G_1 & a_{13} \\ a_{21} & G_2 & a_{23} \\ a_{31} & G_3 & a_{33} \end{vmatrix} \tag{12}$$

$$\bar{\Delta}_3 = \begin{vmatrix} a_{11} & a_{12} & G_1 \\ a_{21} & a_{22} & G_2 \\ a_{31} & a_{32} & G_3 \end{vmatrix} \tag{13}$$

When expanded the determinants can be written

$$\bar{\Delta}_1 = G_1 f_{11}(P) - G_2 f_{12}(P) + G_3 f_{13}(P) \tag{11a}$$

$$\bar{\Delta}_2 = -G_1 f_{21}(P) + G_2 f_{22}(P) - G_3 f_{23}(P) \tag{12a}$$

$$\bar{\Delta}_3 = G_1 f_{31}(P) - G_2 f_{32}(P) + G_3 f_{33}(P) \tag{13a}$$

where the minors $f_{ij}(P)$ are

$$f_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad f_{12} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad f_{13} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$f_{21} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad f_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \qquad f_{23} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$f_{31} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \qquad f_{32} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad f_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

After expanding, there result

$$\begin{aligned} f_{11}(P) = \Bigg\{ & \left[K_Y{}^2\mu_c\left(4\mu_c - C_{Z_{D\alpha}}\right)\right] P^3 \\ & + \left[C_{m_q}\left(\frac{1}{4} C_{Z_{D\alpha}} - \mu_c\right) - C_{m_{D\alpha}}\left(\frac{1}{4} C_{Z_q} + \mu_c\right) - 2C_{Z_\alpha}\mu_c K_Y{}^2\right] P^2 \\ & + \left[C_{m_\theta}\left(\frac{1}{2} C_{Z_{D\alpha}} - 2\mu_c\right) - C_{m_\alpha}\left(\frac{1}{2} C_{Z_q} + 2\mu_c\right) + \frac{1}{2}\left(C_{Z_\alpha}C_{m_q} - C_{Z_\theta}C_{m_{D\alpha}}\right)\right] P \\ & + \left(C_{Z_\alpha}C_{m_\theta} - C_{Z_\theta}C_{m_\alpha}\right)\Bigg\} \end{aligned} \qquad (14)$$

$$f_{12}(P) = \left\{\left(-C_{X_{D\alpha}}\mu_c K_Y^2\right)P^3 + \left[-2C_{X_\alpha}\mu_c K_Y^2 + \frac{1}{4}\left(C_{X_{D\alpha}}C_{m_q} - C_{X_q}C_{m_{D\alpha}}\right)\right]P^2 + \frac{1}{2}\left(C_{X_\alpha}C_{m_q} + C_{X_{D\alpha}}C_{m_\theta} - C_{X_\theta}C_{m_{D\alpha}} - C_{X_q}C_{m_\alpha}\right)P + \left(C_{X_\alpha}C_{m_\theta} - C_{X_\theta}C_{m_\alpha}\right)\right\} \tag{15}$$

$$f_{13}(P) = \left\{\left[C_{X_{D\alpha}}\left(\frac{1}{4}C_{Z_q} + \mu_c\right) + C_{X_q}\left(\mu_c - \frac{1}{4}C_{Z_{D\alpha}}\right)\right]P^2 + \left[C_{X_\alpha}\left(\frac{1}{2}C_{Z_q} + 2\mu_c\right) + C_{X_\theta}\left(2\mu_c - \frac{1}{2}C_{Z_{D\alpha}}\right) + \frac{1}{2}\left(C_{X_{D\alpha}}C_{Z_\theta} - C_{X_q}C_{Z_\alpha}\right)\right]P + \left(C_{X_\alpha}C_{Z_\theta} - C_{X_\theta}C_{Z_\alpha}\right)\right\} \tag{16}$$

$$f_{21}(P) = \left\{\left(-2C_{Z_{u'}}\mu_c K_Y^2\right)P^2 - \left[C_{m_{u'}}\left(\frac{1}{2}C_{Z_q} + 2\mu_c\right) - \frac{1}{2}C_{Z_{u'}}C_{m_q}\right]P - \left(C_{Z_\theta}C_{m_{u'}} - C_{Z_{u'}}C_{m_\theta}\right)\right\} \tag{17}$$

$$
\begin{aligned}
f_{22}(P) = \Bigg\{ & \left(4\mu_c{}^2 K_Y{}^2\right)P^3 \\
& - \left[\mu_c\left(2C_{X_{u'}}K_Y{}^2 + C_{m_q}\right)\right]P^2 \\
& + \left[\frac{1}{2}\left(C_{m_q}C_{X_{u'}} - C_{X_q}C_{m_{u'}}\right) - 2C_{m_\theta}\mu_c\right]P \\
& + \left(C_{X_{u'}}C_{m_\theta} - C_{m_{u'}}C_{X_\theta}\right)\Bigg\}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
f_{23}(P) = \Bigg\{ & \left[-\mu_c\left(C_{Z_q} + 4\mu_c\right)\right]P^2 \\
& - \left[-C_{X_{u'}}\left(\frac{1}{2}C_{Z_q} + 2\mu_c\right) + 2C_{Z_\theta}\mu_c + \frac{1}{2}C_{Z_{u'}}C_{X_q}\right]P \\
& - \left(C_{X_\theta}C_{Z_{u'}} - C_{X_{u'}}C_{Z_\theta}\right)\Bigg\}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
f_{31}(P) = \Bigg\{ & \left[\frac{1}{2}C_{m_{D\alpha}}C_{Z_{u'}} + C_{m_{u'}}\left(2\mu_c - \frac{1}{2}C_{Z_{D\alpha}}\right)\right]P \\
& + \left(C_{Z_{u'}}C_{m_\alpha} - C_{m_{u'}}C_{Z_\alpha}\right)\Bigg\}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
f_{32}(P) = \Bigg[& -\left(C_{m_{D\alpha}}\mu_c\right)P^2 \\
& - \left(2\mu_c C_{m_\alpha} - \frac{1}{2}C_{X_{u'}}C_{m_{D\alpha}} + \frac{1}{2}C_{m_{u'}}C_{X_{D\alpha}}\right)P \\
& - \left(C_{m_{u'}}C_{X_\alpha} - C_{X_{u'}}C_{m_\alpha}\right)\Bigg]
\end{aligned} \tag{21}
$$

$$f_{33}(P) = \left\{\left[\mu_c\left(4\mu_c - C_{Z_{D\alpha}}\right)\right]P^2 + \left[C_{X_{u'}}\left(\frac{1}{2}C_{Z_{D\alpha}} - 2\mu_c\right) - 2C_{Z_\alpha}\mu_c - \frac{1}{2}C_{Z_{u'}}C_{X_{D\alpha}}\right]P + \left(C_{X_{u'}}C_{Z_\alpha} - C_{Z_{u'}}C_{X_\alpha}\right)\right\} \tag{22}$$

The solution of equations (8a), (8b), and (8c), which will result in a time history of $\bar{u}'$, $\bar{\alpha}$, and $\bar{\theta}$, respectively, as a function of s_c, can be obtained from the Heaviside expansion theorem when there are simple poles (reference 6). This expansion theorem is an efficient method of finding the inverse Laplace transform of the quotient of two polynomials $\frac{f(p)}{F(p)}$. If, for example,

$$\bar{u}' = \frac{\bar{\Delta}_1}{\bar{\Delta}} = \frac{f(p)}{F(p)}$$

where $f(p)$ and $F(p)$ are polynomials with no common factors and the degree of $f(p)$ is lower than that of $F(p)$, then for the case of simple poles and distinct roots

$$u' = \sum_{n=1}^{m} \frac{f(\lambda_n)}{F'(\lambda_n)} e^{\lambda_n s_c}$$

where λ_n are the linear and distinct roots of $F(p)$ set equal to zero. The Heaviside expansion theorem is modified as indicated in reference 6 if any of the roots of $F(p) = 0$ are repeated linear factors. It is important to note that the expression for the Heaviside expansion theorem given here is different from the expression given in reference 4 because of the different transforms of functions that are used in the Heaviside operational calculus and Laplace transformation. However, if a problem is consistently followed through by either one of these two operational methods, identical solutions will be obtained.

The application of the inversion theorem of Laplace transformation to the solution of equations (8a), (8b), and (8c) by computing residues is shown in appendix C.

