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TRANSFORMATIONS OF THE HODOGRAPH FLOW EQUATION AND THE INTRODUCTION OF TWO GENERALIZED POTENTIAL FUNCTIONS

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TRANSFORMATIONS OF THE HODOGRAPH FLOW EQUATION AND THE INTRODUCTION OF TWO GENERALIZED POTENTIAL FUNCTIONS

By Luigi Crocco[1]

SUMMARY

It has been shown that the hodograph equations of motion can be derived in a symmetrical form by the choice of the velocity and the mass velocity as independent variables. The equations obtained by the use of the velocity potential, the stream function, or their transforms as the unknown function are of the same general form and therefore can be treated in the same manner.

Particular sets of solutions have been studied independently of the gas law adopted and some properties of the series obtained by means of these sets have been discussed. Approximate gas laws for which the solutions of the hodograph equations can be easily found have been briefly discussed.

The equations have been further transformed so as to have as independent variables the complex velocity and the complex mass velocity. Two new generalized potential functions can then be introduced that satisfy very compact equations. From these functions, all the quantities concerned with the representation of the motion can be derived by means of formulas independent of the gas law adopted. By means of the generalized potential functions some developments have been performed with the approximate Chaplygin–Von Kármán–Tsien law.

An approximate transonic method has also been suggested.

INTRODUCTION

From a purely mathematical point of view, the ordinary hodograph equations for the stream function or for the velocity-potential function and the equations relating them to the physical coordinates are sufficient for the study of two-dimensional isentropic flows. However, from a more physical point of view they are not very elegant because of their lack of symmetry in contrast with the symmetry of the corresponding relations for the incompressible case.

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Now, the equations that define the velocity potential ϕ and the stream function ψ are

$$\left.\begin{aligned} \phi_n &= 0 \\ \phi_s &= w \\ \psi_s &= 0 \\ \psi_n &= \frac{\rho}{\rho_0} w = m(w) \end{aligned}\right\} \qquad (1)$$

where the subscripts denote the differential quotients with respect to the element of streamline ds or the element of normal dn, obtained from ds by a counter-clockwise rotation of 90°, and w, $\rho(w)$, and $\rho_0 = \rho(0)$ represent, respectively, the velocity, density, and stagnation density. Equations (1), which are symmetrical with respect to ds and dn if $\rho = \rho_0$, conserve their property of symmetry for variable ρ if the mass velocity m is considered in some way the counterpart of w. If the hodograph equations for ϕ and ψ (or for other functions) can be expressed so as to make w and m (instead of the relation connecting them) appear explicitly, the equations will then have a symmetrical form that can be interesting not only from a formal point of view but also from the fact that it can give rise to many possible developments, some of which are illustrated in the present paper. In particular, it is possible to choose as new independent variables the complex velocity and mass velocity and to introduce a new generalized potential function satisfying a very compact equation from which ϕ and ψ, their Legendre transforms X and ω, and the physical coordinates x and y can be deduced by simple differentiations.

HODOGRAPH EQUATIONS

The hodograph equations can be directly deduced as follows. If N and S (fig. 1) are the normal and the subnormal to the streamline and θ is the direction of motion at a point P

$$z = x + iy = e^{i\theta}(S + iN) \tag{2a}$$

$$dz = e^{i\theta}\left[dS - N\,d\theta + i(dN + S\,d\theta)\right]$$

$$= e^{i\theta}(ds + i\,dn) = e^{i\theta}\left(\frac{d\phi}{w} + i\,\frac{d\psi}{m}\right) \tag{2b}$$

where the defining equations (1) have been used in the last step. It follows from equations (2) that

$$\left.\begin{aligned} d\phi &= w(dS - N\,d\theta) \\ d\psi &= m(dN + S\,d\theta) \end{aligned}\right\} \tag{3}$$

These are two relations between exact differentials and therefore can be written as

$$\left.\begin{aligned} dS &= S_w\,dw + S_\theta\,d\theta = \frac{1}{w}\,\phi_w\,dw + \left(\frac{1}{w}\,\phi_\theta + N\right)d\theta \\ dN &= N_m\,dm + N_\theta\,d\theta = \frac{1}{m}\,\psi_m\,dm + \left(\frac{1}{m}\,\psi_\theta - S\right)d\theta \end{aligned}\right\} \tag{4}$$

where m and w are two related variables so that the meaning of partial differentiation with respect to m is

$$\frac{\partial}{\partial m} = \frac{dw}{dm}\frac{\partial}{\partial w} = \frac{1}{m^2}\frac{\partial}{\partial w}$$

Since dS and dN are exact differentials, it follows from equations (4) that

$$\frac{\partial}{\partial\theta}\left(\frac{1}{w}\,\phi_w\right) = \frac{\partial}{\partial w}\left(\frac{1}{w}\,\phi_\theta + N\right)$$

and

$$\frac{\partial}{\partial\theta}\left(\frac{1}{m}\,\psi_m\right) = \frac{\partial}{\partial m}\left(\frac{1}{m}\,\psi_\theta - S\right)$$

so that

$$\left.\begin{aligned} w^2N_w &= \phi_\theta \\ m^2S_m &= -\psi_\theta \end{aligned}\right\} \qquad (5)$$

But from equations (4)

$$mN_m = \psi_m \qquad \text{or} \qquad mN_w = \psi_w$$

and

$$wS_w = \phi_w \qquad \text{or} \qquad wS_m = \phi_m$$

Hence, putting $-w^2 \frac{\partial}{\partial w}$ equal to $\frac{\partial}{\partial\left(\frac{1}{w}\right)}$ and performing the same transformation for m results in the following equations:

$$\left.\begin{aligned} -\phi_\theta &= \frac{1}{m}\frac{\partial\psi}{\partial\left(\frac{1}{w}\right)} \\ \psi_\theta &= \frac{1}{w}\frac{\partial\phi}{\partial\left(\frac{1}{m}\right)} \end{aligned}\right\} \tag{6}$$

which yield the well-known Chaplygin equations for ϕ and ψ in a symmetrical form.

It is seen from equations (4) that

$$S_\theta = \frac{1}{w}\phi_\theta + N$$

and

$$N_\theta = \frac{1}{m}\psi_\theta - S$$

and, with the aid of equations (5),

$$S_\theta = wN_w + N = \frac{\partial(wN)}{\partial w}$$

and

$$-N_\theta = mS_m + S = \frac{\partial(mS)}{\partial m}$$

These equations can be satisfied by putting, respectively,

$$S = \chi_w \quad , \quad wN = \chi_\theta \quad , \quad N = \omega_m \quad , \quad mS = -\omega_\theta \tag{7}$$

Equations (7) are consistant if

$$\chi_\theta = w\omega_m \quad , \quad -\omega_\theta = m\chi_w \tag{8}$$

a symmetrical system of equations in χ and ω very similar to equations (6).[2]

From equations (3) and (7) it is deduced that

$$d\phi = w\ d\chi_w - \chi_\theta\ d\theta = d(w\chi_w - \chi)$$

and

$$d\psi = m\ d\omega_m - \omega_\theta\ d\theta = d(m\omega_m - \omega)$$

Hence, to within an unessential constant,

$$\left.\begin{aligned} \phi &= w\chi_w - \chi \\ \psi &= m\omega_m - \omega \end{aligned}\right\} \tag{9}$$

which give ϕ and ψ in terms of χ and ω.[3] The functions χ and ω are of course the Legendre transforms of ϕ and ψ considered as functions of the physical coordinates.

The functions χ and ω are distinguished by the fact that once a solution of equations (8) is known all the other functions concerning

[2]Equations (8) have already been written in the present form by Bateman (reference 1).

[3]Relations (9) already have been derived in the present form by Bateman and Pérés (references 1 and 2).

the physical representation of motion can be derived by simple differentiations, ϕ and ψ being obtained from equations (9) and z from equations (2) and (7). However, if ϕ and ψ are the known functions satisfying equations (6), integrations are necessary to deduce the other quantities. After integration and determination of the constants so as to satisfy equation (8), the following explicit expressions for X and ω are obtained from equations (9):

$$X = -w\left[\int_{1/w_r}^{1/w} \phi(w_1,\theta)d\left(\frac{1}{w_1}\right) + \frac{1}{w_r}\int_{\theta_r}^{\theta} \phi(w_r,\theta_1)\sin(\theta - \theta_1)d\theta_1 \right.$$

$$\left. - \frac{1}{m_r}\int_{\theta_r}^{\theta} \psi(w_r,\theta_1)\cos(\theta - \theta_1)d\theta_1 + C_1 \cos\theta + C_2 \sin\theta\right]$$

$$\omega = -m\left[\int_{1/m_r}^{1/m} \psi(w_1,\theta)d\left(\frac{1}{m_1}\right) + \frac{1}{m_r}\int_{\theta_r}^{\theta} \psi(w_r,\theta_1)\sin(\theta - \theta_1)d\theta_1 \right.$$

$$\left. + \frac{1}{w_r}\int_{\theta_r}^{\theta} \phi(w_r,\theta_1)\cos(\theta - \theta_1)d\theta_1 - C_1 \sin\theta + C_2 \cos\theta\right]$$

where w_r and θ_r are two arbitrary reference quantities, $m_r = m(w_r)$, and C_1 and C_2 are two arbitrary constants with no influence on ϕ and ψ. It is readily deduced with the aid of equations (2) and (7) that

$$z = e^{i\theta}\left[\frac{\phi(w,\theta)}{w} + i\,\frac{\psi(w,\theta)}{m} - \int_{1/w_r}^{1/w} \phi(w_1,\theta)d\left(\frac{1}{w_1}\right) - i\int_{1/m_r}^{1/m} \psi(w_1,\theta)d\left(\frac{1}{m_1}\right)\right]$$

$$- i\int_{\theta_r}^{\theta}\left[\frac{\phi(w_r,\theta_1)}{w_r} + i\,\frac{\psi(w_r,\theta_1)}{m_r}\right]e^{i\theta_1}d\theta_1 - (C_1 + iC_2)$$

By differentiation,

$$dz = e^{i\theta}\left(\frac{d\phi}{w} + i\,\frac{d\psi}{m}\right)$$

which agrees with equation (2b).

From equations (6) and (8) the hodograph equations for the functions χ, ω, ϕ, and ψ can be easily shown to be as follows:

$$w \frac{\partial}{\partial m}\left(m \frac{\partial \chi}{\partial w}\right) + \frac{\partial^2 \chi}{\partial \theta^2} = 0 \qquad \text{(10a)}$$

$$m \frac{\partial}{\partial w}\left(w \frac{\partial \omega}{\partial m}\right) + \frac{\partial^2 \omega}{\partial \theta^2} = 0 \qquad \text{(10b)}$$

$$\frac{1}{m} \frac{\partial}{\partial\left(\frac{1}{w}\right)}\left[\frac{1}{w} \frac{\partial \phi}{\partial\left(\frac{1}{m}\right)}\right] + \frac{\partial^2 \phi}{\partial \theta^2} = 0 \qquad \text{(10c)}$$

$$\frac{1}{w} \frac{\partial}{\partial\left(\frac{1}{m}\right)}\left[\frac{1}{m} \frac{\partial \psi}{\partial\left(\frac{1}{w}\right)}\right] + \frac{\partial^2 \psi}{\partial \theta^2} = 0 \qquad \text{(10d)}$$

Each of these equations reduces to the Laplace equation if $m = w$.

The following equations are obtained from equations (10) with w as independent variable:

$$w^2 \frac{\partial^2 \chi}{\partial w^2} + \left(1 - M^2\right) w \frac{\partial \chi}{\partial w} + \left(1 - M^2\right) \frac{\partial^2 \chi}{\partial \theta^2} = 0$$

$$w^2 \frac{\partial^2 \omega}{\partial w^2} + \left[1 + M^2 - w \frac{d}{dw} \log\left(1 - M^2\right)\right] w \frac{\partial \omega}{\partial w} + \left(1 - M^2\right) \frac{\partial^2 \omega}{\partial \theta^2} = 0$$

$$w^2 \frac{\partial^2 \phi}{\partial w^2} + \left[1 - M^2 - w \frac{d}{dw} \log\left(1 - M^2\right)\right] w \frac{\partial \phi}{\partial w} + \left(1 - M^2\right) \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$w^2 \frac{\partial^2 \psi}{\partial w^2} + \left(1 + M^2\right) w \frac{\partial \psi}{\partial w} + \left(1 - M^2\right) \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

where M, the Mach number, is defined by (see equation (15))

$$1 - M^2 = \frac{d \log m}{d \log w}$$

For every particular law $m(w)$; that is, for every $\rho(w)$, explicit equations are obtained. For $M < 1$ (subsonic flow) the equations are of the elliptic type; for $M > 1$ (supersonic flow) the equations are of the hyperbolic type.

The hodograph equations are frequently transformed so as to simplify the second-order terms. Thus, if for $\frac{dm}{dw} > 0$,

$$d\lambda = \left(\frac{dm}{w}\frac{dw}{m}\right)^{1/2} = \left[w\ d\left(\frac{1}{m}\right) m\ d\left(\frac{1}{w}\right)\right]^{1/2} = \left(d \log w\ d \log m\right)^{1/2} \tag{11}$$

$$\left.\begin{aligned} \alpha &= \left(\frac{dm/w}{dw/m}\right)^{1/4} = \left(\frac{m^2}{w^2}\frac{d \log m}{d \log w}\right)^{1/4} = \left(\frac{dm^2}{dw^2}\right)^{1/4} \\ \beta &= \left(\frac{w\ d\left(\frac{1}{m}\right)}{m\ d\left(\frac{1}{w}\right)}\right)^{1/4} = \left(\frac{w^2}{m^2}\frac{d \log m}{d \log w}\right)^{1/4} = \left(\frac{d\left(\frac{1}{m^2}\right)}{d\left(\frac{1}{w^2}\right)}\right)^{1/4} \end{aligned}\right\} \tag{12}$$

then

$$\frac{dm}{w} = \alpha^2 d\lambda$$

$$\frac{dw}{m} = \frac{1}{\alpha^2}\, d\lambda$$

$$w\ d\left(\frac{1}{m}\right) = -\beta^2 d\lambda$$

$$m\ d\left(\frac{1}{w}\right) = -\frac{1}{\beta^2}\, d\lambda$$

and equations (10) are transformed into

$$\phi_{\lambda\lambda} + \phi_{\theta\theta} - 2\phi_\lambda \frac{d \log \beta}{d\lambda} = 0 \tag{13a}$$

$$\psi_{\lambda\lambda} + \psi_{\theta\theta} + 2\psi_\lambda \frac{d \log \beta}{d\lambda} = 0 \tag{13b}$$

$$\chi_{\lambda\lambda} + \chi_{\theta\theta} + 2\chi_\lambda \frac{d \log \alpha}{d\lambda} = 0 \tag{13c}$$

$$\omega_{\lambda\lambda} + \omega_{\theta\theta} - 2\omega_\lambda \frac{d \log \alpha}{d\lambda} = 0 \tag{13d}$$

If $\phi = \beta\phi_*$, $\psi = \frac{1}{\beta}\psi_*$, $\chi = \frac{1}{\alpha}\chi_*$, and $\omega = \alpha\omega_*$, the equations for ϕ_*, ψ_*, χ_*, and ω_* are readily shown to be

$$\left.\begin{aligned}
&\phi_{*\lambda\lambda} + \phi_{*\theta\theta} - \phi_*\beta \frac{d^2}{d\lambda^2}\left(\frac{1}{\beta}\right) = 0 \\
&\psi_{*\lambda\lambda} + \psi_{*\theta\theta} - \psi_* \frac{1}{\beta}\frac{d^2\beta}{d\lambda^2} = 0 \\
&\chi_{*\lambda\lambda} + \chi_{*\theta\theta} - \chi_* \frac{1}{\alpha}\frac{d^2\alpha}{d\lambda^2} = 0 \\
&\omega_{*\lambda\lambda} + \omega_{*\theta\theta} - \omega_*\alpha \frac{d^2}{d\lambda^2}\left(\frac{1}{\alpha}\right) = 0
\end{aligned}\right\} \tag{14}$$

For $\frac{dm}{dw} < 0$ analogous transformations can be performed with the introduction of $d\lambda_1$, α_1, and β_1 defined in the same way as $d\lambda$, α, and β with $-dm$ instead of dm. It is seen that λ represents some kind of an integral mean between $\log w$ and $\log m$. Introducing the sound velocity $a = \left(\frac{dp}{d\rho}\right)^{1/2}$ and the Mach number $M = \frac{w}{a}$ leads to the following equations deduced from the Bernoulli equation $dp + \rho w \, dw = 0$:

$$\left.\begin{aligned} a^2 + w^2 \frac{d \log w}{d \log \rho} &= 0 \\ \frac{d \log \rho}{d \log w} &= -M^2 \\ \frac{d \log m}{d \log w} &= 1 - M^2 \end{aligned}\right\} \tag{15}$$

Then for $M < 1$,

$$d\lambda = \sqrt{1 - M^2} \, d \log w \tag{16a}$$

$$\alpha = \left(\frac{\rho}{\rho_0} \sqrt{1 - M^2}\right)^{1/2} \tag{16b}$$

$$\beta = \left(\frac{\rho_0}{\rho} \sqrt{1 - M^2}\right)^{1/2} \tag{16c}$$

There is obtained a corresponding set of equations for $d\lambda_1$, α_1, and β_1 when $M > 1$ by simply replacing $1 - M^2$ by $M^2 - 1$.

