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AN INVESTIGATION BY THE HODOGRAPH METHOD OF FLOW THROUGH A SYMMETRICAL NOZZLE WITH LOCALLY SUPERSONIC REGIONS

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SUMMARY

The flow of a compressible fluid through a channel having locally supersonic regions is studied by using the Tricomi equation in the hodograph variables as an approximation in the sonic region to the equation of flow of an irrotational, inviscid gas. It is shown that this is equivalent to studying the flow of a gas having a pressure-density relation matching the isentropic relation to the third derivative at the sonic point. A one-parameter family of solutions of the Tricomi equation is used which provides symmetrical accelerated-decelerated flows. The variation of this parameter alters the Mach number at the center of the throat, the velocity distribution and gradient along the center streamline, as well as the shape of the channel, that is, the curvature of the bounding streamline.

As specific examples, flows are computed having Mach number equal to unity and to 0.86 at the center of the throat section. Constant-velocity lines are plotted and it is found that the velocity gradient becomes zero at three places along each streamline outside of a limiting streamline for values of the parameter greater than zero ($M < 1$ at center of throat section). For the parameter equal to zero ($M = 1$ at center of throat section), the velocity gradient along the streamlines and the curvature is discontinuous at all points of the two characteristics which meet the center streamline.

Other solutions to the Tricomi equation are discussed which may be used to formulate channel flows. The exact nature of these flows has not yet been investigated.

INTRODUCTION

In the investigations of the effect of free-stream Mach number on the lift and drag of an airfoil in a uniform stream, it was found that

the lift did not begin to decrease or the drag increase until the free-stream velocity exceeded the critical free-stream velocity, that is, the velocity for which the local velocity of sound is just attained at some point of the body. Thus it appeared that, for free-stream velocities in between the critical free-stream velocity and the velocity for which the lift characteristics changed rapidly with Mach number, there might exist about the body a locally supersonic region which contained no shocks, although such shock-free flows have never been observed experimentally. A similar phenomenon occurs in the study of flows through a gradual constriction. When the exit pressure of a nozzle is decreased, the mean velocity at the narrowest cross section and also the total mass flow are increased. However, when the mean velocity at the constriction reaches the local velocity of sound, the mass flow reaches a maximum; and the flow then changes from an accelerated-decelerated flow to one in which the fluid is always accelerated in passing through the nozzle. Further decreases in the exit pressure fail to increase the mass flow but only introduce shocks into the supersonic side of the nozzle to satisfy the conditions of conservation of mass, momentum, and energy. One question which occurs regarding such a process is: Do there occur isolated regions of shock-free supersonic flow in the neighborhood of the walls before the mass flow reaches the maximum and the flow becomes purely accelerated?

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SYMBOLS

x,y	Cartesian coordinates $(z = x + iy)$
u,v	velocity components in x- and y-direction, respectively
γ	ratio of specific heats
$\beta = \frac{1}{\gamma - 1}$	
α_1	Mach angle
$\tau = q^2/2\beta$	($\tau = 1$ corresponds to limiting speed)
$\vec{q}$	velocity vector
q	magnitude of velocity vector

θ inclination of velocity vector with x-axis

$$T_1 = 2\tau(1 - \tau)^{-\beta}$$

$$T_2 = \frac{1 - (2\beta + 1)\tau}{2\tau(1 - \tau)^{\beta+1}}$$

c local speed of sound normalized to speed of sound in still air

ρ density of fluid normalized to stagnation density

v specific volume

p pressure of fluid corresponding to density ρ

p_s critical pressure (pressure at $s = 0$)

M Mach number (q/c)

$$s = \frac{2.1120s^*}{1 + 1.2304s^*}$$

$$s^* = \int_{\tau}^{(2\beta+1)^{-1}} \frac{(1 - \tau)^{\beta}}{2\tau} d\tau$$

$$s_o = \left(\frac{9}{4}\lambda^2\right)^{1/3}$$

φ velocity potential

ψ stream function

$$\psi' = \omega_1\psi = (1.45320 - 0.84664s)\psi$$

φ' conjugate potential to ψ' (by equations (6))

Ψ symmetrical stream function given by equation (19)

ψ_1 stream function given by equation (15)

$$\alpha_o = \sqrt{\frac{4}{9} s^3 - \lambda^2}$$

$$\alpha = \sqrt[4]{\left(\frac{4}{9} s^3 - \lambda^2 + \theta^2\right)^2 + 4\lambda^2\theta^2}$$

$$\omega = \tan^{-1} \frac{-2\lambda\theta}{\theta^2 + \frac{4}{9}\left(s^3 - s_o{}^3\right)}$$

$$\beta_1 = \left[\left(\alpha \cos \frac{\omega}{2} - \theta\right)^2 + \left(\alpha \sin \frac{\omega}{2} + \lambda\right)^2\right]^{\frac{1}{6}}$$

$$\Omega = \tan^{-1} \frac{\alpha \sin \frac{\omega}{2} + \lambda}{\alpha \cos \frac{\omega}{2} - \theta}$$

$$\delta = \tan^{-1} \frac{\lambda}{\alpha_o}$$

$$\sigma = \sqrt{\frac{4}{9} s^3 + \theta^2}$$

$$\mu = \sqrt{(\theta - i\lambda)^2 + \frac{4}{9} s^3} - (\theta - i\lambda)$$

EARLY INVESTIGATIONS OF LOCALLY SUPERSONIC REGIONS

The first attempt to construct a flow through a channel having locally supersonic regions was made by Taylor in 1931 (reference 1). He expanded the velocity potential in a double power series about some central point of a two-dimensional nozzle whose walls consisted of two arcs of circles. Taylor was able to obtain several flows which were symmetric in the velocity about the line of centers of the two circles. The values of the maximum velocity on the straight streamline for these flows were bounded by a certain velocity for which the local velocity of sound on the wall was just barely exceeded. From this result,

Taylor concluded that the potential flow through a channel with locally supersonic regions was impossible. Görtler (reference 2), however, felt that Taylor's conclusion was unfounded, for if more terms of the power series were taken and the boundary conditions on the wall satisfied to more terms in the series expansion, then the difficulty which Taylor encountered in determining the coefficients might not exist. Görtler, using much of Taylor's computations, satisfied one less relation of the boundary conditions than Taylor and thereby had two free parameters to vary instead of one. One of these parameters represented the velocity at the center axis of the throat of the nozzle, and the other, the velocity gradient along the center axis at this point. By varying the velocity gradient Görtler obtained a sequence of channels which exhibited locally supersonic regions near the channel walls at the constriction. As the velocity gradient was increased, the locally supersonic region gradually extended toward the axis. When the local velocity of sound was attained at the axis, the flow became identical to Taylor's unsymmetrical flow through a Laval nozzle. Görtler showed that the locally supersonic region cannot extend to the center axis if the velocity gradient is to be continuous along the center streamline. When the sonic velocity is just attained on the axis, but not exceeded, then the condition of continuity of the gradient will require a zero gradient at the sonic point. Under this condition the following theorem, proved by Görtler, shows that the sonic line must be a straight line and the curvature of the wall at the throat must be zero.

> Theorem: Let an inviscid flow of a compressible gas be given which is symmetrical about the $y = 0$ axis and is analytic in the neighborhood of a point $x = x_0$ and $y = 0$. Further, suppose that the local velocity of sound is attained at this point of the center axis and that the velocity gradient vanishes; that is, $u(x_0,0) = 1$ and $\frac{\partial u}{\partial x}(x_0,0) = 0$. Then it follows necessarily for all y's in the neighborhood that $u(x_0,y) \equiv 1$, $\frac{\partial u}{\partial x}(x_0,y) = 0$, and $v(x_0,y) \equiv 0$.

Flows having locally supersonic regions in which the sonic velocity is just reached on the center axis, therefore, will not be analytic in the neighborhood of the center-axis point of the throat. Such a flow has been computed by Tomotika and Tamada and a similar flow having symmetry about both horizontal and vertical axes is given in this work. The velocity gradient in both flows has a jump at the center of the nozzle.

The work of Tomotika and Tamada (reference 3), using an approximate gas law, and of Frankl (reference 4), Görtler (reference 2), Oswatitsch and Rothstein (reference 5), Fox and Southwell (reference 6), and others, using power series, already indicates strongly that solutions

which have locally supersonic regions for the exact flow equations do exist. Recognition must be given, of course, to the lack of knowledge about the convergence of the power series, and to the fact that the Tricomi equation which the Japanese authors use approximates the actual gas only in a small neighborhood of the sonic velocity. Yet, the fact that both methods give flows having locally supersonic regions clearly points toward a similar result from the exact equation for an isentropic compressible, inviscid fluid. But the existence of these solutions by no means settles the matter. It is to be noted that "existence" is used in the mathematical sense. Actually, all experimental results (Liepmann (reference 7) and Frankl (reference 4)) show that irrotational flow breaks down soon after the appearance of the lower critical Mach number (Tsien and Kuo (reference 8)). None of the experiments shows locally supersonic regions of any appreciable extent without shocks.

GENERAL PROPERTIES OF A LOCALLY SUPERSONIC REGION

Frankl (reference 9) and Nikolsky and Taganoff (reference 10) have made some general investigations of the properties of the locally supersonic regions without shocks. Frankl (reference 9) showed that the symmetric flow over an obstacle which is symmetric about two perpendicular axes is uniquely determined by the boundary conditions on the portions of the arcs DFD' and EGE' in figure 1 which lie outside of the region defined by the two pairs of characteristics BD and BE and B'D' and B'E' and the conditions at infinity. The segments of the body DE and D'E' between the characteristics then depend upon the rest of the body and the free-stream Mach number. Thus it appears that symmetric flow with supersonic regions does not exist for an arbitrary body, although for a given free-stream Mach number there might well be a large class of profiles for which such flows do exist.

The results of Frankl can be shown more rigorously and easily by making use of the following uniqueness theorem proved by Tricomi (reference 11):

> In a domain bounded by a curve in the subsonic region with its termini on the sonic line in the hodograph plane and two intersecting characteristics drawn from these termini, the solution of the Tricomi equation is uniquely determined by the values prescribed on the subsonic arc and on one of the characteristic segments bounding the domain.

The character of the flow through a nozzle having locally supersonic regions is shown in the hodograph plane in figure 2. If the value of the stream function is given on the arc BCD and the segment of the

axis OAB and the characteristic ED, then the solution is given uniquely inside the region BCDEOB. In the physical plane, this means that the solution is given uniquely in the region A'B'C'D'E'O'A' (fig. 3) since the arc OA in figure 2 maps into the line $\theta = 0$ (O'A' in fig. 3) the line AB into the center streamline, and the characteristics DE and EO into the Mach lines D'E' and E'O'. The same thing can be shown for A'O'G'H'I'J'A' since symmetry is assumed. Since the values of θ, q, and ψ are known on the two Mach lines E'O' and O'G', then the solution is determined inside the quadrilateral E'L'G'O'. Thus the flow field is entirely determined by the boundary conditions on the arcs C'F' and M'I', and by the velocity distribution on the center streamline JB' and on C'B' and I'J'.

The fact that, for an inviscid flow which has locally supersonic regions, the boundary cannot be chosen in an arbitrary manner actually has little significance in the flow of real gases. In such flows, there exists a boundary layer near the walls and a pertinent question regarding locally supersonic regions may be stated as follows: Does the boundary layer adapt itself so that locally supersonic regions can occur without shocks, or are shocks always present in locally supersonic regions?

Nikolsky and Taganoff have discovered some of the properties which a locally supersonic region in an inviscid flow free of shock waves must have. Briefly these properties are as follows:

(1) The slope of the contour at any given point in the locally supersonic region is the arithmetic mean of the slopes of the velocity vector at the points on the transition line which are the termini of the characteristics originating from the given point of the contour.

(2) If one moves along the transition line in such a way that the region of subsonic velocities lies on the left, then the velocity vector will always rotate clockwise.

(3) If, in the supersonic zone, a segment of a characteristic of one family is given such that the characteristics of the other family originating from points of this segment extend to the transition line, then the inclination of the velocity vector and its magnitude are monotonic functions along the given segment of the characteristic.

(4) The magnitude of the velocity along a rectilinear section of the profile in a supersonic zone must decrease in the direction of flow.

(5) Prandtl-Meyer flow between two characteristics of one family, with the characteristics of the other family rectilinear, cannot be completely realized up to the transition line in the finite region.

(6) Characteristics of the first family which originate from points of a rectilinear section of the profile cannot extend to the transition line, but must instead terminate in a compression shock.

Nikolsky and Taganoff also discuss the possibility of the Mach lines forming an envelope in the supersonic zone. They show that if the solution to the flow over a body is found which satisfies the boundary conditions then the Jacobian of the transformation from the hodograph plane to the physical plane cannot vanish on the boundary, since the curvature of the streamline must be zero when $J = 0$. By using the property of statement (3) above they show that if the Jacobian does not vanish on the boundary then it cannot vanish in the locally supersonic region in the flow field outside the body. They give as the criterion of breakdown the relation

$$\frac{dq}{d\theta} = q \tan \alpha_1$$

where q is the velocity magnitude, θ the angle of inclination of the contour, and α_1 the Mach angle associated with the local Mach number. What Nikolsky and Taganoff failed to notice, however, is that the above equation is the condition that the streamline representing the body be tangent to a characteristic in the hodograph plane. If the derivatives with respect to θ and q of the stream function are finite, then the streamline will have infinite curvature at the point where the relation above is satisfied. What Nikolsky and Taganoff have actually shown may be stated more precisely in the following form:

> Theorem: If, along the portion of a streamline lying in a local supersonic region of flow, there exists no point at which the curvature becomes infinite, then there cannot exist any limit line inside the supersonic zone lying between the given streamline and that streamline which is just tangent to the transition line.

However, if one considers the continuation of the flow pattern then one may find a streamline for which the curvature is infinite and hence the Jacobian $J = 0$. Nikolsky and Taganoff were unable to show that $J = 0$ on the transition line was an impossibility under the conditions of the theorem above. Friedrichs (reference 12), however, showed that, under the hypotheses of the theorem above, infinite acceleration and infinite curvature of the streamlines could not occur on the transition line between the two streamlines mentioned in the theorem. In investigating the proximity to breakdown of a potential flow over an obstacle, it seems advisable to study the continuation of the flow into the region

inside the body and investigate how near to the body streamline lie the cusps of a limit line. If limit lines lie near the body, then it is probable that a slight change in the boundary or an increase in the free stream would result in an inability to satisfy the boundary conditions.