Lateral Motion

The nondimensional linearized NACA standard equations of motion are:

Sideslip:

$$2\mu_b\left(D_b\beta + D_b\psi\right) = C_{Y_\delta}\delta + C_{Y_\beta}\beta + \frac{1}{2}C_{Y_p}D_b\phi + C_L\phi + \left(C_L \tan\gamma\right)\psi + \frac{1}{2}C_{Y_r}D_b\psi$$

Roll:

$$2\mu_b\left(K_X{}^2D_b{}^2\phi + K_{XZ}D_b{}^2\psi\right) = C_{l_\delta}\delta + C_{l_\beta}\beta + \frac{1}{2}C_{l_p}D_b\phi + \frac{1}{2}C_{l_r}D_b\psi$$

Yaw:

$$2\mu_b\left(K_Z{}^2D_b{}^2\psi + K_{XZ}D_b{}^2\phi\right) = C_{n_\delta}\delta + C_{n_\beta}\beta + \frac{1}{2}C_{n_p}D_b\phi + \frac{1}{2}C_{n_r}D_b\psi \qquad (23)$$

It is important to note that from the standpoint of mechanics, K_{XZ} should be defined as $K_{XZ} = -\left({K_{Z_o}}^2 - {K_{X_o}}^2\right)\sin\eta\cos\eta$ in a right-hand system of axes. However, the definition of K_{XZ} as presented in the symbol list is used in the paper to conform with recent NACA standard equations of motion.

The Laplace transformation is demonstrated for the case in which the control-surface motion can be simulated by the sine function

$$\delta = \delta_m \sin as_b \qquad (24)$$

where δ_m is the amplitude. (In most cases δ_m is assumed to be 1.) Rearranging and substituting for δ give:

$$\left.\begin{aligned}
&\left(2\mu_b D_b - C_{Y_\beta}\right)\beta - \left(\frac{1}{2} C_{Y_p} D_b + C_L\right)\phi + \left(2\mu_b D_b - \frac{1}{2} C_{Y_r} D_b - C_L \tan \gamma\right)\psi = C_{Y_\delta}\delta_m \sin as_b \\
&-C_{l_\beta}\beta + \left(2\mu_b K_X{}^2 D_b{}^2 - \frac{1}{2} C_{l_p} D_b\right)\phi - \left(\frac{1}{2} C_{l_r} D_b - 2\mu_b K_{XZ} D_b{}^2\right)\psi = C_{l_\delta}\delta_m \sin as_b \\
&-C_{n_\beta}\beta - \left(\frac{1}{2} C_{n_p} D_b - 2\mu_b K_{XZ} D_b{}^2\right)\phi + \left(2\mu_b K_Z{}^2 D_b{}^2 - \frac{1}{2} C_{n_r} D_b\right)\psi = C_{n_\delta}\delta_m \sin as_b
\end{aligned}\right\} \quad (23a)$$

If the initial conditions (for $s_b = 0$) are β_o, ϕ_o, ψ_o, p_o, and r_o and the trim position is assumed as zero (thus $\delta_o = 0$), the Laplace transformation (with P substituted for p to avoid confusion with angular velocity) for equation (23a) can be written as follows:[1]

$$\left.\begin{aligned}
&\left(2\mu_b P - C_{Y_\beta}\right)\bar{\beta} - \left(\frac{1}{2} C_{Y_p} P + C_L\right)\bar{\phi} + \left(2\mu_b P - \frac{1}{2} C_{Y_r} P - C_L \tan \gamma\right)\bar{\psi} = C_{Y_\delta}\delta_m \frac{a}{P^2 + a^2} \\
&\qquad + 2\mu_b\beta_o - \frac{1}{2} C_{Y_p}\phi_o + \left(2\mu_b - \frac{1}{2} C_{Y_r}\right)\psi_o \\
&-C_{l_\beta}\bar{\beta} + \left(2\mu_b K_X{}^2 P^2 - \frac{1}{2} C_{l_p} P\right)\bar{\phi} - \left(\frac{1}{2} C_{l_r} P - 2\mu_b K_{XZ} P^2\right)\bar{\psi} = C_{l_\delta}\delta_m \frac{a}{P^2 + a^2} \\
&\qquad + 2\mu_b K_X{}^2\left(P\phi_o + p_o\right) - \frac{1}{2} C_{l_p}\phi_o - \frac{1}{2} C_{l_r}\psi_o + 2\mu_b K_{XZ}\left(P\psi_o + r_o\right) \\
&-C_{n_\beta}\bar{\beta} - \left(\frac{1}{2} C_{n_p} P - 2\mu_b K_{XZ} P^2\right)\bar{\phi} + \left(2\mu_b K_Z{}^2 P^2 - \frac{1}{2} C_{n_r} P\right)\bar{\psi} = C_{n_\delta}\delta_m \frac{a}{P^2 + a^2} \\
&\qquad - \frac{1}{2} C_{n_p}\phi_o + 2\mu_b K_{XZ}\left(P\phi_o + p_o\right) + 2\mu_b K_Z{}^2\left(P\psi_o + r_o\right) - \frac{1}{2} C_{n_r}\psi_o
\end{aligned}\right\} \quad (25)$$

Equations (25) can be expressed in shorter form as

$$\left.\begin{aligned}
b_{11}\bar{\beta} + b_{12}\bar{\phi} + b_{13}\bar{\psi} &= E_1 \\
b_{21}\bar{\beta} + b_{22}\bar{\phi} + b_{23}\bar{\psi} &= E_2 \\
b_{31}\bar{\beta} + b_{32}\bar{\phi} + b_{33}\bar{\psi} &= E_3
\end{aligned}\right\} \quad (25a)$$

---

[1]For practical engineering purposes a simplified case is of interest, namely response to a horizontal gust ψ_o, while $\beta_o = \phi_o = p_o = r_o = \delta = 0$.

where

$$
\left.
\begin{aligned}
b_{11} &= 2\mu_b P - C_{Y_\beta} \\
b_{12} &= -\left(\frac{1}{2} C_{Y_p} P + C_L\right) \\
b_{13} &= \left(2\mu_b P - \frac{1}{2} C_{Y_r} P - C_L \tan \gamma\right) \\
b_{21} &= -C_{l_\beta} \\
b_{22} &= \left(2\mu_b K_X{}^2 P^2 - \frac{1}{2} C_{l_p} P\right) \\
b_{23} &= -\left(\frac{1}{2} C_{l_r} P - 2\mu_b K_{XZ} P^2\right) \\
b_{31} &= -C_{n_\beta} \\
b_{32} &= -\left(\frac{1}{2} C_{n_p} P - 2\mu_b K_{XZ} P^2\right) \\
b_{33} &= \left(2\mu_b K_Z{}^2 P^2 - \frac{1}{2} C_{n_r} P\right)
\end{aligned}
\right\}
\tag{26}
$$

$$
\begin{aligned}
H_1 &= \frac{C_{Y_\delta} \delta_m a + \left[2\mu_b \beta_o - \frac{1}{2} C_{Y_p} \phi_o - \left(\frac{1}{2} C_{Y_r} - 2\mu_b\right) \psi_o\right] (P^2 + a^2)}{(P^2 + a^2)} \\
&= \frac{h_1(P)}{P^2 + a^2} \\
&= \frac{h_1(P)}{(P + ia)(P - ia)}
\end{aligned}
\tag{27a}
$$

$$H_2 = \frac{C_{l_\delta}\delta_m a + \left[2\mu_b K_X{}^2\left(P\phi_o + p_o\right) - \frac{1}{2} C_{l_p}\phi_o - \frac{1}{2} C_{l_r}\psi_o + 2\mu_b K_{XZ}\left(P\psi_o + r_o\right)\right]\left(P^2 + a^2\right)}{P^2 + a^2}$$

$$= \frac{h_2(P)}{P^2 + a^2}$$

$$= \frac{h_2(P)}{(P + ia)(P - ia)} \tag{27b}$$

$$H_3 = \frac{C_{n_\delta}\delta_m a + \left[-\frac{1}{2} C_{n_p}\phi_o + 2\mu_b K_{XZ}\left(P\phi_o + p_o\right) + 2\mu_b K_Z{}^2\left(P\psi_o + r_o\right) - \frac{1}{2} C_{n_r}\psi_o\right]\left(P^2 + a^2\right)}{P^2 + a^2}$$