Approximate Methods

It is seen from equations (13) that for constant α or β these equations reduce to the Laplace equation. The first possibility is to be rejected for subsonic motion because, as equation (16) shows, it gives ρ as an increasing function of M. The second one is the well-known Kármán–Tsien approximation (references 3 and 4)

$$\beta^2 = \frac{\rho_0}{\rho} \sqrt{1 - M^2} = \sqrt{K} \tag{17}$$

where K is a constant. This equation reduces to the Chaplygin approximation for K = 1 (reference 5 and derived studies).

Some considerations that will be useful in a subsequent section are now introduced. With the help of equations (12), equation (17) can be immediately integrated to obtain

$$\left.\begin{aligned} \frac{1}{m^2} &= K\left(\frac{1}{w^2} + H\right) \\ \frac{{\rho_0}^2}{\rho^2} &= K\left(1 + Hw^2\right) \end{aligned}\right\} \tag{18}$$

where H is an arbitrary constant. Integrating now equation (16a) and the Bernoulli equations (15), there follows, respectively,

$$\lambda = \log \frac{w}{1 + \sqrt{1 + Hw^2}} + \text{Constant} = -\log\left(\frac{1}{w} + \frac{1}{m\sqrt{K}}\right) + \text{Constant} \tag{19a}$$

and

$$\frac{p}{\rho_0} = -\frac{1}{KH} \frac{\rho_0}{\rho} + \text{Constant} \tag{19b}$$

From equations (15), for a real isentropic gas,

$$\frac{\rho_0^{\,2}}{\rho^2} = \left(1 - \frac{w^2}{w_{max}^{\,2}}\right)^{-\frac{2}{\gamma-1}} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{2}{\gamma-1}} \tag{20a}$$

and

$$\frac{d}{dw^2}\left(\frac{\rho_0^{\,2}}{\rho^2}\right) = \frac{\rho_0^{\,2}}{\rho^2} \frac{1}{a^2} \tag{20b}$$

where $w_{max} = a_0\left(\frac{2}{\gamma - 1}\right)^{1/2}$ is the maximum velocity at zero density, a_0 is the stagnation sound velocity, and γ is the adiabatic index.

A comparison between equations (18) and (20) is shown in figure 2 where the law, equation (18), is represented by a straight line. Chaplygin takes it as the tangent to the graph of equation (20) at $w = 0$, so that $K = 1$ and $H = \frac{1}{a_0^{\,2}}$. In the Kármán-Tsien method[4] the tangent is taken at $w = w_\infty$, the speed at infinity, so that

$$\left.\begin{aligned}
K &= \frac{\rho_0^{\,2}}{\rho_\infty^{\,2}}\left(1 - M_\infty^{\,2}\right) = \left(1 + \frac{\gamma - 1}{2} M_\infty^{\,2}\right)^{\frac{2}{\gamma-1}}\left(1 - M_\infty^{\,2}\right) \\
KH &= \frac{\rho_0^{\,2}}{\rho_\infty^{\,2}} \frac{1}{a_\infty^{\,2}} \\
H &= \frac{1}{a_\infty^{\,2}\left(1 - M_\infty^{\,2}\right)}
\end{aligned}\right\} \tag{21}$$

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[4]This method has been often presented in a less coherent form, as the constants of equations (18) and (19) are determined for different conditions, though the formula for the correction of pressure coefficients is not affected by this incoherence.

Observe that for $K \neq 1$ the value of ρ at $w = 0$ differs from the exact stagnation density ρ_0. It is easily seen that the value for KH in equations (21) gives the slope of the true isentropic relation at a point corresponding to the conditions at infinity, but, as this depends only on the value of KH, it is seen also that Kármán's condition is satisfied for every parallel to the tangent at $w = w_\infty$ (fig. 2), that is, for the KH value given by equation (21) but for different values of K. This suggests the possibility of improving the mean approximation of the Kármán–Tsien method by an appropriate choice of K. The Kármán–Tsien correction formula for the pressure coefficient has then to be modified. The modified formula, independent of the constant of integration in equation (19a), is then

$$C_p = \frac{C_{p_0}}{\sqrt{M_\infty^2 + \frac{\rho_\infty^2}{\rho_0^2} K \left[\frac{\rho_\infty}{\rho_0}\sqrt{K} + \frac{C_{p_0}}{2}\left(\sqrt{M_\infty^2 + \frac{\rho_\infty^2}{\rho_0^2} K} - \frac{\rho_\infty}{\rho_0}\sqrt{K}\right)\right]}}$$

which reduces to the Kármán–Tsien correction formula for the value of K given by equation (21). Application of the modified formula with some value of K between $\frac{\rho_0^2}{\rho_\infty^2}\left(1 - M_\infty^2\right)$ and 1 gives values of C_p in better agreement with experimental values.

For supersonic motion the hodograph equations reduce to the simplest hyperbolic equation (wave equation) for $\alpha_1 = \text{Constant}$ or $\beta_1 = \text{Constant}$, with α_1 and β_1 given by equations (12) with $-dm$ instead of dm or by equation (16) with $M^2 - 1$ in place of $1 - M^2$. The second possibility is now to be rejected because the resulting value of ρ increases with M. The first possibility gives

$$\alpha_1^2 = \left(-\frac{dm^2}{dw^2}\right)^{1/2} = \frac{\rho}{\rho_0}\sqrt{M^2 - 1} = \sqrt{K_1}$$

that is, after integration[5]

$$m^2 = K_1 \left(H_1 - w^2\right)$$

$$\frac{\rho^2}{{\rho_0}^2} = K_1 \left(\frac{H_1}{w^2} - 1\right)$$

$$w^2 = \frac{K_1 H_1 {\rho_0}^2}{K_1 {\rho_0}^2 + \rho^2}$$

The constants K_1 and H_1 can be determined so as to satisfy Kármán's condition:

$$K_1 = \frac{{\rho_\infty}^2}{{\rho_0}^2} \left({M_\infty}^2 - 1\right)$$

$$K_1 H_1 = \frac{{\rho_\infty}^2}{{\rho_0}^2} {a_\infty}^2 {M_\infty}^4$$

$$H_1 = \frac{{a_\infty}^2 {M_\infty}^4}{{M_\infty}^2 - 1}$$

With α_1 = Constant, equations (13c) and (13d) (modified for $M > 1$) reduce to the simple wave equation. The general solution can therefore be represented by, for instance,

$$\chi = f(\lambda_1 \pm \theta) \tag{22}$$

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[5] Pérés has already indicated a law of this kind (reference 2).

Other laws for which the hodograph equations reduce to the Laplace equation for $M < 1$ (or to the wave equations for $M > 1$) are immediately deduced from equations (14), (or from the corresponding equation for the supersonic case) as the laws which make one of the four quantities α, $1/\alpha$, β, and $1/\beta$, (or those corresponding for $M > 1$) linear in λ (or in λ_1). For the true isentropic gas the curves of α, β, $1/\alpha$, and $1/\beta$ as functions of λ are shown in figure 3. For $w = 0$, $\lambda = -\infty$, and $\alpha = \beta = 1$ the Chaplygin and the Kármán-Tsien approximations replace the true shape by a horizontal line which can give an approximation not too bad even for $w = 0$. Approximate laws for which equations (14) reduce to the Laplace equation are represented by arbitrary (and generally not horizontal) straight lines. It is seen that for $\lambda = -\infty$ these lines diverge hopelessly from the true law. Hence these laws do not appear to be convenient for the approximate representation in a large range of velocity. Nevertheless they can possibly have application when the variations of velocity from a mean value (for instance, the value at infinity) are small. In this case it is possible to achieve a better approximation than with the Kármán-Tsien method by taking as the approximate law the tangent to one of the curves of figure 3. The resulting approximate p,ρ curve will have a contact of second order with the real isentropic.

For the supersonic case, α_1, β_1, $1/\alpha_1$, and $1/\beta_1$, as functions of λ_1, are shown in figure 4 for $\gamma = 1.4$. An interesting possibility is given by the curve of α_1, which can be well approximated by a straight line between $M = 2$ and $M = 10$. Hence in this range the exact hodograph equation in $\chi_{*1} = \alpha_1 \chi$ differs very little from the simple wave equation in λ_1 and θ. Therefore the general solution of the supersonic motion in the said range of M is approximately

$$\chi = \frac{1}{\alpha_1} f(\lambda_1 \pm \theta)$$

where α_1 and λ_1 are given by the true isentropic law. This approximation seems to be better than the approximation given by equation (22).

Exact Solutions of The Hodograph Equations

The power set.– Many studies have been developed on two sets of particular solutions of the hodograph equation in ϕ or ψ, the so-called power set and exponential set, characterized by the fact that for the incompressible case they reduce respectively to the natural powers of the logarithms of the complex velocity and those of the complex velocity itself. The symmetrical form of equations (10) makes it possible to present these solutions in a form that seems interesting.

By the introduction (though it is not strictly necessary) of four auxiliary quantities $\tilde{\omega}$, $\tilde{\chi}$, $\tilde{\psi}$, and $\tilde{\phi}$, satisfying in the same order equations (10), and some supplementary conditions, the four complex quantities

$$F = \chi + i\tilde{\omega}$$

$$\tilde{F} = \tilde{\chi} + i\omega$$

$$G = \phi + i\tilde{\psi}$$

and

$$\tilde{G} = \tilde{\phi} + i\psi$$

can be made to satisfy not only equations (10) in the same order but also the relations

$$mF_w = i\tilde{F}_\theta \tag{23a}$$

$$w\tilde{F}_m = iF_\theta \tag{23b}$$

$$\frac{1}{w} G_{1/m} = -i\tilde{G}_\theta \tag{23c}$$

and

$$\frac{1}{m} \tilde{G}_{1/w} = -iG_\theta \tag{23d}$$

corresponding to equations (6), (8), and the relations

$$\left.\begin{aligned} G &= wF_w - F \\ \tilde{G} &= m\tilde{F}_m - \tilde{F} \end{aligned}\right\} \tag{24}$$

corresponding to equations (9). Conversely F and $\tilde{F}$ are given by formulas corresponding to those written in section entitled "Hodograph Equations" for χ and ω:

$$F = -w\left[\int_{1/w_r}^{1/w} G(w,\theta)\,d\left(\frac{1}{w}\right) + \frac{1}{w_r}\int_{\theta_r}^{\theta} G(w_r,\theta_1)\sin(\theta - \theta_1)\,d\theta_1 + \frac{i}{m_r}\int_{\theta_r}^{\theta} \tilde{G}(w_r,\theta_1)\cos(\theta - \theta_1)\,d\theta_1\right] \tag{25}$$

and a similar equation for $\tilde{F}$ with m and w, G and $\tilde{G}$ interchanged.

The quantities $\tilde{\chi}$, $\tilde{\omega}$, $\tilde{\phi}$, and $\tilde{\psi}$ can be determined so that they reduce to χ, ω, ϕ, and ψ for the incompressible case $m = w$; hence the equations

$$F_1 = \tilde{F}_1 = \chi_1 + i\omega_1$$

and

$$G_1 = \tilde{G}_1 = \phi_1 + i\psi_1$$

satisfy the Laplace equations

$$\Delta F_1 = 0$$

and

$$\Delta G_1 = 0$$

$\left(\text{where } \Delta = \dfrac{\partial^2}{\partial(\log w)^2} + \dfrac{\partial^2}{\partial\theta^2}\right)$ to which equations (10) reduce when $m = w$.

An operator $((\ ))$ is now defined as that which, when applied to $\left(\log \frac{w}{w_r}\right)^h$ (w_r being an arbitrary reference velocity), transforms it into

$$\left(\left(\left(\log \frac{w}{w_r}\right)^h\right)\right) = h! \int_{w_r}^{w} \frac{dw_1}{m_1} \int_{m_r}^{m_1} \frac{dm_2}{w_2} \int_{w_r}^{w_2} \frac{dw_3}{m_3} \ . \ . \ . = h!I(w,m,w_r,m_r) \tag{26a}$$

for $h = 1, 2, \ldots$, the integration being repeated h times; then, the indication of the lower limit of integration being omitted for brevity,[6]

$$\left(\left(\left(\log \frac{m}{m_r}\right)^h\right)\right) = h! \int_{m_r}^{m} \frac{dm_1}{w_1} \int_{w_r}^{w_1} \frac{dw_2}{m_2} \int_{m_r}^{m_2} \frac{dm_3}{w_3} \ . \ . \ . = h!I(m,w) \tag{26b}$$

$$\left(\left(\left(\log \frac{w_r}{w}\right)^h\right)\right) = h! \int_{1/w_r}^{1/w} m_1 \, d\left(\frac{1}{w_1}\right) \int_{1/m_r}^{1/m_1} w_2 \, d\left(\frac{1}{m_2}\right) \ . \ . \ . = h!I\left(\frac{1}{w},\frac{1}{m}\right) \tag{26c}$$

$$\left(\left(\left(\log \frac{m_r}{m}\right)^h\right)\right) = h! \int_{1/m_r}^{1/m} w_1 \, d\left(\frac{1}{m_1}\right) \int_{1/w_r}^{1/w_1} m_2 \, d\left(\frac{1}{w_2}\right) \ . \ . \ . = h!I\left(\frac{1}{m},\frac{1}{w}\right) \tag{26d}$$

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[6]Formulas (26c) and (26d) could preferably be written as operations on powers of $\log \dfrac{1/w}{1/w_r}$ and $\log \dfrac{1/m}{1/m_r}$ and are different from the operations given by equations (26a) and (26b) on powers of $-\log \frac{w}{w_r}$ and $-\log \frac{m}{m_r}$.

From these equations there is deduced

$$\left.\begin{aligned} m \frac{d}{dw} I_h(w,m) &= I_{h-1}(m,w) \\ m \frac{d}{dw}\left(\left(\left(\log \frac{w}{w_r}\right)^h\right)\right) &= h\left(\left(\left(\log \frac{m}{m_r}\right)^{h-1}\right)\right) \end{aligned}\right\} \qquad (27)$$

and the analogous equations obtained by interchanging w and m, and/or by changing w into 1/w and m into 1/m.

It is then easily verified that, with $I_0 = 1$, each of the following functions

$$\left.\begin{aligned} F &= \left(\left(\left(\log \frac{w}{w_r} \mp i\theta\right)^n\right)\right) \\ \tilde{F} &= \left(\left(\left(\log \frac{m}{m_r} \mp i\theta\right)^n\right)\right) \\ G &= \left(\left(\left(\log \frac{m_r}{m} \pm i\theta\right)^n\right)\right) \\ \tilde{G} &= \left(\left(\left(\log \frac{w_r}{w} \pm i\theta\right)^n\right)\right) \end{aligned}\right\} \qquad (28)$$

is a solution of the corresponding equation,[7] the signs having been selected so as to satisfy also equations (23). The choice of the upper or lower sign does not affect the values of ϕ, ψ, χ, and ω derived from equations (28) so that for the solution of the flow problem it is

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[7]This kind of solution has been first discovered by Bergman (see for instance reference 6) and by Bers and Gelbart (reference 7). The present form is new and more symmetrical.

sufficient to retain, for instance, only the upper one;[8] nevertheless in some cases it can be useful to consider solutions with both signs.

The solution represented by the values, equations (28), of G and $\tilde{G}$ does not coincide with the one corresponding to the values, equations (28), of F and $\tilde{F}$ for it differs from the values of G and $\tilde{G}$ derived from F and $\tilde{F}$ by means of equations (24). This can be shown as follows: From the recurrence formula, which is easy to verify,

$$I_h(w,m) - \frac{w}{m} I_{h-1}(m,w) = \int_{1/m_r}^{1/m} w_1 \, d\left(\frac{1}{m_1}\right) \int_{1/w_r}^{1/w_1} m_2 \, d\left(\frac{1}{w_2}\right) \left[I_{h-2}(w_2,m_2) - \frac{w_2}{m_2} I_{h-3}(m_2,w_2)\right]$$

and from the values, directly deduced with $I_0 = 1$,

$$I_1(w,m) - \frac{w}{m} I_0(m,w) = -I_1\left(\frac{1}{m},\frac{1}{w}\right) - \frac{w_r}{m_r}$$

and

$$I_2(w,m) - \frac{w}{m} I_1(m,w) = I_2\left(\frac{1}{m},\frac{1}{w}\right) + \frac{m_r}{w_r} I_1\left(\frac{1}{m},\frac{1}{w}\right)$$

There is deduced with the aid of equations (27)

$$\left(1 - w \frac{d}{dw}\right) I_h(w,m) = I_h(w,m) - \frac{w}{m} I_{h-1}(m,w)$$

$$= (-1)^h I_h\left(\frac{1}{m},\frac{1}{w}\right) + \frac{1}{2}\left[(-1)^h\left(\frac{w_r}{m_r} + \frac{m_r}{w_r}\right) - \left(\frac{w_r}{m_r} - \frac{m_r}{w_r}\right)\right] I_{h-1}\left(\frac{1}{m},\frac{1}{w}\right) \tag{29}$$

and the analogous relations obtained after interchanging the variables.

