NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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CRITICAL COMBINATIONS OF BENDING, SHEAR, AND TRANSVERSE COMPRESSIVE STRESSES FOR BUCKLING OF INFINITELY LONG FLAT PLATES

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SUMMARY

Three-dimensional interaction surfaces are presented for the computation of elastic buckling stresses for an infinitely long flat plate subjected to combinations of bending, shear, and transverse compression in its plane - a loading approximating that occurring in a shear web. Surfaces are presented for two sets of edge conditions: both edges simply supported and lower edge simply supported, upper edge clamped. Present results are in good agreement with data for one-load and two-load limiting cases previously published.

INTRODUCTION

A loading that occurs in the shear webs of thin wings of aircraft is a combination of bending, shear, and transverse compression, the transverse compression being induced by spanwise bending of the covers. The buckling strength of an unstiffened infinitely long flat plate under such a loading is computed approximately in the present paper by the minimum-potential-energy method.

The assumptions made for the analysis are that the plate is elastic and infinitely long, and that the bending moment, shear, and transverse compression are constant along the length of the plate. The lower edge is assumed to be simply supported and the upper edge, either simply supported, elastically restrained against rotation, or clamped. The neutral axis for bending stress is assumed to be halfway between the upper and lower edges.

The results of the analysis are given in the form of interaction curves and the details of the solutions are given in the appendixes. A comparison of the present results with existing analytical data for one-load and two-load conditions is made. No previous data are known, however, for a supported or restrained flat plate subjected to combinations of bending, shear, and transverse compression.

SYMBOLS

a,b arbitrary coefficients used with subscripts

D plate flexural stiffness per unit width, inch-pounds $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$

b width of plate, inches

E Young's modulus of elasticity, pounds per square inch

k_B, k_T, k_C nondimensional buckling stress coefficients under system of combined loadings: $\left(k_B = \frac{\sigma_B b^2 t}{\pi^2 D}\right)$; $\left(k_T = \frac{\tau b^2 t}{\pi^2 D}\right)$; $\left(k_C = \frac{\sigma_C b^2 t}{\pi^2 D}\right)$

m,n,p,i,j integers, also used as subscripts (single prime (') and double primes (") are used with m, n, p, and i to indicate odd and even integers, respectively)

N_x force per unit length acting in middle plane of plate in x-direction, pounds per inch

N_y force per unit length acting in middle plane of plate in y-direction, pounds per inch

N_{xy} shear force per unit length acting in middle plane of plate in x- and y-directions, pounds per inch

R_B, R_T, R_C buckling stress ratios: $\left(R_B = \frac{\sigma_B}{(\sigma_B)_{cr}}\right)$; $\left(R_T = \frac{\tau}{\tau_{cr}}\right)$; $\left(R_C = \frac{\sigma_C}{(\sigma_C)_{cr}}\right)$

S	stiffness per unit length of elastic restraining medium, inch-pounds per inch per quarter radian
t	thickness of plate, inches
w	deflection normal to plane of plate, inches
x,y	coordinates
β	ratio of half wave length of buckles to plate width (λ/b)
ϵ	nondimensional restraint coefficient $\left(4\frac{Sb}{D}\right)$; 0 for simply supported edges, ∞ for clamped edges
γ_1,γ_2	Lagrangian multipliers
λ	half wave length of buckles in x-direction, inches
σ_B	bending stress at buckling at edge of plate under system of combined loadings, pounds per square inch
τ,σ_C	shear and transverse compressive buckling stresses, respectively, under system of combined loadings, pounds per square inch
$(\sigma_B)_{cr},\tau_{cr},(\sigma_C)_{cr}$	plate buckling stresses due to each type of loading applied alone, pounds per square inch
θ_1,θ_2 and ϕ_1,ϕ_2	arbitrary coefficients
μ	Poisson's ratio

RESULTS AND DISCUSSION

Results in chart form are presented for the case of both edges simply supported and for the case of lower edge simply supported, upper edge clamped. The solution, however, for the case of lower edge simply supported, upper edge elastically restrained against rotation is given in the form of a determinantal buckling equation. The loading and edge conditions considered herein are shown in figure 1.

Both edges simply supported.- Combinations of bending, shear, and transverse compressive stresses which cause buckling of an unstiffened infinitely long flat plate with both edges simply supported were calculated in the manner described in appendix A.

The results are shown by a three-dimensional interaction surface in terms of the stress ratios R_B, R_T, and R_C in figure 2. Buckling occurs for any stress combination which corresponds to a point on or outside the interaction surface. In order to convert a combination of values of R_B, R_T, and R_C into the corresponding stresses, the buckling stress for each type of loading applied alone must be known. These buckling stresses, as given by the present solution, are as follows:

For pure bending

$$\left(\sigma_B\right)_{cr} = 23.90 \frac{\pi^2 D}{b^2 t} \tag{1}$$

for pure shear

$$\tau_{cr} = 5.36 \frac{\pi^2 D}{b^2 t} \tag{2}$$

and for pure transverse compression

$$\left(\sigma_C\right)_{cr} = \frac{\pi^2 D}{b^2 t} \tag{3}$$

(Equation (1) checks the value for $\left(\sigma_B\right)_{cr}$ given by Schuette and McCulloch in fig. 9 of reference 1; equation (2) checks the value for τ_{cr} given by Timoshenko on p. 361 of reference 2; equation (3) is essentially the Euler column buckling equation.)

The interaction surface is symmetric about the planes $R_T = 0$ and $R_B = 0$. The shape of the interaction surface is suggested by its traces on planes corresponding to constant values of 0, 0.5, and 0.8 for the stress ratios. (For example, the shaded planes in fig. 2 correspond to $R_C = 0.5$ and $R_T = 0.5$.) The flat portion of the interaction surface at $R_C = 1.0$ indicates that appreciable bending and shear stress may be applied to the plate without reducing the critical transverse compressive stress. In the region $R_C = 1.0$, the plate buckles essentially as an Euler column.

In figures 3, 4, and 5 the interaction surface is described by two-dimensional plots which are more suitable for obtaining quantitative information. The calculations indicate that where a sharp change in slope occurs in an interaction curve, with the curve becoming vertical (for example, the curves for $R_\tau = 0$ and 0.5 in fig. 4 and $R_B = 0.8$ in fig. 5), the buckle wave length undergoes a sudden transition from some finite length when $R_C < 1$ to an infinite length when $R_C = 1$. On the other hand, a gradual transition to verticality in an interaction curve (for example, the curves for $R_C = 1.0$ in fig. 3 and $R_B = 0$ and 0.5 in fig. 5) indicates a gradual transition to an infinitely long buckle wave length.

Additional information regarding buckle wave length is given in table I together with a tabulation of the critical combinations of stress coefficients.

As noted previously, the one-load limiting-case solutions are in good agreement with previously published data. In addition, the interaction curves for two components of loading check with existing data: The curve for $R_\tau = 0$ in figure 4 agrees well with figure 3 of reference 3. The curve for $R_B = 0$ in figure 5 is practically identical with the curve for $\epsilon = 0$ in figure 3(b) of reference 4. No known data, however, are available to check the present shear-bending interaction curve (fig. 3).

Lower edge simply supported, upper edge clamped.- Combinations of bending, shear, and transverse compressive stresses which cause buckling of an infinitely long flat plate with the lower edge simply supported and the upper edge clamped were calculated in the manner described in appendix B.

The results are shown by a three-dimensional interaction surface in terms of the stress ratios R_B, R_τ, and R_C in figure 6. In order to convert a combination of values of R_B, R_τ, and R_C into their corresponding buckling stresses, the buckling stress corresponding to each loading applied separately must be known. These single-load buckling stresses are as follows:

For pure bending

$$\left(\sigma_B\right)_{cr} = 39.96 \frac{\pi^2 D}{b^2 t} \tag{4}$$

for pure shear

$$\tau_{cr} = 6.637 \frac{\pi^2 D}{b^2 t} \qquad (5)$$

and for pure transverse compression

$$(\sigma_C)_{cr} = 2.045 \frac{\pi^2 D}{b^2 t} \qquad (6)$$

The value of the numerical coefficient in equation (4) is about 5 percent lower than the value given in figure 9 of reference 1 because more terms are used in the present solution of the deflection function. The value of the numerical coefficient of equation (5) is about $4\frac{1}{4}$ percent lower than the value given by the approximation suggested in reference 5. This approximation is determined as the geometric mean of the two values of the coefficient determined for an infinitely long flat plate with both edges simply supported and with both edges clamped. Equation (6) reduces to the Euler column buckling equation given on page 89 of reference 2 when the conversion from plate stiffness to column stiffness is made.

The general nature of the interaction surface is similar to that for the plate with simply supported edges (fig. 2) with some significant differences. Because of the unsymmetric edge conditions, the surface is not symmetric about the $R_B = 0$ plane since positive and negative bending moments have different effects, but the curve is symmetric about the $R_\tau = 0$ plane. The perfect flatness at $R_C = 1.0$, which was observed in the interaction surface for the case in which both edges are simply supported (fig. 2), does not occur in the present case. The surface, however, is sufficiently flat in the region of $R_C = 1.0$ to indicate that, if an exact solution could be obtained, it would probably lead to a perfectly flat portion.

The bulging out of the interaction surface indicates that the application of a positive bending moment (positive directions shown in fig. 1(a)) actually increases the buckling strength of the plate with regard to the other two types of load. This increase in buckling strength indicates that the beneficial effect of the tension on the simply supported edge is greater than the detrimental effect of an equal compression on the clamped edge.

Projections on the coordinate planes of the traces of the planes shown in figure 6 are shown in figures 7, 8, and 9. Quantitative information is more readily available from these figures.

Some information regarding buckle wave length is given in table II together with a tabulation of critical combinations of stress coefficients.

As noted previously, the single-loading limiting-case solutions are in good agreement with previously published data; however, no direct comparison for the interaction of two components of loading can be made for the condition of lower edge simply supported and upper edge clamped, because this condition apparently has not been studied previously for the particular loading cases considered herein.

Lower edge simply supported, upper edge elastically restrained.- No calculated results are presented for the case of lower edge simply supported, upper edge elastically restrained; however, the stability determinant derived in appendix A and given in table III can be used directly for solutions. The rotational-restraint parameter ϵ appears only in the term T_{16}.

CONCLUDING REMARKS

Three-dimensional interaction surfaces have been presented for the computation of buckling stresses for an infinitely long flat plate subjected to combinations of bending, shear, and transverse compression in its plane - a loading approximating that occurring in a shear web. Reduction of the present solution to cases of one load or combinations of two loads gives results in good agreement with previously published data. The interaction surfaces have been presented for two sets of boundary conditions, namely, both edges simply supported and lower edge simply supported, upper edge clamped. A theoretical solution has also been derived for the case in which the lower edge is simply supported and the upper edge is elastically restrained against rotation, but no computed results are presented for this case.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
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APPENDIX A

THEORETICAL SOLUTION FOR UPPER EDGE EITHER SIMPLY SUPPORTED OR ELASTICALLY RESTRAINED

Details are presented of a minimum-potential-energy solution for the buckling stresses of an infinitely long flat plate, with the lower edge simply supported and the upper edge either simply supported or elastically restrained against rotation, subjected to combinations of bending, shear, and transverse compression in the plane of the plate. The deflection function used satisfies, term by term, the conditions of zero deflection and zero moment at the lower edge and zero deflection at the upper edge. Lagrangian multipliers are used to satisfy the rotational boundary condition at the upper edge.