APPROXIMATE HODOGRAPH METHOD FOR TRANSONIC FLOW INVOLVING SOLUTIONS TO THE TRICOMI EQUATION

In this discussion an approximation to the compressible-flow hodograph equations will be used in which the Tricomi equation replaces the usual second-order linear equation for the stream function. In the notation of reference 13, the equations

$$\left.\begin{aligned} \varphi_\theta &= T_1 \psi_\tau \\ \varphi_\tau &= -T_2 \psi_\theta \end{aligned}\right\} \tag{1}$$

where $T_1 = 2\tau(1 - \tau)^{-\beta}$ and $T_2 = \dfrac{1 - (2\beta + 1)\tau}{2\tau(1 - \tau)^{\beta+1}} = \dfrac{1 - M^2}{\rho}$ represent an exact linearization of the equations of irrotational motion of an inviscid, compressible fluid. If a new variable

$$s^* = \int_\tau^{(2\beta+1)^{-1}} \frac{(1 - \tau)^\beta}{2\tau}\, d\tau$$

is introduced into equation (1), the equations

$$\left.\begin{aligned} \varphi_\theta &= -\psi_{s^*} \\ \varphi_{s^*} &= \omega^*(s^*)\psi_\theta \end{aligned}\right\} \tag{2}$$

are obtained, where $\omega^*(s^*) = \frac{1 - M^2}{\rho^2}$. Then the following differential equation for ψ results when φ is eliminated from equation (2):

$$\psi_{s^*s^*} + \omega^*(s^*)\psi_{\theta\theta} = 0 \tag{3}$$

The general forms for two simultaneous differential equations in two independent variables which reduce to the Tricomi equation have been developed by Loewner (reference 14). One such form will be considered here. Let ψ and φ be defined by the relations

$$\left.\begin{aligned} \omega_1\psi_s &= \frac{1}{\omega_1}\varphi_\theta \\ \frac{1}{\omega_1}\varphi_s &= s\omega_1\psi_\theta \end{aligned}\right\} \tag{4}$$

and let a transformation of the dependent variables be given by

$$\left.\begin{aligned} \psi' &= \omega_1\psi \\ \varphi'_s &= -\frac{1}{\omega_1}\varphi_s \\ \varphi'_\theta &= -\frac{1}{\omega_1}\varphi_\theta + \frac{d\omega_1}{ds}\psi \end{aligned}\right\} \tag{5}$$

On eliminating φ' from equation (5), it can be seen that, for compatibility with equation (4), the quantity ω_1 must take the form $\omega_1 = as + b$. When equations (5) are substituted into equation (4) the results are

$$\left.\begin{aligned} \psi'_s &= \varphi'_\theta \\ \varphi'_s &= -s\psi'_\theta \end{aligned}\right\} \tag{6}$$

Elimination of φ' from the above equations leads to the Tricomi equation

$$\psi'_{ss} + s\psi'_{\theta\theta} = 0 \tag{7}$$

Now equation (4) may be written in the form of equation (3) by the introduction of the independent variable s^* defined by

$$s^* = \int \frac{ds}{\omega_1^2} + \text{Constant} = \frac{s}{b(as + b)}$$

if

$$\omega^* = \frac{b^6 s^*}{(1 - abs^*)^5}$$

The function ω^* can be made to approximate $\frac{1 - M^2}{\rho^2}$ in the neighborhood of the velocity of sound best by determining the two constants so that the first two derivatives of ω^* and $\frac{1 - M^2}{\rho^2}$ coincide. From this, with $\gamma = 1.4$, the results

$$\left.\begin{aligned} \omega^* &= \frac{9.4204 s^*}{(1 + 1.2304 s^*)^5} \\ s &= \frac{2.1120 s^*}{(1 + 1.2304 s^*)} \end{aligned}\right\} \tag{8}$$

and

$$\psi = (1.45320 - 0.84664s)^{-1}\psi' \tag{9}$$

are obtained. The relation between s and s^* is shown in figure 4 and tabulations of s, q, $1/q$, $1/\rho$, s^*, and ω_1 are given in tables I and II.

EXACT EQUATION OF STATE LEADING TO THE TRICOMI EQUATION

This reduction, which consisted essentially of changes in the dependent and independent variables and an approximation of $\frac{1 - M^2}{\rho^2}$ to the second-order term in its power-series expansion in terms of s^*, may be regarded as producing the exact equation of flow for a gas having a somewhat altered pressure-density relation. (This would correspond to viewing the Kármán-Tsien or the Chaplygin approximation for the hodograph relations as the exact equations for a gas having a pressure-density relation of the form $p = A + \frac{B}{\rho}$.) Using Bernoulli's equation in differential form, the equation

$$\frac{1 - M^2}{\rho^2} = v^2 + \frac{dv}{ds^*} \tag{10}$$

is obtained, where $v = 1/\rho$. If the approximation $\omega^*(s^*)$ for $\frac{1 - M^2}{\rho^2}$ is now used, a differential equation for v is obtained whose solution yields the relation for ρ as a function of s^* for the new gas. Although the equation is nonlinear, it is of the Riccati type, and can be solved either by expressing v as a power series in s^* or in closed form by making a suitable substitution for the dependent variable. Since the behavior of the flow near $s^* = 0$ is desired, the power-series expansion was developed and it was found that the specific volume could be expressed as

$$\frac{1}{\rho} = 1.5774 - 2.4883s^* + 8.6354(s^*)^2 - 30.175(s^*)^3 + 88.024(s^*)^4 - 223.32(s^*)^5 + \ldots \tag{11}$$

The first four terms agree with a similar expansion of $1/\rho$ for the isentropic pressure-density relation with $\gamma = 1.4$.

Using the relation between s^* and q (equation (11)) and a Maclaurin expansion for $1/q$, the series expression for the velocity is

$$\frac{1}{q} = 1.0954 + 1.7280s^* + 1.7199(s^*)^2 - 3.9340(s^*)^4 + 6.7097(s^*)^5 - 9.5639(s^*)^6 + \ldots \tag{12}$$

From this and Bernoulli's equation an expression may be obtained for the pressure as follows:

$$p - p_s = 0.8333s^* - 1.3145(s^*)^2 + 0.6912(s^*)^3 - 0.3816(s^*)^4 + 1.0939(s^*)^5 - 1.3948(s^*)^6 + \ldots \quad (13)$$

Equations (11) and (13) then constitute the pressure-density relation leading to the Tricomi equation. These expressions give agreement with the usual isentropic relations up to the third derivative of p with respect to ρ at the sonic velocity.

However, the use of a power series to obtain $1/\rho$ is hampered by the fact that the series converges very slowly. As an alternative method, the Riccati equation may be solved in closed form. As usual this involves a substitution of the form $v = \frac{1}{\rho} = \frac{u'}{u}$ in equation (10), which yields

$$(1 + cs^*)^5 u'' = d \times s^* u$$

where $c = 1.2304$ and $d = 9.4204$ are substituted for convenience. Letting

$$u = (1 + cs^*)w(\eta)$$

where

$$\eta = \frac{d^{1/3} s^*}{1 + cs^*}$$

leads to the differential equation $w_{\eta\eta} - \eta w = 0$. This is the well-known Airy equation and a linear combination of its solutions together

with boundary conditions on $1/\rho$, that is, at $s^* = 0$ and $\frac{1}{\rho} = 1.57744$, provides the result

$$\frac{1}{\rho} = \frac{1.2304}{1 + 1.2304s^*} + \frac{2.4176}{(1 + 1.2304s^*)^2} \frac{\left[\xi^{2/3} I_{-2/3}(\xi) - 0.1963\xi^{2/3} I_{2/3}(\xi)\right]}{\left[\xi^{1/3} I_{1/3}(\xi) - 0.1963\xi^{1/3} I_{-1/3}(\xi)\right]}$$

where

$$\xi = \left(\frac{1.6118s^*}{1 + 1.2304s^*}\right)^{3/2}$$

and $I_\nu(\xi)$ is a Bessel function with imaginary argument in the notation of Watson (reference 15). The quantity ξ becomes imaginary for supersonic values of speed ($s^* < 0$), and if $\tilde{\xi}$ is defined by $\tilde{\xi} = \left(\frac{-1.6118s^*}{1 + 1.2304s^*}\right)^{3/2}$ the relation for $1/\rho$ in the supersonic range is

$$\frac{1}{\rho} = \frac{1.2304}{1 + 1.2304s^*} + \frac{2.4176}{(1 + 1.2304s^*)^2} \frac{\left[\tilde{\xi}^{2/3} J_{-2/3}(\tilde{\xi}) - 0.1963\tilde{\xi}^{2/3} J_{2/3}(\tilde{\xi})\right]}{\left[-\tilde{\xi}^{1/3} J_{1/3}(\tilde{\xi}) - 0.1963\tilde{\xi}^{1/3} J_{-1/3}(\tilde{\xi})\right]}$$

where $J_\nu(\tilde{\xi})$ is the Bessel function of the first kind. The functions q and M may also be computed in the same manner and they are found to have the form

$$q = \frac{-0.1667}{(1 + 1.2304s^*)\left[\xi^{1/3} I_{1/3}(\xi) - 0.1963\xi^{1/3} I_{-1/3}(\xi)\right]}$$

$$M^2 = 1 - \frac{9.4204s^*\rho^2}{(1 + 1.2304s^*)^5}$$

Thus again the state quantities have been found for the present hypothetical gas in terms of s^*. This pressure-density relation has been compared graphically with the usual isentropic relation for $\gamma = 1.4$ (see fig. 5). The curves are almost coincident in the range $\rho = 0.42$ to $\rho = 0.72$. This corresponds to a Mach number range of 1.35 to 0.85. For values of s^* near zero, the curves plotted for Mach number as a function of s^* for the two pressure-density relations agree quite closely (see fig. 6). For large values of s^*, however, the approximation curve departs from the usual isentropic curve $p = \rho^\gamma/\gamma$, since the slope of the hypothetical pressure-velocity relation decreases more rapidly than that for the isentropic curve (see fig. 5).

HODOGRAPH SOLUTIONS YIELDING ACCELERATED-DECELERATED CHANNEL FLOW

Solutions for the Tricomi equation (7) which produce an unsymmetrical accelerated-decelerated channel have been given by Tomotika and Tamada (reference 3), Falkovich (reference 16), and Ehlers (reference 13). Consider a solution which has the form

$$\psi_0(s,\theta) = \frac{1}{\omega_1}\left[\left(\sqrt{\theta^2 + \frac{4}{9}s^3} - \theta\right)^{1/3} - \left(\sqrt{\theta^2 + \frac{4}{9}s^3} + \theta\right)^{1/3}\right] \tag{14}$$

Tomotika and Tamada noticed that this function remained a solution when $(\theta - i\lambda)$ is substituted for θ. In fact, if the function

$$\psi_1(s,\theta) = \operatorname{Re}\frac{1}{\omega_1}\left\{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3} - \left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{1/3}\right\} \tag{15}$$

is considered, a stream function results which has a branch-point singularity of the one-half power at $s = \left(\frac{9}{4}\lambda^2\right)^{1/3} = s_o$ and $\theta = 0$. This branch point in the subsonic stream may be said to "generate" a channel flow with locally supersonic regions having a Mach number at the narrowest cross section determined by λ. Of the other properties of the function it may be observed that, when $s > s_o$ and $\theta = 0$, then $\psi_1 = 0$. It would appear that $\psi_1(s,\theta)$ has a line of one-third-order branch points along $s = 0$. However, this branch point may be eliminated by writing the function in the form

$$\omega_1\psi_1 = \mathrm{Re}\left\{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3} - \frac{\left[(\theta - i\lambda)^2 + \frac{4}{9}s^3 - (\theta - i\lambda)^2\right]^{1/3}}{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3}}\right\} \tag{16}$$

$$\omega_1\psi_1 = \mathrm{Re}\left\{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3} - \frac{\left(\frac{4}{9}\right)^{1/3}s}{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3}}\right\}$$

$$= \left[\beta_1 - \frac{\left(\frac{4}{9}\right)^{1/3}s}{\beta_1}\right]\cos\frac{\Omega}{3} \tag{17}$$

where

$$\left.\begin{aligned}
\beta_1 &= \left[\left(\alpha \cos\frac{\omega}{2} - \theta\right)^2 + \left(\alpha \sin\frac{\omega}{2} + \lambda\right)^2\right]^{1/6} \\
\alpha &= \sqrt[4]{\left(\frac{4}{9}s^3 - \lambda^2 + \theta^2\right)^2 + 4\lambda^2\theta^2} \\
\Omega &= \tan^{-1}\frac{\alpha \sin\frac{\omega}{2} + \lambda}{\alpha \cos\frac{\omega}{2} - \theta} \\
\omega &= \tan^{-1}\frac{-2\lambda\theta}{\frac{4}{9}s^3 - \lambda^2 + \theta^2}
\end{aligned}\right\} \qquad (18)$$

Since β_1 does not vanish (the choice of sign for the one-half-order branch point is such that this is insured) it can be seen that the singularities along $s = 0$ are removable. The observation may be made that

$$\Psi_1(s,\theta) + \Psi_1(s,-\theta) = \bar{\Psi}(s,\theta) \qquad (19)$$

is a symmetrical function of θ, and it is this function which shall be used as the stream function. The function $\bar{\Psi}$ has been plotted in figure 7 for $\lambda = 0.050912$. This corresponds to $s = 0.18$ at the center of the channel or a maximum Mach number on the center streamline of 0.86. For sufficiently large negative values of s (supersonic flow), $\frac{\partial\bar{\Psi}}{\partial\theta}$ becomes zero at a point above the $\theta = 0$ axis as well as at $\theta = 0$. This indicates the flow in the supersonic region will have streamlines along which the velocity gradient changes sign more than once. This can be shown from the following analysis. A series expansion of $\bar{\Psi}_\theta$ for constant s about $\theta = 0$ has the form

$$\bar{\Psi}_\theta = \left(\bar{\Psi}_{\theta\theta}\right)_{\theta=0}\theta + O\left(\theta^2\right)$$

From the hodograph, it is apparent that for $s > 0.18$ the derivative Ψ_θ is positive for $\theta > 0$. Along a streamline for which Ψ_θ does not vanish for a value of $\theta > 0$, the value of $\Psi_{\theta\theta}$ then will be positive in the neighborhood of the line $\theta = 0$ since $\Psi_\theta \geqq 0$ for $\theta \geqq 0$. The limiting streamline of those along which the velocity gradient does not change sign more than once is the streamline which passes through the point on the s-axis for which $\Psi_{\theta\theta}$ vanishes. This was found to be $\Psi = 0.762$ for $s = -0.132$. For $s < -0.132$ and hence $\Psi > 0.762$ the value of Ψ_θ for $\theta > 0$ in the neighborhood of the line $\theta = 0$ is negative; for $s > -0.132$ and $\Psi < 0.762$, Ψ_θ is positive. (See fig. 8(a).) The resulting effects on the constant-velocity curves in the physical plane are shown in figure 8(b). It is apparent from the figure that $\frac{\partial \Psi}{\partial \theta} = 0$ does not occur along the characteristics passing through $s = 0$ and $\theta = 0$. This remark is made at this point since in the limiting case as $\lambda \longrightarrow 0$ this characteristic does become the locus of the changes in sign of velocity gradient. This is discussed in more detail in the section Limiting Case When Sonic Velocity Is Just Reached on Center of Streamline.