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[8]In this case for $m = w$ the four solutions, equations (28), reduce (to within a multiplicative constant) to $\left(\log \frac{w}{w_r} - i\theta\right)^n$, that is to the power set for the incompressible case.

Application of equations (24) to equations (28) yields the following equation:

$$-G = \left(1 - w\frac{\partial}{\partial w}\right)\left(\left(\left(\log\frac{w}{w_r} \mp i\theta\right)^n\right)\right)$$

$$= (-1)^n\left(\left(\left(\log\frac{m_r}{m} \pm i\theta\right)^n\right)\right) + (-1)^n\frac{n}{2}\left(\frac{w_r}{m_r} + \frac{m_r}{w_r}\right)\left(\left(\left(\log\frac{m_r}{m} \pm i\theta\right)^{n-1}\right)\right)$$

$$-\frac{n}{2}\left(\frac{w_r}{m_r} - \frac{m_r}{w_r}\right)\left(\left(\left(\log\frac{m_r}{m} \mp i\theta\right)^{n-1}\right)\right)$$

and a similar equation for $-\tilde{G}$, if w is interchanged with m, with the $\pm$ sign. This G differs from the elementary solution, equations (28), although it is a linear combination of such elementary solutions, a fact holding also for the incompressible case. In that case, however, the expression for G contains only an elementary solution with the upper sign if F is so, for then $w_r = m_r$.

The physical coordinates are easily deduced; for by equations (7)

$$\left.\begin{aligned} S + i\tilde{N} &= F_w = \frac{i}{m}\tilde{F}_\theta \\ \tilde{S} + iN &= \tilde{F}_m = \frac{i}{w}F_\theta \end{aligned}\right\} \tag{30}$$

where $\tilde{S}$ and $\tilde{N}$ are auxiliary quantities connected with $\tilde{\chi}$ and $\tilde{\omega}$ by relations similar to equations (7). Hence

$$z = e^{i\theta}(S + iN)$$

with

$$S = \frac{1}{2}\left(F_w + \overline{F}_w\right) = \frac{n}{2m}\left[\left(\left(\left(\log\frac{m}{m_r} \mp i\theta\right)^{n-1}\right)\right) + \left(\left(\left(\log\frac{m}{m_r} \pm i\theta\right)^{n-1}\right)\right)\right]$$

$$= \frac{n}{m}R\left(\left(\left(\log\frac{m}{m_r} - i\theta\right)^{n-1}\right)\right)$$

$$iN = \frac{1}{2}\left(\tilde{F}_m - \tilde{\tilde{F}}_m\right) = \pm\frac{n}{2w}\left[\left(\left(\left(\log\frac{w}{w_r} \mp i\theta\right)^{n-1}\right)\right) - \left(\left(\left(\log\frac{w}{w_r} \pm i\theta\right)^{n-1}\right)\right)\right]$$

$$= i\,\frac{n}{w}\,I\left(\left(\left(\log\frac{w}{w_r} - i\theta\right)^{n-1}\right)\right)$$

It is now again verified that, as already observed, the choice of the sign does not affect the results concerning the values of ϕ, ψ, and z but only the introduced auxiliary functions. In subsequent work, therefore, only the upper sign is retained.

Linear combinations, and in some cases infinite series of the elementary solutions, equations (28), are still solutions of the corresponding equations.

<u>Infinite series in the power set</u>.– If, say, F is given by an infinite series, then developing and inverting the order of summations gives

$$F = \sum_{n=0}^{\infty} \alpha_n\left(\left(\left(\log\frac{w}{w_r} - i\theta\right)^n\right)\right) = \sum_{n=0}^{\infty}\alpha_n n!\sum_{h=0}^{n} I_h(w,m)\,\frac{(-i\theta)^{n-h}}{(n-h)!}$$

$$= \sum_{h=0}^{\infty} I_h(w,m)\sum_{n=h}^{\infty}\frac{n!}{(n-h)!}\,\alpha_n(-i\theta)^{n-h} = \sum_{h=0}^{\infty} I_h(w,m)\left(\frac{d^hA}{d\zeta^h}\right)_{\zeta=-i\theta} \qquad (31)$$

where

$$A(\zeta) = \sum_{0}^{\infty}\alpha_n\zeta^n$$

represents the function corresponding to this power series.

Now, independently of the convergence of this power series it is readily verified by differentiation, that equation (31), whenever it converges, is a solution of the equation in F (see equation (10a)). Similarly when it converges, the series

$$\tilde{F} = \sum_{h=0}^{\infty} I_h(m,w)\left(\frac{d^h A}{d\zeta^h}\right)_{\zeta=-i\theta} \tag{32}$$

obtained from equation (31) by simply interchanging w and m satisfies the equation in $\tilde{F}$ (see equations (10b)). For G and $\tilde{G}$ there are the analogous solutions

$$\left.\begin{aligned} G &= \sum_{h=0}^{\infty} I_h\left(\frac{1}{m},\frac{1}{w}\right)\left[\frac{d^h B(\zeta)}{d\zeta^h}\right]_{\zeta=i\theta} \\ \tilde{G} &= \sum_{h=0}^{\infty} I_h\left(\frac{1}{w},\frac{1}{m}\right)\left[\frac{d^h B(\zeta)}{d\zeta^h}\right]_{\zeta=i\theta} \end{aligned}\right\} \tag{33}$$

Now, let G and $\tilde{G}$ be deduced from F and $\tilde{F}$ by means of equations (24), with the help of equation (29). There results the following expression for G:

$$-G = \sum_{k=0}^{\infty} I_{2k}\left(\frac{1}{m},\frac{1}{w}\right)\left[\frac{d^{2k}}{d\zeta^{2k}}\left(A - \frac{w_r}{m_r}\frac{dA}{d\zeta}\right)\right]_{\zeta=-i\theta}$$

$$- \sum_{k=0}^{\infty} I_{2k+1}\left(\frac{1}{m},\frac{1}{w}\right)\left[\frac{d^{2k+1}}{d\zeta^{2k+1}}\left(A - \frac{m_r}{w_r}\frac{dA}{d\zeta}\right)\right]_{\zeta=-i\theta}$$

and a similar expression for $\tilde{G}$ after w and m, w_r and m_r have been interchanged.

The values of ϕ and ψ deduced from these expressions for G and $\tilde{G}$ are the same as those derived from equations (33) if

$$B(\zeta) = -A(-\zeta) - \frac{1}{2}\left(\frac{w_r}{m_r} + \frac{m_r}{w_r}\right)\frac{dA(-\zeta)}{d\zeta} + \frac{1}{2}\left(\frac{w_r}{m_r} - \frac{m_r}{w_r}\right)\frac{d\bar{A}(\zeta)}{d\zeta}$$

The formula for F can be written in a different form:

$$F = \sum_{h=0}^{\infty} i^h I_h(w,m) \left[a_1^{(h)}(\theta) + i a_2^{(h)}(\theta)\right]$$

with

$$A(-i\theta) = a_1(\theta) + i a_2(\theta)$$

and

$$B(i\theta) = b_1(\theta) + i b_2(\theta)$$

It follows that

$$G = \sum_{h=0}^{\infty} (-i)^h I_h\left(\frac{1}{m}, \frac{1}{w}\right) \left[b_1^{(h)}(\theta) + i b_2^{(h)}(\theta)\right]$$

with

$$b_1(\theta) = -a_1(\theta) - \frac{w_r}{m_r} a_2'(\theta)$$

and

$$b_2(\theta) = -a_2(\theta) + \frac{m_r}{w_r} a_1'(\theta)$$

There are similar equations for $\tilde{F}$ and $\tilde{G}$.

Hence for $w = w_r$

$$\left.\begin{aligned} \chi_r &= a_1(\theta) \\ m_r\left(\frac{\partial \chi}{\partial w}\right)_r &= -a_2'(\theta) \\ \omega_r &= a_2(\theta) \\ w_r\left(\frac{\partial \omega}{\partial m}\right)_r &= a_1'(\theta) \end{aligned}\right\} \qquad (34)$$

so that A and all the solutions of equations (31) and (32) are determined by the values of χ and its radial derivative on the circle $w = w_r$ of the hodograph plane or by the corresponding values for ω. Similar statements hold for B, ϕ, and ψ and solutions of equations (33). Hence the solutions written depend upon two arbitrary functions and, in their region of convergence, represent the general solution of the hodograph equations. Naturally they do not give any indication of the behaviour of the corresponding solutions at $w = 0$; for as $m \rightarrow w \rightarrow 0$, $|I_h| \rightarrow \infty$ as $|\log w|^h$ so that the origin is certainly outside the region of convergence.

In fact, if the solutions must be regular at $w = 0$, only one of the functions $a(\theta)$ and $b(\theta)$ can be chosen arbitrarily, and other representations of the solutions are needed to determine the other. The region of convergence of the series, equations (31), (32), and (33), depends on the form of $A(\zeta)$ and $B(\zeta)$. However, a general idea of its shape can be given by making very general assumptions about these functions and $m(w)$.

Let $r(\theta)$ be less than the distance in the ζ-plane between the point $-i\theta$ and the nearest singularity of $A(\zeta)$ and let $A_{max}(\theta)$

be the upper bound of $|A(\zeta)|$ on the circle with center $-i\theta$ and radius r. Then the Cauchy's inequality gives

$$\left|\frac{d^h A(\zeta)}{d\zeta^h}\right|_{\zeta=-i\theta} \leqq h!\,\frac{A_{max}(\theta)}{r(\theta)^h} \tag{35}$$

and a similar expression for B, with B_{max} and r_1 in place of A_{max} and r. For the I_h, if

$$\left.\begin{aligned}
\left(\frac{1}{m}\frac{dw}{d\log w}\right)_{max} &= \left(\frac{w}{m}\right)_{max} = \left(\frac{\rho_0}{\rho}\right)_{max} = a \\
\left|\frac{1}{w}\frac{dm}{d\log w}\right|_{max} &= |m'|_{max} = \left(\frac{\rho}{\rho_0}\left|1 - M^2\right|\right)_{max} = b \\
\left[m\frac{d(1/w)}{d\log(1/w)}\right]_{max} &= \left(\frac{m}{w}\right)_{max} = \left(\frac{\rho}{\rho_0}\right)_{max} = c \\
\left|w\frac{d(1/m)}{d\log(1/w)}\right|_{max} &= \left(\frac{w^2}{m^2}|m'|\right)_{max} = \left(\frac{\rho_0}{\rho}\left|1 - M^2\right|\right)_{max} = d
\end{aligned}\right\} \tag{36}$$

are the upper bounds of the written quantities between w and w_r, equations (26) give for even values of h

$$\left.\begin{aligned}
\left|I_h(w,m)\right| &< \frac{1}{h!}\left(\sqrt{ab}\left|\log\frac{w}{w_r}\right|\right)^h \\
\left|I_h\left(\frac{1}{w},\frac{1}{m}\right)\right| &< \frac{1}{h!}\left(\sqrt{cd}\left|\log\frac{w}{w_r}\right|\right)^h
\end{aligned}\right\} \tag{37}$$

and the same limitations for $I_h(m,w)$ and $I_h(1/m,1/w)$.

For odd values of h the upper bounds for $\left|I_h(w,m)\right|$ and $\left|I_h(m,w)\right|$ are obtained from the corresponding expressions in equations (37) by multiplying them with $(a/b)^{1/2}$ and $(b/a)^{1/2}$, respectively, and those for $\left|I_h(1/w,1/m)\right|$ and $\left|I_h(1/m,1/w)\right|$ are obtained by multiplying the expressions in equations (37) by $(c/d)^{1/2}$ and $(d/c)^{1/2}$, respectively.

Consideration of the series (31) shows that the terms of the series

$$\sum_{k=0}^{\infty} \left|I_{2k}(w,m)\right| \left|\frac{d^{2k}A(\zeta)}{d\zeta^{2k}}\right|_{\zeta=-i\theta} + \sum_{k=0}^{\infty} \left|I_{2k+1}(w,m)\right| \left|\frac{d^{2k+1}A(\zeta)}{d\zeta^{2k+1}}\right|_{\zeta=-i\theta}$$

are less than the corresponding terms of the geometric series

$$A_{max}\left(1 + \frac{a}{r}\left|\log\frac{w}{w_r}\right|\right)\sum_{k=0}^{\infty}\left(\frac{ab}{r^2}\left|\log\frac{w}{w_r}\right|^2\right)^k$$

which converges when the ratio is less than 1.

Hence the series (31) and (as it may be deduced in the same way) (32) converge absolutely in the region

$$(ab)^{1/2}\left|\log\frac{w}{w_r}\right| < r(\theta) \tag{38a}$$

Similarly, the series (33) converge absolutely for

$$(cd)^{1/2}\left|\log\frac{w}{w_r}\right| < r_1(\theta) \tag{38b}$$

Now since ab and cd are functions of w and w_r, and since r and r_1 are quantities which increase with the distance from the singularities of A and B, the general shape of the region of convergence in the hodograph plane is a curved strip, which contains the circle $w = w_r$, and whose width will be a minimum when $-i\theta$ is nearest to a singularity of $A(\zeta)$ or $B(\zeta)$.

If a singularity lies on the imaginary axis, that is, if $A(-i\theta)$ or $B(i\theta)$, and their derivates have a singularity for some value of θ, the corresponding width of the region of convergence will be zero. This happens, for instance, when the reference velocity w_r is the velocity at infinity of the flow round a body.

An observation of some interest is that as, for plausible laws, ρ/ρ_0 is $O(1)$ for $M = 1$ then $(ab)^{1/2}$ and $(cd)^{1/2}$ (equations (36)) are $O\left(\left|1 - M^2\right|^{1/2}\right)$ or $O\left(\left|1 - M_r{}^2\right|^{1/2}\right)$ (the larger of the two) for M and M_r near 1.

Hence equations (38) show that for given $A(\zeta)$ and $B(\zeta)$ the width of the region of convergence is the greatest and the rapidity of convergence the best near the sonic line $M = 1$. Therefore it is believed that the solution represented by series (31), (32), and (33) may have applications in the solution of transonic problems, naturally in combination with other methods converging in the rest of the field of motion.

Finally observe that the development of equation (31) can be handled differently so as to obtain a power series in θ:

$$F = \sum_{h=0}^{\infty} \frac{(-i\theta)^h}{h!} \sum_{n=h}^{\infty} n!\alpha_n I_{n-h}(w,m)$$

which by use of equation (27) and similar expressions and with

$$f_1(w) = \sum_{n=0}^{\infty} n!\alpha_n I_n(w,m) \quad \text{and} \quad f_2(w) = \sum_{n=0}^{\infty} n!\alpha_n I_n(m,w) \quad \text{becomes}$$

$$F = \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k}}{(2k)!} \left[w \frac{d}{dm} \left(m \frac{d}{dw}\right)\right]^k f_1(w) + \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k+1}}{(2k+1)!} w \frac{d}{dm} \left[m \frac{d}{dw}\left(w \frac{d}{dm}\right)\right]^k f_2(w)$$

Similarly,

$$\tilde{F} = \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k}}{(2k)!} \left[m \frac{d}{dw} \left(w \frac{d}{dm}\right)\right]^k f_2(w) + \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k+1}}{(2k+1)!} m \frac{d}{dw} \left[w \frac{d}{dm}\left(m \frac{d}{dw}\right)\right]^k f_1(w)$$

The functions F and $\tilde{F}$, whenever the series converge, satisfy the corresponding equations and the relations (23), as can be directly verified, with arbitrary f_1 and f_2. Analogous solutions hold for G and $\tilde{G}$ with $i\theta$ in place of $-i\theta$, $1/m$ and $1/w$ in place of w and m, and two arbitrary functions $g_1(w)$ and $g_2(w)$. For real values of f_1 and f_2 the real and imaginary parts of F and $\tilde{F}$ are obtained directly; so that the equation

$$\chi + i\omega = \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k}}{(2k)!}\left[w \frac{d}{dm}\left(m \frac{d}{dw}\right)\right]^k f_1(w)$$

$$+ \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k+1}}{(2k + 1)!} m \frac{d}{dw}\left[w \frac{d}{dm}\left(m \frac{d}{dw}\right)\right]^k f_1(w)$$

depends only on $f_1(w)$; correspondingly, g_1 and g_2 become real and the equation

$$\phi + i\psi = \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!}\left\{\frac{1}{m} \frac{d}{d(1/w)}\left[\frac{1}{w} \frac{d}{d(1/m)}\right]\right\}^k g_1(w)$$

$$+ \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k + 1)!} \frac{1}{w} \frac{d}{d(1/m)}\left\{\frac{1}{m} \frac{d}{d(1/w)}\left[\frac{1}{w} \frac{d}{d(1/m)}\right]\right\}^k g_1(w)$$

depends only on $g_1(w)$.