Upper Edge Elastically Restrained

Deflection function.- The deflection of a plate subjected to the loads shown in figure 1(a) is assumed to be of the form

$$w = \sin\frac{\pi x}{\lambda}\sum_{n=1,2,3,\ldots}^{\infty} a_n \sin\frac{n\pi y}{b} + \cos\frac{\pi x}{\lambda}\sum_{n=1,2,3,\ldots}^{\infty} b_n \sin\frac{n\pi y}{b} \tag{A1}$$

where a_n and b_n are constants. The rotation θ of the upper edge $(y = b)$ is assumed to be expressed as

$$\theta = \theta_1 \sin\frac{\pi x}{\lambda} + \theta_2 \cos\frac{\pi x}{\lambda} \tag{A2}$$

where θ_1 and θ_2 are constants.

The deflection function represented by equation (A1) satisfies, term by term, the boundary conditions

$$w_{y=0} = 0 \tag{A3}$$

$$w_{y=b} = 0 \tag{A4}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0 \tag{A5}$$

The compatibility condition

$$\left(\frac{\partial w}{\partial y}\right)_{y=b} = \theta_{y=b} \tag{A6}$$

is satisfied by the deflection function provided the following constraining relationships are satisfied:

$$\left.\begin{aligned} \theta_1 &= \frac{\pi}{b} \sum_{n=1,2,3,\ldots}^{\infty} na_n(-1)^n \\ \theta_2 &= \frac{\pi}{b} \sum_{n=1,2,3,\ldots}^{\infty} nb_n(-1)^n \end{aligned}\right\} \tag{A7}$$

<u>Potential-energy expression</u>.- The potential energy F for an infinitely long flat plate, with the lower edge simply supported and the upper edge elastically restrained against rotation, buckling under stresses in its plane is (from reference 6 and p. 325 of reference 2)

$$F = \frac{D}{2}\int_{y=0}^{b}\int_{x=0}^{\lambda}\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1 - \mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy +$$

$$2\int_{x=0}^{\lambda} S\left(\theta^2\right)_{y=0}\,dx + \frac{1}{2}\int_{y=0}^{b}\int_{x=0}^{\lambda}\left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 +\right.$$

$$\left.2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy \tag{A8}$$

where, for the present case,

$$N_x = \left(1 - \frac{2y}{b}\right)\sigma_B t$$

$$N_y = -\sigma_C t$$

$$N_{xy} = \tau t$$

Substituting expression (A1) for w and expression (A2) for θ into the energy expression (A8) and performing the indicated integrations gives

$$F = \frac{\pi^4}{4} \sum_{n=1,2,3,\ldots}^{\infty} \left(\frac{1}{\beta^3} + \frac{2n^2}{\beta} + \beta n^4\right)\left(a_n^2 + b_n^2\right) + \frac{b^2\epsilon\beta}{2}\left(\theta_1^2 + \theta_2^2\right) +$$

$$4\frac{k_B\pi^2}{\beta}\left[\sum_{m=1,2,3,\ldots}^{\infty} \sum_{n=1,2,3,\ldots}^{\infty} a_m a_n \frac{mn}{\left(m^2 - n^2\right)^2} + \right.$$

$$\left.\sum_{m=1,2,3,\ldots}^{\infty} \sum_{n=1,2,3,\ldots}^{\infty} b_m b_n \frac{mn}{\left(m^2 - n^2\right)^2}\right] +$$

$$\frac{k_C\pi^4\beta}{4}\left(\sum_{n=1,2,3,\ldots}^{\infty} a_n^2 n^2 + \sum_{n=1,2,3,\ldots}^{\infty} b_n^2 n^2\right) +$$

$$4k_T\pi^3 \sum_{m=1,2,3,\ldots}^{\infty} \sum_{n=1,2,3,\ldots}^{\infty} a_m b_n \frac{mn}{m^2 - n^2} \qquad \text{(A9)}$$

$(m \pm n$ always odd$)$

where

k_B, k_T, k_C desired stress coefficients

ϵ rotational-restraint coefficient

$\beta = \frac{\lambda}{b}$

Minimization by the method of Lagrangian multipliers.- By the principle of minimum potential energy, F (equation (A9)) must be minimized with respect to all independent undetermined deflection parameters in this equation. Inasmuch as the parameters a_n, b_n, θ_1, and θ_2 are not independent but are related through the constraining relationships (A7), this minimization can be accomplished by means of the method of Lagrange's undetermined multipliers. The function

$$G = F - \gamma_1\left[\theta_1 - \frac{\pi}{b}\sum_{n=1,2,3,\ldots}^{\infty} a_n n(-1)^n\right] - \gamma_2\left[\theta_2 - \frac{\pi}{b}\sum_{n=1,2,3,\ldots}^{\infty} b_n n(-1)^n\right] \tag{A10}$$

is set up. The potential energy F is a minimum when

$$\frac{\partial G}{\partial a_j} = \frac{\partial G}{\partial b_j} = \frac{\partial G}{\partial \theta_1} = \frac{\partial G}{\partial \theta_2} = \frac{\partial G}{\partial \gamma_1} = \frac{\partial G}{\partial \gamma_2} = 0 \tag{A11}$$

$$(j = 1, 2, 3, \ldots)$$

Performing the differentiation on the parameters a_n and b_n gives

$$\frac{\partial G}{\partial a_i} = 2a_i\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)i^2 + \beta i^4\right] + \frac{16k_B}{\pi^2\beta}\sum_{m=1,2,3,\ldots}^{\infty}\frac{2mia_m}{\left(m^2 - i^2\right)^2} - \frac{16k_T}{\pi}\sum_{m=1,2,3,\ldots}^{\infty}\frac{b_m mi}{m^2 - i^2} + \frac{\gamma_1 \pi i(-1)^i}{b} = 0 \tag{A12}$$

$$(m \pm i \text{ must be odd})$$
$$(i = 1, 2, 3, \ldots)$$

$$\frac{\partial G}{\partial b_i} = 2b_i\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)i^2 + \beta i^4\right] + \frac{16k_B}{\pi^2\beta}\sum_{m=1,2,3,\ldots}^{\infty}\frac{2mib_m}{\left(m^2 - i^2\right)^2} - \frac{16k_T}{\pi}\sum_{m=1,2,3,\ldots}^{\infty}\frac{a_m mi}{m^2 - i^2} + \frac{\gamma_2 \pi i(-1)^i}{b} = 0 \tag{A13}$$

$$(m \pm i \text{ must be odd})$$
$$(i = 1, 2, 3, \ldots)$$

Minimizing with respect to θ_1 and θ_2 gives

$$\frac{\partial G}{\partial \theta_1} = \frac{4b^2 \epsilon \beta \theta_1}{\pi^4} - \gamma_1 = 0 \tag{A14}$$

$$\frac{\partial G}{\partial \theta_2} = \frac{4b^2 \epsilon \beta \theta_2}{\pi^4} - \gamma_2 = 0 \tag{A15}$$

Minimizing with respect to the Lagrangian multipliers gives the following original constraining relationships:

$$\frac{\partial G}{\partial \gamma_1} = -\theta_1 + \frac{\pi}{b} \sum_{n=1,2,3,\ldots}^{\infty} a_n n(-1)^n = 0 \tag{A16}$$

$$\frac{\partial G}{\partial \gamma_2} = -\theta_2 + \frac{\pi}{b} \sum_{n=1,2,3,\ldots}^{\infty} b_n n(-1)^n = 0 \tag{A17}$$

Equations (A12) to (A17) constitute an infinite set of homogeneous simultaneous equations in the unknowns a_n, b_n, γ_1, γ_2, θ_1, and θ_2. These equations can have other solutions than the trivial one of $a_n, b_n, \ldots = 0$ only if the determinant of the coefficients of the unknowns is zero. The vanishing of this determinant is therefore the condition for buckling; stress combinations which cause the determinant to vanish cause buckling of the plate.

Instead of using equations (A12) to (A17) directly, it is advantageous first to combine them and then to reduce the system of equations to a simplified form. This simplification is possible because each of the equations represented by equation (A12) contains either a single odd-subscript deflection coefficient (a_n, b_n) or a single even-subscript deflection coefficient (a_n, b_n); the same fact is true of equations expressed by equation (A13). This fact affords the possibility of solving for and eliminating each even-subscript deflection coefficient in terms of all the odd-subscript coefficients; the reverse is also true. The number of equations required for an accurate solution is thus reduced approximately by one-half. The simplification just described is carried through in detail in the following discussion.

When odd and even values of the subscripts are designated by single and double primes, respectively, the set of equations (A12) may be broken into the following subsets:

$$a_{i'}\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(i')^2 + \beta(i')^4\right] + \frac{16k_B}{\pi^2\beta}\sum_{m''=2,4,6,\ldots}^{\infty}\frac{m''i'a_{m''}}{\left[(m'')^2 - (i')^2\right]^2} +$$

$$\frac{8k_T}{\pi}\sum_{m''=2,4,6,\ldots}^{\infty}\frac{b_{m''}m''i'}{(m'')^2 - (i')^2} - \frac{\gamma_1\pi i'}{2b} = 0 \qquad \text{(A12)'}$$

$$(i' = 1, 3, 5, \ldots)$$

$$a_{i''}\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(i'')^2 + \beta(i'')^4\right] + \frac{16k_B}{\pi^2\beta}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{m'i''a_{m'}}{\left[(m')^2 - (i'')^2\right]^2} +$$

$$\frac{8k_T}{\pi}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{b_{m'}m'i''}{(m')^2 - (i'')^2} + \frac{\gamma_1\pi i''}{2b} = 0 \qquad \text{(A12)''}$$

$$(i'' = 2, 4, 6, \ldots)$$

Similarly, equations (A13) can be written as

$$b_{i'}\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(i')^2 + \beta(i')^4\right] + \frac{16k_B}{\pi^2\beta}\sum_{m''=2,4,6,\ldots}^{\infty}\frac{b_{m''}m''i'}{\left[(m'')^2 - (i')^2\right]^2} -$$

$$\frac{8k_T}{\pi}\sum_{m''=2,4,6,\ldots}^{\infty}\frac{a_{m''}m''i'}{(m'')^2 - (i')^2} - \frac{\gamma_2\pi i'}{2b} = 0 \qquad \text{(A13)'}$$

$$(i' = 1, 3, 5, \ldots)$$

$$b_{i''}\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(i'')^2 + \beta(i'')^4\right] + \frac{16k_B}{\pi^2\beta}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{b_{m'}m'i''}{\left[(m')^2-(i'')^2\right]^2} - \frac{8k_T}{\pi}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{a_{m'}m'i''}{(m')^2-(i'')^2} + \frac{\gamma_2\pi i''}{2b} = 0 \qquad \text{(A13)}'$$