The breakdown of isentropic, potential flows is usually associated with the appearance of limit lines. Some remarks have already been made about criterions for breakdown, but here it will be sufficient to remark that Tollmein (reference 17) and Tsien (reference 18) have shown that inviscid potential flows cannot be continued across limit lines. Limit lines occur when the Jacobian of the transformation from the hodograph to the physical plane

$$J\left(\frac{x,y}{s,\theta}\right) = \frac{\omega_1^2}{\rho q^2}\left(s\Psi_\theta^2 + \Psi_s^2\right) \tag{20}$$

becomes zero. This may only occur for $s < 0$ since s, Ψ_θ, and Ψ_s are real and the Jacobian is always positive for $s > 0$. Geometrically, a necessary and sufficient condition that limit lines occur is that the streamlines and the characteristics be tangent in the hodograph plane, provided the derivatives Ψ_θ and Ψ_s remain finite. As a matter of fact in this case the limit lines are the locus of the points of tangency.

The hodograph of the function Ψ was tested graphically for limit lines by seeking points of tangency of the streamlines using a "portable" characteristic curve, drawn on tracing paper and moved across the supersonic field. From figure 7 it is apparent that no limit lines occur for

that portion of the s,θ field which has been computed. Using this test, it is not certain that limit lines do not appear somewhere else in the supersonic portion of the flow, but since the choice of bounding streamline has so far been left open this is not an important problem.

CALCULATION OF PHYSICAL COORDINATES

To obtain the coordinates of the flow in the physical plane the condition of irrotationality and continuity may be put into the following form:

$$d\varphi = u\,dx + v\,dy$$

$$\frac{i}{\rho}\,d\psi = (-v\,dx + u\,dy)i$$

These equations may then be written as

$$dz = \frac{e^{i\theta}}{q}\left(d\varphi + \frac{i}{\rho}\,d\psi\right) \tag{21}$$

Expressing φ and ψ as functions of s and θ, and then replacing the partial derivatives of φ with those of ψ by equation (4), the following differential expression for the complex position coordinate is obtained:

$$dz = \frac{1}{q}\,e^{i\theta}\left(-\omega_1{}^2\psi_s + \frac{i}{\rho}\,\psi_\theta\right)d\theta + \left(s\omega_1{}^2\psi_\theta + \frac{i}{\rho}\,\psi_s\right)ds \tag{22}$$

With this formula it is possible to show that if ψ is an even function of θ the flow pattern in the physical plane is symmetric. For this purpose, it is first of all necessary to prove that the line $\theta = 0$ in the hodograph plane maps into a straight line in the physical plane. Setting $\theta = 0$ and $d\theta = 0$ in equation (22) leads to

$$dz = \frac{1}{q}\left(s\omega_1{}^2\psi_\theta + \frac{i}{\rho}\,\psi_s\right)_{\theta=0} ds \tag{23}$$

The stream function $\bar{\Psi}$ in equation (19) is an even function of θ; hence $\bar{\Psi}_\theta \equiv 0$ for $\theta = 0$. Then

$$dz = \frac{i}{\rho q} \bar{\Psi}_s \Big]_{\theta=0} ds = \frac{i}{\rho q} d\bar{\Psi} \Big]_{\theta=0} \tag{24}$$

This is purely imaginary; and $\theta = 0$ then is seen to map into the y-axis. If the integration for the streamlines is carried out along each streamline from the point $\theta = 0$, the obvious differential relations

$$d\Psi = 0 = \Psi_s \, ds + \Psi_\theta \, d\theta$$

$$ds = - \frac{\Psi_\theta}{\Psi_s} d\theta$$

lead to the simple formula, where ψ = Constant,

$$z = - \int_0^\theta \frac{\omega_1^2}{q} e^{i\theta} \left(\Psi_s + \frac{s\Psi_\theta^2}{\Psi_s} \right) d\theta \tag{25}$$

Since (see preceding section) the Jacobian of the transformation from the s,θ- to the x,y-plane is never zero, and since it may be represented as

$$J\left(\frac{x,y}{s,\theta}\right) = \frac{\omega_1^2}{\rho q^2} \left(\Psi_s^2 + s\Psi_\theta^2 \right)$$

Equation (25) becomes

$$z = - \left(\int_0^\theta \frac{\rho q e^{i\theta}}{\Psi_s} J \, d\theta \right)_{\psi=\text{Constant}}$$

or

$$x = \left(- \int_0^{\theta} \frac{\rho q \cos \theta \, J}{\Psi_s} \, d\theta \right)_{\psi=\text{Constant}}$$

$$y = \left(- \int_0^{\theta} \frac{\rho q \sin \theta \, J}{\Psi_s} \, d\theta \right)_{\psi=\text{Constant}}$$

Now for Ψ in equation (19), the relation $\rho(\theta)q(\theta) = \rho(-\theta)q(-\theta)$ along any single streamline, since Ψ_s and Ψ are even in θ, implies that the same value of s occurs for θ and $-\theta$. Thus for x the integrand is a product of four even functions of θ: J, ρq, Ψ_s, and $\cos \theta$. Therefore, x itself is odd. In a similar manner, y is seen to be an even function of θ. Since x is an odd function of θ for a given value of s and y is an even function of θ, then the channel is symmetric about the line $\theta = 0$.

The mapping from the hodograph plane back to the physical plane may be accomplished by integrating equations (21) or (22) along either constant ψ or constant s lines. Since either one of these integrations must be carried out by numerical or graphical methods, the choice depends on the method involving the least amount of labor.

If the integration is performed along a streamline, then from equation (21) the physical coordinates are found by integrating

$$dz = \frac{1}{q} e^{i\theta} \, d\varphi$$

If φ were known as a function of s and θ along each streamline this would yield a very simple method for computing the flow. However, the differential $d\varphi$ may be replaced by the differential of φ' with an additional term involving ψ and ω_1. Multiplying the second of equations (5) by ds and the third by $d\theta$ and adding the two relations lead to

$$d\varphi' = - \frac{1}{\omega_1} \, d\varphi + \frac{d\omega_1}{ds} \, \psi \, d\theta$$

Thus the physical coordinates are found by integrating

$$dz = -\frac{\omega_1}{q} e^{i\theta} d\varphi' + \frac{a\omega_1}{q} e^{i\theta}\psi \, d\theta \tag{26}$$

along ψ = Constant.

Now the function φ' can be computed from equations (6) when ψ' is known. Differentiating ψ_1' with respect to s gives

$$\psi_{1s}' = \mathrm{Re}\left\{\frac{1}{3}\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{-2/3}\left[\frac{2s^2}{3\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}}\right] - \frac{1}{3}\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{-2/3}\left[\frac{2s^2}{3\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}}\right]\right\}$$

This expression may be simplified by multiplying the first term inside the braces by unity in the form

$$\frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{2/3}}{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{2/3}}$$

and by multiplying the second term by a similar factor with the positive sign before $(\theta - i\lambda)$ changed to minus. This leads to

$$\psi_{1s}' = \operatorname{Re} \frac{1}{3}\left(\frac{9}{4}\right)^{2/3} \left\{ \frac{2}{3} \frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{2/3}}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} - \right.$$

$$\left. \frac{2}{3} \frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{2/3}}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} \right\}$$

Since $\psi_{1s}' = \varphi_\theta'$, the potential φ' becomes

$$\varphi' = \frac{1}{3}\left(\frac{9}{4}\right)^{2/3} \operatorname{Re} \left\{ \left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{2/3} + \right.$$

$$\left. \left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{2/3} \right\} + f(s) \qquad (28)$$

The arbitrary function of s results from the integration with respect to θ. That $f(s) = 0$ will be readily apparent for the functions ψ' and φ' must obey the relation

$$\varphi_s' = -s\psi_\theta$$

Thus, differentiating equation (28) with respect to s yields

$$\varphi_s' = \frac{1}{3}\left(\frac{9}{4}\right)^{2/3} \operatorname{Re} \left\{ \frac{2}{3}\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{-1/3} \frac{2s^2}{3\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} + \frac{2}{3}\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{-1/3} \frac{2s^2}{3\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} \right\}$$

Employing the same procedure as on ψ_s' leads to

$$\varphi_s' = \operatorname{Re} \frac{1}{3} \left\{ s \frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3}}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} + \frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{1/3}}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} \right\}$$

The expression on the right side can be readily recognized as $-s\psi_\theta'$ as required. The potential φ' corresponding to ψ' given by equation (17) is then

$$\varphi' = \frac{1}{3}\left[\frac{s^2}{\beta_1^2} + \left(\frac{9}{4}\right)^{2/3}\beta_1^2\right] \cos\frac{2\Omega}{3} \tag{29}$$

If for constant s the function ψ' is an even function of θ, then φ' is an odd function. This is apparent from equation (6) above, since differentiation of an odd function of θ with respect to θ results in an even function. Thus, the conjugate potential Φ' corresponding to Ψ is

$$\Phi' = \varphi'(s,\theta) - \varphi'(s,-\theta)$$

$$= \frac{1}{3}\left[\frac{s^2}{\beta_1^2} + \left(\frac{9}{4}\right)^{2/3}\beta_1^2\right]\left[\cos\frac{2\Omega}{3} - \cos\frac{2}{3}\left(\pi - \frac{\Omega}{3}\right)\right]$$

$$= \frac{\sqrt{3}}{3}\left[\frac{s^2}{\beta_1^2} + \left(\frac{9}{4}\right)^{2/3}\beta_1^2\right]\sin\left(\frac{\pi}{3} - \frac{2\Omega}{3}\right) \qquad (30)$$

Here use has been made of the fact that β_1 is an even function of θ. In order to perform the actual integration, the streamline and constant φ' lines must be plotted in the hodograph plane. Then for each streamline the quantities $(\omega_1/q)\cos\theta$ and $(\omega_1/q)\sin\theta$ are plotted against Φ' and also against θ. The integrations then are performed with the aid of a planimeter.

A symmetric channel flow has been computed for $s_o = 0.18$ ($\lambda = 0.0509117$), which corresponds to a Mach number of 0.86 at the center of the throat section (see figs. 8(a) and 8(b)).

VELOCITY DISTRIBUTION ON CENTER STREAMLINE $\Psi = 0$

The streamline $\Psi = 0$ must be treated separately. The position coordinates are computed by integrating equation (22), which for this case reduces simply to

$$\frac{dx}{ds} = \frac{s\omega_1^2}{q}\Psi_\theta$$

since $\Psi_s = 0$ and $d\theta \equiv 0$. The derivative with respect to θ is found by differentiating the stream function in the form of equation (15). This leads to

$$\psi_{1\theta} = -\frac{Re}{3\omega_1}\left\{\frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3}}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} + \frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^{1/3}}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}}\right\} \tag{31}$$

or

$$\psi_{1\theta} = -\frac{1}{3\omega_1}Re\left\{\frac{\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3}}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}} + \frac{\left(\frac{4}{9}\right)^{1/3}s}{\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3}\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]^{1/3}}\right\} \tag{32}$$

Finally,

$$\psi_{1\theta} = -\frac{1}{3\omega_1\alpha}\left[\beta_1 \cos\left(\frac{\Omega}{3} - \frac{\omega}{2}\right) + \frac{\left(\frac{4}{9}\right)^{1/3}s}{\beta_1}\cos\left(\frac{\Omega}{3} + \frac{\omega}{2}\right)\right] \tag{33}$$

The quantities α, ω, β_1, and Ω are defined in equation (18). If θ goes to zero through positive values with $s > \left(\frac{9}{4}\lambda^2\right)^{1/3}$, then $\omega = 2\pi$ and

$$\psi_{1\theta}(0) = \frac{1}{3}\frac{2}{\omega_1\alpha_o}\left(\frac{4}{9}s^3\right)^{1/6}\cos\left(\frac{\pi}{3} - \frac{\delta}{3}\right)$$

where $\delta = \tan^{-1}\frac{\lambda}{\alpha_o}$ and $\alpha_o = \sqrt{\frac{4}{9}s^3 - \lambda^2}$. If θ goes to zero through negative values of θ for $s > \left(\frac{9}{4}\lambda^2\right)^{1/3}$, then $\omega = 0$ and

$$\psi_{1\theta}(-0) = -\frac{2}{3\omega_1\alpha_o}\left(\frac{4}{9}s^3\right)^{1/6}\cos\frac{\delta}{3}$$

The derivative of the stream function for the symmetrical flow then becomes

$$\Psi_\theta = \frac{\partial}{\partial\theta}\psi(\theta) + \frac{\partial}{\partial\theta}\psi(-\theta)$$

$$= \psi_\theta(\theta) - \psi_\theta(-\theta)$$

which for $\theta = 0$ reduces to

$$\left.\begin{aligned}\Psi_\theta &= \frac{2}{3\omega_1\alpha_o}\left(\frac{4}{9}s^3\right)^{1/6}\left[\cos\left(\frac{\pi}{3} - \frac{\delta}{3}\right) + \cos\frac{\delta}{3}\right] \\ \Psi_\theta &= \left(\frac{2}{3}\right)^{4/3}\frac{\sqrt{3}}{\omega_1\alpha_o}s^{1/2}\cos\left(\frac{\pi}{6} - \frac{\delta}{3}\right)\end{aligned}\right\} \quad (34)$$

With this expression substituted for Ψ_θ, the physical coordinates are found by integrating

$$dx = \sqrt{3}\left(\frac{2}{3}\right)^{4/3} \frac{s^{3/2}\omega_1}{q\alpha_0} \cos\left(\frac{\pi}{6} - \frac{\delta}{3}\right) ds \tag{35}$$

Unfortunately, $\alpha_0 = 0$ at $s = \left(\frac{3}{2}\lambda\right)^{2/3}$ and the quantity on the right-hand side becomes infinite so that numerical integration in this form is impossible. This can be remedied by noting that $\frac{d\delta}{ds} = -\frac{3}{2}\frac{\lambda}{s\alpha_0}$. With this substitution for α_0, equation (35) yields

$$\frac{dx}{d\delta} = -2\sqrt{3}\left(\frac{2}{3}\right)^{4/3} \frac{s^{5/2}\omega_1}{\lambda q} \cos\left(\frac{\pi}{6} - \frac{\delta}{3}\right) \tag{36}$$

The integration can be readily performed by plotting the right-hand side against δ and using a planimeter. The velocity distribution along $\Psi = 0$ as well as along the other streamlines is plotted in figure 9 for $\lambda = 0.050912$.