The physical coordinates are then found by means of equations (2) and (7) to be

$$z = x + iy = e^{i\theta}\left(\sum_{k=0}^{\infty} \frac{(-i\theta)^{2k}}{(2k)!} \frac{d}{dw}\left[w \frac{d}{dm}\left(m \frac{d}{dw}\right)\right]^k f_1(w)\right.$$

$$\left.+ \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k+1}}{(2k + 1)!} \frac{d}{dm}\left\{m \frac{d}{dw}\left[w \frac{d}{dm}\left(m \frac{d}{dw}\right)\right]^k\right\} f_1(w)\right)$$

From $\chi + i\omega$ there follows by means of equation (9) an expression for $\phi + i\psi$ which must coincide with the written one if

$$g_1(w) = \left(w\frac{d}{dw} - 1\right)f_1(w)$$

or

$$f_1(w) = -w\int g_1\ d(1/w)$$

From this coincidence the following interesting formulas can be deduced:

$$\left\{\frac{1}{m}\frac{d}{d(1/w)}\left[\frac{1}{w}\frac{d}{d(1/m)}\right]\right\}^k\left(w\frac{d}{dw} - 1\right) = \left(w\frac{d}{dw} - 1\right)\left[w\frac{d}{dm}\left(m\frac{d}{dw}\right)\right]^k$$

and

$$\frac{1}{w}\frac{d}{d(1/m)}\left\{\frac{1}{m}\frac{d}{d(1/w)}\left[\frac{1}{w}\frac{d}{d(1/m)}\right]\right\}^k\left(w\frac{d}{dw} - 1\right) = -\left(m\frac{d}{dm} - 1\right)m\frac{d}{dw}\left[w\frac{d}{dm}\left(m\frac{d}{dw}\right)\right]^k$$

which are easy to verify directly.

The meaning of the written solution is readily found by observing that when the series converge, for $\theta = 0$, then $\chi = f_1(w)$, $\phi = g_1(w)$, $\omega = \psi = 0$, $x = \frac{df_1}{dw}$, and $y = 0$, so that $f_1 = \int x\ dw$, $g_1 = \int w\ dx$, and the whole solution is determined when the "axial" law of distribution of velocity is given.

Hence the solution, under a somewhat different and more explicit form, reduces to the one studied by Lighthill (reference 8) in his work on the transonic flow in symmetrical channels. As Lighthill observed the coefficients of the expansions become infinite at sonic speed (for then $dm = 0$), so that the series diverge in the transonic region. In this region, however, the solution can be found by following the Lighthill's ingenious method, that is, inverting the series giving $i\psi$ or the one giving $\frac{i\omega + i\theta m\chi}{(i\theta)^2}$, since the coefficients of the inverted series are finite at sonic speed. The application of Lighthill's method can be made easier by the present form of the solution.

The exponential set.- If in equations (31), (32), and (33)

$$A(\zeta) = \frac{1}{n^h} A^{(h)}(\zeta) = e^{n\zeta}$$

and

$$B(\zeta) = \frac{1}{n^h} B^{(h)}(\zeta) = e^{n\zeta}$$

with arbitrary n, then the functions

$$\left.\begin{aligned} F &= E_n(w,m)e^{-in\theta} \\ \tilde{F} &= E_n(m,w)e^{-in\theta} \\ G &= E_n\left(\frac{1}{m},\frac{1}{w}\right)e^{in\theta} \\ \tilde{G} &= E_n\left(\frac{1}{w},\frac{1}{m}\right)e^{in\theta} \end{aligned}\right\} \quad (39)$$

with coefficients defined by

$$\left.\begin{aligned} E_n(w,m) &= \sum_{h=0}^{\infty} n^h I_h(w,m) \\ E_n(m,w) &= \sum_{h=0}^{\infty} n^h I_h(m,w) \end{aligned}\right\} \quad (40)$$

and similar equations for the other E_n will be solutions of the corresponding equations (10) and (23) which reduce, for $m = w$ to $\left(we^{-i\theta}/w_r\right)^n$ (equations (10a), (10b), (23a), and (23b)) and to $\left(w_r e^{i\theta}/w\right)^n$ (equations (10c), (10d), (23c), and (23d)), that is, to the exponential set for the incompressible case.

It is immediately verified by means of equation (37) that the series E_n converge for all values of w and w_r for which a, b, c, and d are limited, that is, for which ρ and M are limited and not zero. For plausible gas laws this excludes only the values $w = 0$ and $w = w_{max}$.

It is therefore seen, and easily verified directly, that equations (40) and the two other equations for E_n are, in order, solutions of the ordinary differential equations

$$w \frac{d}{dm}\left(m \frac{d\chi_n}{dw}\right) - n^2\chi_n = 0 \tag{41a}$$

$$m \frac{d}{dw}\left(w \frac{d\omega_n}{dm}\right) - n^2\omega_n = 0 \tag{41b}$$

$$\frac{1}{m} \frac{d}{d(1/w)}\left[\frac{1}{w} \frac{d\phi_n}{d(1/m)}\right] - n^2\phi_n = 0 \tag{41c}$$

$$\frac{1}{w} \frac{d}{d(1/m)}\left[\frac{1}{m} \frac{d\psi_n}{d(1/w)}\right] - n^2\psi_n = 0 \tag{41d}$$

deduced from equations (10) by taking χ, ω, ϕ, and ψ as the product of a sinusoidal factor in $n\theta$ and of the corresponding function χ_n, ω_n, ϕ_n, or ψ_n of w. For the normal isentropic law, equations (41) become the known equations of the hypergeometric type and have been the object of the investigations of many authors. (See, for instance, references 5, 6, 8, 9, 10, 11, and 12.) Equation (41d) has generally been studied with particular regard to those solutions that satisfy the condition that ψ_n/w^n is unity at $w = 0$.

It is immediately verified that the solutions E_n do not satisfy this condition and that the corresponding conditions at $w = w_r$ are:

$$(E_n)_r = 1$$

$$\left[m \frac{d}{dw} E_n(w,m)\right]_r = \left[w \frac{d}{dm} E_n(m,w)\right]_r = \left[\frac{1}{w} \frac{d}{d(1/m)} E_n\left(\frac{1}{m},\frac{1}{w}\right)\right]_r$$

$$= \left[\frac{1}{m} \frac{d}{d(1/w)} E_n\left(\frac{1}{w},\frac{1}{m}\right)\right]_r = n$$

The symmetrical form of equations (41) allows some general relations to be easily derived. Some of these relations, that may be useful for further developments, are now stated briefly.

The E_n can be considered as the superposition of two independent solutions[9] of equations (41):

$$\left.\begin{aligned} C_n(w,m) &= \sum_{k=0}^{\infty} n^{2k} I_{2k}(w,m) \\ S_n(w,m) &= \sum_{k=0}^{\infty} n^{2k+1} I_{2k+1}(w,m) \end{aligned}\right\} \tag{42}$$

[9]Solutions of the kind of E_n, C_n, and S_n were first introduced by Bers and Gelbart, reference 7.

for which the conditions at $w = w_r$ are:

$$(C_n)_r = 1$$

$$\left(\frac{dC_n}{dw}\right)_r = 0$$

$$(S_n)_r = 0$$

$$\left(m\,\frac{dS_n(w,m)}{dw}\right)_r = \left(w\,\frac{dS_n(m,w)}{dm}\right)_r = n$$

They are connected by relations similar to those connecting the exponentials and the hyperbolic cosine and sine:

$$C_{-n} = C_n$$

$$S_{-n} = -S_n$$

$$E_{\pm n} = C_n \pm S_n$$

$$2C_n = E_n + E_{-n}$$

$$2S_n = E_n - E_{-n}$$

$$C_n(w,m)C_n(m,w) - S_n(w,m)S_n(m,w) = 1$$

$$E_n(w,m)E_{-n}(m,w) + E_n(m,w)E_{-n}(w,m) = 2$$

$$m\,\frac{dC_n(w,m)}{dw} = nS_n(m,w)$$

$$m\,\frac{dS_n(w,m)}{dw} = nC_n(m,w)$$

$$m\,\frac{dE_n(w,m)}{dw} = nE_n(m,w)$$

Similar relations hold when w and m are replaced by their inverse values.

All the solutions of equations (41) can be represented by linear combinations of C_n and S_n, but these functions (as all the series in I_h) are not suited to give the behaviour of solutions near $w = 0$. Since this behaviour is very important for many physical applications, it is necessary to follow a different method of investigation: Let

$$F = \left(X_n e^{-i\theta}\right)^n$$

$$\tilde{F} = \left(\tilde{X}_n e^{-i\theta}\right)^n$$

$$G = \left(Y_n e^{i\theta}\right)^n$$

and

$$\tilde{G} = \left(\tilde{Y}_n e^{i\theta}\right)^n$$

be solutions of the corresponding equations (10) which reduce to the exponential set for the incompressible case. Then $X_n{}^n$, $\tilde{X}_n{}^n$, $Y_n{}^n$, and $\tilde{Y}_n{}^n$ are solutions of equations (41). Now, if for negative values of n the last two must coincide with ϕ_{-n} and ψ_{-n} (such that, for instance, $\frac{\psi_{-n}}{w^{-n}} = 1$ at $w = 0$) and if for positive values of n the first two must coincide with the corresponding χ_n and ω_n, it is seen that X_n/w, $\tilde{X}_n/m$, mY_n, and $w\tilde{Y}_n$ must be equal to unity at $w = 0$. It is now shown that this is possible for all values of n except some exceptional values: Let

$$\left.\begin{aligned}
R_n &= m \frac{d \log X_n}{dw} \\
\tilde{R}_n &= w \frac{d \log \tilde{X}_n}{dm} \\
T_n &= \frac{1}{w} \frac{d \log Y_n}{d(1/m)} \\
\tilde{T}_n &= \frac{1}{m} \frac{d \log \tilde{Y}_n}{d(1/w)}
\end{aligned}\right\} \tag{43}$$

It is immediately seen from equations (23) that R_n must satisfy Riccati's equation

$$\frac{1}{n} m \frac{dR_n}{dw} = \frac{dm^2}{dw^2} - R_n{}^2 \tag{44}$$

and that $\tilde{R}_n$, T_n, and $\tilde{T}_n$ must satisfy the corresponding equations with m and w interchanged or inverted. Furthermore, these quantities are connected by the following relations:

$$\tilde{R}_n = \frac{1}{R_n} \tag{45a}$$

$$\tilde{T}_n = \frac{1}{T_n} \tag{45b}$$

$$T_{-n} = \frac{nm - R_n w}{nwR_n - m} \tag{45c}$$

It is deduced from equations (43) that

$$\frac{X_n}{w} = \exp\left[-\int_0^w \left(\frac{1}{w} - \frac{R_n}{m}\right) dw\right] \tag{46a}$$

$$\frac{\tilde{X}_n}{m} = \exp\left[\int_0^m \left(\frac{\tilde{R}_n}{w} - \frac{1}{m}\right) dm\right] \tag{46b}$$

$$mY_n = \exp\left[-\int_0^m \frac{wT_n - m}{m^2} dm\right] \tag{46c}$$

$$w\tilde{Y}_n = \exp\left[\int_0^w \frac{w - m\tilde{T}_n}{w^2} dw\right] \tag{46d}$$

so that these quantities are equal to unity and analytic at $w = 0$ if all the integrands are analytic there. It can be shown in fact that, if $\frac{m}{w} = \frac{\rho}{\rho_0}$ is an analytical function of w near $w = 0$, the integrands of equations (46a) and (46b) are zero and analytic at $w = 0$ for all values of n except negative integral and half-integral values (for only the negative integers if $\frac{\rho}{\rho_0}$ is an analytic function of w^2); while the integrands of equations (46c) and (46d) are zero and analytic at $w = 0$ for all values of n except positive integral and half-integral values greater than 1 (for only the positive integers greater than 1, if $\frac{\rho}{\rho_0}$ is analytic in w^2).

These results reduce to the well-known results when the equations are hypergeometric. In this case Lighthill (reference 8) has given the most complete discussion of the solutions of the equation in ψ_n and deduced important theorems, some of which may possibly be generalized following the present method. It can be seen that the exclusion of the pole at $n = -1$ for the equations in ϕ_n and ψ_n is a general property, which does not hold for the equations in χ_n and ω_n.

For $n = 1$ the solutions of equation (44) (and of the analogous equations) and the corresponding solutions of equations (24) are:

$$R_1 = \frac{m}{w} = T_1 \tag{47a}$$

$$\tilde{R}_1 = \frac{w}{m} = \tilde{T}_1 \tag{47b}$$

$$F = X_1 e^{-i\theta} = we^{-i\theta} \tag{47c}$$

$$\tilde{F} = \tilde{X}_1 e^{-i\theta} = me^{-i\theta} \tag{47d}$$

$$G = Y_1 e^{i\theta} = \frac{1}{m} e^{i\theta} \tag{47e}$$

$$\tilde{G} = \tilde{Y}_1 e^{i\theta} = \frac{1}{w} e^{i\theta} \tag{47f}$$

It is seen from equations (24) that the values of G and $\tilde{G}$ (hence of ϕ and ψ) corresponding to equations (47a) and (47b) are identically zero. In fact, it is seen from equations (2) and (3) that this solution represents merely a displacement of the origin of the physical coordinates. Equations (47e) and (47f) are more interesting as they coincide with the well-known Ringleb solution (reference 13). The corresponding values of F and $\tilde{F}$ are determined by equation (25) (and the analogous equations). Thus,

$$\left.\begin{aligned} F &= -we^{i\theta}\left(\int_{1/w_r}^{1/w} \frac{1}{m_1}\, d(1/w_1) + \frac{1}{2w_r m_r}\right) \\ \tilde{F} &= -me^{i\theta}\left(\int_{1/m_r}^{1/m} \frac{1}{w_1}\, d(1/m_1) + \frac{1}{2w_r m_r}\right) \end{aligned}\right\} \tag{48}$$

plus a constant multiple of $we^{-i\theta}$ and $me^{-i\theta}$. The coefficients of $e^{i\theta}$ in these formulas, together with w and m respectively, represent two independent solutions of equations (41a) and (41b) for $n^2 = 1$. The operators $-1 + w\frac{d}{dw}$ and $-1 + m\frac{d}{dm}$ which generally allow the deduction of two independent solutions of the equations in ϕ_n and ψ_n from two independent solutions of the equations in χ_n and ω_n, suffer an exception for $n = 1$ as they produce only one set of solutions; that is, $1/m$ and $1/w$, respectively; for when applied to w and m the result is zero. This exceptional case is explained in the section entitled "A new set." When $n = 1$ and $R_1 = \frac{m}{w}$ equation (45c) becomes indeterminate, but the corresponding value of T_{-1} can be deduced from this relation as the limiting value for $n \rightarrow 1$ of the indeterminate expression. It follows from equation (44) that with the condition $R_n = 1$ at $w = 0$

$$R_n = n\left[\frac{m}{w} - \frac{2(n-1)}{w^{2n}} \int_0^w m_1 w_1^{2(n-1)} dw_1\right] + O\left[(n-1)^2\right]$$

Hence, it is deduced that

$$T_{-1} = \lim_{n \to 1} \frac{nm - wR_n}{nwR_n - m} = \frac{m \int_0^w m_1 \, dw_1}{w \int_0^m w_1 \, dm_1} \tag{49}$$

and

$$Y_{-1} = \frac{m}{2 \int_0^m w_1 \, dm_1}$$

Similar expressions hold for $\tilde{T}_{-1}$ and $\tilde{Y}_{-1}$. Hence the equations

$$\left.\begin{aligned} (Y_{-1})^{-1} &= \phi_1 = \frac{2}{m} \int_0^m w_1 \, dm_1 \\ (\tilde{Y}_{-1})^{-1} &= \psi_1 = \frac{2}{w} \int_0^w m_1 \, dw_1 \end{aligned}\right\} \tag{50}$$

represent the second solution of equations (41c) and (41d), independent of $Y_1 = \frac{1}{m}$ and $\tilde{Y}_1 = \frac{1}{w}$ and reducing to zero at $w = 0$. The expressions $\phi_1 e^{-i\theta}$ and $\psi_1 e^{-i\theta}$ represent a kind of motion between two parallel walls. These solutions could be directly obtained by inverting and exchanging the variables in equations (48) and putting $w_r = m_r = 0$.