$$= \frac{h_3(P)}{P^2 + a^2}$$

$$= \frac{h_3(P)}{(P + ia)(P - ia)} \tag{27c}$$

Now equations (25) or equations (25a) represent three simultaneous algebraic equations which will be solved for $\overline{\beta}$, $\overline{\phi}$, and $\overline{\psi}$ by the method of determinants

$$\overline{\beta} = \frac{\overline{\Delta}_1}{\overline{\Delta}} \tag{28a}$$

$$\overline{\phi} = \frac{\overline{\Delta}_2}{\overline{\Delta}} \tag{28b}$$

$$\overline{\psi} = \frac{\overline{\Delta}_3}{\overline{\Delta}} \tag{28c}$$

where the determinants are

$$\bar{\Delta} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} \tag{29}$$

$$\bar{\Delta}_1 = \begin{vmatrix} H_1 & b_{12} & b_{13} \\ H_2 & b_{22} & b_{23} \\ H_3 & b_{32} & b_{33} \end{vmatrix} \tag{30}$$

$$\bar{\Delta}_2 = \begin{vmatrix} b_{11} & H_1 & b_{13} \\ b_{21} & H_2 & b_{23} \\ b_{31} & H_3 & b_{33} \end{vmatrix} \tag{31}$$

$$\bar{\Delta}_3 = \begin{vmatrix} b_{11} & b_{12} & H_1 \\ b_{21} & b_{22} & H_2 \\ b_{31} & b_{32} & H_3 \end{vmatrix} \tag{32}$$

If the values of equations (26) are substituted into equation (29) and the determinant $\bar{\Delta}$ is expanded, a quartic equation is obtained

$$\bar{\Delta} = AP^4 + BP^3 + CP^2 + DP + E = 0 \tag{33}$$

which generally has a pair of complex conjugate roots and two real roots

$$P_{1,2} = -a \pm ib \text{ (Dutch-roll oscillation)}$$

$$P_3 = -c \text{ (spiral mode)}$$

$$P_4 = -d \text{ (rolling subsidence)}$$

Thus

$$\bar{\Delta} = (P + a - ib)(P + a + ib)(P + c)(P + d) \tag{33a}$$

The coefficients of equation (33) are

$$A = 8\mu_b^3\left(K_X^2K_Z^2 - K_{XZ}^2\right)$$

$$B = -2\mu_b^2\left[K_X^2\left(2K_Z^2C_{Y_\beta} + C_{n_r}\right) + K_Z^2C_{l_p} - K_{XZ}\left(C_{n_p} + C_{l_r} + 2K_{XZ}C_{Y_\beta}\right)\right]$$

$$C = \mu_b\left[K_X^2\left(C_{Y_\beta}C_{n_r} + 4\mu_bC_{n_\beta} - C_{Y_r}C_{n_\beta}\right) + K_Z^2\left(C_{Y_\beta}C_{l_p} - C_{Y_p}C_{l_\beta}\right)\right.$$

$$- K_{XZ}\left(C_{Y_\beta}C_{l_r} + C_{Y_\beta}C_{n_p} + 4\mu_bC_{l_\beta} - C_{Y_r}C_{l_\beta} - C_{Y_p}C_{n_\beta}\right)$$

$$\left.+ \frac{1}{2}\left(C_{l_p}C_{n_r} - C_{n_p}C_{l_r}\right)\right]$$

$$D = \left\{-2\mu_b C_L\left[\tan\gamma\left(K_X{}^2 C_{n_\beta} - K_{XZ} C_{l_\beta}\right) + K_Z{}^2 C_{l_\beta} - K_{XZ} C_{n_\beta}\right]\right.$$

$$+ \frac{1}{4}\left[C_{Y_\beta}\left(C_{n_p} C_{l_r} - C_{l_p} C_{n_r}\right) + C_{l_\beta}\left(C_{n_r} C_{Y_p} - C_{Y_r} C_{n_p}\right)\right.$$

$$\left.\left. + C_{n_\beta}\left(C_{Y_r} C_{l_p} - C_{Y_p} C_{l_r}\right)\right] + \mu_b\left(C_{l_\beta} C_{n_p} - C_{n_\beta} C_{l_p}\right)\right\}$$

$$E = \frac{C_L}{2}\left[\tan\gamma\left(C_{n_\beta} C_{l_p} - C_{l_\beta} C_{n_p}\right) + C_{l_\beta} C_{n_r} - C_{n_\beta} C_{l_r}\right]$$

The development of the determinants (30), (31), and (32) gives

$$\overline{\Delta}_1 = H_1 F_{11}(P) - H_2 F_{12}(P) + H_3 F_{13}(P) \qquad (30a)$$

$$\overline{\Delta}_2 = -H_1 F_{21}(P) + H_2 F_{22}(P) - H_3 F_{23}(P) \qquad (31a)$$

$$\overline{\Delta}_3 = H_1 F_{31}(P) - H_2 F_{32}(P) + H_3 F_{33}(P) \qquad (32a)$$

where the minors are

$$F_{11} = \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} \qquad F_{12} = \begin{vmatrix} b_{12} & b_{13} \\ b_{32} & b_{33} \end{vmatrix} \qquad F_{13} = \begin{vmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \end{vmatrix}$$

$$F_{21} = \begin{vmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{vmatrix} \qquad F_{22} = \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix} \qquad F_{23} = \begin{vmatrix} b_{11} & b_{13} \\ b_{21} & b_{23} \end{vmatrix}$$

$$F_{31} = \begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix} \qquad F_{32} = \begin{vmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{vmatrix} \qquad F_{33} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

The expansions yield

$$F_{11}(P) = 4\mu_b{}^2\left(K_X{}^2K_Z{}^2 - K_{XZ}{}^2\right)P^4 - \mu_b\left[K_X{}^2C_{n_r} + K_Z{}^2C_{l_p} - K_{XZ}\left(C_{n_p} + C_{l_r}\right)\right]P^3 + \frac{1}{4}\left(C_{l_p}C_{n_r} - C_{l_r}C_{n_p}\right)P^2 \tag{34}$$

$$F_{12}(P) = \mu_b\left[-K_{XZ}\left(4\mu_b - C_{Y_r}\right) - K_Z{}^2C_{Y_p}\right]P^3 + \left[\frac{1}{4}\left(C_{Y_p}C_{n_r} - C_{Y_r}C_{n_p}\right) - \mu_b\left(2K_Z{}^2C_L - 2K_{XZ}C_L \tan\gamma - C_{n_p}\right)\right]P^2 + \frac{C_L}{2}\left(C_{n_r} - C_{n_p}\tan\gamma\right)P \tag{35}$$

$$F_{13}(P) = \left[\mu_bK_X{}^2\left(-4\mu_b + C_{Y_r}\right) - \mu_bK_{XZ}C_{Y_p}\right]P^3 + \left\{\frac{1}{4}\left(C_{l_r}C_{Y_p} - C_{Y_r}C_{l_p}\right) + \mu_b\left[2C_L\left(K_X{}^2\tan\gamma - K_{XZ}\right) + C_{l_p}\right]\right\}P^2 + \frac{C_L}{2}\left(C_{l_r} - C_{l_p}\tan\gamma\right)P \tag{36}$$

$$F_{21}(P) = -2\mu_b\left(K_Z{}^2C_{l_\beta} - K_{XZ}C_{n_\beta}\right)P^2 + \frac{1}{2}\left(C_{l_\beta}C_{n_r} - C_{n_\beta}C_{l_r}\right)P \tag{37}$$

$$F_{22}(P) = \left\{4\mu_b K_Z{}^2 P^3 - \mu_b\left(2K_Z{}^2 C_{Y_\beta} + C_{n_r}\right)P^2 + \left[\frac{1}{2}\left(C_{n_r}C_{Y_\beta} - C_{Y_r}C_{n_\beta}\right) + 2\mu_b C_{n_\beta}\right]P - C_L \tan\gamma\, C_{n_\beta}\right\} \qquad (38)$$

$$F_{23}(P) = \left[4\mu_b{}^2 K_{XZ} P^3 + \mu_b\left(-2K_{XZ}C_{Y_\beta} - C_{l_r}\right)P^2 + \frac{1}{2}\left(C_{l_r}C_{Y_\beta} - C_{Y_r}C_{l_\beta} + 4\mu_b C_{l_\beta}\right)P - C_L \tan\gamma\, C_{l_\beta}\right] \qquad (39)$$

$$F_{31}(P) = \left[2\mu_b\left(-K_{XZ}C_{l_\beta} + K_X{}^2 C_{n_\beta}\right)P^2 + \frac{1}{2}\left(C_{n_p}C_{l_\beta} - C_{n_\beta}C_{l_p}\right)P\right] \qquad (40)$$

$$F_{32}(P) = \left[4\mu_b{}^2 K_{XZ} P^3 + \mu_b\left(-2K_{XZ}C_{Y_\beta} - C_{n_p}\right)P^2 + \frac{1}{2}\left(C_{n_p}C_{Y_\beta} - C_{n_\beta}C_{Y_p}\right)P - C_L C_{n_\beta}\right] \qquad (41)$$

$$F_{33}(P) = \left[4\mu_b{}^2 K_X{}^2 P^3 - \mu_b\left(2K_X{}^2 C_{Y_\beta} + C_{l_p}\right)P^2 + \frac{1}{2}\left(C_{Y_\beta}C_{l_p} - C_{Y_p}C_{l_\beta}\right)P - C_L C_{l_\beta}\right] \qquad (42)$$