CURVES OF CONSTANT VELOCITY

The curves $q = \text{Constant}$ in the physical plane are found by integrating the relation

$$\frac{dz}{d\theta} = e^{i\theta}\left(-\frac{\omega_1^2\Psi_s}{q} + \frac{i}{\rho q}\Psi_\theta\right)_{q=\text{Constant}} \tag{37}$$

To study the nature of the curve in the neighborhood of the vertical axis of symmetry $x = 0$ ($\theta = 0$ and $s < 0.18$) it is convenient to use a power-series expansion about $\theta = 0$. The coefficients of such a series are readily obtained by differentiating successively the expression on the right with respect to θ and then setting $\theta = 0$. The expansion for z becomes

$$z - z_o = \left(\frac{dz}{d\theta}\right)_{\theta=0}\theta + \frac{1}{2!}\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0}\theta^2 + \frac{1}{3!}\left(\frac{d^3z}{d\theta^3}\right)_{\theta=0}\theta^3 + \cdots$$

$$= \left(-\frac{\omega_1^2}{q}\Psi_s\right)_{\theta=0}\theta + \frac{i}{2}\left(\frac{1}{\rho q}\Psi_{\theta\theta}\right)_{\theta=0}\theta^2 +$$

$$\frac{1}{6q}\left(-\omega_1^2\Psi_{s\theta\theta} + \omega_1^2\Psi_s - \frac{2}{\rho}\Psi_{\theta\theta}\right)_{\theta=0}\theta^3 + O(i\theta^4) \qquad (38)$$

since Ψ_θ and $\Psi_{s\theta}$ are zero at $\theta = 0$. The parametric representation of q = Constant in the neighborhood of $\theta = 0$ then has the form

$$x = A\theta + B\theta^3 + O(\theta^5)$$

$$y = C\theta^2 + O(\theta^4)$$

The slope of the line q = Constant at the axis of symmetry is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2C\theta + O(\theta^3)}{A + O(\theta^2)}$$

and $\frac{dy}{dx} = 0$ at $\theta = 0$. Now the curvature is given by

$$K = \frac{d^2y}{dx^2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2}$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2}\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \left[\frac{2C}{A} + O(\theta)\right]\left[\frac{1}{A} + O(\theta^2)\right]$$

which for $\theta = 0$ reduces to

$$K = \frac{2C}{A^2} = \left(\frac{q\Psi_{\theta\theta}}{\rho\omega_1{}^4\Psi_s{}^2}\right)_{\theta=0}$$

The sign of the curvature at the point $\theta = 0$ depends upon the sign of C which in turn depends on the sign of $\Psi_{\theta\theta}$. Now, the second derivative $\Psi_{\theta\theta}$ becomes zero for $s = -0.132$ and $\theta = 0$. For $s > -0.132$, the quantity $\Psi_{\theta\theta}$ is greater than zero and the curvature of the $s = \text{Constant}$ lines is convex to the center streamline of the channel (see fig. 8(b)). For $s < -0.132$, the quantity $\Psi_{\theta\theta}$ is less than zero for $\theta = 0$ and the curvature of the constant s line is concave to the center streamline of the channel. The curves $s = \text{Constant}$ for $0.18 > s > -0.132$ show a minimum along $x = 0$, while the curves $s = \text{Constant}$ for $s < -0.132$ exhibit a maximum at $x = 0$, and two minimums symmetrically placed about the y-axis. Along the streamlines which cut these latter curves, the fluid is accelerated to a maximum velocity and then is decelerated slightly to a minimum velocity at the throat. The fluid is accelerated to the maximum velocity before being decelerated finally to subsonic velocities again. This behavior is apparent from the velocity distribution along the streamlines $\Psi = 0.832$ and 0.920 in figure 9.

The curves $s = \text{Constant}$ for $s > 0.18$ which intersect the x-axis at 90° are concave to the incoming flow in the neighborhood of the straight streamline at the entrance side of the vertical axis of symmetry and convex to the incoming flow on the exit side. Further from the center streamline and toward the wall, however, the curvature changes and the constant-velocity lines become convex to the incoming flow on the entrance side and concave to the incoming flow on the downstream side of the vertical axis of symmetry (see fig. 8(b)).

Since the stream function has a branch point at $s = 0.18$ and $\theta = 0$, the above analysis cannot be applied to the curve of constant velocity $s = 0.18$. To investigate the nature of this curve in the neighborhood of the center streamline, it is necessary to obtain the expansion of equation (37) for small values of θ. The derivative $\Psi_{1\theta}$ is found from equation (33). Using the fact that $\Omega(-\theta) = \pi - \Omega(\theta)$ and $\omega(-\theta) = 2\pi - \omega(\theta)$ leads to

$$\Psi_\theta = -\frac{\sqrt{3}}{3\omega_1\alpha}\left[\beta_1 \cos\left(\frac{\Omega}{3} - \frac{\omega}{2} - \frac{\pi}{6}\right) + \frac{\left(\frac{4}{9}\right)^{1/3} s}{\beta_1} \cos\left(\frac{\Omega}{3} + \frac{\omega}{2} + \frac{\pi}{6}\right)\right] \tag{39}$$

where Ω, α, and ω are defined in equations (18). In a similar way by using equation (27) for $\psi_{1s}'(\theta)$ the following equations are obtained:

$$\left.\begin{aligned} \Psi_s &= \frac{1}{\omega_1}\left[\psi_{1s}'(\theta) + \psi_{1s}'(-\theta)\right] - \frac{1}{\omega_1}\frac{d\omega_1}{ds}\Psi \\ \Psi_s &= -\frac{2}{9\omega_1}\frac{\sqrt{3}}{\alpha}\left[\frac{s^2}{\beta_1^2}\sin\left(\frac{2\Omega}{3} + \frac{\omega}{2} - \frac{\pi}{3}\right) - \right. \\ &\qquad \left.\left(\frac{9}{4}\right)^{2/3}\beta_1^2 \sin\left(\frac{2\Omega}{3} - \frac{\omega}{2} - \frac{\pi}{3}\right)\right] - \frac{a}{\omega_1}\Psi \end{aligned}\right\} \tag{40}$$

Letting $s = 0.18$ in equations (39) and (40) leads to some simplification since for this case $\frac{4}{9}s^3 = \lambda^2$. Accordingly, for values of θ positive and near zero,

$$\left.\begin{aligned} \alpha &= \theta^{1/2}\left(\theta^2 + 4\lambda^2\right)^{1/4} \approx (2\lambda\theta)^{1/2} + O\left(\theta^{5/2}\right) \\ \omega &= \tan^{-1}\left(\frac{-2\lambda}{\theta}\right) \approx \frac{3}{2} + \frac{\theta}{2\lambda} + O\left(\theta^2\right) \\ \Omega &= \tan^{-1}\frac{\sqrt{\frac{\alpha^2 - \theta^2}{2}} + \lambda}{-\sqrt{\frac{\alpha^2 + \theta^2}{2}} - \theta} \approx \frac{\pi}{2} + \left(\frac{\theta}{\lambda}\right)^{1/2} + O\left(\theta^{3/2}\right) \\ \beta_1 &= \left[\left(\sqrt{\frac{\alpha^2 + \theta^2}{2}} + \theta\right)^2 + \left(\sqrt{\frac{\alpha^2 + \theta^2}{2}} + \lambda\right)^2\right]^{1/6} \\ \beta_1 &= \lambda^{1/3}\left[1 + \frac{1}{3}\left(\frac{\theta}{\lambda}\right)^{1/2} - \frac{1}{18}\left(\frac{\theta}{\lambda}\right) + O\left(\theta^{3/2}\right)\right] \end{aligned}\right\} \tag{41}$$

The approximate values of the above quantities lead finally to

$$\Psi_\theta = \frac{1}{\sqrt{3}} \frac{\theta^{-1/2}}{\omega_1 \lambda^{1/6}} \left[1 + \frac{5}{36} \frac{\theta}{\lambda} + O\left(\theta^{3/2}\right) + \ldots \right]$$

$$= 0.72902\theta^{-1/2} \left[1 + 2.728\theta + O\left(\theta^{3/2}\right) + \ldots \right] \tag{42}$$

$$\Psi_s = - \frac{\sqrt{3}\lambda^{1/6}}{2} \left(\frac{4}{9}\right)^{1/3} \frac{\theta^{-1/2}}{\omega_1} - \left[\frac{2a}{\sqrt{3}\omega_1^2} \lambda^{-1/6} + \frac{7\sqrt{3}}{36} \left(\frac{4}{9}\right)^{1/3} \frac{1}{\omega_1 \lambda} \right] \theta^{1/2}$$

$$= -0.3093\theta^{-1/2} - 5.877\theta^{1/2} + O\left(\theta^{3/2}\right) \tag{43}$$

With these expansions and equation (37), the slope of the curve $s = 0.18$ in the neighborhood of $\theta = 0$ is given by

$$\frac{dy}{dx} = 1.948 - 39.53\theta + O(\theta^2)$$

The numerical value of the slope at $\theta = 0$ is finally

$$\frac{dy}{dx} = 1.948$$

Now the curvature is proportional to the second derivative $\frac{d^2y}{dx^2}$ which for the present case is given by

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) \frac{d\theta}{dx} \propto \theta^{1/2} + O(\theta)$$

The curvature of the $s = 0.18$ line is seen to be zero where it cuts the center streamline.

LIMITING CASE WHEN SONIC VELOCITY IS JUST REACHED ON CENTER OF STREAMLINE

Setting $\lambda = 0$ in the stream function in equation (15) leads directly to the stream function considered for the flow through a Laval nozzle in reference 13. From this solution it is possible, however, to construct a channel having locally supersonic regions by the appropriate choice of branches. Tomotika and Tamada (reference 3) constructed a channel having $M = 1$ at the center streamline using a solution in the ψ,φ-plane. Their solution for this limiting case can be shown to be the same as equation (14) with $\lambda = 0$ when expressed in the hodograph variables. Now equation (6) when expressed in φ' and ψ' as the independent variables becomes

$$\left.\begin{aligned} \theta_{\varphi'} &= s_{\psi'} \\ \theta_{\psi'} &= -s s_{\varphi'} \end{aligned}\right\} \tag{44}$$

If s and φ' were replaced by $-s$ and $-\varphi'$ the equation takes the form given by Tomotika and Tamada (reference 3) for $K = \frac{1}{2}$. Their solutions for equation (44) are then

$$\left.\begin{aligned} \frac{s}{2} &= -z - 2(\psi')^2 \\ \frac{\theta}{2} &= 2\psi'\left[z - 2\varphi' + \frac{2}{3}(\psi')^2\right] \end{aligned}\right\} \tag{45}$$

where z must satisfy

$$(z - 2\mu)^2(z + \mu) = a^3$$

and $\mu = (\psi')^2 - \varphi'$. When $a = 0$ it is quite simple to eliminate z and φ' from the equations above and to obtain ψ' as a function

of s and θ. Setting the factor $z - 2\mu$ equal to zero and eliminating φ' and z with the aid of equation (45) leads to

$$(\psi')^3 + \frac{3}{16}\psi' s + \frac{3}{64}\theta = 0$$

The resulting function ψ is

$$2\left(\frac{2}{3}\right)^{1/3}\psi = -\frac{1}{2\omega_1}\left(\theta - \sqrt{\theta^2 + \frac{4}{9}s^3}\right)^{1/3} - \frac{1}{2\omega_1}\left(\theta + \sqrt{\theta^2 + \frac{4}{9}s^3}\right)^{1/3} \tag{46}$$

Similarly, the relation $z + \mu = 0$ yields

$$(\psi')^3 + \frac{3\psi' s}{4} - \frac{3\theta}{8} = 0$$

whose solution leads to the stream function

$$2\left(\frac{3}{2}\right)^{1/3}\psi = \frac{1}{\omega_1}\left(\theta - \sqrt{\theta^2 + \frac{4}{9}s^3}\right)^{1/3} + \frac{1}{\omega_1}\left(\theta - \sqrt{\theta^2 + \frac{4}{9}s^3}\right)^{1/3} \tag{47}$$

Equations (46) and (47) express the two branches of the stream function which go to make up the accelerated-decelerated flow through a nozzle. Equation (46) may be obtained from equation (47) by inserting unity in the form of $e^{2\pi i}$ inside of the parentheses on the right-hand side of equation (47) and taking the real part. For the flow computed by Tomotika and Tamada, figure 10 shows the regions of the hodograph plane in which the two branches represented by equations (46) and (47) were used.

To obtain a flow which is symmetric about the line $\theta = 0$ as well as about the straight streamline the stream function given by

$$\Psi'(\theta) = \frac{2}{3}\left[\psi(\theta) + \psi(-\theta)\right] \tag{48}$$

shall be considered. In this formula $\psi(\theta)$ is given by equation (47) and $\psi(-\theta)$ by equation (46) if $\theta > 0$ and $\theta^2 > \frac{4}{9}(-s)^3$. In the region between the characteristic $\theta = \pm\frac{2}{3}(-s)^{3/2}$, $\psi(\theta)$ and $\psi(-\theta)$ are both given by equation (47). The choice of the factor $2/3$ was made so that, in the region of the upper quarter plane $\theta > 0$ and $s > 0$ and the section in the left quarter plane where $\theta > \frac{2}{3}(-s)^{3/2}$, the stream function is simply given by

$$\Psi = \frac{1}{\omega_1}\left[\left(\theta - \sqrt{\theta^2 + \frac{4}{9}s^3}\right)^{1/3} + \left(\theta + \sqrt{\theta^2 + \frac{4}{9}s^3}\right)^{1/3}\right] \tag{49}$$

The physical plane for this portion of the hodograph has already been computed by Ehlers (reference 13).