If $n \to \infty$, equation (44) shows that $R_n^{\,2} \to R_\infty^{\,2} = \frac{dm^2}{dw^2}$. Similarly the equations

$$\left.\begin{aligned} \tilde{R}_\infty^{\,2} &= \frac{1}{R_\infty^{\,2}} = \frac{dw^2}{dm^2} \\ \tilde{T}_{-\infty}^{\,2} &= \frac{1}{T_{-\infty}^{\,2}} = \frac{d(1/m^2)}{d(1/w^2)} \end{aligned}\right\} \tag{51}$$

coincide with $1/\alpha^4$ and $1/\beta^4$ (equations (12)) and can be explicitly calculated by equation (16). Then equations (46) show that

$$X_\infty = \tilde{X}_\infty = Y_{-\infty}^{\;-1} = \tilde{Y}_{-\infty}^{\;-1} = e^\lambda \tag{52}$$

where $\lambda = \log w - \int_0^w \left[1 - (d \log m/d \log w)^{1/2}\right] d \log w$ coincides with the value deduced by integrating the $d\lambda$ given by equation (11) and determining the constant of integration so that $\frac{e^\lambda}{w} = 1$ at $w = 0$. Hence $e^{n\lambda}$ is the subsonic asymptotic value of χ_n, ω_n, ϕ_n, and ψ_n for $n \to \infty$.

The solutions of equations (41) just discussed are bound to the solutions, equations (42), by simple relations. For instance,

$$\left.\begin{aligned}\frac{\chi_n(w)}{\chi_n(w_r)} &= C_n(w,m) + R_n(w_r)S_n(w,m)\\ \frac{\psi_n(w)}{\psi_n(w_r)} &= C_n\left(\frac{1}{w},\frac{1}{m}\right) - \tilde{T}_n(w_r)S_n\left(\frac{1}{w},\frac{1}{m}\right)\end{aligned}\right\} \quad (53)$$

as can be verified by controlling the identity of the conditions at $w = w_r$. Hence as $w \to 0$ and $|C_n|$ and $|S_n| \to \infty$

$$\lim_{w \to 0} \frac{C_n(w,m)}{S_n(w,m)} = \lim_{w \to 0} \frac{S_n(m,w)}{C_n(m,w)} = -R_n(w_r)$$

$$\lim_{w \to 0} \frac{C_n\left(\frac{1}{m},\frac{1}{w}\right)}{S_n\left(\frac{1}{m},\frac{1}{w}\right)} = \lim_{w \to 0} \frac{S_n\left(\frac{1}{w},\frac{1}{m}\right)}{C_n\left(\frac{1}{w},\frac{1}{m}\right)} = T_n(w_r)$$

The following interesting expansions are deduced by applying equations (53) to the solutions in closed form obtained for $n = \pm 1$:

$$\left.\begin{aligned}w &= w_r C_1(w,m) + m_r S_1(w,m)\\ w\int_{1/w_r}^{1/w} \frac{1}{m_1}\, d(1/w_1) &= -\frac{1}{w_r}\, S_1(w,m)\end{aligned}\right\} \quad (54)$$

(and the corresponding expansions with the variables interchanged or inverted).

Finally, it should be observed that differentiating (for instance) equation (41a) with respect to w yields

$$\frac{1}{m}\frac{d}{dw}\left[w\frac{d}{d(1/m)}\left(\frac{dX_n}{dw}\right)\right] + (n^2 - 1)\frac{dX_n}{dw} = 0$$

This equation is of the same general form as equations (41) with only one of the variables inverted and $1 - n^2$ in place of n^2; it can therefore be treated in the same way as equations (41).

Hence two particular solutions similar to equations (42),

$$C_{\sqrt{1-n^2}}\left(\frac{1}{m},w\right) = \sum_{k=0}^{\infty}(1 - n^2)^k I_{2k}\left(\frac{1}{m},w\right)$$

and

$$S_{\sqrt{1-n^2}}\left(\frac{1}{m},w\right) = \sum_{k=0}^{\infty}(1 - n^2)^{k+\frac{1}{2}} I_{2k+1}\left(\frac{1}{m},w\right)$$

can be defined through the integrals I_h given by the formula (26) by simply replacing the present variables. For $n^2 > 1$ the second series is imaginary and must be divided by i to obtain a real solution. The general solution for dX_n/dw is given by a linear combination of these solutions, and the general solution for X_n is obtained by integrating and adding an approximate constant. Now, this must coincide with the one in terms of $C_n(w,m)$ and $S_n(w,m)$. It is then easily derived that

$$C_{\sqrt{1-n^2}}\left(\frac{1}{m},w\right) = \frac{m_r}{m}C_n(m,w) + \frac{w_r}{m}\frac{S_n(m,w)}{n}$$

and

$$\frac{S_{\sqrt{1-n^2}}\left(\frac{1}{m}, w\right)}{\sqrt{1 - n^2}} = -\frac{1}{m_r m} \frac{S_n(m, w)}{n}$$

Analogous relations obtained by interchanging and inverting the variables also hold. For $n^2 = 1$ these relations give equations (54) as a particular case.

Other interesting relations can be deduced in the same way.

Infinite series in the exponential set.— The series in $\phi_n e^{-in\theta}$ and $\psi_n e^{-in\theta}$ have been used (as series in $\chi_n e^{-in\theta}$ and $\omega_n e^{-in\theta}$ could be) by many authors in the case of the normal isentropic law (references 5, 6, 9, and 12).

They seem to have their natural field of application in the problem of two-dimensional gas jets, as Chaplygin first showed in his classical memoir.

The application to flows around bodies seems to be more difficult, especially for flows with circulation. The difficulty arises first from the presence in the hodograph plane of a singularity at $w = w_\infty$ and from the ensuing necessity of employing more than one series development in the exponential set with different sequences of n (as appears already in the incompressible case) with added eventual terms in other sets, and of insuring that the different series are the continuations of each other.[10] This can be achieved (although in a not very simple way) by putting the condition of continuity of the solutions and of their derivatives on the the transition curves (often circles), as has been done by Tsien and Kuo (reference 12) and as the author himself has done in an unpublished work in a somewhat different way, but it is believed that the main obstacle to this method arises from the difficulty of insuring that the body will have a closed contour when a circulation is present. In fact if the so-called "natural" series (that is a series having the same coefficients as in a chosen incompressible case with circulation) is used in one part of

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[10]This difficulty is avoided in the method by Bergman (reference 6), which uses a different type of expansion and uses the series in the exponential set only as eventual auxiliary series.

the hodograph plane (so that a basic series from which the coefficients of the other series will be deduced by the foresaid method is obtained), the resulting body will be closed only in the limiting incompressible case. In a tentative method the author has tried to obtain the closing-up of the contour by taking the coefficients of the basic series as simple functions of $R_n(w_r)$ (or of the other quantities in equations (43), w_r generally coinciding with w_∞) containing an arbitrary parameter. These simple functions reduce to the coefficients of the "natural" series when, for vanishing w_r, R_n becomes unity. The arbitrary parameter is then so determined that the contour closes up. However, because of the necessity of using, to express this condition, different series connected by intricate relations, this method seems to be very complicated.[11] In the method studied by the author, series of the kind given by equations (31), (32), and (33) (that can be put in simple relation with series in the exponential set) could be used, especially for the condition in transonic and supersonic regions where the series in the exponential set cease to be useful.

It is worthwhile to mention here that the demonstration of the convergence of the series in the exponential set (that Chaplygin first deduced in the hypergeometric case in a somewhat complicated way) can be obtained very simply and under very wide assumptions for $m(w)$ by using the properties of the functions defined by equations (43). Taking, for instance, the series in $X_n e^{-in\theta} = (X_n e^{-i\theta})^n$, it is immediately seen that $X_1 > X_\infty$; for, by equations (47) and (51),

$$\frac{R_\infty}{R_1} = (d \log m/d \log w)^{1/2} = (1 - M^2)^{1/2} < 1$$

and

$$\log (X_1/X_\infty) = \int_0^w \left[1 - (R_\infty/R_1)\right] d \log w > 0$$

[11]The problem of the closed contour has been solved in a very elegant way by Lighthill (reference 8), who discovered a very simple development converging in all the field (subsonic and transonic) and gave the conditions for the closing-up of the flow behind the body.

Moreover, for plausible $m(w)$,

$$R_1 = \frac{\rho}{\rho_0}$$

and

$$R_\infty = (1 - M^2)^{1/2} \frac{\rho}{\rho_0}$$

are decreasing functions of w, so that $\frac{dR_1}{dw} < 0$ and $\frac{dR_\infty}{dw} < 0$. Taking now a value of n greater than 1 it is immediately seen from equations (44) that if, for some value of w, $R_n \geqq R_1$, then $\frac{-d(R_n - R_1)}{dw} > 0$, and if $R_n \leqq R_\infty$ then $\frac{-d(R_\infty - R_n)}{dw} > 0$. Hence if one of the two conditions is verified for some value of w, then for decreasing w the value of R_n will diverge more and more from the value of R_1 and of R_∞, so that it cannot be equal to unity at $w = 0$. Hence, if $R_n = 1$ at $w = 0$, for other values of w, $R_\infty < R_n < R_1$ and from equations (46) $X_\infty < X_n < X_1$.

The following limitations for $\chi_n = X_n^n$ are found by use of equations (47) and (52), for $n > 1$:

$$e^{n\lambda} < \chi_n < w^n$$

Naturally, these limitations hold only for real values of R_∞ and λ, hence for $M \leqq 1$. In the same way it is proved that, for $n > 1$,

$$m^n < \omega_n < e^{n\lambda}$$

The analogous demonstration for ϕ_n and ψ_n requires the assumption that $T_{-\infty}^{\;2} = \frac{d(1/w^2)}{d(1/m^2)}$ must be an increasing function of w. This does not seem to be too restrictive a condition, for $T_{-\infty}$ must in any case be unity at $w = 0$ and infinity at $M = 1$. According to this assumption it can be directly verified that

$$\frac{d}{dw}\left[\frac{mw}{\left[1 + \left(\frac{d \log m}{d \log w}\right)\right]^{1/2}}\right] - m = -\frac{m^2}{\left[1 + \left(\frac{d \log m}{d \log w}\right)^{1/2}\right]^2} \frac{d}{dw}\left[\frac{d(1/m^2)}{d(1/w^2)}\right] > 0$$

and

$$m - \frac{d}{dw}\left(\frac{mw^2}{m + w}\right) = \frac{m^3}{(m + w)^2}\left[1 - \frac{d(1/m^2)}{d(1/w^2)}\right] > 0$$

By integrating these inequalities between 0 and w it follows easily that

$$\frac{m}{w}\int_0^w m_1 \, dw_1 \geq \int_0^m w_1 \, dm_1 \geq \left(\frac{d \log m}{d \log w}\right)^{1/2} \int_0^w m_1 \, dw_1$$

the sign of equality holding only at $w = 0$. Hence equations (49) and (51) show that, excluding $w = 0$, $1 < T_{-1} < T_{-\infty}$; and the equation corresponding to (44) shows that T_{-1}, like $T_{-\infty}$, is an increasing function of w. Now for $-n < -1$ from the equation in T_{-n}, by a reasoning identical to that developed for R_n, it can be proved that $1 < T_{-1} < T_{-n} < T_{-\infty}$ and that $\frac{1}{m} > Y_{-1} > Y_{-n} > Y_{-\infty}$. Hence for $\phi_n = Y_{-n}{}^{-n}$ the following bounds hold, for $n > 1$:

$$m^n < \left(\frac{2}{m}\int_0^m w_1 \, dm_1\right)^n < \phi_n < e^{n\lambda}$$

Similarly, for ψ_n

$$e^{n\lambda} < \psi_n < \left(\frac{2}{w}\int_0^w m_1 \, dw_1\right)^n < w^n$$

The convergence of the Chaplygin (or other) series can be immediately demonstrated by means of these bonds.

For negative integral (eventually half-integral) values of n, for the reasons discussed in the section entitled "The exponential set," the condition that R and T are unity at $w = 0$ is not sufficient to determine the solution. However, it is possible to give supplementary conditions, which are omitted for brevity, such that the resulting solutions of equations (41) may be used to construct series converging in all the subsonic hodograph field exterior to a given circle.

A new set.- Here, only briefly mentioned, is a different set of solutions of equations (23) in closed form. It has been observed in the section entitled "The exponential set" that the solutions ϕ_1 and ψ_1 given by equations (50) cannot be derived from the solutions of the equations in χ_n and ω_n for $n^2 = 1$.

Conversely, if for given $G = \phi_n e^{-in\theta}$ and $\tilde{G} = \psi_n e^{-in\theta}$ of the exponential set values of F and $\tilde{F}$ are deduced by means of equation (25), or the analogous equation for $\tilde{F}$, the solutions obtained are still of the corresponding exponential set for all values of n but 1.

In this case, writing instead of $G = \phi_1 e^{-i\theta}$ and $\tilde{G} = \psi_1 e^{-i\theta}$ the more general formulas

$$G = \frac{e^{-i\theta}}{m}\left(\int_{m_r}^{m} w_1 \, dm_1 + \frac{w_r m_r}{2}\right)$$

and

$$\tilde{G} = \frac{e^{-i\theta}}{w}\left(\int_{w_r}^{w} m_1 \, dw_1 + \frac{w_r m_r}{2}\right)$$

corresponding to equations (48), it follows from equation (25), with $\theta_r = 0$, that

$$F = -we^{-i\theta}\left[\int_{1/w_r}^{1/w} \frac{1}{m_1}\, d\left(\frac{1}{w_1}\right)\left(\int_{m_r}^{m_1} w_2\, dm_2 + \frac{w_r m_r}{2}\right) + \frac{i\theta}{2}\right]$$

and

$$\tilde{F} = -me^{-i\theta}\left[\int_{1/m_r}^{1/m} \frac{1}{w_1}\, d\left(\frac{1}{m_1}\right)\left(\int_{w_r}^{w_1} m_2\, dw_2 + \frac{w_r m_r}{2}\right) + \frac{i\theta}{2}\right]$$

are new solutions of the corresponding equations in (23), which are not included in any of the sets already discussed.

Then if the variables are inverted and interchanged and the sign of θ is changed, it is seen that

$$G = -\frac{1}{m}\, e^{i\theta}\left[\int_{m_r}^{m} w_1\, dm_1 \left(\int_{1/w_r}^{1/w_1} \frac{1}{m_2}\, d\left(\frac{1}{w_2}\right) + \frac{1}{2w_r m_r}\right) - \frac{i\theta}{2}\right]$$

and

$$\tilde{G} = -\frac{1}{w}\, e^{i\theta}\left[\int_{w_r}^{w} m_1\, dw_1 \left(\int_{1/m_r}^{1/m_1} \frac{1}{w_2}\, d\left(\frac{1}{m_2}\right) + \frac{1}{2w_r m_r}\right) - \frac{i\theta}{2}\right]$$

satisfy the corresponding equations in (23); hence the respective real and imaginary parts satisfy the equations for ϕ and ψ.

Now applying again equation (25) to the last expressions, other solutions of the equations in F and $\tilde{F}$ are found, from which by changing again the variables new solutions G and $\tilde{G}$ are found. Hence it is seen that the repeated application of the described process generates a new set of solutions F and $\tilde{F}$, and G and $\tilde{G}$.

The first terms of the corresponding incompressible set of G_i, which can be easily obtained by repeated application of the formula to which equation (25) reduces for $m = w$

$$F_i = -V \int_{w_r}^{V} G_i(V)\ d\left(\frac{1}{V}\right)$$

(where $V = we^{-i\theta}$) and successive inversion of V, are

$$V/2$$

$$-\frac{1}{2V} \log \frac{V}{w_r}$$

$$-\frac{V}{4}\left[\log \frac{V}{w_r} + \frac{1}{2}\left(\frac{{w_r}^2}{V^2} - 1\right)\right]$$

$$-\frac{1}{8V}\left[\log \frac{V}{w_r}\left(1 + \log \frac{V}{w_r}\right) - \frac{1}{2}\left(\frac{V^2}{{w_r}^2} - 1\right)\right]$$

. . . . .

THE GENERALIZED POTENTIAL FUNCTIONS

In the preceding sections, it has been shown that the symmetrical form, obtained by making the velocity and mass velocity (and not the connection between them) appear explicitly in the hodograph equations, gives rise to an interesting general treatment of these equations. It has been seen that the complex functions F and G, of which the real parts are χ and ϕ, are connected by symmetrical relations to the functions $\mathfrak{F}$ and $\mathfrak{G}$, having ω and ψ as imaginary parts. In this section, it will be shown that all these complex functions can be deduced by simple differentiations from a unique function Φ, called the generalized potential function.

A second generalized potential function Ψ is also introduced, with interesting properties.