$$(i'' = 2, 4, 6, \ldots)$$

Solving equations (A12)' for $a_{i'}$ and substituting into equations (A12)" gives

$$a_{i''}\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(i'')^2 + \beta(i'')^4\right] + \frac{16k_B}{\pi^2\beta}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{m'i''}{\left[(m')^2-(i'')^2\right]^2}\left(\frac{1}{\frac{1}{\beta^3}+\left(\frac{2}{\beta}-\beta k_C\right)(m')^2+\beta(m')^4}\left\{-\frac{16k_B}{\pi^2\beta}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{p''m'a_{p''}}{\left[(p'')^2-(m')^2\right]^2} - \frac{8k_T}{\pi}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{b_{p''}p''m'}{(p'')^2-(m')^2} + \frac{\gamma_1\pi m'}{2b}\right\}\right) + \frac{8k_T}{\pi}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{b_{m'}m'i''}{(m')^2-(i'')^2} + \frac{\gamma_1\pi i''}{2b} = 0 \qquad \text{(A12)}'''$$

$$(i'' = 2, 4, 6, \ldots)$$

Substituting the value of $b_{i'}$ obtained from equations (A13)' into (A12)''' gives the following equations in which all the deflection coefficients have even subscripts:

$$
\begin{aligned}
& a_{i''}\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(i'')^2 + \beta(i'')^4\right] + \\
& \frac{16k_B}{\pi^2\beta}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{m'i''}{\left[(m')^2-(i'')^2\right]^2}\left(\frac{1}{\frac{1}{\beta^3}+\left(\frac{2}{\beta}-\beta k_C\right)(m')^2+\beta(m')^4}\left\{-\frac{16k_B}{\pi^2\beta}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{p''m'a_{p''}}{\left[(p'')^2-(m')^2\right]^2} - \right.\right. \\
& \left.\left.\frac{8k_T}{\pi}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{b_{p''}p''m'}{(p'')^2-(m')^2} + \frac{\gamma_1\pi m'}{2b}\right\}\right) + \\
& \frac{8k_T}{\pi}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{m'i''}{(m')^2-(i'')^2}\left(\frac{1}{\frac{1}{\beta^3}+\left(\frac{2}{\beta}-\beta k_C\right)(m')^2+\beta(m')^4}\left\{-\frac{16k_B}{\pi^2\beta}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{b_{p''}p''m'}{\left[(p'')^2-(m')^2\right]^2} + \right.\right. \\
& \left.\left.\frac{8k_T}{\pi}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{a_{p''}p''m'}{(p'')^2-(m')^2} + \frac{\gamma_2\pi m'}{2b}\right\}\right) + \frac{\gamma_1\pi i''}{2b} = 0 \qquad \text{(A18)}
\end{aligned}
$$

$$(i'' = 2, 4, 6, \ldots)$$

Treating equations (A13)' and (A13)" in a similar manner gives

$$
\begin{aligned}
&b_{i''}\left[\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(i'')^2 + \beta(i'')^4\right] + \\
&\frac{16k_B}{\pi^2\beta}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{m'i''}{\left[(m')^2 - (i'')^2\right]^2}\left(\frac{1}{\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(m')^2 + \beta(m')^4}\left\{-\frac{16k_B}{\pi^2\beta}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{b_{p''}p''m'}{\left[(p'')^2 - (m')^2\right]^2} + \right.\right. \\
&\left.\left.\frac{8k_T}{\pi}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{a_{p''}p''m'}{(p'')^2 - (m')^2} + \frac{\gamma_2\pi m'}{2b}\right\}\right) - \\
&\frac{8k_T}{\pi}\sum_{m'=1,3,5,\ldots}^{\infty}\frac{m'i''}{(m')^2 - (i'')^2}\left(\frac{1}{\frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_C\right)(m')^2 + \beta(m')^4}\left\{-\frac{16k_B}{\pi^2\beta}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{p''m'a_{p''}}{\left[(p'')^2 - (m')^2\right]^2} - \right.\right. \\
&\left.\left.\frac{8k_T}{\pi}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{b_{p''}p''m'}{(p'')^2 - (m')^2} + \frac{\gamma_1\pi m'}{2b}\right\}\right) + \frac{\gamma_2\pi i''}{2b} = 0
\end{aligned}
\qquad \text{(A19)}
$$

$$(i'' = 2, 4, 6, \ldots)$$

Eliminating θ_1 between equations (A14) and (A16) and then eliminating all values of $a_{n'}$ by means of equations (A12)' gives

$$-\frac{4b\epsilon\beta}{\pi^3}\sum_{m'=1,3,5,\ldots}^{\infty} m'\left[\frac{1}{\frac{1}{\beta^3}+\left(\frac{2}{\beta}-\beta k_C\right)(m')^2+\beta(m')^4}\right]\left\{-\frac{16k_B}{\pi^2\beta}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{p''m'a_{p''}}{\left[(p'')^2-(m')^2\right]^2}-\frac{8k_T}{\pi}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{b_{p''}p''m'}{(p'')^2-(m')^2}+\frac{\gamma_1\pi m'}{2b}\right\}+\frac{4b\epsilon\beta}{\pi^3}\sum_{p''=2,4,6,\ldots}^{\infty}a_{p''}p''-\gamma_1=0 \qquad \text{(A14)'}$$

Similarly, eliminating θ_2 between equations (A15) and (A17) and then eliminating all the values of $b_{n'}$ by means of equations (A13') gives

$$-\frac{4b\epsilon\beta}{\pi^3}\sum_{m'=1,3,5,\ldots}^{\infty} m'\left[\frac{1}{\frac{1}{\beta^3}+\left(\frac{2}{\beta}-\beta k_C\right)(m')^2+\beta(m')^4}\right]\left\{-\frac{16k_B}{\pi^2\beta}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{b_{p''}p''m'}{\left[(p'')^2-(m')^2\right]^2}+\frac{8k_T}{\pi}\sum_{p''=2,4,6,\ldots}^{\infty}\frac{a_{p''}p''m'}{(p'')^2-(m')^2}+\frac{\gamma_2\pi m'}{2b}\right\}+\frac{4b\epsilon\beta}{\pi^3}\sum_{p''=2,4,6,\ldots}^{\infty}b_{p''}p''-\gamma_2=0 \qquad \text{(A15)'}$$

Solution by stability determinant.- Equations (A18), (A19), (A14)', and (A15)' constitute a system of homogeneous simultaneous equations containing as unknowns γ_1, γ_2, and all the even-subscript deflection coefficients. Equating the determinant of these equations to zero gives the buckling criterion in the form of a determinantal equation.

In order to determine the combinations of k_B, k_T, and k_C which satisfy the equations, the values of two of the stress coefficients and β can be substituted into the determinant of coefficients, and the value of the third stress coefficient which satisfies the determinantal equation can thus be obtained. A graphical minimization of the third stress coefficient with respect to β gives the combinations of minimum buckling stress coefficients.

Upper Edge Simply Supported

Solution by stability determinant.- The stability determinant derived in the preceding section and given in table III can be used for the solution when both edges are simply supported. The last two rows of the determinant are first multiplied through by the restraint coefficient ϵ. Equating ϵ to zero (for simple support) then reduces all the terms in these two rows to zero except for the elements on the principal diagonal. The subdeterminant consisting of the first six rows and first six columns of the original determinant can then be factored out and equated to zero. Numerical calculations yielding the results in table I were made by using the first four rows and columns of this factored determinant. Half wave lengths of buckling are given in the table together with the critical combinations of k_B, k_T, and k_C which satisfy the determinantal equation. These calculations are the basis of figures 2 to 5.

Special calculations for $R_C = 1$.- Consideration of the physics of the problem suggests the possible existence of a flat portion in the interaction surface at $R_C = 1$ and parallel to the R_BR_T plane. In order to investigate this region, $k_C = 1.0$ was substituted into the fourth-order determinant, and critical combinations of k_B and k_T were obtained. These combinations led to the curve for $R_C = 1.0$ in figure 3, which checks closely with the results of reference 4.

APPENDIX B

THEORETICAL SOLUTION FOR UPPER EDGE CLAMPED

Details are presented of a minimum-potential-energy solution for the buckling stresses of an infinitely long flat plate, with the lower edge simply supported and the upper edge clamped, subjected to combinations of bending, shear, and transverse compression in the plane of the plate. Calculations were first made by using the stability determinant based on elastically restrained edge deflection functions given in the previous appendix, with $\epsilon = \infty$. After the calculations were completed, however, it was discovered that greater accuracy for the same amount of work was obtainable by using a deflection function which satisfies, term by term, the zero-slope boundary condition at the upper edge as well as the zero-deflection, zero-moment conditions at the lower edge. The zero-deflection condition at the upper edge was satisfied by use of Lagrangian multipliers. This solution was used to calculate salient points on the interaction surface (values given in table II) which were then used to adjust the originally calculated interaction surfaces. The adjusted values appear in figures 6 to 9.

Solution Based on Elastically Restrained Edge Deflection Function

The simplified eighth-order stability determinant derived in appendix A and given in table III was used to calculate the interaction surface of a plate, with the lower edge simply supported and the upper edge clamped, by setting the rotational restraint coefficient ϵ equal to ∞ and by determining the values of k_B, k_T, k_C, and β which cause the determinant of coefficients a_n, b_n, γ_1, and γ_2 to vanish. The results of these calculations have been adjusted on the basis of the calculations derived in the next section.

Solution Based on Clamped-Edge Deflection Function

Deflection function.- The deflection of the clamped-edge plate shown in figure 1 is assumed to be of the form

$$w = \sin \frac{\pi x}{\lambda} \sum_{n=1,3,5,\ldots}^{\infty} a_n \sin \frac{n\pi y}{2b} + \cos \frac{\pi x}{\lambda} \sum_{n=1,3,5,\ldots}^{\infty} b_n \sin \frac{n\pi y}{2b} \quad \text{(B1)}$$

where the series coefficients a_n and b_n are constants. The deflection ϕ of the upper edge is assumed to be expressed as

$$\phi = \phi_1 \sin \frac{\pi x}{\lambda} + \phi_2 \cos \frac{\pi x}{\lambda} \tag{B2}$$

where ϕ_1 and ϕ_2 are constants.