As indicated in the section entitled "Longitudinal Motion" the solution of equations (28a), (28b), and (28c), which will result in a time history of β, ϕ, and ψ, respectively, as a function of s_b, can be obtained from the Heaviside expansion theorem (reference 6) or by computing residues as shown in appendix C.

General Remarks

The method presented in this paper can also be applied to cases in which the airplane is equipped with automatic pilots acting on elevator, rudder, and ailerons. Each automatic pilot is characterized by additional equations of motion. Thus, generally speaking, there are four simultaneous equations for longitudinal motion and five simultaneous equations for lateral motion (two automatic pilots). As before, the Laplace transformation is applied to these equations and the problem is treated according to the method indicated in this paper.

Some transforms of simulating functions for control-surface motion are presented in appendix D and simplified methods of computation of Laplace transformation are given in appendix E.

CONCLUDING REMARKS

The application of the Laplace transformation to the solution of the lateral and longitudinal stability equations has been presented. The expressions for the time history of the motion in response to a sinusoidal control motion were derived for the general case in which the initial conditions, initial displacements and initial velocities, were assumed different from zero.

Ryan Aeronautical Company
Lindbergh Field, San Diego, Calif., July 22, 1949

APPENDIX A

HISTORICAL SKETCH

A short historical sketch on the development of the operational calculus and its application to airplane dynamics is presented.

The fundamentals for the theory of small oscillations about a steady state of motion were developed in 1877 by Routh (references 8 and 9). Then as early as 1903 Bryan applied the mathematical equations of motion of a rigid body to the disturbed motion of an airplane (reference 10). In the following years the mathematical theory remained fundamentally in the form proposed by Bryan, but the method of application was changed as the result of the development of experimental research by the NACA.

During those years many scientists were working on the problems of dynamic stability, not only in the United States but also in Great Britain, France, Belgium, Germany, and other countries. In 1927 the equations of motion were first expressed in dimensionless form by Glauert (reference 11). Jones, Bairstow, Zimmerman, and Millikan (references 1, 2, 12, 13, and 14) also dealt with dynamic stability and their work is well-known to the average engineer in this country.

The need for a means of describing the response of the system (mathematically similar to the system used herein) to the applied disturbance was realized by electrical engineers many years ago. In 1899 Heaviside, impelled by this need, contributed a significant development. In his electromagnetic theory he originally devised his operational calculus for the solution of ordinary linear differential equations with constant coefficients and some of the partial differential equations of applied mathematics. The principles of this method are illustrated in reference 15.

The significance of Heaviside's contributions were not recognized in his lifetime because of the inadequacy of the mathematical treatment and the obscurity of his papers. Bromwich, making use of the theory of functions of a complex variable, explained and established the validity of Heaviside's methods. Bromwich's method consisted of finding the solution of a given differential equation, with initial and boundary conditions, in the form of a complex integral over a suitable path; the choice of the integrand and contour is sometimes difficult. Further research by Carson, Carslaw and Jaeger, and Doetsch (references 16 to 18) resulted in the application of the Laplace transformation to the differential equation. Finally, Doetsch recognized fully the value of the "inversion theorem" for the Laplace transformation. Thus the Laplace transformation is an important step forward in operational mathematics. A complete treatment of the subject of Laplace transformation can be found in references 6, 7, and 17. For some time now it has been recognized that by applying the Laplace transform a better substitute for Heaviside operational methods can be obtained.

Among the early attempts to apply operational calculus to the problems of stability and control was a very fundamental work well-known in this country (reference 4). This paper deals with lateral motion and applies Heaviside operational calculus. Later several papers were written on the dynamic response of the aircraft which also made use of the Heaviside method. Some dealt with tail load variations due to elevator motion (for example, see reference 19). Others dealt with stability with free controls (reference 20), stick forces in maneuvers (reference 21), and the behavior of the airplane equipped with automatic control (reference 22).

APPENDIX B

ILLUSTRATIVE EXAMPLES OF APPLICATION OF LAPLACE TRANSFORMATION

Example I

Example I illustrates the application of Laplace transform to the equation

$$\left(D^2 - 3D + 2\right)x = e^{at} \qquad (t < 0)$$

The initial conditions when $t = 0$ are

$$x = x_o$$

$$Dx = x_1$$

Table I (transform 3) shows that the transform of

$$x(t) = e^{at}$$

is

$$\bar{x}(p) = \frac{1}{(p - a)}$$

Thus the Laplace transformation of the given equation is ($n = 2$ when equation (2) is applied)

$$\left(p^2 - 3p + 2\right)\bar{x} = \frac{1}{p - a} + \left(px_o + x_1\right) - 3x_o$$

Example II

Example II applies the Laplace transform to the equation

$$(D^3 - 2D^2 + D)x = 4 \qquad (t > 0)$$

The initial conditions when $t = 0$ are

$$x_o = 1$$

$$x_1 = 2$$

$$x_2 = -2$$

Table I (transform 1) gives the transform for

$$x(t) = 4$$

as

$$\bar{x}(p) = \frac{4}{p}$$

If the rules of equation (2) are applied, the transform can be written $(n = 3)$ as

$$(p^3 - 2p^2 + p)\bar{x} = \frac{4}{p} + \left(p^2x_o + px_1 + x_2\right) - 2(px_o + x_1) + x_o$$

$$= \frac{4}{p} + (p^2 + 2p - 2) - 2(p + 2) + 1$$

$$= \frac{p^3 - 5p + 4}{p}$$

APPENDIX C

THE INVERSION THEOREM OF LAPLACE TRANSFORMATION

By means of this theorem $x(t)$ can be obtained from its transform $\overline{x}(P)$. If

$$\overline{x}(P) = \int_0^\infty e^{-Pt} x(t)dt \qquad R(P) > 0$$

then

$$x(t) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \overline{x}(\lambda)e^{\lambda t}d\lambda \tag{C1}$$

where γ is a constant greater than the real part of all singularities of $\overline{x}(\lambda)$ and

$$\int_{\gamma-i\infty}^{\gamma+i\infty} = \lim_{w \to \infty} \int_{\gamma-iw}^{\gamma+iw}$$

The path $(\gamma - i\infty, \gamma + i\infty)$ may be replaced by a circle C containing all the poles of the integrand. Then $x(t)$ is equal to $2\pi i$ times the sum of residues at these poles. The method of evaluating the residues is shown at the end of this appendix.

Example I

Let

$$\overline{x} = \frac{P^3 + P - 4}{P^2 - 2P - 3}$$

The inversion theorem, with λ substituted for P in the equation, gives

$$x = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\lambda^3 + \lambda - 4}{\lambda^2 + 2\lambda - 3} e^{\lambda t} d\lambda$$

The denominator has two roots −3 and 1. The residues must be evaluated at two simple poles at $\lambda = 1$ and at $\lambda = -3$ and then summed in order to obtain x.