Let the complex velocity and mass velocity be defined by

$$\left.\begin{aligned} V &= we^{-i\theta} \\ W &= m(w)e^{-i\theta} = \frac{\rho}{\rho_0} V \end{aligned}\right\} \tag{55}$$

and observe that they are bound by the condition that their ratio $\frac{W}{V} = \frac{\rho}{\rho_0}$ must be real and equal to a prescribed function of w, or that w must be a prescribed function $\Omega(W/V)$ of the relative density. When these conditions are satisfied, the two moduli of equations (55) will be connected by $|W| = m(|V|)$. Hence equations (55) can be written

$$V = \Omega\left(\frac{W}{V}\right)e^{-i\theta}$$

$$W = \frac{W}{V}\Omega\left(\frac{W}{V}\right)e^{-i\theta}$$

so that

$$e^{i\theta} = \frac{1}{V}\,\Omega\left(\frac{W}{V}\right)$$

$$w = \Omega\left(\frac{W}{V}\right)$$

and

$$m = \frac{W}{V}\,\Omega\left(\frac{W}{V}\right)$$

It is now seen that if W and V are not bound by the foresaid conditions and are independent, these relations can be used to define generalized complex values of $\theta(V,W)$, $w(V,W)$, and $m(V,W)$ and therefore of all the related quantities. Hence the equation

$$1 - M^2 = \frac{d \log m}{d \log w} = 1 - \frac{d \log \rho}{d \log w} = 1 - \frac{d \log (W/V)}{d \log \Omega(W/V)} \tag{56}$$

defines a complex Mach number $M = M(W/V)$ which, if the physical conditions concerning V and W are satisfied, reduces to the real $M(\rho/\rho_0)$ and can therefore be immediately deduced without the help of equation (56) by simply replacing in the expression $M(\rho/\rho_0)$ the real variable ρ/ρ_0 by the generally complex variable W/V.

Now the hodograph equations, considered for complex values of the variables, can be transformed by taking V and W as new independent variables. From equations (55),

$$\frac{\partial V}{\partial w} = e^{-i\theta}$$

$$\frac{\partial V}{\partial m} = e^{-i\theta} \frac{dw}{dm}$$

$$i \frac{\partial V}{\partial \theta} = we^{-i\theta}$$

$$\frac{\partial W}{\partial w} = e^{-i\theta} \frac{dm}{dw}$$

$$\frac{\partial W}{\partial m} = e^{-i\theta}$$

and

$$i \frac{\partial W}{\partial \theta} = me^{-i\theta}$$

so that

$$\left.\begin{aligned} m\frac{\partial}{\partial w} &= W\left(\frac{\partial}{\partial V} + \frac{dm}{dw}\frac{\partial}{\partial W}\right) \\ w\frac{\partial}{\partial m} &= V\left(\frac{dw}{dm}\frac{\partial}{\partial V} + \frac{\partial}{\partial W}\right) \\ i\frac{\partial}{\partial \theta} &= V\frac{\partial}{\partial V} + W\frac{\partial}{\partial W} \end{aligned}\right\} \tag{57}$$

Hence equations (23a) and (23b) become

$$W\left(F_V + \frac{dm}{dw}F_W\right) = V\tilde{F}_V + W\tilde{F}_W$$

and

$$V\left(\frac{dw}{dm}\tilde{F}_V + \tilde{F}_W\right) = VF_V + WF_W$$

where subscripts denote partial derivatives. These are satisfied if

$$F_V = \tilde{F}_W \tag{58a}$$

$$V\tilde{F}_V = W\frac{dm}{dw}F_W \tag{58b}$$

or

$$V^2\tilde{F}_V = \frac{d\ \log\ m}{d\ \log\ w}W^2F_W \tag{58c}$$

The first of these equations can be satisfied if a function Φ is introduced such that

$$\left.\begin{aligned} F &= \Phi_W \\ \tilde{F} &= \Phi_V \end{aligned}\right\} \tag{59}$$

and the second will also be satisfied if Φ is a solution of the equation

$$V^2\Phi_{VV} = (1 - M^2)W^2\Phi_{WW} \tag{60}$$

where $1 - M^2$ is the function of W/V defined by equation (56).

The meaning of equations (59) and (60) is the following. If a $\Phi(V,W)$ satisfying equation (60) is known, and after Φ_V and Φ_W are calculated, the right values of V and W are introduced (that is, such that W/V is real and $|W| = m(|V|)$), then

$$\chi = \text{R.P.}(\Phi_W)$$

$$\omega = \text{I.P.}(\Phi_V)$$

will be solutions of equations (8).

Observe that if in equation (60) the two complex variables are replaced by the real variables θ and w (and $m(w)$) by the inverse relations of equations (57), the resulting equation (in a complex Φ) remains unchanged by exchanging w and m. The functions F and $\tilde{F}$ (and χ and ω) can be deduced by relations containing the derivatives of Φ with respect to w, m, and $i\theta$.

Equations (23c) and (23d) can be treated in the same way and the result is found that if $\Psi(1/V,1/W)$ is a solution of

$$\frac{1}{V^2}\frac{\partial^2\Psi}{\left(\partial\frac{1}{V}\right)^2} = (1 - M^2)\frac{1}{W^2}\frac{\partial^2\Psi}{\left(\partial\frac{1}{W}\right)^2} \tag{61}$$

then the functions

$$\left.\begin{aligned} G &= \frac{\partial\Psi}{\partial\frac{1}{V}} \\ \tilde{G} &= \frac{\partial\Psi}{\partial\frac{1}{W}} \end{aligned}\right\} \tag{62}$$

will be, upon substitution of the right values of W and V, solutions of equations (23c) and (23d), so that the functions

$$\phi = \text{R.P.}\left(\frac{\partial\Psi}{\partial\frac{1}{V}}\right)$$

and

$$\psi = \text{I.P.}\left(\frac{\partial\Psi}{\partial\frac{1}{W}}\right)$$

will be solutions of equations (6).

Naturally ϕ and ψ can also be related to Φ. From equations (24), with the help of equations (57) and (58), it follows that

$$\left.\begin{aligned} G &= \frac{V^2}{W}\,\Phi_{VV} + V\Phi_{VW} - \Phi_W = V^2 \frac{\partial}{\partial V}\left(\frac{\Phi_W}{V} + \frac{\Phi_V}{W}\right) \\ \tilde{G} &= W\Phi_{VW} + \frac{W^2}{V}\,\Phi_{WW} - \Phi_V = W^2 \frac{\partial}{\partial W}\left(\frac{\Phi_W}{V} + \frac{\Phi_V}{W}\right) \end{aligned}\right\} \tag{63}$$

Hence, comparison with equations (62) shows that to within an unessential constant,

$$\Psi = -\left(\frac{\Phi_W}{V} + \frac{\Phi_V}{W}\right) \tag{64}$$

The physical coordinates are deduced by the following relations which may be obtained from equations (30) by use of equations (55), (57), and (58):

$$\left.\begin{aligned} S + i\tilde{N} &= e^{-i\theta}\left(\frac{V}{W}\,\Phi_{VV} + \Phi_{VW}\right) \\ \tilde{S} + iN &= e^{-i\theta}\left(\Phi_{VW} + \frac{W}{V}\,\Phi_{WW}\right) \end{aligned}\right\} \tag{65}$$

After assigning to W and V their correct values, separating the real from the imaginary part, and using equations (2), it can be shown explicitly that

$$z = \Phi_{VW} + \frac{1}{2W}\left(V\Phi_{VV} + \overline{V\Phi}_{VV}\right) + \frac{1}{2V}\left(W\Phi_{WW} - \overline{W\Phi}_{WW}\right)$$

which can also be written

$$z = \frac{1}{V}\,\text{R.P.}\left[V\left(V\frac{\partial}{\partial V} + W\frac{\partial}{\partial W} + 1\right)\frac{\Phi_V}{W}\right] + \frac{i}{W}\,\text{I.P.}\left[W\left(V\frac{\partial}{\partial V} + W\frac{\partial}{\partial W} + 1\right)\frac{\Phi_W}{V}\right] \qquad (66)$$

By differentiating this equation, there can be deduced an expression of dz which must coincide with the expression (2), that is with

$$dz = \frac{d\phi}{V} + i\,\frac{d\psi}{W} \qquad (67)$$

when $d\phi$ and $d\psi$ are obtained by differentiating the expression derived from equations (63)

$$\phi + i\psi = \text{R.P.}\left[V^2\frac{\partial}{\partial V}\left(\frac{\Phi_W}{V} + \frac{\Phi_V}{W}\right)\right] + i\,\text{I.P.}\left[W^2\frac{\partial}{\partial W}\left(\frac{\Phi_W}{V} + \frac{\Phi_V}{W}\right)\right] \qquad (68)$$

It can be shown that the agreement exists if Φ is a solution of equation (60), and if W/V is real. It seems therefore that the comparison of the values of z calculated directly from equation (66), or deduced by integration of equation (67) (using equation (68)), may constitute an interesting check on the accuracy of approximate methods.

Observe that $\Phi = c_1V + c_2W + c_3VW$ is a particular solution of equation (60), the simple meaning of which is that c_1 constitutes an additive constant of ω and of $-\psi$, c_2 the same for χ and $-\phi$, and c_3 (complex) represents a general displacement of the origin of the physical coordinates.

Particular sets of solutions of equations (60) and (61) are easily deduced from the sets studied in the preceding sections by integrating the relations (59) and (62) and introducing the expressions of w, m, and θ as functions of V and W.

In the incompressible case, $V = W$, $1 - M^2 = 1$ and equations (60) and (61) reduce to identities satisfied by every function of V. The written relations reduce then to

$$F_i(V) = \Phi_i'(V)$$

$$G_i(V) = -V^2\Psi_i'(V) = V\Phi_i''(V) - \Phi_i'(V) = V^2 \frac{d}{dV}\left(\frac{\Phi_i'(V)}{V}\right)$$

$$z_i(V) = F_i'(V) = \Phi_i''(V) = e^{i\theta}(S + iN) = w\,\frac{S + iN}{V}$$

If the solution corresponding to the incompressible flow around a given body is known in the physical plane, $\Phi_i(V)$ can be deduced from these relations. The profile of the body can, for instance, be defined by a relation $S = P(N)$ between the subnormal and the normal for $\psi = 0$ (and by giving this value to the corresponding streamline).

Hence for $\text{I.P.}(G_i) = 0$, that is, for $\text{I.P.}(V\Phi_i'') = \text{I.P.}(\Phi_i')$ it follows that

$$wS = \text{R.P.}(V\Phi_i'') = wP\left[\frac{1}{w}\,\text{I.P.}(\Phi_i')\right]$$

The corresponding relation for the compressible case, which can be easily deduced from the preceding formulas, is

for $\text{I.P.}\left[W^2 \frac{\partial}{\partial W}\left(\frac{\Phi_W}{V} + \frac{\Phi_V}{W}\right)\right] = 0,$

$$\text{R.P.}\left(W\Phi_{VW} + V\Phi_{VV}\right) = mP\left[\frac{1}{m}\ \text{I.P.}(\Phi_V)\right]$$

after the correct values for V and W are introduced. The function P will be the same in both cases if the profile is unchanged.

The equations (60) and (61) are of the same general kind. It is possible to pass from the one to the other not only by a substitution like equation (64) but also by simply putting

$$\left.\begin{aligned} \Phi &= VW\Phi_* \\ \Psi &= \frac{1}{VW}\Psi_* \end{aligned}\right\} \qquad (69)$$

since, as can be immediately verified, Φ_* must satisfy equation (61) and Ψ_*, equation (60). The equations may be transformed in many ways by changing the two independent variables. One of these transformations is obtained by taking as new independent variables (K being a constant):

$$\left.\begin{aligned} \xi &= \frac{1}{V} + \frac{1}{W\sqrt{K}} \\ \eta &= \frac{1}{V} - \frac{1}{W\sqrt{K}} \end{aligned}\right\} \qquad (70)$$

The equation (61) in Ψ (or Φ_*) is thus transformed into

$$\Psi_{\xi\eta} = -\frac{1}{4}\left[1 - (1 - M^2)\frac{V^2}{KW^2}\right]\left(\Psi_{\xi\xi} - 2\Psi_{\xi\eta} + \Psi_{\eta\eta}\right) \tag{71}$$

where the first factor of the right-hand side is a function of W/V, hence of η/ξ, since

$$\frac{W}{V} = \frac{1}{\sqrt{K}}\frac{1 + (\eta/\xi)}{1 - (\eta/\xi)}$$

A similar transformation may be performed on equation (60) by putting $\xi = V + W$ and $\eta = V - W$.

Other interesting transformations are obtained by taking $\xi = \log V + \epsilon(W/V)$ and $\eta = \delta(W/V)$ and choosing in different ways the functions ϵ and δ. In this case the variability of the η variable can be restricted to the real field. Particular cases are obtained by taking, for instance, $\epsilon = 0$, $\epsilon = \log (W/V)$, $\delta = 1 - M^2$ and $\delta = w(W/V)$ $\left(\text{since } \dfrac{d \log w}{d \log (V/W)} = \dfrac{1}{M^2}\right)$. An important case is the one for which $\xi = \lambda - i\theta$ and $\eta = \lambda$, where λ is the same as in equation (11) or equation (52); this is obtained by taking $\dfrac{d\epsilon}{d \log (V/W)} = \dfrac{-1}{\left(1 + \sqrt{1 - M^2}\right)}$ and $\dfrac{d\delta}{d \log (V/W)} = \dfrac{\sqrt{1 - M^2}}{M^2}$. If this transformation is performed, an equation in Φ_* and Ψ (or one in Φ and Ψ_*) is deduced that may be used to obtain directly solutions of the kind obtained by Bergman (reference 6) and Lighthill (reference 8).

Finally, let equation (60) be written with the actual isentropic law. In this case

$$\frac{\rho}{\rho_0} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{1-\gamma}}$$

Hence, consideration of the observation following equation (56) and substitution of the resulting value of $M(W/V)$ in equation (60), yields:

$$V^2\Phi_{VV} = \left\{1 - \frac{2}{\gamma - 1}\left[\left(\frac{V}{W}\right)^{\gamma-1} - 1\right]\right\}W^2\Phi_{WW}$$

There is a corresponding equation for equation (61). For real W/V the corresponding values of $1 - M^2$ are shown in figure 5 for some values of γ with $\frac{W}{V} = \frac{\rho}{\rho_0}$ as abscissa. It is interesting to observe that if γ is in the actual range for gases, its value does not seem to affect to a great extent the shape of the curves, especially in the subsonic range. The factor $1 - M^2$ is linear in W/V for $\gamma = 0$ and in V/W for $\gamma = 2$; for other values of γ in the actual range it is not far from a straight line in the subsonic range.

The curve $\gamma = -1$, that is $1 - M^2 = \frac{W^2}{V^2}$ is also represented in the figure, and corresponds to Chaplygin's approximation. The Kármán-Tsien approximation corresponds to $1 - M^2 = \frac{KW^2}{V^2}$ where the constant K (see equations (17) and (21)) is so chosen that at infinity $(\rho = \rho_\infty)$ $1 - M^2$ will take the value $1 - M_\infty^2$ given by the true law.

THE CHAPLYGIN-KÁRMÁN-TSIEN CASE

For the Chaplygin-Kármán-Tsien approximation the right-hand side of equation (71) (or equation (69)) becomes zero so that the general solutions of equations (60) and (61) are

$$\left.\begin{aligned} \Phi &= VW\left[f_1(\xi) + f_2(\eta)\right] \\ \Psi &= e_1(\xi) + e_2(\eta) \end{aligned}\right\} \qquad (72)$$

with e_1, e_2, f_1, and f_2 arbitrary functions of the variables ξ and η defined by equations (70). The value $K = 1$ corresponds to Chaplygin's approximation; $K = (1 - M_\infty{}^2)\dfrac{\rho_0{}^2}{\rho_\infty{}^2}$, to the Kármán-Tsien approximation.[12] As it has been observed in the section entitled "Approximate Methods" it can be convenient to chose K-values between the two.

The solutions (72) can be interpreted in two different ways. First they can be regarded as the exact solutions of the corresponding equations for a gas satisfying the ideal law (18).

With the use of equations (70) for ξ and η, this law can be written $|\xi||\eta| = -H$; or for real W/V, hence for real ξ/η,

$$|\xi\eta| = \xi\bar{\eta} = \eta\bar{\xi} = -H \tag{73}$$

This relation allows the expression of Ψ (and Φ_*) as the sum of two arbitrary functions of ξ and $\bar{\xi}$ or of η and $\bar{\eta}$.

In the second interpretation, equations (72) are considered as approximate solutions of the equations for the actual law of gases. In this case

$$|\xi\eta| = \xi\bar{\eta} = \eta\bar{\xi} = \frac{1}{w^2}\left(1 - \frac{\rho_0{}^2}{\rho^2 K}\right) \tag{74}$$

is no longer a constant, but is a function of w.

[12]As observed in footnote 4 the actual presentation of this method is more coherent than the usual one, as the constants H and K of formula (18) are here deduced for a single reference condition; namely, the infinite point.