The deflection function (B1) satisfies, term by term, the boundary conditions

$$w_{y=0} = 0 \tag{B3}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0 \tag{B4}$$

$$\left(\frac{\partial w}{\partial y}\right)_{y=b} = 0 \tag{B5}$$

The compatibility condition

$$w_{y=b} = \phi_{y=b} = 0 \tag{B6}$$

is satisfied by the deflection function provided the following constraining relationships are satisfied:

$$\left.\begin{aligned} \phi_1 &= \sum_{n=1,3,5,\ldots}^{\infty} a_n(-1)^{\frac{n-1}{2}} = 0 \\ \phi_2 &= \sum_{n=1,3,5,\ldots}^{\infty} b_n(-1)^{\frac{n-1}{2}} = 0 \end{aligned}\right\} \tag{B7}$$

Potential-energy expression.- The potential energy F for an infinitely long flat plate, with the lower edge simply supported and the upper edge clamped, buckling under stresses in its plane (p. 325 of reference 2) is

$$F = \frac{D}{2}\int_{y=0}^{b}\int_{x=0}^{\lambda}\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1 - \mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy +$$

$$\frac{1}{2}\int_{y=0}^{b}\int_{x=0}^{\lambda}\left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy \qquad \text{(B8)}$$

where, for the present case,

$$N_x = \left(1 - \frac{2y}{b}\right)\sigma_B t$$

$$N_y = -\sigma_C t$$

$$N_{xy} = \tau t$$

Substituting the expression (B1) for w into the energy expression (B8) and performing the indicated integrations gives the expression to be minimized as

$$\begin{aligned}
F = {} & k_B\pi^2\beta\left(\sum_{n=1,3,5,\ldots}^{\infty}\frac{{a_n}^2}{n^2} + \sum_{n=1,3,5,\ldots}^{\infty}\frac{{b_n}^2}{n^2}\right) + 4k_B\pi^2\beta\left[\sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{n+m}{2}\ \text{odd}\right)}}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}\frac{a_m a_n}{(n+m)^2} - {}\right. \\
& \sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{n-m}{2}\ \text{odd}\right)}}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}\frac{a_m a_n}{(n-m)^2} + \sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{n+m}{2}\ \text{odd}\right)}}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}\frac{b_m b_n}{(n+m)^2} - \left.\sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{n-m}{2}\ \text{odd}\right)}}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}\frac{b_m b_n}{(n-m)^2}\right] + {} \\
& \frac{k_C\pi^4\beta^3}{16}\left(\sum_{n=1,3,5,\ldots}^{\infty}n^2{a_n}^2 + \sum_{n=1,3,5,\ldots}^{\infty}n^2{b_n}^2\right) + k_\tau\pi^3\beta^2\sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}a_n b_m\left[\frac{2mn}{n^2 - m^2} - (-1)^{\frac{n+m}{2}}\right] - {} \\
& \frac{\pi^4}{4}\left[\frac{1}{\beta}\left(\sum_{n=1,3,5,\ldots}^{\infty}{a_n}^2 + \sum_{n=1,3,5,\ldots}^{\infty}{b_n}^2\right) + \frac{\beta}{2}\left(\sum_{n=1,3,5,\ldots}^{\infty}n^2{a_n}^2 + \sum_{n=1,3,5,\ldots}^{\infty}n^2{b_n}^2\right) + {}\right. \\
& \left.\frac{\beta^3}{16}\left(\sum_{n=1,3,5,\ldots}^{\infty}n^4{a_n}^2 + \sum_{n=1,3,5,\ldots}^{\infty}n^4{b_n}^2\right)\right]
\end{aligned} \qquad \begin{matrix}\text{(B9)} \\ (m^2 \neq n^2)\end{matrix}$$

where

k_B, k_τ, k_C desired stress coefficients

$\beta = \frac{\lambda}{b}$

Minimization by the method of Lagrangian multipliers.- By the principle of minimum potential energy, F (equation (B9)) must be minimized with respect to all independent undetermined deflectional parameters in this equation. Inasmuch as the parameters a_n, b_n, ϕ_1, and ϕ_2 are not independent but are related through the constraining relationship (B7), this minimization can be accomplished by means of the method of Lagrange's undetermined multipliers. The function

$$G = F - \gamma_1 \sum_{n=1,3,5,\ldots}^{\infty} a_n(-1)^{\frac{n-1}{2}} - \gamma_2 \sum_{n=1,3,5,\ldots}^{\infty} b_n(-1)^{\frac{n-1}{2}} \tag{B10}$$

is set up. The value of F will be a minimum when

$$\frac{\partial G}{\partial a_j} = \frac{\partial G}{\partial b_j} = \frac{\partial G}{\partial \gamma_1} = \frac{\partial G}{\partial \gamma_2} = 0 \tag{B11}$$

Performing the differentiation on the parameters a_n and b_n gives

$$\frac{\partial G}{\partial a_i} = 0 = 2k_B\pi^2\beta \frac{a_i}{i^2} + 4k_B\pi^2\beta\left[\sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{i+m}{2}\text{ odd}\right)}}^{\infty} \frac{2a_m}{(i+m)^2} - \sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{i-m}{2}\text{ odd}\right)}}^{\infty} \frac{2a_m}{(i-m)^2}\right] +$$

$$\frac{k_C\pi^4\beta^3}{16} 2a_i i^2 + k_T\pi^3\beta^2 \sum_{m=1,3,5,\ldots}^{\infty} b_m\left[\frac{2im}{i^2 - m^2} - (-1)^{\frac{i+m}{2}}\right] -$$

$$\frac{\pi^4 a_i}{2}\left(\frac{1}{\beta} + \frac{\beta i^2}{2} + \frac{\beta^3 i^4}{16}\right) - \gamma_1(-1)^{\frac{i-1}{2}} \tag{B12}$$

$$(i^2 \neq m^2)$$

and

$$\frac{\partial G}{\partial b_i} = 0 = 2k_B\pi^2\beta\,\frac{b_i}{i^2} + 4k_B\pi^2\beta\left[\sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{i+m}{2}\text{ odd}\right)}}^{\infty}\frac{2b_m}{(i+m)^2} - \sum_{\substack{m=1,3,5,\ldots \\ \left(\frac{i-m}{2}\text{ odd}\right)}}^{\infty}\frac{2b_m}{(i-m)^2}\right] +$$

$$\frac{k_C\pi^4\beta^3}{16}\,2i^2b_i + k_\tau\pi^3\beta^2\sum_{m=1,3,5,\ldots}^{\infty}a_m\left[\frac{2im}{m^2 - i^2} - (-1)^{\frac{i+m}{2}}\right] -$$

$$\frac{\pi^4 b_i}{2}\left(\frac{1}{\beta} + \frac{\beta i^2}{2} + \frac{\beta^3 i^4}{16}\right) - \gamma_2(-1)^{\frac{i-1}{2}} \qquad \text{(B13)}$$

$$(i^2 \neq m^2)$$

Minimizing equation (B10) with respect to the Lagrangian multipliers gives the constraining relationships

$$\frac{\partial G}{\partial \gamma_1} = 0 = -\sum_{m=1,3,5,\ldots}^{\infty} a_m(-1)^{\frac{m-1}{2}} \qquad \text{(B14)}$$

$$\frac{\partial G}{\partial \gamma_2} = 0 = -\sum_{m=1,3,5,\ldots}^{\infty} b_m(-1)^{\frac{m-1}{2}} \qquad \text{(B15)}$$

Equations (B12) to (B15) constitute an infinite set of homogeneous simultaneous equations containing the coefficients a_n, b_n, γ_1, and γ_2. The simultaneous solution of this set of equations gives the combinations of buckling stress coefficients, the accuracy of the solution depending on the number of equations solved simultaneously.

Solution by stability determinant.- Equations (B12) to (B15) constitute a system of homogeneous simultaneous equations containing the unknown deflection coefficients a_n and b_n. Equating the determinant of these equations to zero gives the buckling criterion in the form of a determinantal equation. In order to determine the nontrivial combinations of k_B, k_τ, and k_C which satisfy the equations, values of two of the stress coefficients and β can be substituted into the determinant of coefficients and the value of the third stress coefficient which satisfies the determinantal equation can thus be obtained. A graphical minimization of the third stress coefficient with respect to β gives the combinations of minimum buckling-stress coefficients. The results of the calculations based on the 14th-order determinant of coefficients (table IV) are given in table II.

The calculations for the infinitely long flat plate with the lower edge simply supported and the upper edge clamped, based on the stability determinant for the elastically restrained upper edge (table III), were graphically adjusted on the basis of calculations given in table II and are the basis of figures 6 to 9.

REFERENCES

1. Schuette, Evan H., and McCulloch, James C.: Charts for the Minimum-Weight Design of Multiweb Wings in Bending. NACA TN 1323, 1947.

2. Timoshenko, S.: Theory of Elastic Stability. McGraw-Hill Book Co., Inc., 1936.

3. Grossman, Norman: Elastic Stability of Simply Supported Flat Rectangular Plates under Critical Combinations of Transverse Compression and Longitudinal Bending. Jour. Aero. Sci., vol. 16, no. 5, May 1949, pp. 272-276.

4. Batdorf, S. B., and Houbolt, John C.: Critical Combinations of Shear and Transverse Direct Stress for an Infinitely Long Flat Plate with Edges Elastically Restrained against Rotation. NACA Rep. 847, 1946. (Formerly NACA ARR L4L14.)

5. Stowell, Elbridge Z.: Critical Shear Stress of an Infinitely Long Flat Plate with Equal Elastic Restraints against Rotation along the Parallel Edges. NACA ARR 3K12, 1943.

6. Lundquist, Eugene E., and Stowell, Elbridge Z.: Critical Compressive Stress for Flat Rectangular Plates Supported along All Edges and Elastically Restrained against Rotation along the Unloaded Edges. NACA Rep. 733, 1942. (Formerly NACA ACR, May 1941.)

TABLE I

CALCULATED BUCKLING STRESS COEFFICIENTS AND HALF-WAVE-LENGTH RATIOS

FOR INFINITELY LONG FLAT PLATE WITH BOTH EDGES SIMPLY SUPPORTED

k_B	k_T	k_C	λ/b
23.90	0	0	0.7
22.82	2.00	0	.8
18.38	4.00	0	.9
9.60	5.14	0	1.1
0	5.36	0	1.2
22.66	0	.5	.7
21.50	2.00	.5	.8
18.50	3.26	.5	1.1
14.10	4.00	.5	1.2
0	4.53	.5	1.5
21.82	0	.8	.7
20.32	2.00	.8	.8
17.00	3.10	.8	1.3
12.00	3.68	.8	1.8
0	3.89	.8	1.8
0	0	1.0	∞
21.22	0	1.0	.9
20.70	1.00	1.0	.9
19.27	2.00	1.0	1.0
17.79	2.50	1.0	1.2
15.30	2.85	1.0	2.0
14.14	2.88	1.0	∞

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TABLE II

CALCULATED BUCKLING STRESS COEFFICIENTS AND HALF-WAVE-LENGTH RATIOS FOR INFINITELY LONG FLAT PLATE WITH LOWER EDGE SIMPLY SUPPORTED AND UPPER EDGE CLAMPED

k_B	k_τ	k_C	λ/b
39.96	0	0	0.5
36.51	4.700	0	.6
20.00	7.963	0	1.1
0	6.637	0	1.2
38.30	0	1.0225	.5
0	4.818	1.0225	1.6
37.25	0	1.636	.5
35.80	2.700	1.636	.6
20.00	4.578	1.636	2.8
0	3.193	1.636	2.3
36.52	0	1.922	.6
20.00	2.789	1.922	6.0
0	1.832	1.922	4.0
0	0	2.045	∞