Example 2

Consider the simultaneous equations

$$(3p + 2)\bar{x} + P\bar{y} = \frac{1}{P}$$

$$P\bar{x} + (4P + 3)\bar{y} = 0$$

which yield

$$\bar{x} = \frac{4P + 3}{P(P + 1)(11P + 6)}$$

$$\bar{y} = \frac{1}{(11P + 6)(P + 1)}$$

Application of the inversion theorem of Laplace transformation gives the solution

$$x = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{4\lambda + 3}{\lambda(\lambda + 1)(11\lambda + 6)} e^{\lambda t} d\lambda$$

$$y = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{(11\lambda + 6)(\lambda + 1)} d\lambda$$

The sums of residues on each pole for x and y give the solution as a function of time.

The inversion theorem for Laplace transformation is now applied to equations (8a), (8b), and (8c). If λ is substituted for P, new determinants Δ, Δ_1, Δ_2, and Δ_3 are obtained from $\overline{\Delta}$, $\overline{\Delta}_1$, $\overline{\Delta}_2$, and $\overline{\Delta}_3$. A time history of u', α, and θ as a function of the parameter s_c is obtained when the inversion theorem is applied to these equations (the path angle γ is then also determined as $\theta = \alpha + \gamma$):

$$u' = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Delta_1}{\Delta} e^{\lambda s_c} d\lambda \tag{C2}$$

$$\alpha = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Delta_2}{\Delta} e^{\lambda s_c} d\lambda \tag{C3}$$

$$\theta = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Delta_3}{\Delta} e^{\lambda s_c} d\lambda \tag{C4}$$

In order to evaluate the integrals (C2), (C3), and (C4), the values of all residues for each integral must be found and summed. This procedure is demonstrated on equation (C2). The parameter λ is substituted for P, equation (10a) is substituted for $\overline{\Delta}$, and equation (11a) is substituted for $\overline{\Delta}_1$. The auxiliary substitutions for $G_1(\lambda)$, $G_2(\lambda)$, and $G_3(\lambda)$ are obtained from equations (7a), (7b), and (7c). The values for $f_{11}(\lambda)$, $f_{12}(\lambda)$, and $f_{13}(\lambda)$ are obtained from equations (14), (15), and (16) and λ is again substituted for P. Then equation (C2) can be written

$$u' = \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\left[g_1(\lambda)f_{11}(\lambda) - g_2(\lambda)f_{22}(\lambda) + g_3(\lambda)f_{33}(\lambda)\right] e^{\lambda s_c} d\lambda}{(\lambda + ia)(\lambda - ia)(\lambda + b - ic)(\lambda + b + ic)(\lambda + d - ie)(\lambda + d + ie)} \qquad \text{(C5)}$$

The residues R must be evaluated for six simple poles: R_1 at pole at $\lambda = -ia$, R_2 at pole at $\lambda = ia$, R_3 at pole at $\lambda = -b + ic$, R_4 at pole at $\lambda = -b - ic$, R_5 at pole at $\lambda = -d + ie$, and R_6 at pole at $\lambda = -d - ie$. Finally $u' = \Sigma R$. With the use of the relations

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

the u' terms can be collected and expressed as sine and cosine terms, as illustrated in the numerical example. If any function other than the sine function were used to simulate elevator motion, there would be no changes in equations (11a), (12a), and (13a) (for $\bar{\Delta}_1$, $\bar{\Delta}_2$, and $\bar{\Delta}_3$) but the expressions for the functions G_1, G_2, G_3 (equations (7a), (7b), and (7c)) would be different. (See appendix D.)

Numerical Example

Let

$$u' = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda s_c}\left(\lambda^2 + 1.4773122\lambda + 134.4908846\right)d\lambda}{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)}$$

The roots of the quartic are

$$\lambda_{1,2} = -0.739517 \pm 11.56351i$$

$$\lambda_{3,4} = -0.016582 \pm 0.044189561i$$

Substituting these values for λ_1, λ_2, λ_3, and λ_4 gives

$$u' = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda s_c}\left(\lambda^2 + 1.4773122\lambda + 134.4908846\right)d\lambda}{(\lambda + 0.739517 - 11.56351i)(\lambda + 0.739517 + 11.56351i)(\lambda + 0.016582 - 0.044189561i)(\lambda + 0.016582 + 0.044189561i)}$$

In this example there are four simple poles. First the residue is computed at the pole

$$\lambda_1 = -0.739517 + 11.56351i$$

Then

$$R_1 = \frac{e^{\lambda_1 s_c}\left(\lambda_1{}^2 + 1.4773122\lambda_1 + 134.4908846\right)}{(\lambda_1 + 0.739517 + 11.56351i)(\lambda_1 + 0.016582 - 0.044189561i)(\lambda_1 + 0.016582 + 0.044189561i)}$$

Substituting the value for λ_1 in the last equation yields

$$R_1 = \frac{e^{\lambda_1 s_c}\left[-133.1676468 - 17.10280966i + 1.4773122(-0.739517 + 11.56351i) + 134.4908846\right]}{-3080.28381i + 386.65737}$$

$$= \frac{e^{\lambda_1 s_c}(0.2307403 - 0.01991i)}{-3080.28381i + 386.65737}$$

With the use of the algebraic manipulation of complex numbers

$$R_1 = e^{\lambda_1 s_c} \frac{(0.2307403 - 0.01991i)(3080.28381i + 386.65737)}{(-3080.28381i + 386.65737)(3080.28381i + 386.65737)}$$

$$= e^{\lambda_1 s_c} \frac{(0.1505458 + 0.70304735i)}{9637.653}$$

$$= (0.00001562 + 0.00007294i)e^{(-0.739517+11.56351i)s_c}$$

At pole $\lambda_2 = -0.739517 - 11.56351i$ the sign of imaginary part is changed, thus

$$R_2 = (0.00001562 - 0.00007294i)e^{(-0.739517-11.56351i)s_c}$$

At pole $\lambda_3 = -0.016582 + 0.04418956i$

$$R_3 = e^{\lambda_3 s_c} \frac{\left(\lambda_3{}^3 + 1.4773122\lambda_3 + 134.4908846\right)}{(\lambda_3 + 0.739517 - 11.56351i)(\lambda_3 + 0.739517 + 11.56351i)(\lambda_3 + 0.0165824 + 0.04418956i)}$$

Substituting for λ_3 yields

$$R_3 = e^{\lambda_3 s_c} \frac{\left[-0.0016775 - 0.0014655 + 1.4773122(-0.016582 + 0.04418956i) + 134.4908846\right]}{11.817384i - 0.0056074823}$$

Following the same procedure as that used for the λ_1 root gives

$$R_3 = e^{\lambda_3 s_c} \frac{(134.46471 + 0.06381628i)(-11.817384i - 0.0056074823)}{(11.817384i - 0.0056074823)(-11.817384i - 0.0056074823)}$$

$$= e^{\lambda_3 s_c} \frac{(-0.000133316 + 1589.0214703i)}{-139.650596}$$

$$= (0.0000009546 - 11.37855i)e^{(-0.016582+0.04418956i)s_c}$$

At pole $\lambda_4 = -0.016582 - 0.04418956i$ the sign of the imaginary part is changed, thus

$$R_4 = (0.0000009546 + 11.37855i)e^{(-0.016582-0.04418956i)s_c}$$

Now $u' = R_1 + R_2 + R_3 + R_4$. Substituting the values for R_1, R_2, R_3, and R_4 yields

$$u' = (0.00001562 + 0.000072948i)e^{(-0.739517+11.5635i)s_c}$$
$$+ (0.00001562 - 0.000072948i)e^{(-0.739517-11.5635i)s_c}$$
$$+ (0.0000009546 - 11.37855i)e^{(-0.016582+0.04418956i)s_c}$$
$$+ (0.0000009546 + 11.37855i)e^{(-0.016582-0.04418956i)s_c}$$

$$u' = e^{-0.739517s_c}\left(0.00001562e^{11.5635is_c} + 0.00001562e^{-11.5635is_c}\right.$$
$$\left. + 0.000072948ie^{11.5635is_c} - 0.000072948ie^{-11.5635is_c}\right)$$
$$+ e^{-0.016582s_c}\left(0.0000009546e^{-0.04418956is_c}\right.$$
$$+ 0.0000009546e^{-0.04418956is_c} - 11.37855ie^{0.04418956is_c}$$
$$\left. + 11.37855ie^{-0.04418956is_c}\right)$$

With the use of the relations

$$e^{ix} + e^{-ix} = 2 \cos x$$

$$e^{ix} - e^{-ix} = 2i \sin x$$

the value of u' can be expressed as

$$u' = e^{-0.739517s_c}(0.00003124 \cos 11.5635s_c - 0.000145896 \sin 11.5635s_c)$$
$$+ e^{-0.016582s_c}(0.0000019092 \cos 0.04418956s_c + 22.75710 \sin 0.04418956s_c)$$

The roots must be computed very exactly to several decimals; otherwise the computation by the method of residues does not check to zero.