The relation for the stream function in the region between the characteristics $\theta = \pm\frac{2}{3}(-s)^{3/2}$ is

$$\Psi = \frac{4}{3\omega_1}\left(\frac{2}{3}\right)^{1/3}(-s)^{1/2}\left[\cos\frac{\delta(\theta)}{3} + \cos\frac{\delta(-\theta)}{3}\right] \tag{50}$$

where

$$\delta = \cos^{-1}\frac{\theta}{\frac{2}{3}(-s)^{3/2}}$$

$$\delta(\theta) = \pi - \delta(-\theta)$$

$$\theta > 0$$

Hence, with the aid of a simple trigonometric identity the stream function

$$\Psi = \frac{2^{7/3}3^{1/6}}{3\omega_1}(-s)^{1/2}\cos\left(\frac{\pi}{6} - \frac{\delta}{3}\right) \tag{51}$$

is obtained. The streamlines in the hodograph plane for the entire flow given by equation (48) are shown in figure 11(a).

To compute the coordinates for the curves of constant speed near the y-axis the relation

$$z = \int_{\theta=0}^{\theta} \frac{1}{q} e^{i\theta}\left(-\omega_1{}^2\Psi_s + \frac{1}{\rho}\Psi_\theta\right) d\theta$$

was employed. By a MacLaurin expansion, for small values of θ, the x coordinate can be computed from

$$x = \frac{1}{q}\left(-\frac{\omega_1{}^2}{2s} + \omega_1\frac{d\omega_1}{ds}\right)\Bigg]_{\theta=0}\Psi_o\theta + O(\theta^3)$$

where Ψ_o is the value of the stream function when $\theta = 0$. The y coordinate is computed from

$$z - z_o = \int \frac{1}{q} e^{i\theta}\left(d\varphi + i\frac{d\Psi}{\rho}\right)$$

For small values of θ, this is approximated by

$$y - y_o = \frac{1}{\rho q}\left(\Psi - \Psi_o\right) + O(\theta\varphi)$$

where $\varphi = 0$ at $\theta = 0$. The flow in the physical plane is shown in figure 11(b).

The following paragraphs will give a discussion of the flow pattern. The derivatives Ψ_θ and Ψ_s of the stream function in equation (51) become infinite at the line of branch points $\theta \pm \frac{2}{3}(-s)^{3/2} = 0$. These curves are the characteristics in the hodograph plane which are cusped at the origin of coordinates. As these characteristics are approached along the branch of the stream function given by equation (49) Ψ_θ and Ψ_s remain finite; thus there is a jump in the first derivatives at the characteristics $\theta \pm \frac{2}{3}(-s)^{3/2} = 0$. Furthermore along the $\Psi = 0$ streamline, Ψ_s and Ψ_θ have the following values:

$$\Psi_\theta = \frac{(2/3)^{1/3}}{\omega_1 s}$$

$$\Psi_s \equiv 0$$

The quantity Ψ_θ is seen to be singular at the sonic point.

The effect of the above singularities on the character of the flow will now be investigated. From the equation transforming the solution from the hodograph to the physical plane, namely,

$$(dz)_{\Psi=\text{Constant}} = \frac{\omega_1^2}{q} e^{i\theta}\left(\frac{\Psi_s^2}{\Psi_\theta} + s\Psi_\theta\right) ds$$

and the relation

$$ds = \omega_1^2 \, ds^* = -\frac{\omega_1^2 \rho}{q} dq$$

the velocity gradient along a streamline can be expressed in the following form:

$$\left(\frac{dq}{dl}\right)_{\Psi=k} = -\frac{q^2}{\omega_1^4 \rho}\left(\frac{\Psi_\theta}{\Psi_s^2 + s\Psi_\theta^2}\right)$$

$$= -\frac{1}{\omega_1^2 \rho^2 J}\Psi_\theta$$

where J is the Jacobian of the transformation from the hodograph plane to the physical plane. As the Mach line corresponding to the characteristic $\theta = \frac{2}{3}(-s)^{3/2}$ is approached from the sonic point of the streamline, the velocity gradient approaches a finite value since Ψ_θ and J are both finite. As the same Mach line is approached from the direction of the $\theta = 0$ line, the quantities $J \longrightarrow \infty$ and $\Psi_\theta \longrightarrow \infty$ in such a way that $\left(\frac{dq}{dl}\right)_{\Psi=k}$ is finite but differs in value from $\left(\frac{dq}{dl}\right)_{\Psi=k}$ when approached from the other direction. The line $J = \infty$ then corresponds to a discontinuity in the velocity gradient along the streamline. For the center streamline $\Psi = 0$, the velocity gradient becomes

$$\frac{dq}{dx} = -\frac{q^2}{\omega_1^2 \rho s \Psi_\theta} \tag{52}$$

Now for $\Psi = 0$, $\Psi_\theta \propto \frac{1}{s}$ as $s \longrightarrow 0$. Since $s\Psi_\theta$ has opposite signs on the two sides of the slit corresponding to $\Psi = 0$ in the hodograph plane, and approaches a finite nonzero value as $s \longrightarrow 0$, there is a jump in the velocity gradient given by

$$\left(\frac{dq}{dx}\right)_{s=0} = \frac{(2q^*)^2}{(1.4532)^2 \rho^* \left(\frac{2}{3}\right)^{1/3}}$$

where q^* and ρ^* are the critical velocity and density, respectively.

To study the curvature of the streamlines, the relation

$$\left(\frac{dz}{d\theta}\right)_{\Psi=\text{Constant}} = -\frac{\omega_1^2}{q} e^{i\theta}\left(\Psi_s + \frac{s\Psi_\theta^2}{\Psi_s}\right)$$

is considered. In a manner similar to the computations of the velocity gradient along the streamline, the following result is obtained for the curvature:

$$\left(\frac{d\theta}{dl}\right) = -\frac{1}{\rho q}\frac{\Psi_s}{J} \tag{53}$$

This expression is discontinuous at the line $J = \infty$, since $\frac{\Psi_s}{J}$ approaches different values as the characteristic $(J = \infty)$ is approached from the two directions along the streamline. Thus, the Mach line $(J = \infty)$ is a curve at which the curvature of the streamlines as well as the velocity gradient is discontinuous.

SOME OTHER SOLUTIONS OF THE TRICOMI EQUATION

Tricomi (reference 11) has shown that a single-parameter family of solutions of the differential equation

$$\psi'_{ss} + s\psi'_{\theta\theta} = 0 \tag{54}$$

is given by

$$\psi' = \sigma^n F\left(-n, n + \frac{1}{3}; \frac{2}{3}; \frac{\sigma + \theta}{2\sigma}\right) \tag{55}$$

where $\sigma = \sqrt{\frac{4}{9}s^3 + \theta^2}$. Now it is possible to derive from this formula some solutions in closed form. In place of the hypergeometric function above, consider the following related functions (see p. 286, Whitaker and Watson (reference 19)):

$$\psi'_{1,n} = \sigma^n\left(-\frac{2\sigma}{\sigma + \theta}\right)^{n+1/3} F\left(n + \frac{1}{3}, n + \frac{2}{3}, 2n + \frac{4}{3}; \frac{2\sigma}{\sigma + \theta}\right) \tag{56}$$

and

$$\psi'_{2,n} = \sigma^n\left(-\frac{2\sigma}{\sigma + \theta}\right)^{-n} F\left(-n, -n + \frac{1}{3}; -2n + \frac{2}{3}; \frac{2\sigma}{\sigma + \theta}\right) \tag{57}$$

In $\psi'_{1,n}$ let $n + \frac{2}{3} = m$; and in $\psi'_{2,n}$ let $-n + \frac{1}{3} = m$. Then the solutions become

$$\left.\begin{aligned}\psi'_{1,m} &= \sigma^{m-2/3}\left(-\frac{2\sigma}{\sigma+\theta}\right)^{m-1/3} F\left(m-\frac{1}{3};\ m;\ 2m;\ \frac{2\sigma}{\sigma+\theta}\right)\\ \psi'_{2,m} &= \sigma^{-m+1/3}\left(-\frac{2\sigma}{\sigma+\theta}\right)^{m-1/3} F\left(m-\frac{1}{3};\ m;\ 2m;\ \frac{2\sigma}{\sigma+\theta}\right)\end{aligned}\right\} \tag{58}$$

Thus it is apparent that

$$\psi'_{1,m} = \sigma^{2m-1}\,\psi'_{2,m'} \tag{59}$$

for a given value of m.

In order to see how these solutions may be summed, consider the hypergeometric function written in series form; namely,

$$F\left(m-\frac{1}{3},m;\ 2m;\ z\right) = \frac{\Gamma(2m)}{\Gamma(m)\Gamma\left(m-\frac{1}{3}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+m-\frac{1}{3}\right)\Gamma(m+n)z^n}{\Gamma(2m+n)n!}$$

$$= \frac{\Gamma(2m)}{\Gamma(m)}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+m-\frac{1}{3}\right)z^n(m+n-1)(m+n-2)\ \ldots\ (n+1)}{\Gamma\left(m-\frac{1}{3}\right)(2m+n-1)!}$$

$$= \frac{\Gamma(2m)}{\Gamma(m)}\frac{d}{dz^{m-1}}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+m-\frac{1}{3}\right)}{\Gamma\left(m-\frac{1}{3}\right)}\frac{z^{n+m-1}}{(2m+n-1)!}$$

$$= \frac{\Gamma(2m)}{\Gamma(m)}\frac{d}{dz^{m-1}}\left\{\frac{1}{z^m}\underbrace{\int_0^z\int_0^z\ldots\int_0^z}_{2m-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+m-\frac{1}{3}\right)}{\Gamma\left(m-\frac{1}{3}\right)}\frac{z^n}{n!}\,dz^{2m-1}\right\}$$

The integrand can be recognized now as simply the binomial expansion of

$$(1 - z)^{1/3-m}$$

Thus the expression for ψ'_1 and ψ'_2 becomes

$$\psi'_{1,m} = \frac{\Gamma(2m)}{\Gamma(m)} \sigma^{m-2/3}\left(-\frac{2\sigma}{\sigma + \theta}\right)^{m-1/3} \times$$

$$\left\{\frac{d}{dz^{m-1}}\left[\frac{1}{z^m}\underbrace{\int_0^z \int_0^z \cdots \int_0^z}_{2m-1} (1 - z)^{1/3-m}(dz)^{2m-1}\right]\right\}_{z=\frac{2\sigma}{\sigma+\theta}} \tag{60}$$

$$\psi'_{2,m} = \sigma^{-2m+1}\psi_{1,m'} \tag{61}$$

The integrations involved in the above formula can be very easily performed. Putting $m = 1$ in equation (60) yields

$$\psi'_{1,1} = \sigma^{1/3}\left(-\frac{2\sigma}{\sigma + \theta}\right)^{2/3}\left[\frac{1}{z}\int_0^z (1 - z)^{-2/3}\, dz\right]_{z=\frac{2\sigma}{\sigma+\theta}}$$

$$\psi'_{1,1} = \frac{1}{6}\left[(\theta + \sigma)^{1/3} - (\theta - \sigma)^{1/3}\right]$$

Putting $m = 1$ into equation (61) yields

$$\psi'_{2,1} = \frac{1}{6\sigma}\left[(\theta + \sigma)^{1/3} - (\theta - \sigma)^{1/3}\right]$$

Putting $m = 2$ into equation (60) yields

$$\psi'_{1,2} = \frac{3}{4}\left[(\theta + \sigma)^{4/3} - (\theta - \sigma)^{4/3}\right] - \sigma\left[(\theta + \sigma)^{1/3} - (\theta - \sigma)^{1/3}\right]$$

Now $(\theta + \sigma)^{1/3}$ and $(\theta - \sigma)^{1/3}$ are separate solutions of the differential equation. The functions $\psi'_{2,1}$ can be constructed from solutions of the form $\psi'_{1,1}$ by differentiating with respect to θ. Such a process of successive differentiation yields the following solutions of the Tricomi equation:

$$\left.\begin{aligned}
\psi'_1 &= (\sigma + \theta)^{1/3} \\
\psi'_2 &= \frac{(\sigma + \theta)^{1/3}}{\sigma} \\
\psi'_3 &= \frac{(\sigma + \theta)^{1/3}}{\sigma^2} - \frac{3}{4}\frac{(\sigma + \theta)^{4/3}}{\sigma^3} \\
\psi'_4 &= \frac{27}{28}\frac{(\sigma + \theta)^{7/3}}{\sigma^5} - \frac{9}{4}\frac{(\sigma + \theta)^{4/3}}{\sigma^4} + \frac{(\sigma + \theta)^{1/3}}{\sigma^3} \\
\psi'_5 &= \frac{1}{\sigma^4}(\sigma + \theta)^{1/3} - \frac{9}{2}\frac{1}{\sigma^5}(\sigma + \theta)^{4/3} + \frac{135}{128}\frac{1}{\sigma^6}(\sigma + \theta)^{7/3} - \\
&\quad \frac{81}{56}\frac{1}{\sigma^7}(\sigma + \theta)^{10/3}
\end{aligned}\right\} \quad (62)$$

These five fundamental solutions, together with the set with $-\theta$ replacing θ may be combined to form the solutions corresponding to $m = 1$ in equation (60) and $m = 1, 2, 3,$ and 4 in equation (61), respectively. By equation (59), it is seen that $\sigma\psi'_2$, $\sigma^3\psi'_3$, $\sigma^5\psi'_4$, and $\sigma^7\psi'_5$ are also solutions of the differential equation. This may be verified by substituting them into the differential equation.

The functions ψ_2, ψ_3, ψ_4, and ψ_5 have the property that the point $s = 0$ and $\theta = 0$ is mapped into the point at ∞ in the physical plane. In fact the solutions all become indeterminate at that point. Thus, these solutions may be used to construct flows having a uniform sonic stream at infinity. If θ is replaced by $\theta - i\lambda$ and the real part is considered, then the point $\theta = 0$ and $\frac{2}{3}s^{3/2} = \lambda$ maps into infinity in the physical plane. Tomotika and Tamada used the stream function

$$\psi = \text{Re}\left[\psi_1(\theta - i\lambda) - \psi_1(-\theta + i\lambda) + \psi_2(\theta - i\lambda) + \psi_2(-\theta + i\lambda)\right]$$

to construct the flow over a cusped body having locally supersonic regions.