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TABLE III

STABILITY DETERMINANT FOR INFINITELY LONG FLAT PLATE WITH THE LOWER EDGE SIMPLY SUPPORTED AND THE UPPER EDGE ELASTICALLY RESTRAINED AGAINST ROTATION, SUBJECTED TO BENDING, SHEAR, AND TRANSVERSE COMPRESSIVE STRESSES

$$\begin{vmatrix} \underline{T_1} & 0 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \\ 0 & \underline{T_1} & -T_3 & T_2 & -T_5 & T_4 & -T_7 & T_6 \\ T_2 & -T_3 & \underline{T_8} & 0 & T_9 & T_{10} & T_{11} & T_{12} \\ T_3 & T_2 & 0 & \underline{T_8} & -T_{10} & T_9 & -T_{12} & T_{11} \\ T_4 & -T_5 & T_9 & -T_{10} & \underline{T_{13}} & 0 & T_{14} & T_{15} \\ T_5 & T_4 & T_{10} & T_9 & 0 & \underline{T_{13}} & -T_{15} & T_{14} \\ T_6 & -T_7 & T_{11} & -T_{12} & T_{14} & -T_{15} & \underline{T_{16}} & 0 \\ T_7 & T_6 & T_{12} & T_{11} & T_{15} & T_{14} & 0 & \underline{T_{16}} \end{vmatrix} = 0$$

where

$$T_1 = \pi^2 A_2 - \frac{1024 k_B^2}{\pi^2\beta^2} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)^4} - 256 k_T^2 \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)^2}$$

$$T_2 = -\frac{2048 k_B^2}{\pi^2\beta^2} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)^2(m^2 - 16)^2} - 512 k_T^2 \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)(m^2 - 16)}$$

$$T_3 = -\frac{12288 k_B k_T}{\pi\beta} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)^2(m^2 - 16)^2}$$

$$T_4 = -\frac{3072 k_B^2}{\pi^2\beta^2} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)^2(m^2 - 36)^2} - 768 k_T^2 \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)(m^2 - 36)}$$

$$T_5 = -\frac{49152 k_B k_T}{\pi\beta} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)^2(m^2 - 36)^2}$$

$$T_6 = \pi^2 + \frac{16 k_B}{\beta} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)^2}$$

$$T_7 = 8\pi k_T \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 4)}$$

$$T_8 = \pi^2 A_4 - \frac{4096 k_B^2}{\pi^2\beta^2} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 16)^4} - 1024 k_T^2 \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 16)^2}$$

$$T_9 = -\frac{6144 k_B^2}{\pi^2\beta^2} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 16)^2(m^2 - 36)^2} - 1536 k_T^2 \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 16)(m^2 - 36)}$$

$$T_{10} = -\frac{61440 k_B k_T}{\pi\beta} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 16)^2(m^2 - 36)^2}$$

$$T_{11} = 2\pi^2 + \frac{32 k_B}{\beta} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 16)^2}$$

$$T_{12} = 16\pi k_T \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 16)}$$

$$T_{13} = \pi^2 A_6 - \frac{9216 k_B^2}{\pi^2\beta^2} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 36)^4} - 2304 k_T^2 \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 36)^2}$$

$$T_{14} = 3\pi^2 + \frac{48 k_B}{\beta} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 36)^2}$$

$$T_{15} = 24\pi k_T \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m(m^2 - 36)}$$

$$T_{16} = -\frac{\pi^2}{4} \sum_{m=1,3,5,\ldots}^{\infty} \frac{m^2}{A_m} - \frac{\pi^4}{8\pi\beta}$$

and

$$A_m = \frac{1}{\beta^3} + \left(\frac{2}{\beta} - \beta k_0\right) m^2 + \beta m^4$$

TABLE

STABILITY DETERMINANT FOR INFINITELY LONG FLAT PLATE,

CLAMPED, SUBJECTED TO BENDING, SHEAR,

a_1	b_1	a_3	b_3	a_5	b_5	a_7
$2k_B + \frac{\pi^2\beta^2k_C}{8} - B_1$	0	$-2k_B$	$-\frac{7\pi\beta k_T}{4}$	$\frac{2k_B}{9}$	$\frac{7\pi\beta k_T}{12}$	$-\frac{2k_B}{9}$
0	$2k_B + \frac{\pi^2\beta^2k_C}{8} - B_1$	$-\frac{\pi\beta k_T}{4}$	$-2k_B$	$\frac{17\pi\beta k_T}{12}$	$\frac{2k_B}{9}$	$-\frac{17\pi\beta k_T}{24}$
$-2k_B$	$-\frac{\pi\beta k_T}{4}$	$\frac{2k_B}{9} + \frac{9\pi^2\beta^2k_C}{8} - B_3$	0	$-2k_B$	$-\frac{23\pi\beta k_T}{8}$	$\frac{2k_B}{25}$
$-\frac{7\pi\beta k_T}{4}$	$-2k_B$	0	$\frac{2k_B}{9} + \frac{9\pi^2\beta^2k_C}{8} - B_3$	$\frac{7\pi\beta k_T}{8}$	$-2k_B$	$\frac{41\pi\beta k_T}{20}$
$\frac{2k_B}{9}$	$\frac{17\pi\beta k_T}{12}$	$-2k_B$	$\frac{7\pi\beta k_T}{8}$	$\frac{2k_B}{25} + \frac{25\pi^2\beta^2k_C}{8} - B_5$	0	$-2k_B$
$\frac{7\pi\beta k_T}{12}$	$\frac{2k_B}{9}$	$-\frac{23\pi\beta k_T}{8}$	$-2k_B$	0	$\frac{2k_B}{25} + \frac{25\pi^2\beta^2k_C}{8} - B_5$	$\frac{23\pi\beta k_T}{12}$
$-\frac{2k_B}{9}$	$-\frac{17\pi\beta k_T}{24}$	$\frac{2k_B}{25}$	$\frac{41\pi\beta k_T}{20}$	$-2k_B$	$\frac{23\pi\beta k_T}{12}$	$\frac{2k_B}{49} + \frac{49\pi^2\beta^2k_C}{8} - B_7$
$\frac{2k_B}{25}$	$\frac{49\pi\beta k_T}{40}$	$-\frac{2k_B}{9}$	$-\frac{\pi\beta k_T}{4}$	$\frac{2k_B}{49}$	$\frac{73\pi\beta k_T}{28}$	$-2k_B$
$-\frac{2k_B}{25}$	$-\frac{49\pi\beta k_T}{60}$	$\frac{2k_B}{49}$	$\frac{89\pi\beta k_T}{56}$	$-\frac{2k_B}{9}$	$\frac{7\pi\beta k_T}{48}$	$\frac{2k_B}{81}$
$\frac{2k_B}{49}$	$\frac{97\pi\beta k_T}{84}$	$-\frac{2k_B}{25}$	$-\frac{41\pi\beta k_T}{80}$	$\frac{2k_B}{81}$	$\frac{137\pi\beta k_T}{72}$	$-\frac{2k_B}{9}$
$-\frac{2k_B}{49}$	$-\frac{97\pi\beta k_T}{112}$	$\frac{2k_B}{81}$	$\frac{153\pi\beta k_T}{108}$	$-\frac{2k_B}{25}$	$-\frac{\pi\beta k_T}{4}$	$\frac{2k_B}{121}$
$\frac{2k_B}{81}$	$\frac{161\pi\beta k_T}{144}$	$-\frac{2k_B}{49}$	$-\frac{89\pi\beta k_T}{140}$	$\frac{2k_B}{121}$	$\frac{217\pi\beta k_T}{132}$	$-\frac{2k_B}{25}$
-1	0	1	0	-1	0	1
0	-1	0	1	0	-1	0

where $B_i = \frac{\pi^2}{2}\left(\frac{1}{\beta} + \frac{\beta i^2}{4}\right)^2$

IV

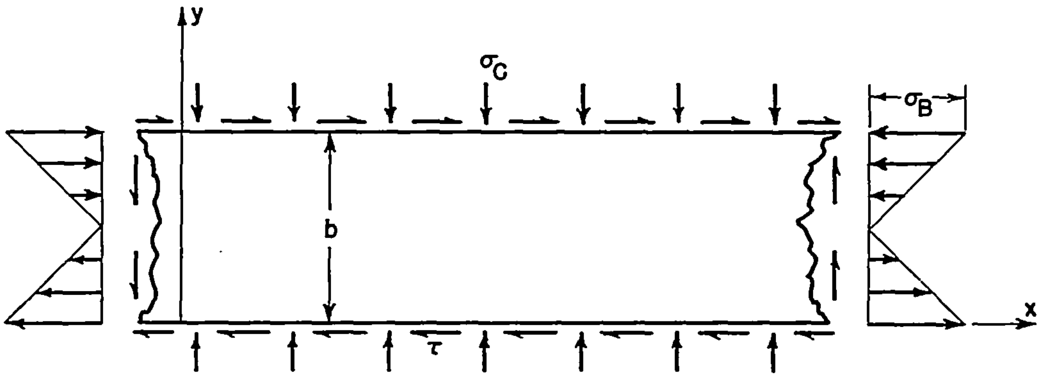
WITH THE LOWER EDGE SIMPLY SUPPORTED AND THE UPPER EDGE