Evaluation of Residues

(a) Simple pole:

The residue at a simple pole of the function $f(z)$ is

$$R = \lim_{z \to a} (z - a)f(z) \tag{C6}$$

(b) Multiple pole:

Let

$$g(z) = (z - a)^m f(z) \tag{C7}$$

Then the residue at the pole of mth order is

$$\begin{aligned} R &= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}g(z)}{dz^{m-1}} \\ &= \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} g(z)\right]_{z=a} \end{aligned} \tag{C8}$$

Example 1 (Simple poles).– Let

$$x = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\lambda^3 + \lambda - 4}{(\lambda^2 - 2\lambda + 2)(\lambda^2 + 2\lambda - 3)} e^{\lambda t} d\lambda$$

The denominator has four roots. There are four simple poles $(m = 1)$ at $\lambda_1 = -3$, $\lambda_2 = 1$, and $\lambda_{3,4} = 1 \pm i$. The computation of the residue at a simple pole $\lambda = -3$ is illustrated. If the term under the integral sign is called $f(\lambda)d\lambda$, then according to equation (C7)

$$g(\lambda) = (\lambda - 1)f(\lambda)$$

Thus the residue is

$$R = \left[\frac{(\lambda - 1)(\lambda^3 + \lambda - 4)e^{\lambda t}}{(\lambda^2 - 2\lambda + 2)(\lambda - 1)(\lambda + 3)}\right]_{\lambda=-3}$$

$$= \frac{(-27 - 3 - 4)e^{-3t}}{(9 + 6 + 2)(-3 - 1)}$$

$$= \frac{-34}{-4 \times 17} e^{-3t} = \frac{1}{2} e^{-3t}$$

<u>Example 2 (Double pole)</u>.— Let

$$x(t) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(\lambda x_0 + x_1 + 4x_0)}{(\lambda + 2)^2} e^{\lambda t} d\lambda$$

There is now a double pole $(m = 2)$ at $\lambda = a = -2$. According to equation (C7)

$$g(\lambda) = (\lambda + 2)^2 f(\lambda)$$

which, when substituted into equation (C8), yields

$$R = \frac{1}{1!}\left[\frac{d}{d\lambda} \frac{(\lambda + 2)^2(\lambda x_0 + x_1 + 4x_0)}{(\lambda + 2)^2} e^{\lambda t}\right]_{\lambda=-2}$$

$$= \left[x_0 e^{\lambda t} + t(\lambda x_0 + x_1 + 4x_0)e^{\lambda t}\right]_{\lambda=-2}$$

$$= x_0 e^{-2t} + t(-2x_0 + x_1 + 4x_0)e^{-2t}$$

$$= \left[x_0 + (2x_0 + x_1)t\right]e^{-2t}$$

Example 3 (Triple pole).— Let

$$x = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\lambda}{(\lambda^2 + a^2)^3} e^{\lambda t} d\lambda$$

There are triple poles at $\lambda = ia$ and at $\lambda = -ia$. If $\lambda = ia$, then equation (C7) gives

$$g(\lambda) = (\lambda - ia)^3 \frac{\lambda}{(\lambda^2 + a^2)^3} e^{\lambda t}$$

which, substituted into equation (C8), gives

$$R_1 = \frac{1}{2!}\left[\frac{d^2}{d\lambda^2} \frac{(\lambda - ia)^3 \lambda e^{\lambda t}}{\left[(\lambda + ia)(\lambda - ia)\right]^3}\right]_{\lambda=ia}$$

$$= -\frac{t}{16a^2}\left(t + \frac{i}{a}\right)e^{iat}$$

Similarly, at $\lambda = -ia$

$$R_2 = -\frac{t}{16a^2}\left(t - \frac{i}{a}\right)e^{-iat}$$

$$x = R_1 + R_2$$

With the use of the relations

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

then

$$x = \frac{t}{8a^3}(\sin at - at \cos at)$$

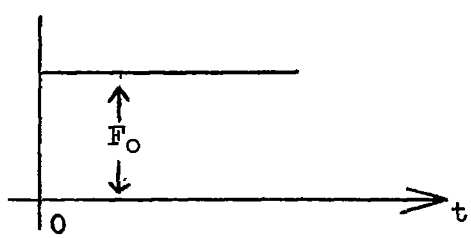
APPENDIX D

TRANSFORMS OF SIMULATING FUNCTIONS FOR CONTROL-SURFACE MOTION

In order to include the control-surface deflection as a function of time (or parameter s_c and s_b) in the equations of motion, the assumed motion must be simulated (for simplicity) by some simple known functions.

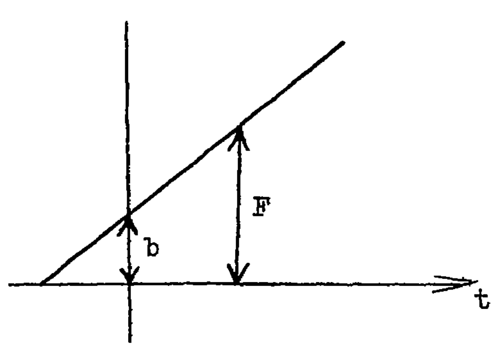
Some examples are given which could be satisfactory in many practical cases. The functions are given here for five examples and their transforms are given in table II.

(1) Step function:



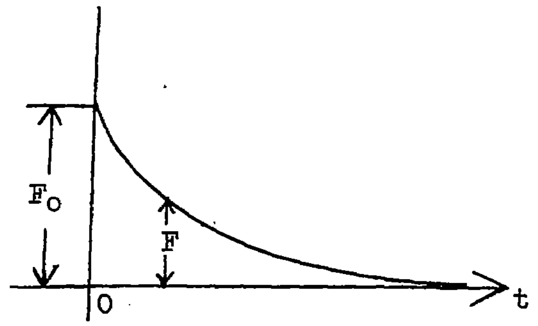
$(F = 0;\ t < 0)$
$(F = F_o;\ t > 0)$

(1a) Straight line:



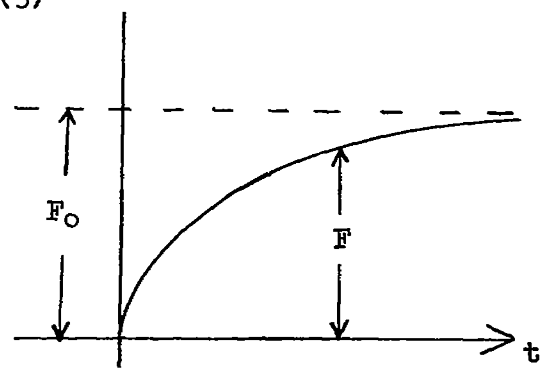
$(F = mt + b)$

(2)



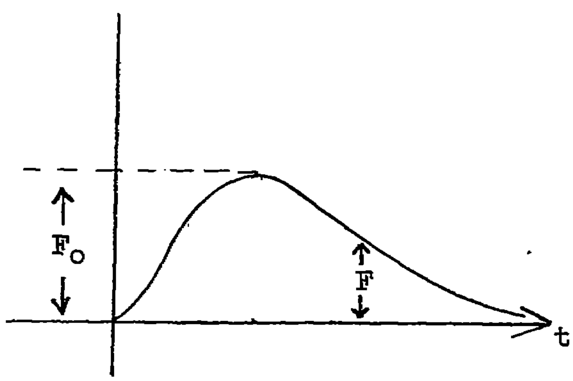
$$\left(F = F_o e^{-at}\right)$$

(3)



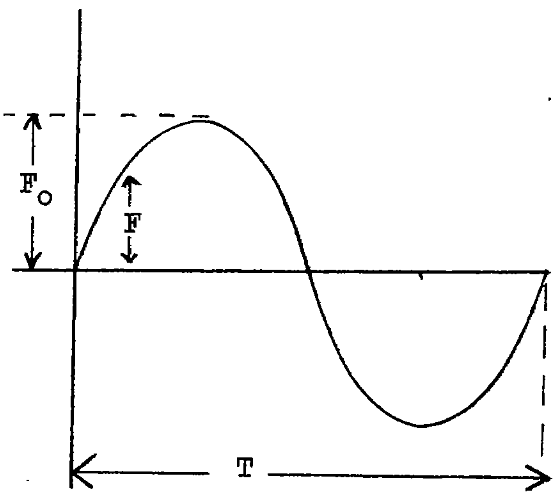
$$\left(F = F_o\left(1 - e^{-at}\right)\right)$$

(4)



$$\left(F = F_o e^{-at}\left(1 - e^{-bt}\right)\right)$$

(5)



$(F = F_o \sin at)$

$\left(a = \frac{2\pi}{T}\right)$

T period

It is often convenient to get the result for unity of control-surface deflection (say one radian); then the result for any arbitrary deflection can be readily computed.