The solutions $\sigma\left[\psi_2(-\theta,s) - \psi_2(\theta,s)\right]$, $\sigma^3\left[\psi_3(-\theta,s) - \psi_3(\theta,s)\right]$, $\sigma^5\left[\psi_4(-\theta,s) - \psi_4(\theta,s)\right]$, and $\sigma^7\left[\psi_5(-\theta,s) - \psi_5(\theta,s)\right]$ may be used to construct channel flows having locally supersonic regions. Let

$$\mu = \left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} - (\theta - i\lambda)\right]$$

Since

$$\left[\sqrt{(\theta - i\lambda)^2 + \frac{4}{9}s^3} + (\theta - i\lambda)\right]^n = \frac{\left(\frac{4}{9}\right)^n s^{3n}}{\mu^n}$$

the appropriate solutions may be written

$$\psi^{(1)} = \frac{1}{\omega_1} \operatorname{Re} \left\{ \frac{3}{4} \left[\mu^{4/3} - \frac{\left(\frac{4}{9}\right)^{4/3} s^4}{\mu^{4/3}} \right] - \sigma(\lambda) \left[\mu^{1/3} - \frac{\left(\frac{4}{9}\right)^{1/3} s}{\mu^{1/3}} \right] \right\}$$

$$\psi^{(2)} = \frac{1}{\omega_1} \operatorname{Re} \left\{ \frac{27}{28} \left[\mu^{7/3} - \frac{\left(\frac{4}{9}\right)^{7/3} s^7}{\mu^{7/3}} \right] - \frac{9}{4} \sigma(\lambda) \left[\mu^{4/3} - \frac{\left(\frac{4}{9}\right)^{4/3} s^4}{\mu^{4/3}} \right] - \sigma^2(\lambda) \left[\mu^{1/3} - \frac{\left(\frac{4}{9}\right)^{1/3} s}{\mu^{1/3}} \right] \right\}$$

$$\psi^{(3)} = \frac{1}{\omega_1} \operatorname{Re} \left\{ \frac{81}{56} \left[\mu^{10/3} - \frac{\left(\frac{4}{9}\right)^{10/3} s^{10}}{\mu^{10/3}} \right] - \frac{135}{128} \sigma(\lambda) \left[\mu^{7/3} - \frac{\left(\frac{4}{9}\right)^{7/3} s^7}{\mu^{7/3}} \right] + \frac{9}{2} \sigma^2(\lambda) \left[\mu^{4/3} - \frac{\left(\frac{4}{9}\right)^{4/3} s^4}{\mu^{4/3}} \right] - \sigma^3(\lambda) \left[\mu^{1/3} - \frac{\left(\frac{4}{9}\right)^{1/3} s}{\mu^{1/3}} \right] \right\}$$

where $\sigma(\lambda) = \sqrt{(\theta - i\lambda)^2 + \frac{4}{9} s^3}$. Expressing the real parts in terms of the magnitudes and arguments of the complex quantities yields

$$\psi^{(1)} = \frac{1}{\omega_1} \left\{ \frac{3}{4} \left[\beta_1^4 - \frac{\left(\frac{4}{9}\right)^{4/3} s^4}{\beta_1^4} \right] \cos \frac{4\Omega}{3} - \alpha \left[\beta_1 - \frac{\left(\frac{4}{9}\right)^{1/3} s}{\beta_1} \right] \cos \left(\frac{\Omega}{3} + \frac{\omega}{2} \right) \right\} \quad (63)$$

$$\psi^{(2)} = \frac{1}{\omega_1}\left\{\frac{27}{28}\left[\beta_1^7 - \frac{\left(\frac{4}{9}\right)^{7/3}s^7}{\beta_1^7}\right]\cos\frac{7\Omega}{3} - \frac{9}{4}\alpha\left[\beta_1^4 - \frac{\left(\frac{4}{9}\right)^{4/3}s^4}{\beta_1^4}\right]\cos\left(\frac{4\Omega}{3}+\frac{\omega}{2}\right) - \alpha^2\left[\beta_1 - \frac{\left(\frac{4}{9}\right)^{1/3}s}{\beta_1}\right]\cos\left(\frac{\Omega}{3}+\omega\right)\right\} \tag{64}$$

and

$$\psi^{(3)} = \frac{1}{\omega_1}\left\{\frac{81}{56}\left[\beta_1^{10} - \frac{\left(\frac{4}{9}\right)^{10/3}s^{10}}{\beta_1^{10}}\right]\cos\frac{10\Omega}{3} - \frac{135}{128}\alpha\left[\beta_1^7 - \frac{\left(\frac{4}{9}\right)^{7/3}s^7}{\beta_1^7}\right]\cos\left(\frac{7\Omega}{3}+\frac{\omega}{2}\right) + \frac{9}{2}\alpha^2\left[\beta_1^4 - \frac{\left(\frac{4}{9}\right)^{4/3}s^4}{\beta_1^4}\right]\cos\left(\frac{4\Omega}{3}+\omega\right) - \alpha^3\left[\beta_1 - \frac{\left(\frac{4}{9}\right)^{1/3}s}{\beta_1}\right]\cos\left(\frac{\Omega}{3}+\frac{3\omega}{2}\right)\right\} \tag{65}$$

If θ goes to zero through positive or negative values for $s > \left(\frac{3}{2}\lambda\right)^{2/3}$ then ω equals 2π or zero, respectively. Hence $\beta_1 = \left(\frac{4}{9}\right)^{1/6} s^{1/2}$ and the stream functions vanish identically. For $s < \left(\frac{3}{2}\lambda\right)^{2/3}$, however, $\omega = \pi$ at $\theta = 0$ and $\beta_1 = \left(\sqrt{\frac{4}{9}s^2 - \lambda^2} + \lambda\right)^{1/3}$ and the stream functions reduce to non-trivial functions of s. Thus, the stream functions in equations (63), (64), and (60) will yield channel flows with locally supersonic regions. It should be pointed out that the solutions given by equation (60) for all integral values of $m > 1$ seem to be reducible to the form

$$\psi_{m'} = \sigma^{2m-1} \frac{\partial^m}{\partial\theta^m}\left[(\sigma + \theta)^{1/3}\right]$$

The solutions for higher values of m than those given in equation (62) have a similar form and may be combined to give channels having locally supersonic regions. It is then apparent that it is possible to approximate quite closely the flow through a given nozzle by taking a sum of these solutions, providing the Jacobian of the transformation to the physical plane does not vanish.

The stream functions given in equations (63), (64), and (65) with $\lambda = 0$ cannot be continued into the supersonic region, since, at $\theta = 0$ and $s = 0$, ψ_θ and ψ_s are both zero and hence the Jacobian vanishes at this point. These solutions, however, may be combined with the solution in equation (14).

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Providence, R. I., March 13, 1950

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TABLE I.- TABULATION OF s, 1/q, q, AND 1/ρ

s	1/q	q	1/ρ
0.34	1.45156	0.68891	1.27356
.26	1.35002	.74073	1.33361
.18	1.26118	.79291	1.39801
.16	1.24065	.80603	1.41545
.14	1.22071	.81920	1.43333
.12	1.20135	.83240	1.45175
.10	1.18252	.84565	1.47078
.08	1.16420	.85896	1.49050
.06	1.14636	.87233	1.51096
.04	1.12897	.88576	1.53223
.02	1.11200	.89928	1.55437
.00	1.09544	.91288	1.57744
-.02	1.07927	.92655	1.60151
-.04	1.06345	.94034	1.62666
-.06	1.04797	.95423	1.65293
-.08	1.03282	.96822	1.68042
-.10	1.01796	.98236	1.70918
-.12	1.00339	.99662	1.73929
-.14	.98908	1.01104	1.77084
-.16	.97502	1.02562	1.80387
-.18	.96120	1.04037	1.83848
-.22	.93420	1.07044	1.91318
-.27	.90152	1.10924	2.01693
-.32	.86986	1.14961	2.13391

TABLE II.- TABULATION OF s, s*, AND ω_1

s	s*	ω_1
0.34	0.20075	1.16534
.26	.14508	1.23307
.18	.09521	1.30080
.16	.08355	1.31774
.15	.07782	1.32620
.14	.07217	1.33467
.12	.06109	1.35160
.10	.05028	1.36854
.08	.03973	1.38547
.06	.02944	1.40240
.04	.01939	1.41933
.02	.00958	1.43627
.00	.0000	1.45320
-.02	-.00936	1.47013
-.04	-.01851	1.48707
-.06	-.02745	1.50400
-.08	-.03619	1.52093
-.10	-.04474	1.53786
-.12	-.05311	1.55480
-.14	-.06129	1.57173
-.16	-.06930	1.58866
-.18	-.07714	1.60560
-.22	-.09233	1.63946
-.27	-.11046	1.68179
-.32	-.12077	1.72412

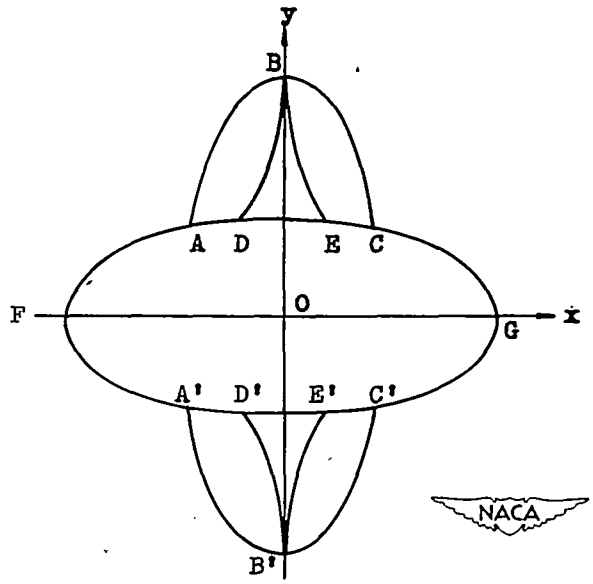


Figure 1.- Local supersonic regions on a symmetrical profile.

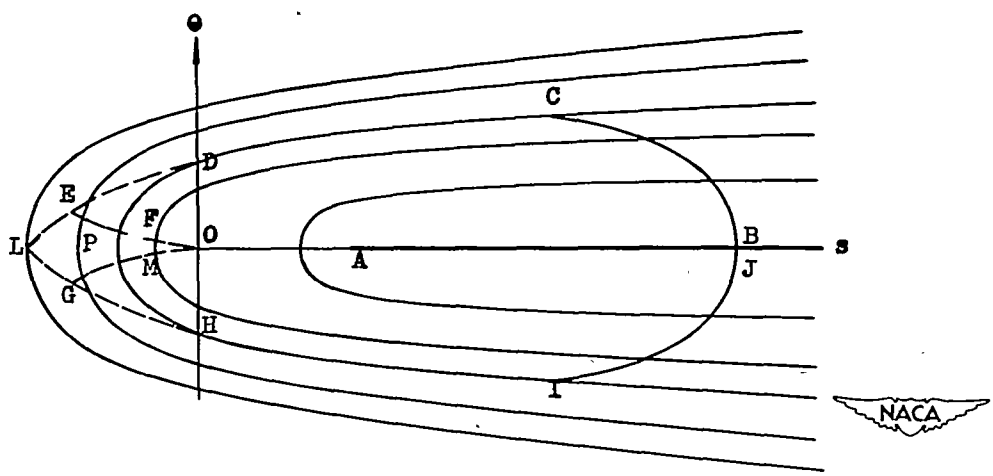


Figure 2.- Character of flow through a nozzle having locally supersonic regions. Hodograph plane.

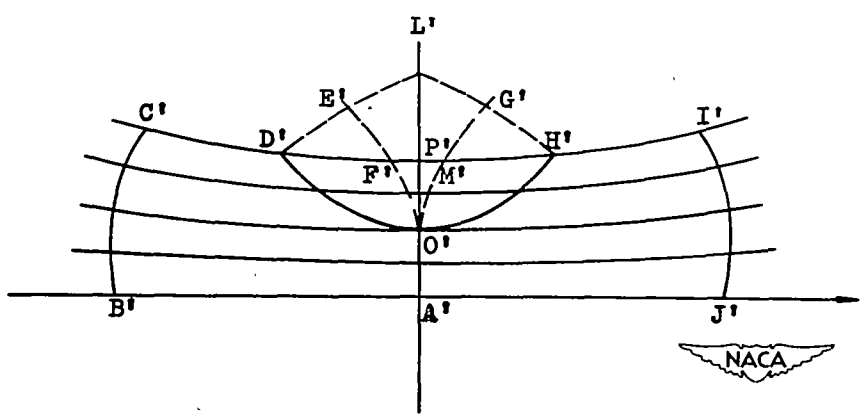


Figure 3.- Character of flow through a nozzle having locally supersonic regions. Physical plane.

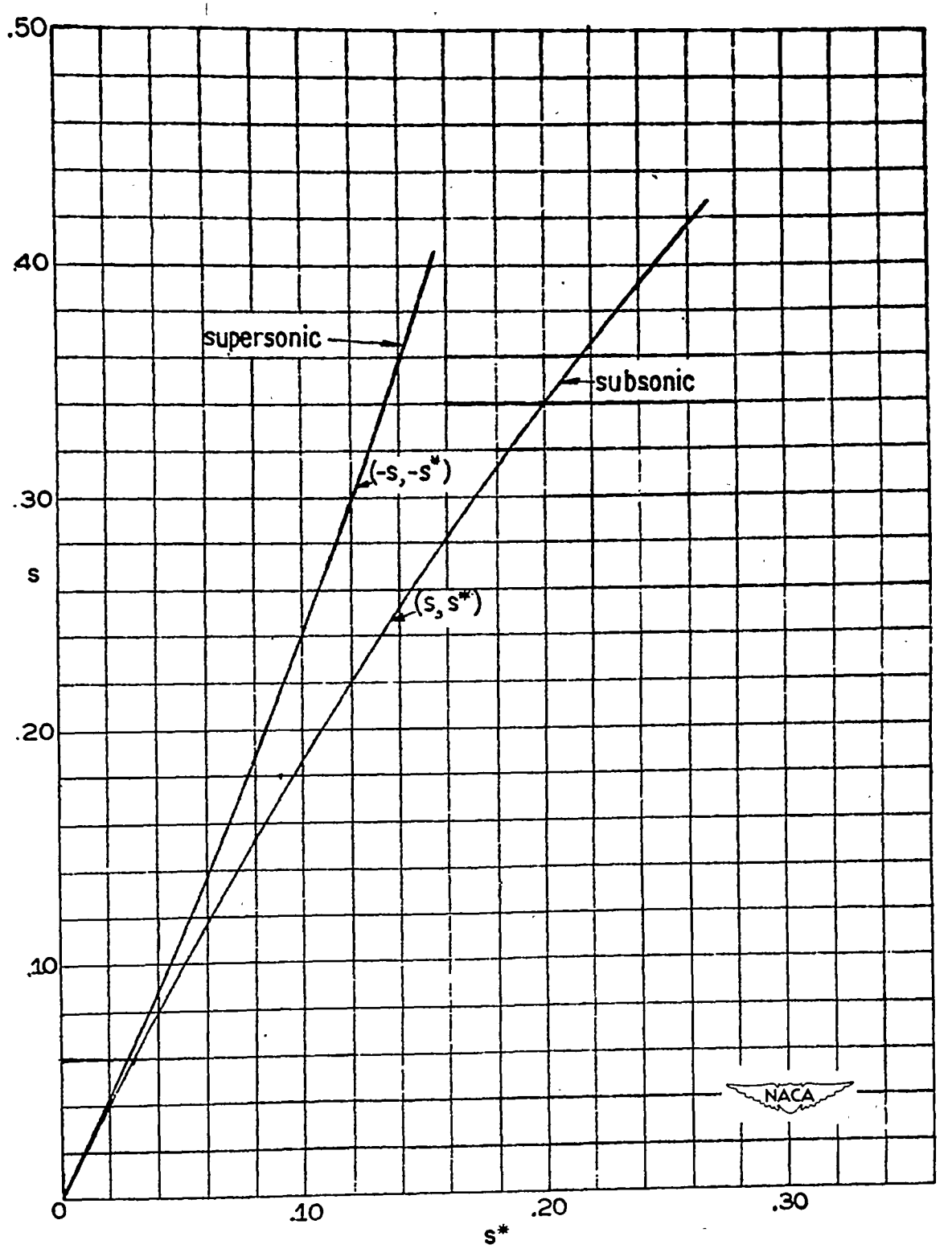


Figure 4.- Relation between s and s*.