AND TRANSVERSE COMPRESSIVE STRESSES

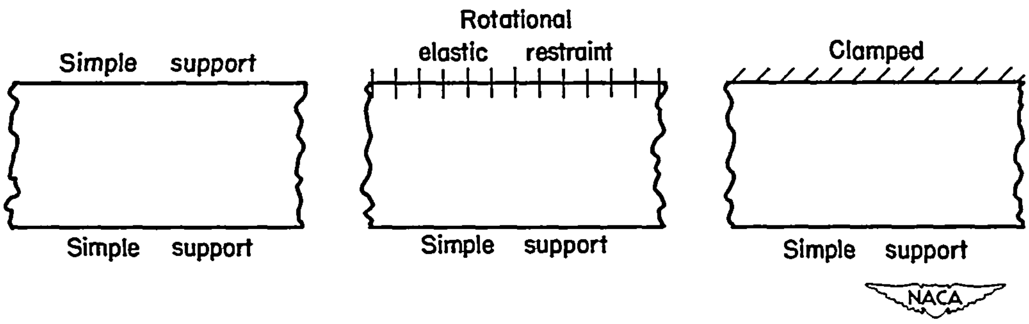
a_9	a_{11}	a_{13}	a_{15}	a_{17}	γ_1	γ_2
$\frac{2k_B}{25}$	$-\frac{2k_B}{25}$	$\frac{2k_B}{49}$	$-\frac{2k_B}{49}$	$\frac{2k_B}{81}$	-1	0
$\frac{49\pi\beta k_T}{40}$	$-\frac{49\pi\beta k_T}{60}$	$\frac{97\pi\beta k_T}{84}$	$-\frac{97\pi\beta k_T}{112}$	$\frac{161\pi\beta k_T}{144}$	0	-1
$-\frac{2k_B}{9}$	$\frac{2k_B}{49}$	$-\frac{2k_B}{25}$	$\frac{2k_B}{81}$	$-\frac{2k_B}{49}$	1	0
$-\frac{\pi\beta k_T}{4}$	$\frac{89\pi\beta k_T}{56}$	$-\frac{41\pi\beta k_T}{80}$	$\frac{153\pi\beta k_T}{108}$	$-\frac{89\pi\beta k_T}{140}$	0	1
$\frac{2k_B}{49}$	$-\frac{2k_B}{9}$	$\frac{2k_B}{81}$	$-\frac{2k_B}{25}$	$\frac{2k_B}{121}$	-1	0
$\frac{73\pi\beta k_T}{28}$	$\frac{7\pi\beta k_T}{48}$	$\frac{137\pi\beta k_T}{72}$	$-\frac{\pi\beta k_T}{4}$	$\frac{217\pi\beta k_T}{132}$	0	-1
$-2k_B$	$\frac{2k_B}{81}$	$-\frac{2k_B}{9}$	$\frac{2k_B}{121}$	$-\frac{2k_B}{25}$	1	0
$\frac{2k_B}{81} + \frac{81\pi^2\beta^2 k_C}{8} - B_9$	$-2k_B$	$\frac{2k_B}{121}$	$-\frac{2k_B}{9}$	$\frac{2k_B}{169}$	-1	0
$-2k_B$	$\frac{2k_B}{121} + \frac{121\pi^2\beta^2 k_C}{8} - B_{11}$	$-2k_B$	$\frac{2k_B}{169}$	$-\frac{2k_B}{9}$	1	0
$\frac{2k_B}{121}$	$-2k_B$	$\frac{2k_B}{169} + \frac{169\pi^2\beta^2 k_C}{8} - B_{13}$	$-2k_B$	$\frac{2k_B}{225}$	-1	0
$-\frac{2k_B}{9}$	$\frac{2k_B}{169}$	$-2k_B$	$\frac{2k_B}{225} + \frac{225\pi^2\beta^2 k_C}{8} - B_{15}$	$-2k_B$	1	0
$\frac{2k_B}{169}$	$-\frac{2k_B}{9}$	$\frac{2k_B}{225}$	$-2k_B$	$\frac{2k_B}{289} + \frac{289\pi^2\beta^2 k_C}{8} - B_{17}$	-1	0
-1	1	-1	1	-1	0	0
0	0	0	0	0	0	0

= 0

NACA



(a) Loading and coordinates.



(b) Edge conditions.

Figure 1.- Schematic description of loading and edge conditions for infinitely long flat plate. Positive directions indicated by arrows.

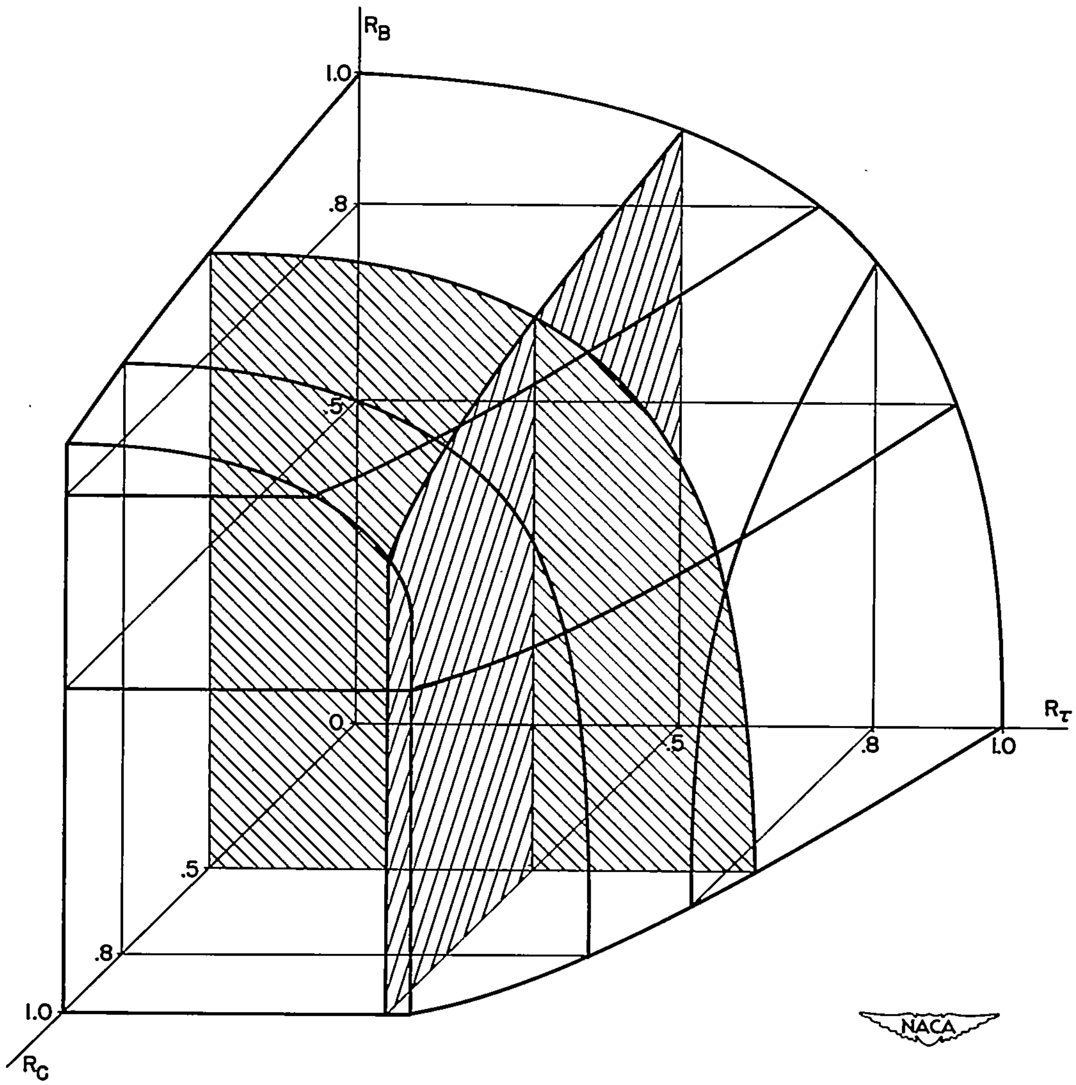


Figure 2.- Interaction surface for buckling of infinitely long flat plate with simply supported edges subjected to bending, shear, and transverse compressive stresses.

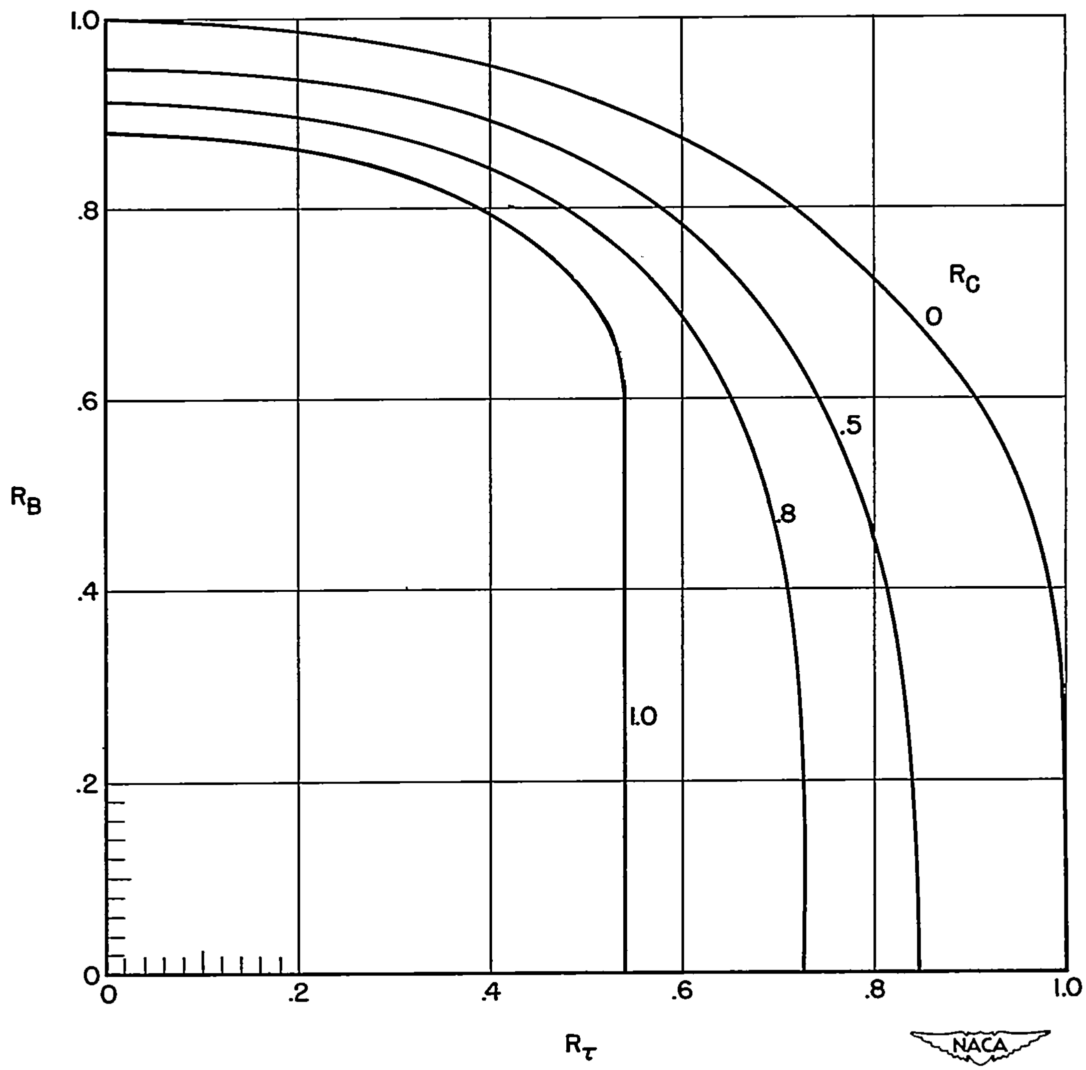


Figure 3.- Shear and bending interaction curves for buckling of an infinitely long flat plate with simply supported edges.