APPENDIX E

SIMPLIFIED METHODS OF COMPUTATION

The computation of the time history of any parameter can be shortened in some cases if the form of the solution and also the value of the function and its first, second, and third derivatives at a given time are known.[1] It is preferable to find these boundary conditions at time $t = 0$ (initial values). On the assumption that the inversion theorem for Laplace transformation gives a parameter y in the form

$$y = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{f(\lambda)e^{\lambda t}d\lambda}{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)\lambda}\ . \ . \ . \qquad \text{(E1)}$$

this method will be applied to the longitudinal and lateral motions.

Longitudinal motion.– The denominator (of the stability equation) has five roots with one root $\lambda_5 = 0$. The other four roots are frequently two conjugate complex pairs:

$$\lambda_{1,2} = \alpha \pm i\beta$$

$$\lambda_{3,4} = \delta \pm i\gamma$$

If equation (1) is assumed to give the solution in the form

$$y = C_o + e^{\alpha t}(C_1 \sin \beta t + C_2 \cos \beta t) + e^{\gamma t}(C_3 \sin \delta t + C_4 \cos \delta t)\ . \ . \ . \qquad \text{(E2)}$$

then $C_o = \lim_{t\to\infty} y$ can be evaluated. Differentiation of equation (E2) gives the derivatives $\dot{y}$, $\ddot{y}$, and $\dddot{y}$.

If the initial values of these functions are known, that is, y_o, $\dot{y}_o$, $\ddot{y}_o$, and $\dddot{y}_o$, at $t = 0$ the constants C_1, C_2, C_3, and C_4 of

[1]The author is indebted for this suggestion to Mr. J. M. Debevoise.

equation (E2) can be obtained by solving four simultaneous algebraic equations which can be written (for $t = 0$) as follows:

$$y_o = C_o + C_2 + C_4 \ . \ . \ . \tag{E3}$$

$$\dot{y}_o = \beta C_1 + \alpha C_2 + \delta C_3 + \gamma C_4 \ . \ . \ . \tag{E4}$$

$$\ddot{y}_o = 2\alpha\beta C_1 + (\alpha^2 - \beta^2)C_2 + 2\gamma\delta C_3 + (\gamma^2 - \delta^2)C_4 \ . \ . \ . \tag{E5}$$

$$\dddot{y}_o = (3\alpha^2\beta - \beta^3)C_1 + (\alpha^3 - 3\alpha\beta^2)C_2 + (3\gamma^2\delta - \delta^3)C_3 + (\gamma^3 - 3\gamma\delta^2)C_4 \ . \ . \ . \tag{E6}$$

The value of $\dddot{y}$ can also be obtained by plotting $\ddot{y}$ against t.

$$\ddot{y} = e^{\alpha t}\left\{\left[(\alpha^2 - \beta^2)C_1 - 2\alpha\beta C_2\right]\sin \beta t + \left[2\alpha\beta C_1 + (\alpha^2 - \beta^2)C_2\right]\cos \beta t\right\} + e^{\gamma t}\left\{\left[(\gamma^2 - \delta^2)C_3 - 2\gamma\delta C_4\right]\sin \delta t + \left[2\gamma\delta C_3 + (\gamma^2 - \delta^2)C_4\right]\cos \delta t\right\}$$

Lateral motion.— For the case of lateral motion the equation has one root $\lambda_5 = 0$, one conjugate complex pair, and two real roots

$$\lambda_{1,2} = \alpha \pm i\beta$$

$$\lambda_3 = \gamma$$

$$\lambda_4 = \delta$$

The time history of any parameter y can be written

$$y = C_o + e^{\lambda t}(C_1 \sin \beta t + C_2 \cos \beta t) + e^{\gamma t}C_3 + e^{\delta t}C_4 \tag{E7}$$

and, as before, $C_o = \lim_{t \to \infty} y$. The initial values of y_o, $\dot{y}_o$, $\ddot{y}_o$, and $\dddot{y}_o$ for $t = 0$ are assumed to be known. Differentiating equation (E7) and substituting for $t = 0$ give four simultaneous algebraic equations from which the constants C_1, C_2, C_3, and C_4 of equation (E2) can be determined:

$$y_o = C_o + C_2 + C_3 + C_4 \ . \ . \ . \tag{E8}$$

$$\dot{y}_o = \beta C_1 + \alpha C_2 + \gamma C_3 + \delta C_4 \ . \ . \ . \tag{E9}$$

$$\ddot{y}_o = 2\alpha\beta C_1 + \left(\alpha^2 - \beta^2\right)C_2 + \gamma^2 C_3 + \delta^2 C_4 \ . \ . \ . \tag{E10}$$

$$\dddot{y}_o = \left(3\alpha^2\beta - \beta^3\right)C_1 + \left(\alpha^3 - 3\alpha\beta^2\right)C_2 + \gamma^3 C_3 + \delta^3 C_4 \ . \ . \ . \tag{E11}$$

If desired the value of $\dddot{y}$ can be obtained by plotting $\ddot{y}$ against time

$$\ddot{y} = e^{\alpha t}\left\{\left[\left(\alpha^2 - \beta^2\right)C_1 - 2\alpha\beta C_2\right]\sin \beta t + \left[2\alpha\beta C_1 + \left(\alpha^2 - \beta^2\right)C_2\right]\cos \beta t\right\}$$
$$+ e^{\gamma t}\gamma^2 C_3 + e^{\delta t}\delta^2 C_4$$

In the cases in which there are simple poles, the method presented in reference 7 can be used. This method is illustrated briefly. It has been seen that any parameter y of the equations of motion (such as α, θ, ψ, ϕ, and so forth) expressed as a function of the operator P can be written as a ratio of two determinants $\overline{A}(P)$ and $\overline{B}(P)$ which are polynomials in P

$$\overline{y}(P) = \frac{\overline{A}(P)}{\overline{B}(P)}$$

Thus the characteristic equation is

$$\overline{B}(P) = 0$$

This equation is a polynomial in P, the highest power of P being q; thus it has q roots: $\lambda_1, \lambda_2, \ldots \lambda_q$. The inversion theorem gives

$$y(t) = \frac{A(t)}{B(t)} = \sum_{k=1}^{k=q} \frac{A(\lambda_k)}{B'(\lambda_k)} e^{\lambda_k t} \qquad (t \geqq 0)$$

where

$$B' = \frac{d}{dP} \overline{B}(P)$$

For multiple poles and special cases, see reference 7 (pp. 152 to 169).

REFERENCES

1. Jones, B. Melvill: Dynamics of the Airplane. The Equations of Motion with Solutions for Small Disturbances from Steady Symmetric Flight. Vol. V of Aerodynamic Theory, div. N, ch. V, secs. 22-31, W. F. Durand, ed., Julius Springer (Berlin), 1935, pp. 135-141.

2. Bairstow, Leonard: Applied Aerodynamics. Second ed., Longmans, Green and Co., 1939.

3. Bryant, L. W., and Williams, D. H.: The Application of the Method of Operators to the Calculation of the Disturbed Motion of an Aeroplane. R. & M. No. 1346, British A.R.C., 1931.

4. Jones, Robert T.: A Simplified Application of the Method of Operators to the Calculation of Disturbed Motions of an Airplane. NACA Rep. 560, 1936.

5. Jones, Robert T.: Calculation of the Motion of an Airplane under the Influence of Irregular Disturbances. Jour. Aero. Sci., vol. 3, no. 12, Oct. 1936, pp. 419-425.