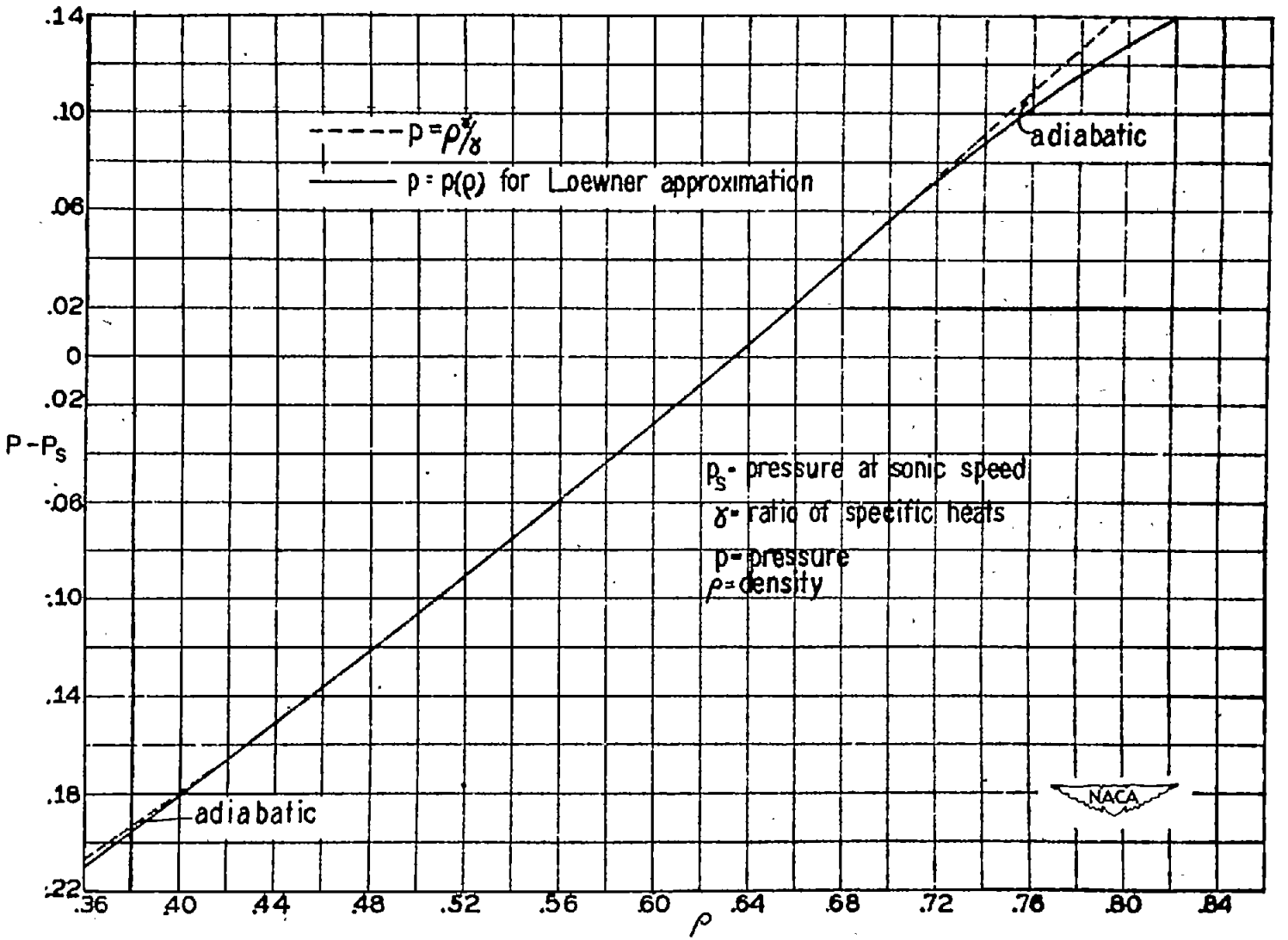


Figure 5.- Pressure-density relation.

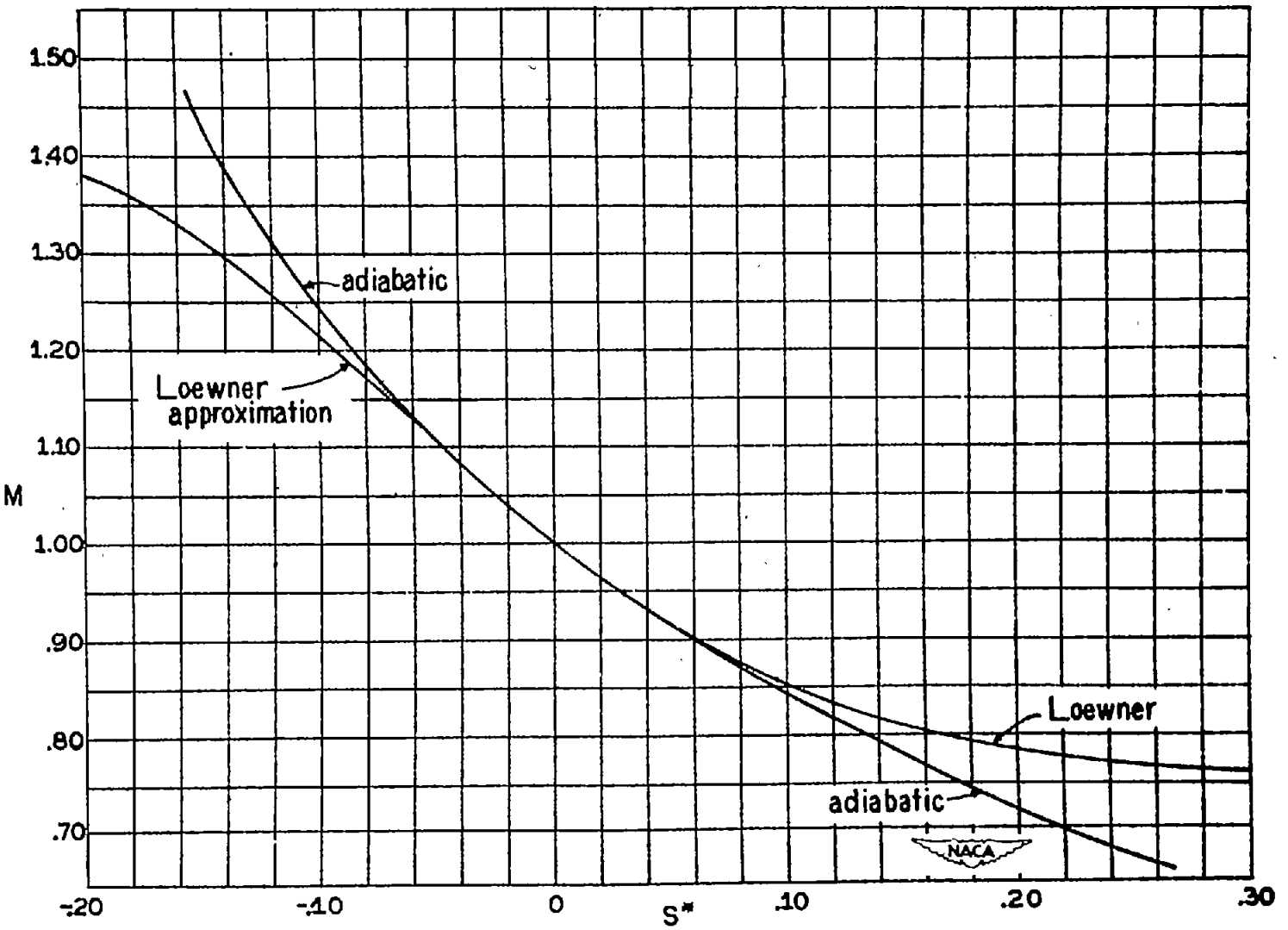


Figure 6.- Mach number as a function of s*.