Naturally the last expression, and the relation (73), do not restrict the independency of ξ and η; as they must be used only after all the formal deductions from equations (72) of the following kind have been performed. From equations (72) and the application of equations (59),

$$F = \Phi_W = V\left(f_1 - \frac{1}{W\sqrt{K}} f_1' + f_2 + \frac{1}{W\sqrt{K}} f_2'\right)$$

$$\tilde{F} = \Phi_V = W\left(f_1 - \frac{1}{V} f_1' + f_2 - \frac{1}{V} f_2'\right)$$

and

$$\frac{\Phi_W}{V} + \frac{\Phi_V}{W} = 2f_1 - \xi f_1' + 2f_2 - \eta f_2'$$

According to equation (64), the last expression coincides with $-\tilde{\Psi}$; hence, by equations (72),

$$e_1 = 2f_1 - \xi f_1'$$

$$e_2 = 2f_2 - \eta f_2'$$

From equations (62),

$$G = \phi + i\tilde{\psi} = \xi f_1'' - f_1' + \eta f_2'' - f_2'$$

and

$$\tilde{G} = \tilde{\phi} + i\psi = \frac{1}{\sqrt{K}}\left(\xi f_1'' - f_1' - \eta f_2'' + f_2'\right)$$

so that

$$\phi + i\sqrt{K}\psi = \xi f_1'' - f_1' + \overline{\eta f_2'' - f_2'} = g_1(\xi) + g_2(\overline{\eta}) \tag{75}$$

g_1 and g_2 being two functions related to f_1 and f_2 by the relations

$$\left.\begin{aligned} g_1' &= \xi f_1''' \\ g_2' &= \eta f_2''' \end{aligned}\right\} \tag{76}$$

Now, from equation (66) for real W/V (hence if V and W have the same argument $-\theta$, and ξ and η the same argument θ):

$$\begin{aligned} z = {} & e^{i\theta}\text{R.P.}\left\{e^{-i\theta}\left[f_1 - \xi f_1' + \frac{1}{2}\xi(\xi + \eta)f_1'' + f_2 - \eta f_2' + \frac{1}{2}\eta(\xi + \eta)f_2''\right]\right\} \\ & + ie^{i\theta}\text{I.P.}\left\{e^{-i\theta}\left[f_1 - \xi f_1' + \frac{1}{2}\xi(\xi - \eta)f_1'' + f_2 - \eta f_2' - \frac{1}{2}\eta(\xi - \eta)f_2''\right]\right\} \\ = {} & f_1 - \xi f_1' + \frac{1}{2}\xi^2 f_1'' + f_2 - \eta f_2' + \frac{1}{2}\eta^2 f_2'' + \frac{1}{2}e^{2i\theta}\overline{\xi\eta(f_1'' + f_2'')} \\ = {} & f_1 - \xi f_1' + \frac{1}{2}\xi^2 f_1'' + f_2 - \eta f_2' + \frac{1}{2}\eta^2 f_2'' + \frac{1}{2}|\xi\eta|\overline{(f_1'' + f_2'')} \end{aligned} \tag{77}$$

This relation gives the physical coordinates as a function of the hodograph coordinates w and θ when $\xi = \left[\frac{1}{w} + \frac{1}{m(w)\sqrt{K}}\right]e^{i\theta}$ and $\eta = \left[\frac{1}{w} - \frac{1}{m(w)\sqrt{K}}\right]e^{i\theta}$ are introduced into it (with the assigned law m(w)).

Differentiating equation (77) yields

$$2\,dz = \xi^2 f_1{}'''d\xi + \eta^2 f_2{}'''d\eta + |\xi\eta|\overline{(f_1{}'''d\xi + f_2{}'''d\eta)} + \overline{(f_1{}'' + f_2{}'')}\,d|\xi\eta|$$

But dz is also given by equation (2), which may be transformed in the following way

$$2\,dz = \frac{2}{V}\,d\phi + i\,\frac{2}{W\sqrt{K}}\sqrt{K}\,d\psi = \xi\,d(\phi + i\sqrt{K}\psi) + \eta\,d(\phi - i\sqrt{K}\psi)$$

or (see equation (75))

$$2\,dz = \xi^2 f_1{}'''d\xi + \eta^2 f_2{}'''d\eta + |\xi\eta|\overline{(f_1{}'''d\xi + f_2{}'''d\eta)}$$

Hence, in accordance with the observation following equation (68), these two expressions for dz coincide only if $|\xi\eta|$ is constant; that is, if the law connecting ξ and η is the law (73) (or (18)), hence the same law as the one for which the equations (60) and (61) admit the solutions (72). This is what is done in the Kármán-Tsien method. If, on the contrary, the alternative interpretation of equation (72) is adopted, the coincidence ceases to exist. In this case, if the law connecting ξ with η is the exact gas law, the error term between the two values of z, that is,

$$\Delta z = \frac{1}{2}\int\overline{(f_1{}'' + f_2{}'')}\,d|\xi\eta|$$

with $|\xi\eta|$ given by equation (74), may be regarded as a measure of the approximation of the approximate solution (72).

Observe that if the law (73) is taken, hence if $\overline{\eta} = -\frac{H}{\xi}$, equation (75) shows that the general solution for $\phi + i\sqrt{K}\psi$ is a function of ξ alone (containing the constant H), that is, it is a function of $\lambda - i\theta$, where λ is given by equation (19), a result in accordance with the Chaplygin's monogeneity conditions.

The expression (77) for z can be also written by using the functions g_1 and g_2 instead of f_1 and f_2 so that (see equation (76)),

$$2z = \int \xi^2 f_1''' d\xi + \int \eta^2 f_2''' d\eta + |\eta\xi| \overline{(f_1'' + f_2'')}$$

$$= \int \xi \, dg_1 + \int \eta \, d\overline{g}_2 + |\eta\xi| \left(\int \overline{\frac{1}{\xi} \, dg_1} + \int \frac{1}{\eta} \, dg_2 \right) \tag{78}$$

If the law (73) is assumed, this equation reduces to

$$2z = \int \xi (dg_1 + dg_2) + \int \overline{\eta (dg_1 + dg_2)}$$

$$= \int \xi \, dg - H \int \overline{\frac{1}{\xi} \, dg} \tag{79}$$

where $g(\xi) = g_1(\xi) + g_2(-H/\xi)$ is the complex potential $\phi + i\sqrt{K}\psi$ in this case. In the incompressible case $H = 0$, $K = 1$, and $2z_i = \int \xi \, dg$ with $\frac{2}{\xi} = V$. Hence Tsien's formula $z = z_i - \frac{H}{4} \int \overline{\left(\frac{dg}{dz_i} \right)^2 dz_i}$ is immediately obtained.

It is well-known that in the case of the flow round a body Tsien's formula generates closed profiles only if circulation is absent. Many authors have studied extentions of the method to the case with circulation. Bers, Germain, and Leray (references 13 and 14) have followed a first way; Lin, Germain, and Gelbart (references 15, 14, and 7) a second way; here a third way of constructing flows around closed profiles with circulation will be shown, based on the subdivision of $g(\xi)$ into $g_1(\xi)$ and $g_2(\xi)$. Let $g_1(V_\infty/V(z_i)) = g_1(\xi/\xi_\infty)$ be the complex potential of an incompressible flow around a closed profile with a circulation Γ. Then, since for ξ near ξ_∞,

$$(\xi/\xi_\infty)^k = 1 + k\left[(\xi/\xi_\infty) - 1\right] + 0\left\{\left[(\xi/\xi_\infty) - 1\right]^2\right\}$$

it follows that

$$\left.\begin{aligned} &\oint \xi \, dg_1 = 2\oint dz_i = 0 \\ &\oint dg_1 = 2\oint \frac{1}{\xi}\, dz_i = -\frac{2}{\xi_\infty^{\ 2}}\oint (\xi - \xi_\infty) dz_i = \Gamma \end{aligned}\right\} \tag{80}$$

where the integrations are performed along any contour in the physical plane enclosing the profile, or around the corresponding contour in the ξ-plane enclosing ξ_∞ (for simplicity, suppose V_∞ and ξ_∞ real). Now generally,

$$\oint \xi^k dz_i = k\xi_\infty^{\ k-1}\oint (\xi - \xi_\infty) dz_i = -\frac{k}{2}\,\xi_\infty^{\ k+1}\,\Gamma \tag{81}$$

so that

$$\oint \frac{1}{\xi}\, dg_1 = 2\oint \frac{1}{\xi^2}\, dz_i = \frac{2}{\xi_\infty}\Gamma \tag{82}$$

and the value of z given by equation (79) for a compressible flow with $g = g_1$ is not one-valued. But if $g = g_1 + g_2$ for the given g_1, a g_2 can be determined in such a way that the corresponding residual terms in equation (79) will compensate the value of the last written integral. If the value of the circulation must remain unchanged, g_2 must be one-valued. This condition is obtained very simply by taking, for instance,

$$g_2 = h\left(\frac{\xi_\infty}{\xi}\right)^n \int \frac{\xi}{\xi_\infty}\, dg_1 = \frac{2h}{\xi_\infty}\left(\frac{\xi_\infty}{\xi}\right)^n z_i$$

The constant h can now be determined so as to obtain the said compensation. From the identity

$$\int \xi^{-r}\, dg_2 = \frac{n}{n + r}\left(\xi^{-r} g_2\right) + \frac{r}{n + r}\int \xi^{-(n+r)}\, d\left(\xi^n g_2\right)$$

and from equation (81) it follows that

$$\oint \xi^{-r}\, dg_2 = \frac{r}{n + r}\oint \xi^{-(n+r)}\, d\left(\xi^n g_2\right)$$

$$= \frac{r}{n + r}\, 2h\xi_\infty{}^{n-1} \oint \xi^{-(n+r)}\, dz_i = rh\xi_\infty{}^{-r}\Gamma$$

Hence for real ξ_∞ and $H = -\xi_\infty \eta_\infty$,

$$\oint \xi\, dg_2 - H \int \overline{\frac{1}{\xi}\, dg_2} = -h(\xi_\infty - \eta_\infty)\Gamma$$

If equation (79) must be one-valued, the last quantity must be equal to (see equation (82))

$$H \int \overline{\frac{1}{\xi}\, dg_1} = -2\eta_\infty \Gamma$$

Hence the equations

$$h = \frac{2\eta_\infty}{\xi_\infty - \eta_\infty}$$

and

$$g_2 = \frac{2\eta_\infty}{\xi_\infty - \eta_\infty}\left(\frac{\xi_\infty}{\xi}\right)^n \int \frac{\xi}{\xi_\infty}\, dg_1 \tag{83}$$

will satisfy the foresaid condition of compensation.

Observe now that the constant factor does not depend on n. Hence the condition is also satisfied if $(\xi_\infty/\xi)^n$ is replaced by $P(\xi_\infty/\xi)/P(1)$, where P is a polymonial in ξ_∞/ξ or an infinite series converging in all the domain of variation of ξ. Hence the function

$$\phi + i\sqrt{K}\psi = g(\xi) = g_1(\xi) + \frac{2\eta_\infty}{\xi_\infty - \eta_\infty}\,\frac{P(\xi_\infty/\xi)}{P(1)} \int \frac{\xi}{\xi_\infty}\, dg_1 \tag{84}$$

where P is an arbitrary function of ξ_∞/ξ, is analytic in all its domain of variation, and will generate a flow around a closed profile with circulation Γ.

Observe that by using equations (70) for real ξ_∞ and η_∞ the expression for the constant h is

$$h = \frac{2\eta_\infty}{\xi_\infty - \eta_\infty} = \frac{\rho_\infty}{\rho_0}\sqrt{K} - 1$$

For the Chaplygin or Kármán-Tsien values of K there is obtained, respectively, the very simple expressions, both vanishing in the incompressible case:

$$h = \frac{\rho_\infty}{\rho_0} - 1$$

and

$$h = \sqrt{1 - M_\infty^{\,2}} - 1$$

Particularly simple forms of equation (84) are obtained by putting $n = 0$ or $n = 1$ in equation (83). For $n = 1$ an expression is obtained which coincides with what becomes the Lighthill solution for $\gamma = -1$ (reference 8).

The solution (84) satisfies the condition of generating solutions around closed profiles with circulation when $|\eta\,\xi| =$ Constant, and z is given by equation (79). Now if $|\eta\,\xi|$ is variable, as given by equation (74), and the equations (72) are considered as approximate solutions of the exact equations (60) and (61), it is still possible to find solutions for which z, given by equation (78), is one-valued. If again $g_1(\xi/\xi_\infty)$ is the complex potential for the incompressible case, satisfying equations (80), with Γ_1 in place of Γ, and if $g_2(\bar{\eta}/\eta_\infty)$ is a function for which the expressions

$$\left.\begin{aligned} \oint \bar{\eta}\, dg_2 &= 0 \\ \oint dg_2 &= \Gamma_2 \end{aligned}\right\} \qquad (85)$$

(analogous to equations (80)) hold where the integrations are performed in the $\bar{\eta}$ plane along the contour corresponding to the one of equations (80), then equation (82) and the analogous relation for g_2 give

$$\oint \frac{1}{\xi} dg_1 = \frac{2\Gamma_1}{\xi_\infty}$$

$$\oint \frac{1}{\bar{\eta}} dg_2 = \frac{2\Gamma_2}{\eta_\infty}$$

so that the sum

$$\oint \overline{\frac{1}{\xi} dg_1} + \oint \frac{1}{\bar{\eta}} dg_2 = 2\left(\frac{\Gamma_1}{\xi_\infty} + \frac{\Gamma_2}{\eta_\infty}\right)$$

will be zero for

$$\Gamma_2 = -\frac{\eta_\infty}{\xi_\infty} \Gamma_1 \tag{86}$$

Hence equation (78) will be one-valued if g_2 satisfies equation (85) with Γ_2 given by equation (86), and

$$\phi + i\sqrt{K}\psi = g_1(\xi) + g_2(\bar{\eta})$$

will represent, when the expressions (70) for ξ and η are replaced by the exact connection between m and w, an approximate solution of the exact equations with circulation

$$\Gamma = \Gamma_1 + \Gamma_2 = \Gamma_1\left(1 - \frac{\eta_\infty}{\xi_\infty}\right)$$

z being given by equation (78), and the error term

$$\Delta z = \frac{1}{2}\int\left(\int\overline{\frac{1}{\xi}\,dg_1} + \int\frac{1}{\eta}\,dg_2\right)d\,|\xi\eta|$$

representing a measure of the approximation obtained.

Clearly the simplest way of satisfying equations (85) and (86) is that of taking[13]

$$g_2(\overline{\eta}) = -\frac{\eta_\infty}{\xi_\infty}\,g_1(\overline{\eta})$$

Naturally this solution still holds when $|\xi\eta|$ = Constant.

An approximate solution for the transonic case is now noted, corresponding to the subsonic Kármán–Tsien approximation. If $M_\infty = 1$, then the Kármán–Tsien value (equation (21)) for K is zero. The

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[13]If the Kármán–Tsien value (equation (21)) is adopted for K, which makes the right-hand side of equation (71) zero at infinity, then

$$-\frac{\eta_\infty}{\xi_\infty} = \frac{1 - \sqrt{1 - M_\infty^2}}{1 + \sqrt{1 - M_\infty^2}} \quad \text{and} \quad 1 - \frac{\eta_\infty}{\xi_\infty} = \frac{2}{1 + \sqrt{1 - M_\infty^2}}$$

corresponding curve of $1 - M^2$ in figure 5 reduces to the horizontal axis $1 - M^2 = 0$. In this case the right-hand sides of equations (60) and (61) are zero, and the respective solutions are

$$\Phi = f_1(W) + Vf_2(W)$$

and

$$\Psi = \frac{1}{V} e_1(W) + e_2(W)$$

where the arbitrary functions involved are bound by the relations

$$-e_1 = f_1'$$

and

$$-e_2 = f_2' + \frac{f_2}{W}$$

Equations (66) and (68) then give

$$z = \frac{1}{W} \text{R.P.}(Wf_2') + \frac{i}{W}\left[\psi + \text{I.P.}(f_2)\right] = f_2' + \frac{i}{W} \text{I.P.}\left[W^2\left(\frac{f_1''}{V} + f_2''\right)\right]$$

$$\phi + i\psi = -\text{R.P.}(f_1') + i\text{I.P.}\left(\frac{W^2}{V} f_1'' + W^2 f_2'' + Wf_2' - f_2\right)$$

The approximate gas law corresponding to $1 - M^2 = 0$ is, by equation (56), $m = \text{Constant}$, therefore ρ_0/ρ is proportional to w. Actually m has a maximum at $M = 1$, and the distance between the streamlines reaches there a minimum. If the approximate law $m = \text{Constant}$ were adopted, the distance between the streamlines would be unchanged throughout the field of motion, and this would give rise to difficulties. But if the solution is considered as an approximate solution of the exact equations in the transonic field (even for $M_\infty \neq 1$ but near 1), then the exact law for $m(w)$ can be introduced in the solutions. This approximate transonic theory seems worthy of development.