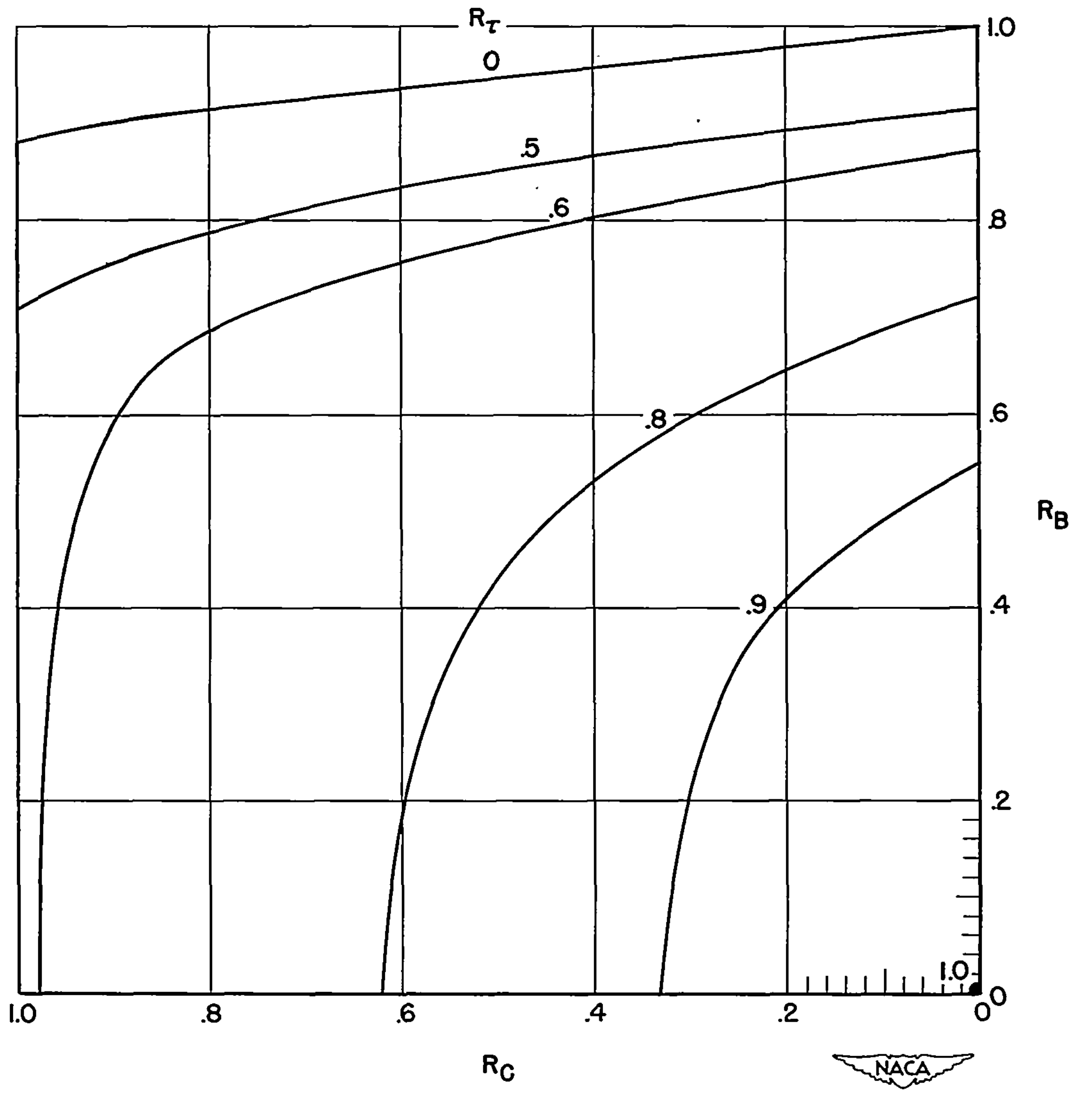


Figure 4.- Transverse compression and bending interaction curves for buckling of an infinitely long flat plate with simply supported edges.

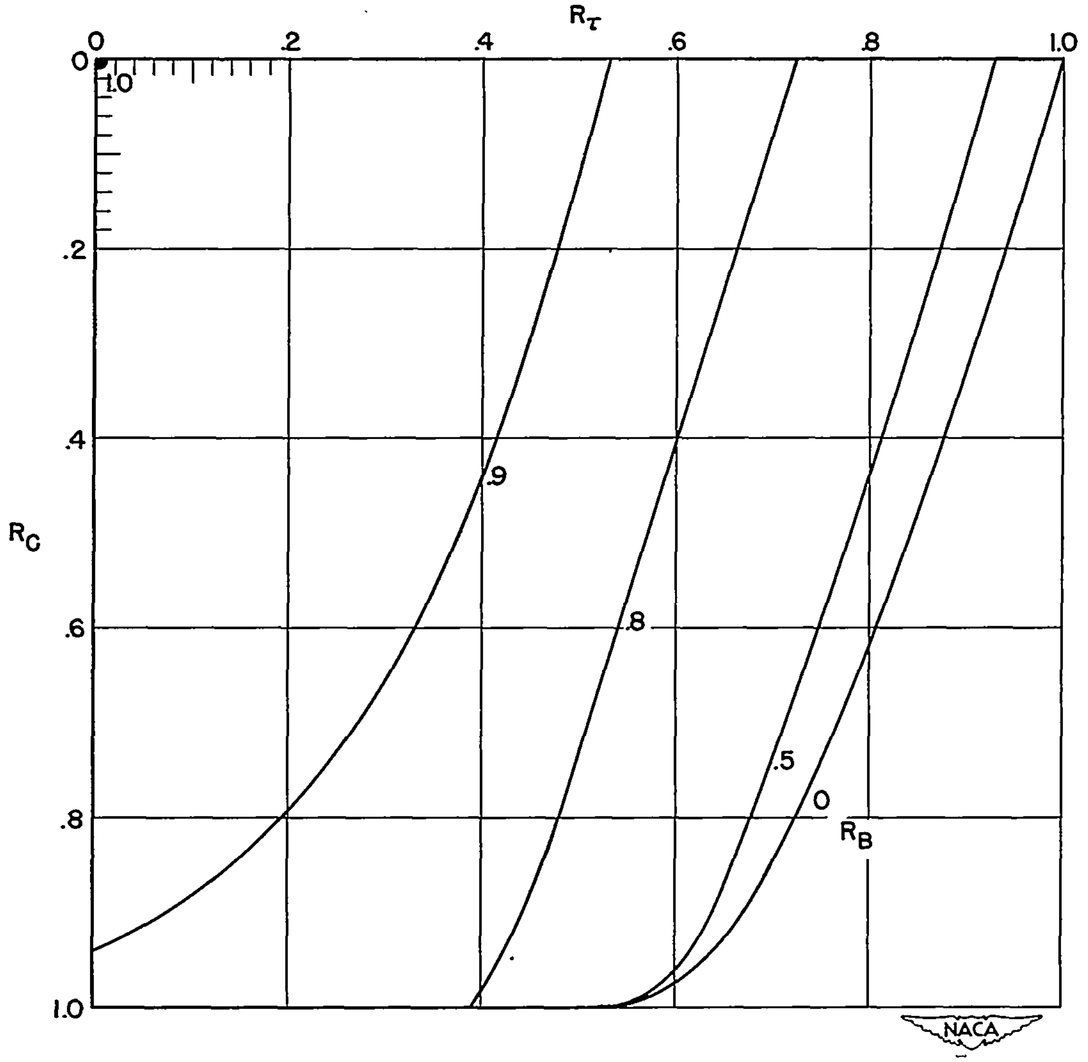


Figure 5.- Transverse compression and shear interaction curves for buckling of an infinitely long flat plate with simply supported edges.

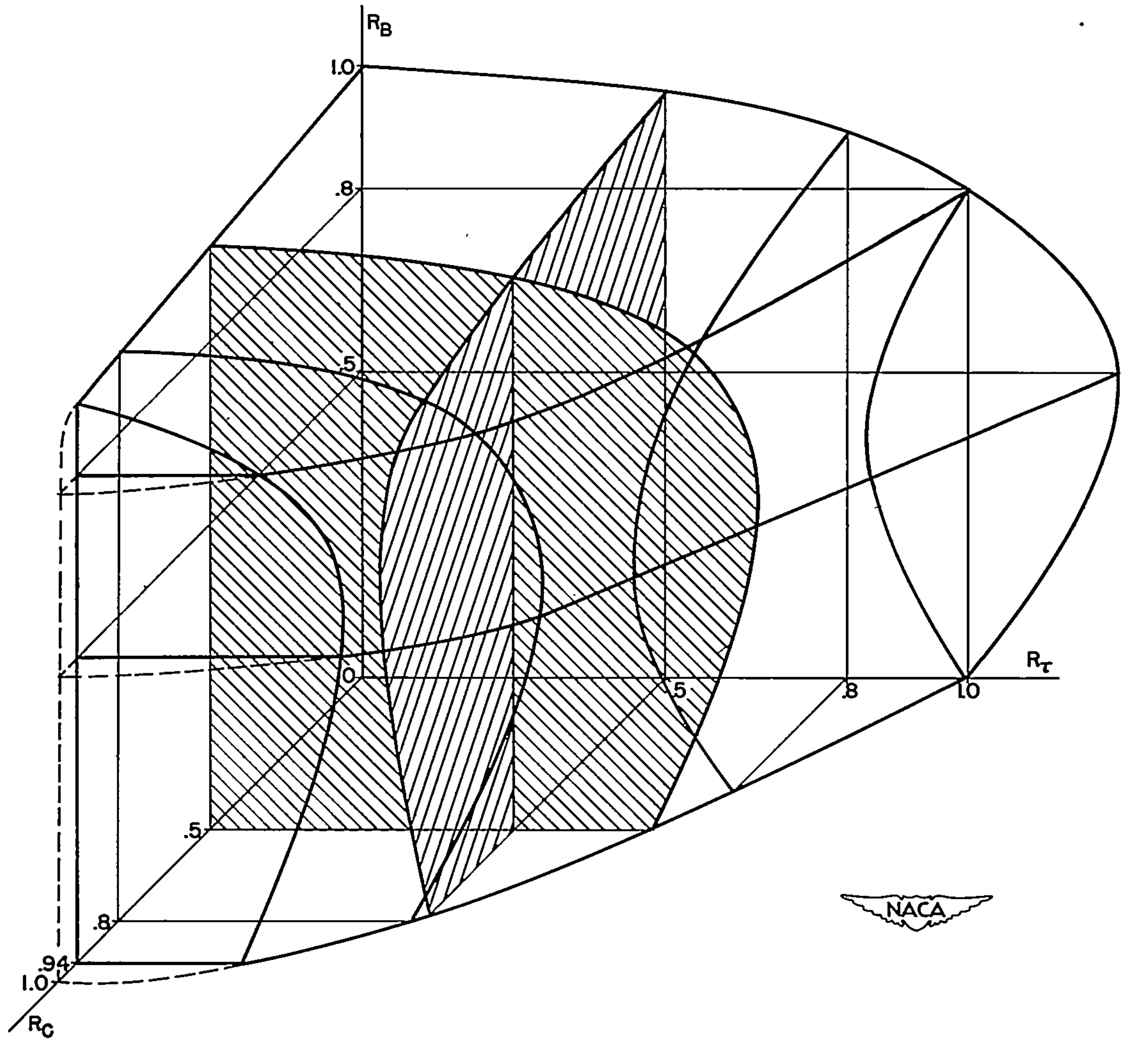


Figure 6.- Interaction surface for buckling of an infinitely long flat plate with lower edge simply supported and upper edge clamped, subjected to bending, shear, and transverse compressive stresses, as shown in figure 1.