6. Churchill, Ruel V.: Modern Operational Mathematics in Engineering. First ed., McGraw-Hill Book Co., Inc., 1944.

7. Gardner, Murray F., and Barnes, John L.: Transients in Linear Systems Studied by the Laplace Transformation, vol. I. John Wiley & Sons, 1942.

8. Routh, E. J.: The Stability of a Given State of Motion. Adams Prize Essay, London, 1877.

9. Routh, E. J.: Advanced Rigid Dynamics, vol. II. The MacMillan Company, 1905.

10. Bryan, G. H.: Stability in Aviation. Macmillan and Co., Ltd., 1911.

11. Glauert, H.: A Non-Dimensional Form of the Stability Equations of an Aeroplane. R. & M. No. 1093, British A.R.C., 1927.

12. Zimmerman, Charles H.: An Analysis of Longitudinal Stability in Power-Off Flight with Charts for Use in Design. NACA Rep. 521, 1935.

13. Zimmerman, Charles H.: An Analysis of Lateral Stability in Power-Off Flight with Charts for Use in Design. NACA Rep. 589, 1937.

14. Millikan, Clark B.: Notes on Dynamic Stability. Lecture Notes AE 251C. Spring term, 1933-1934.

15. Bush, V.: Operational Circuit Analysis. John Wiley and Sons, Inc., 1929.

16. Carson, John R.: Electric Circuit Theory and the Operational Calculus. McGraw-Hill Book Co., Inc., 1926.

17. Carslaw, H. S., and Jaeger, J. C.: Operational Methods in Applied Mathematics. Oxford Univ. Press, 1941.

18. Doetsch, Gustav: Theorie und Anwendung der Laplace-Transformation. Julius Springer (Berlin), 1937.

19. Pearson, Henry A.: Derivation of Charts for Determining the Horizontal Tail Load Variation with Any Elevator Motion. NACA Rep. 759, 1943.

20. Greenberg, Harry, and Sternfield, Leonard: A Theoretical Investigation of Longitudinal Stability of Airplanes with Free Controls Including Effect of Friction in Control System. NACA Rep. 791, 1944.

21. Jones, Robert T., and Greenberg, Harry: Effect of Hinge-Moment Parameters on Elevator Stick Forces in Rapid Maneuvers. NACA Rep. 798, 1944.

22. Jones, Robert T.: A Method for Studying the Hunting Oscillations of an Airplane with a Simple Type of Automatic Control. NACA Rep. 801, 1944.

TABLE I

SIMPLE LAPLACE TRANSFORMS

Transform	$x(t)$	$\overline{x}(P) = \int_0^\infty e^{-Pt}x(t)dt$
1	1	$\frac{1}{P}$
2	$\frac{t^{n-1}}{(n-1)!}$ (n a positive integer)	$\frac{1}{P^n}$
3	$e^{at}(P > Re(a))$	$\frac{1}{P-a}$
4	sin at	$\frac{a}{P^2+a^2}$
5	cos at	$\frac{P}{P^2+a^2}$
6	sin h at$(P > \|a\|)$	$\frac{a}{P^2-a^2}$
7	cos h at$(P > \|a\|)$	$\frac{P}{P^2-a^2}$
8	$\frac{t}{2a}$ sin at	$\frac{P}{(P^2+a^2)^2}$
9	$\frac{1}{2a^3}$(sin at − at cos at)	$\frac{1}{(P^2+a^2)^2}$

NACA

TABLE II

TRANSFORMS OF SIMULATING FUNCTIONS FOR CONTROL-SURFACE MOTION

Type	Control-surface motion determining $x(t)$	Transform for initial condition $\delta_o = 0$ $\bar{x}(P) = \int_0^\infty e^{-Pt}x(t)dt$	Transform for assumption $F_o = 1$ $\bar{x}(P)$
1	$F_o(t)$ step function	$\frac{F_o}{P}$	$\frac{1}{P}$
1a	$mt + b$	$(m + bP)/P^2$	For $b = 0$, m/P^2
2	F_oe^{-at}	$\frac{F_o}{P + a}$	$\frac{1}{P + a}$
3	$F_o(1 - e^{-at})$	$\frac{F_oa}{P(P + a)}$	$\frac{a}{P(P + a)}$
4	$F_oe^{-at}(1 - e^{-bt})$	$\frac{F_ob}{(P + a)(P + a + b)}$	$\frac{b}{(P + a)(P + a + b)}$
5	$F_o \sin at$	$\frac{F_oa}{P^2 + a^2}$	$\frac{a}{P^2 + a^2}$

NACA

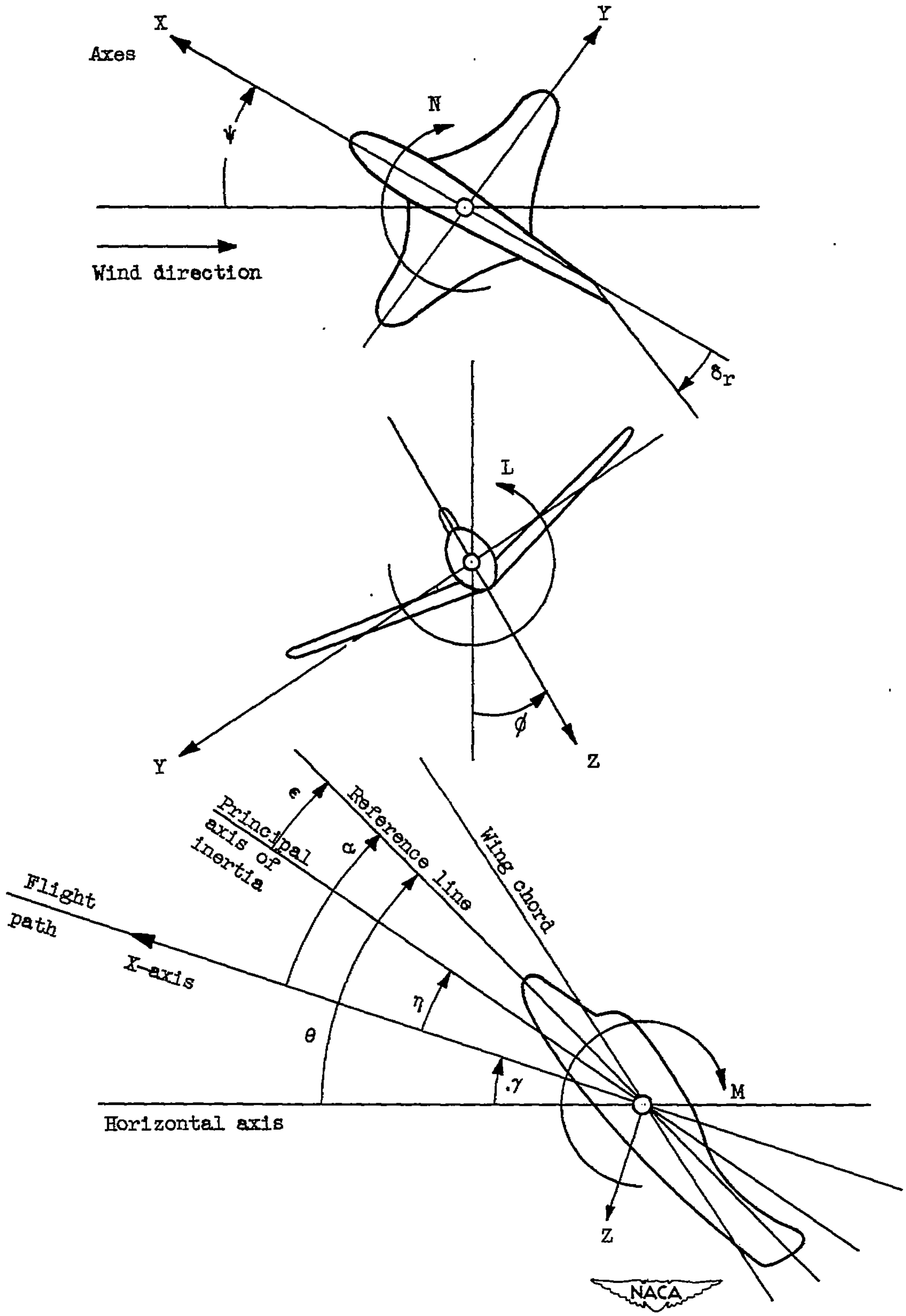


Figure 1.– Axes and notation used. Arrows indicate positive directions of moments, forces, and angular displacements.