September 27, 1949

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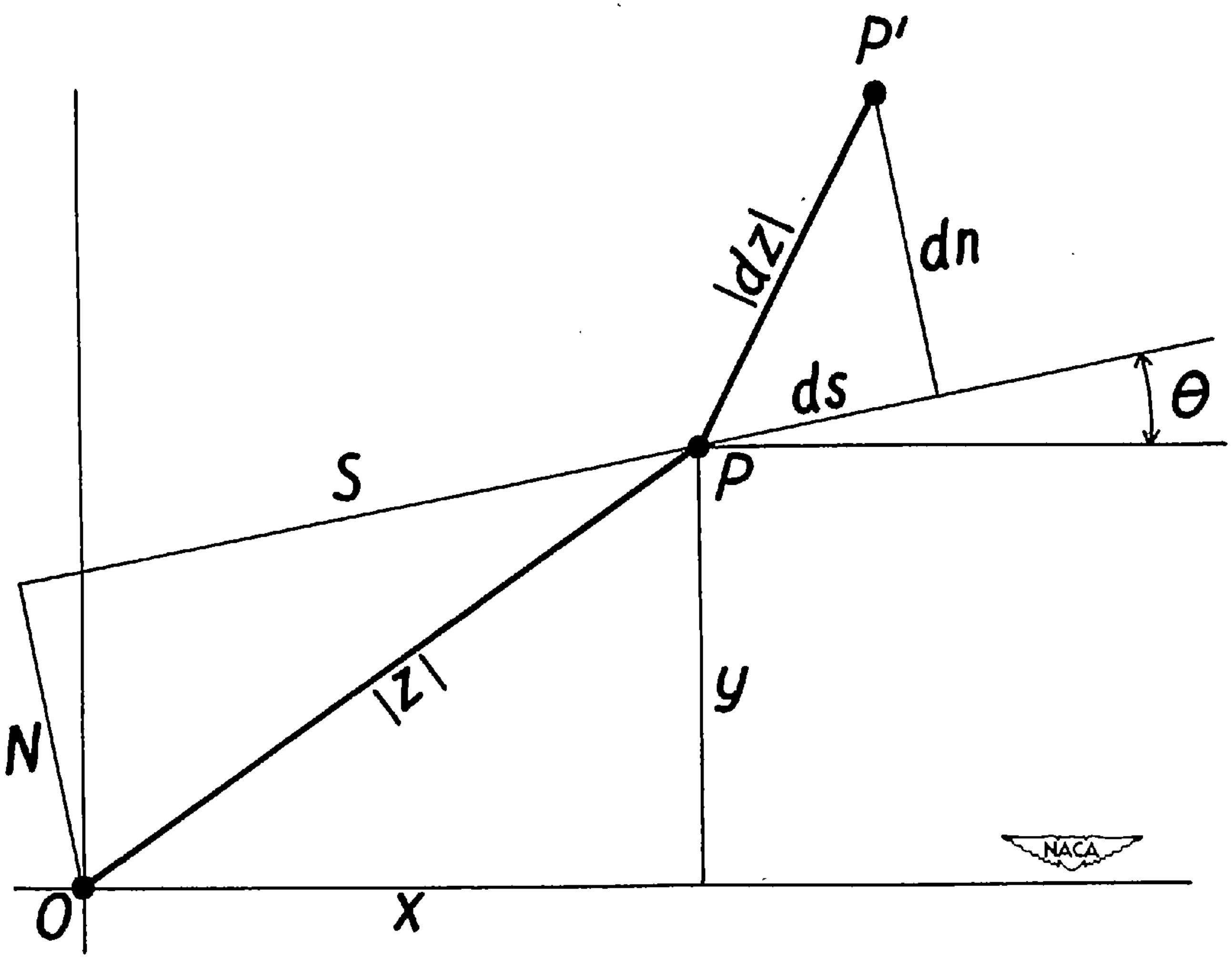

Figure 1.- The plane of flow x,y or S,N.

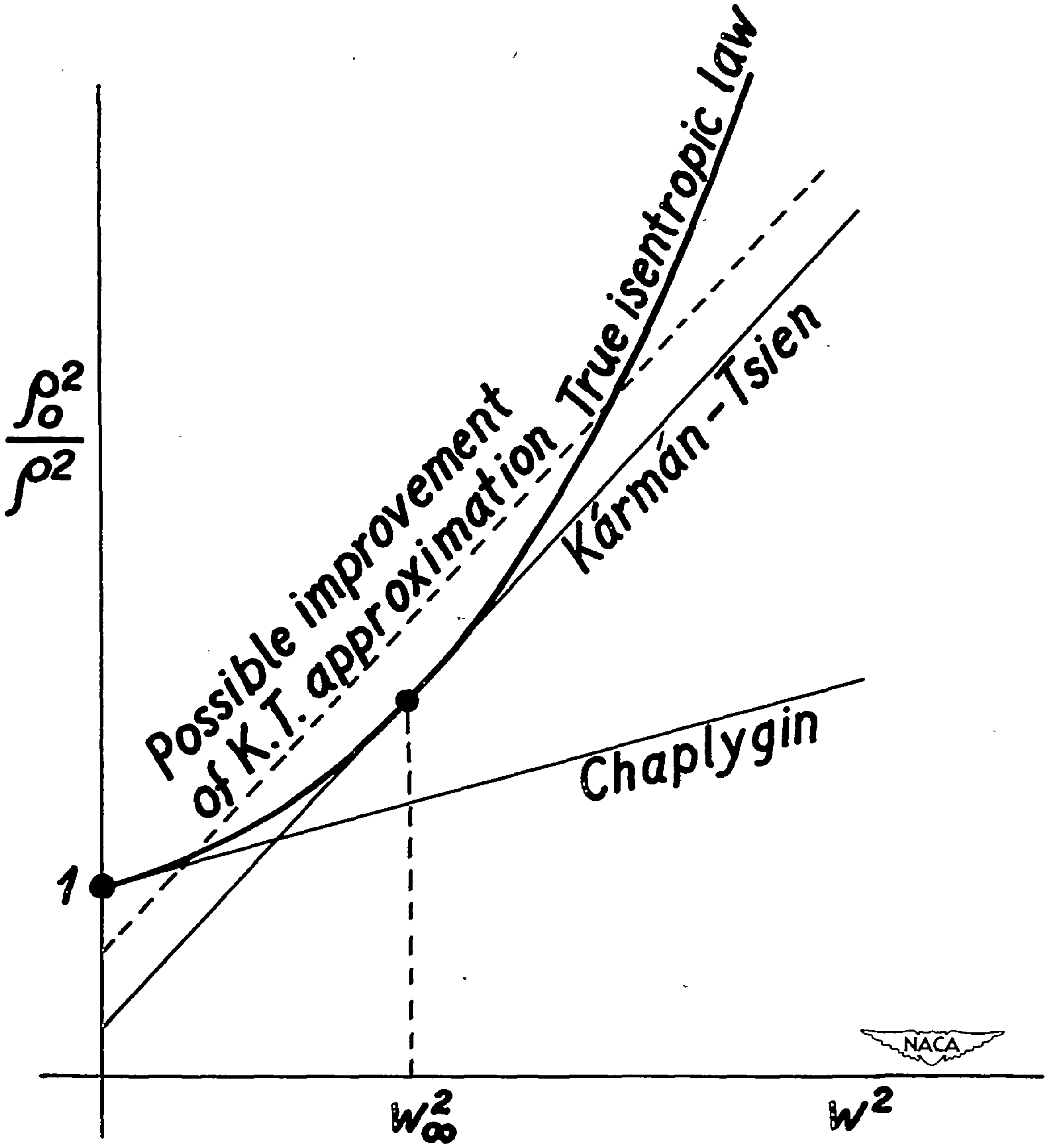

Figure 2.- Comparison of the Kármán-Tsien and Chaplygin approximations with the true isentropic law.

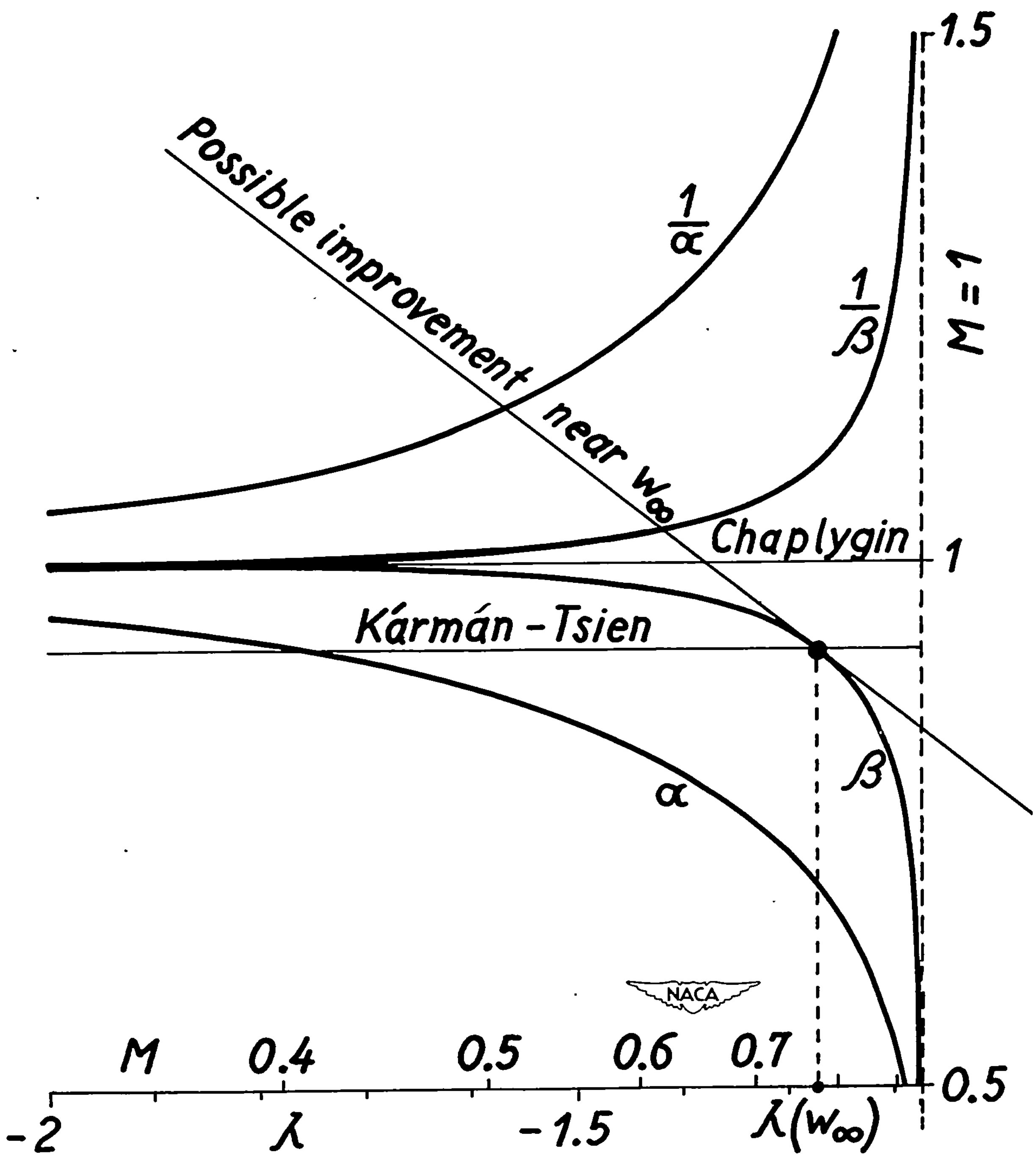

Figure 3.- Variables α and β as functions of λ for isentropic gas.

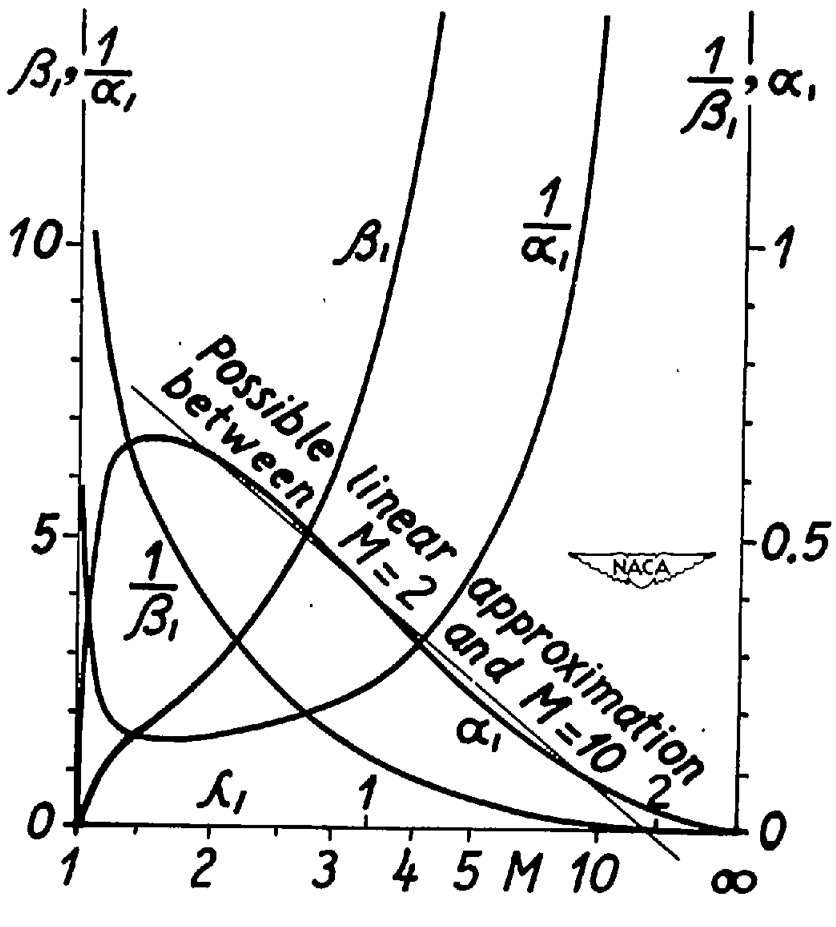

Figure 4.- Variables α_1 and β_1 as functions of λ_1 for $\gamma = 1.4$.

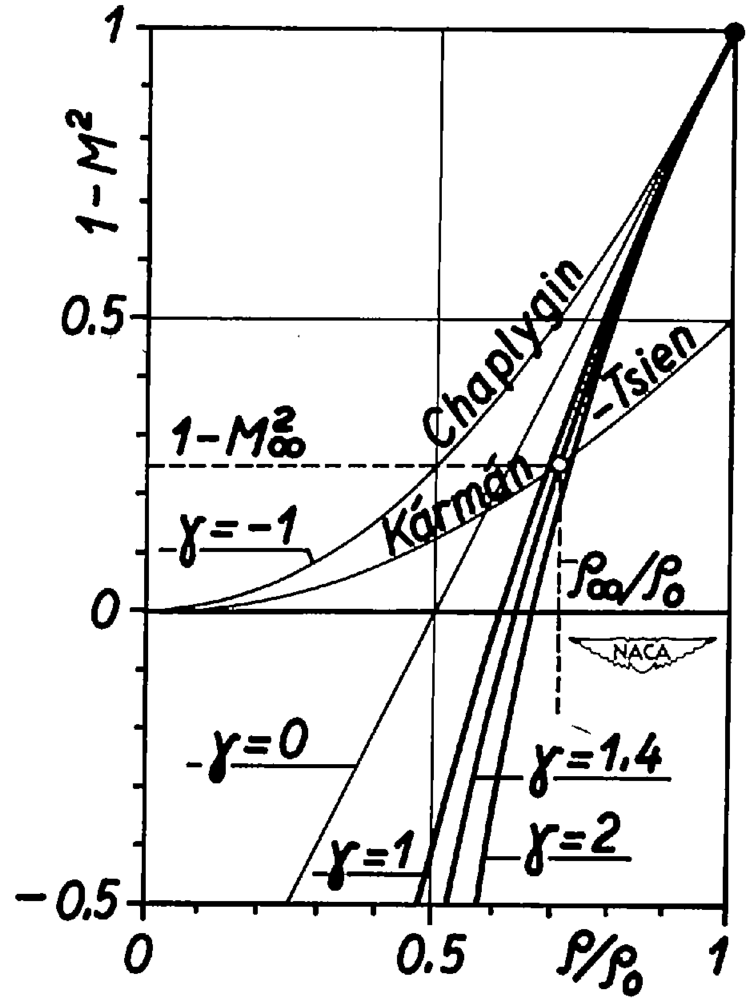

Figure 5.- Comparison of the Chaplygin and the Kármán-Tsien approximations of $1 - M^2$ as a function of ρ/ρ_0 with the exact relation for several values of γ.