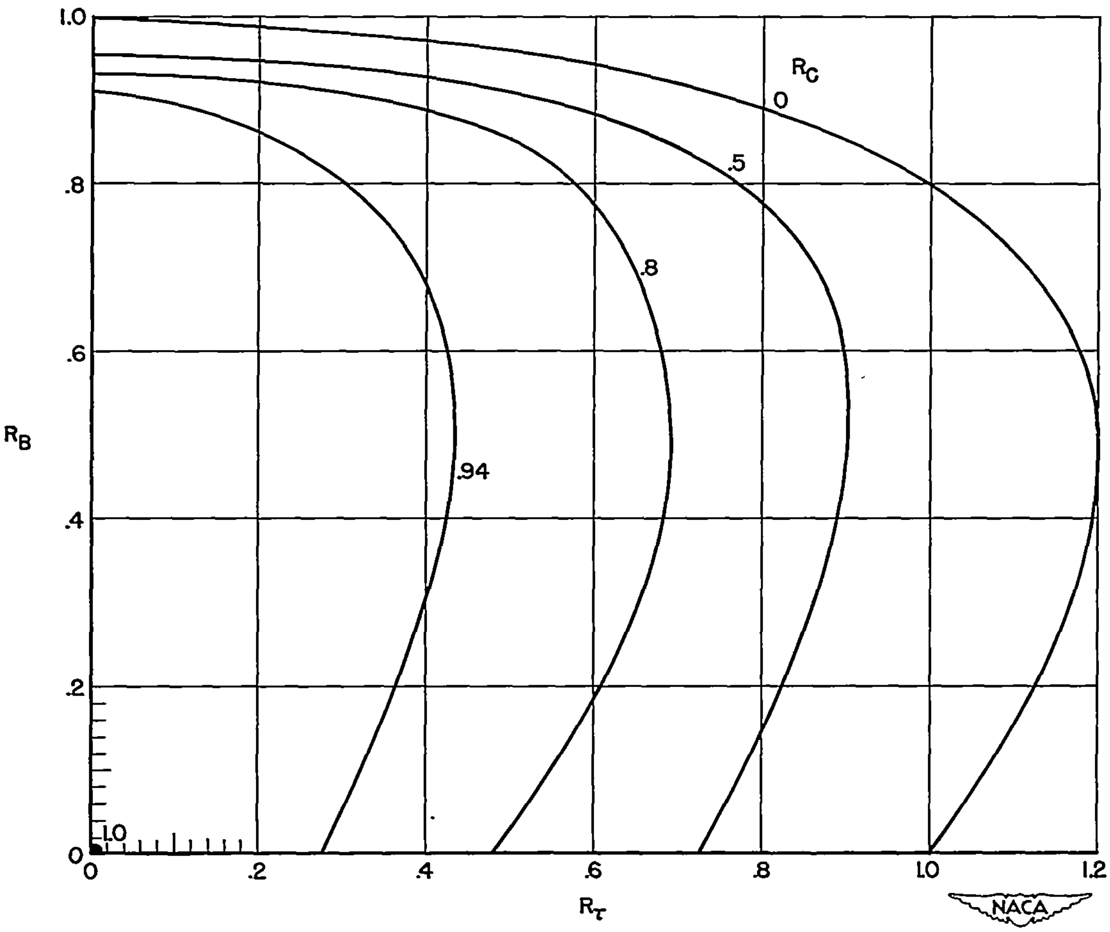


Figure 7.- Shear and bending interaction curves for buckling of an infinitely long flat plate with lower edge simply supported and upper edge clamped.

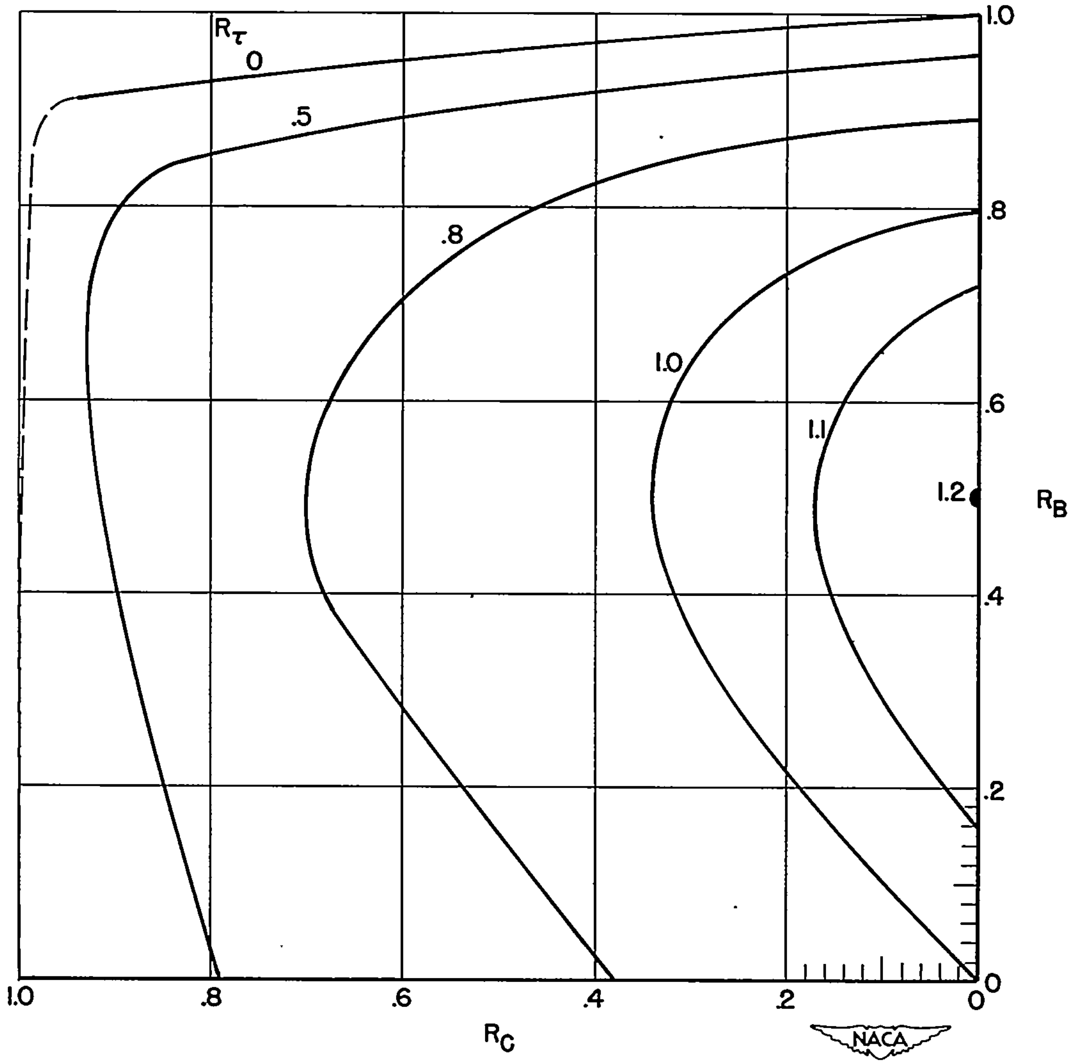


Figure 8.- Transverse compression and bending interaction curves for buckling of an infinitely long flat plate with lower edge simply supported and upper edge clamped.

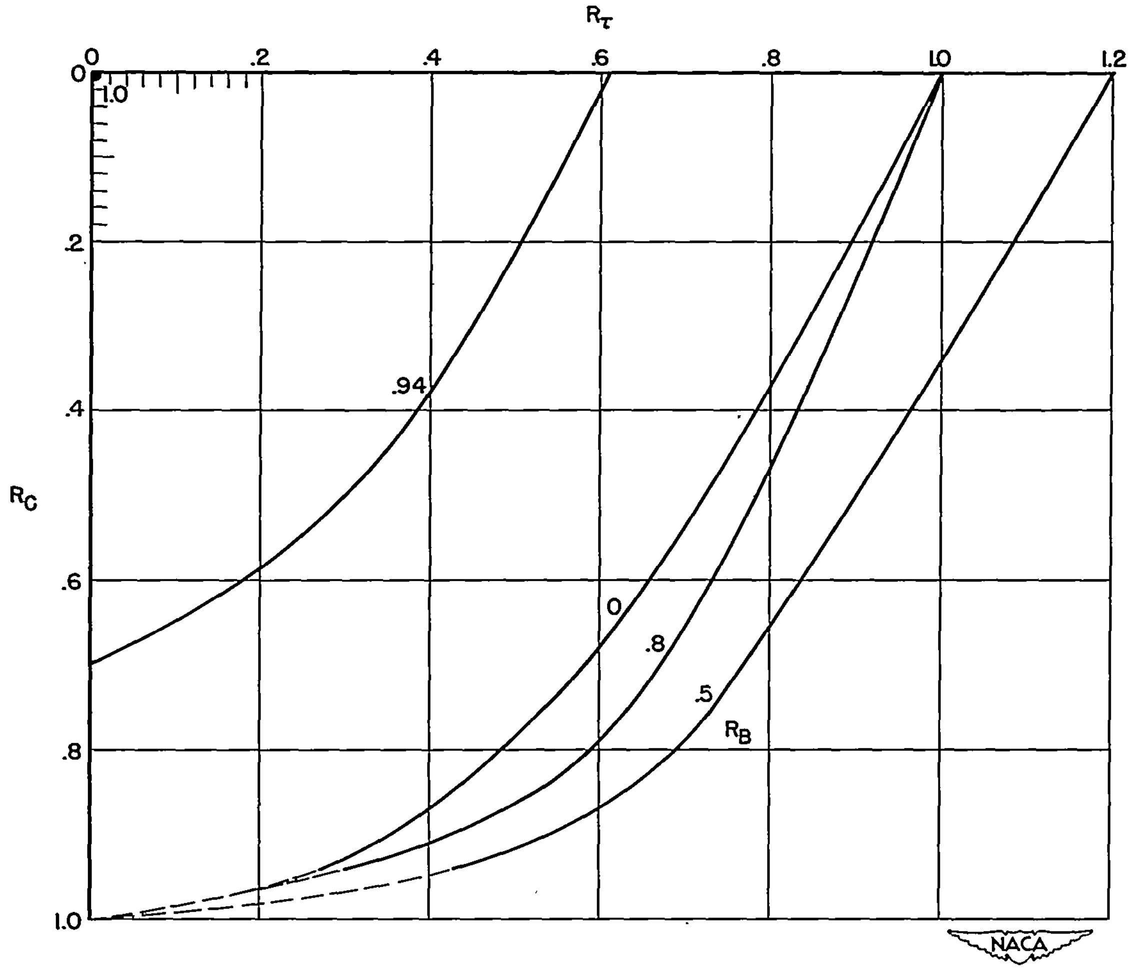


Figure 9.- Transverse compression and shear interaction curves for buckling of an infinitely long flat plate with lower edge simply supported and upper edge clamped.