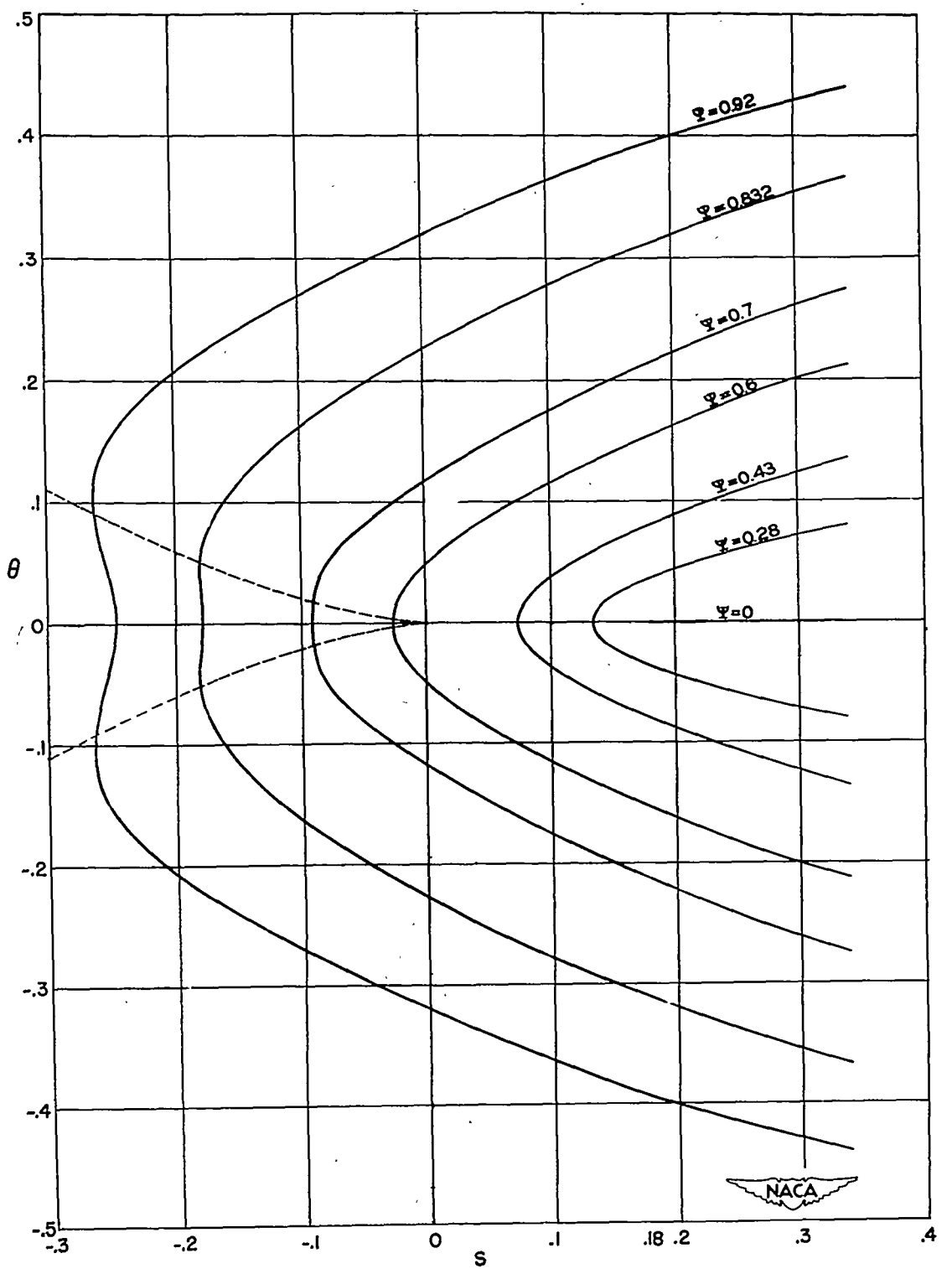
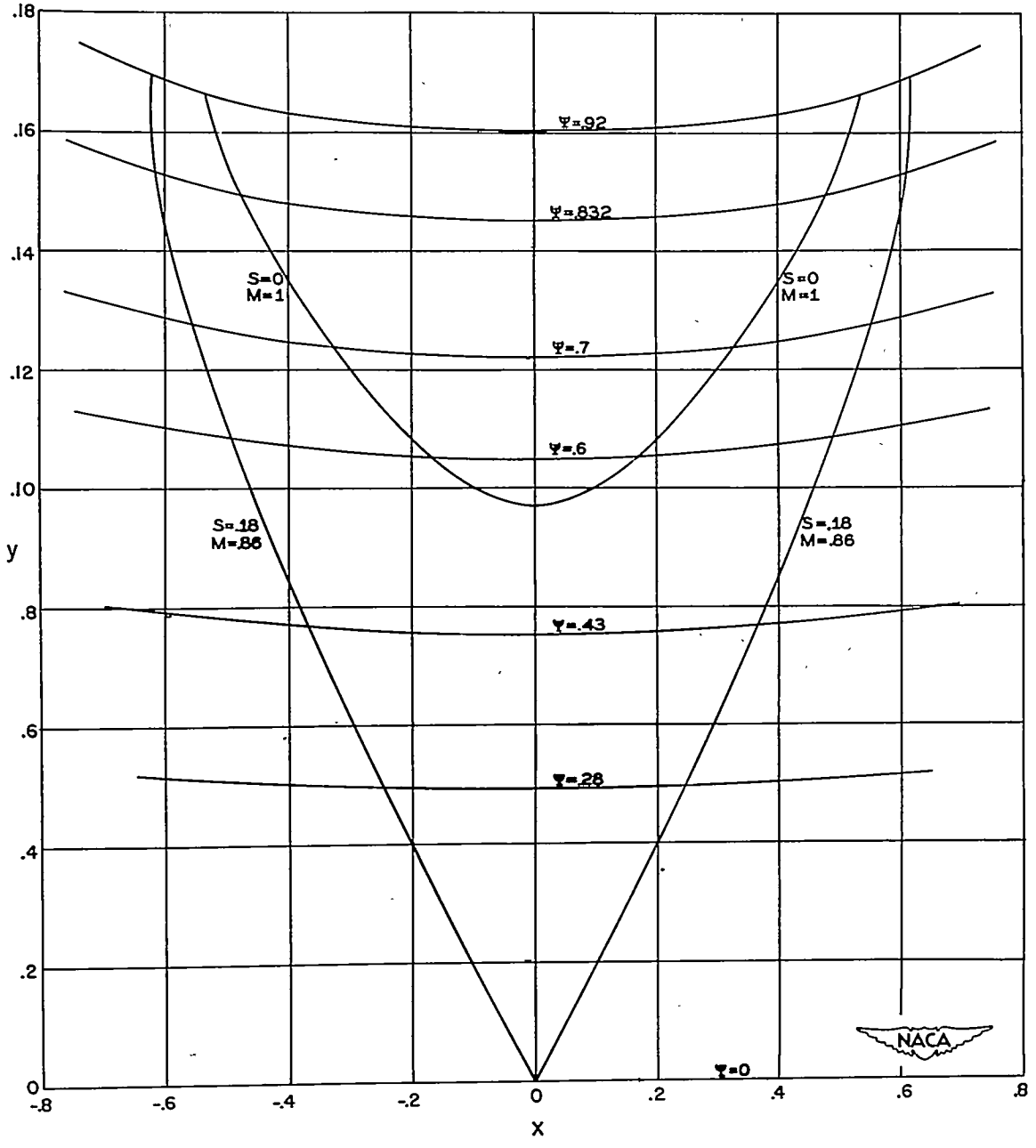
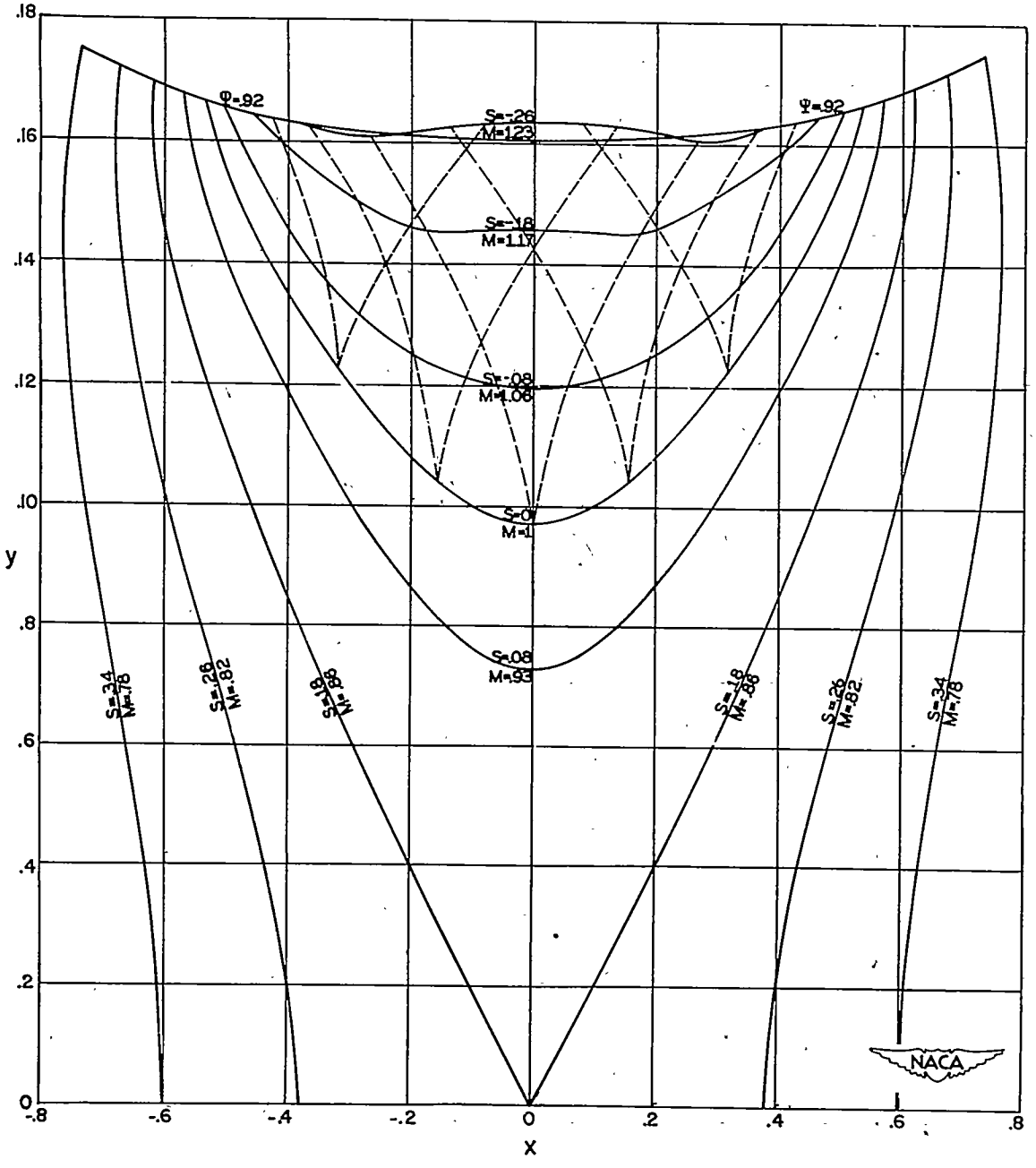


Figure 7.- Hodograph of symmetrical flow through a nozzle for $s_o = 0.18(\lambda = 0.0509117)$. Dashed lines indicate characteristics.



(a) Streamlines and constant-velocity lines.

Figure 8.- Constant-velocity lines in physical plane for $s_o = 0.18(\lambda = 0.0509117)$.



(b) Characteristics (dashed lines) and constant-velocity lines.

Figure 8.- Concluded.

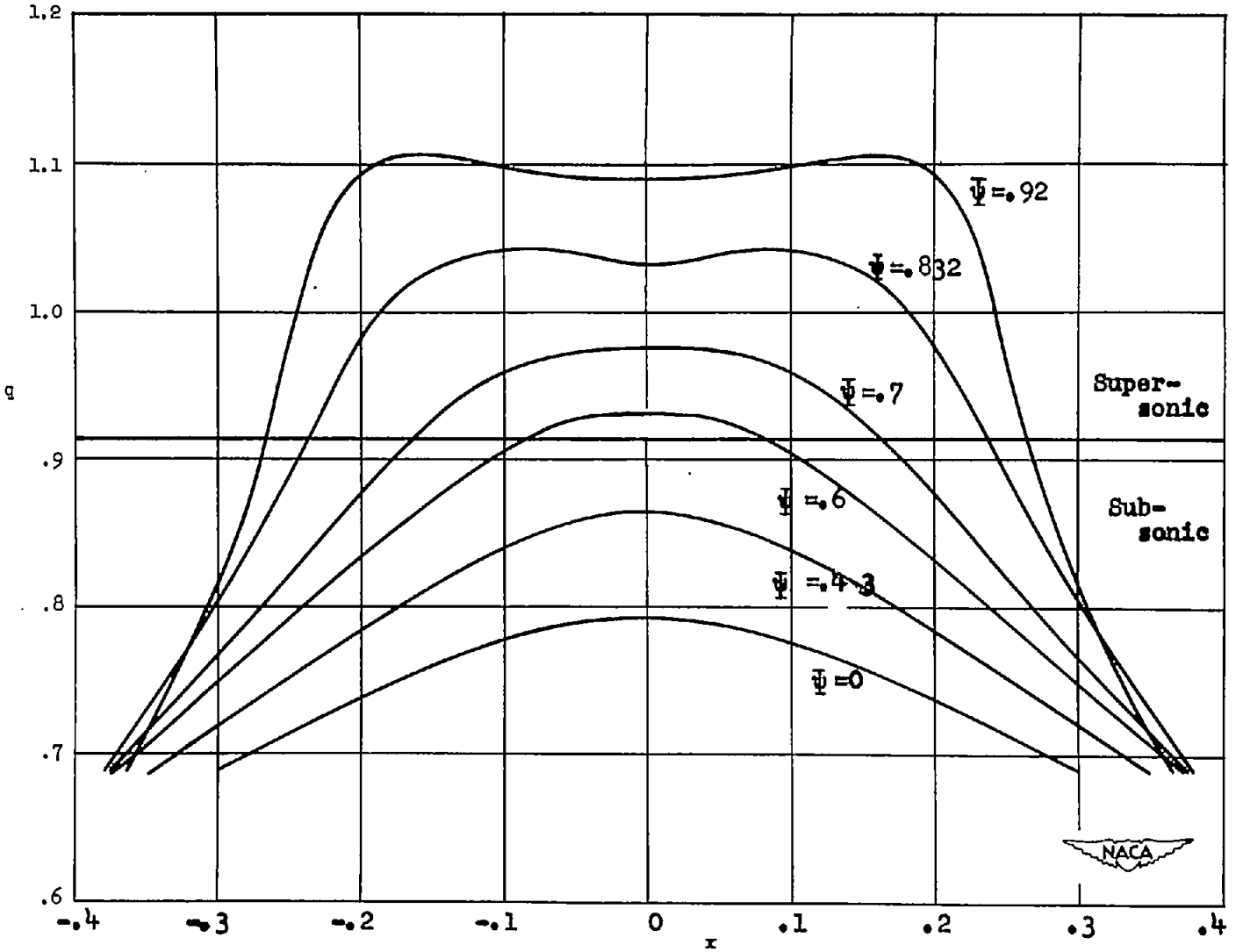


Figure 9.- Velocity distribution on streamlines for $\lambda = 0.050912$.

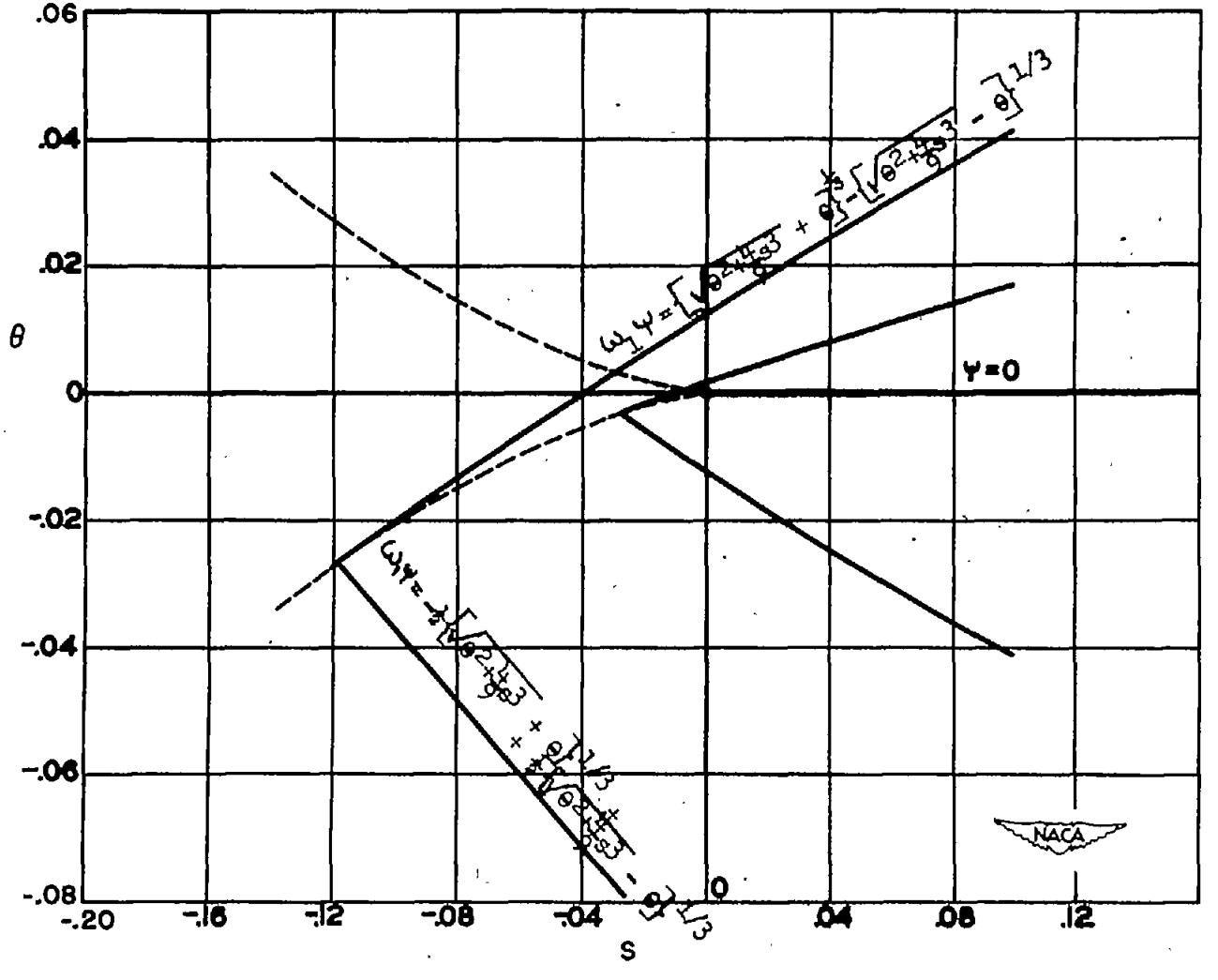
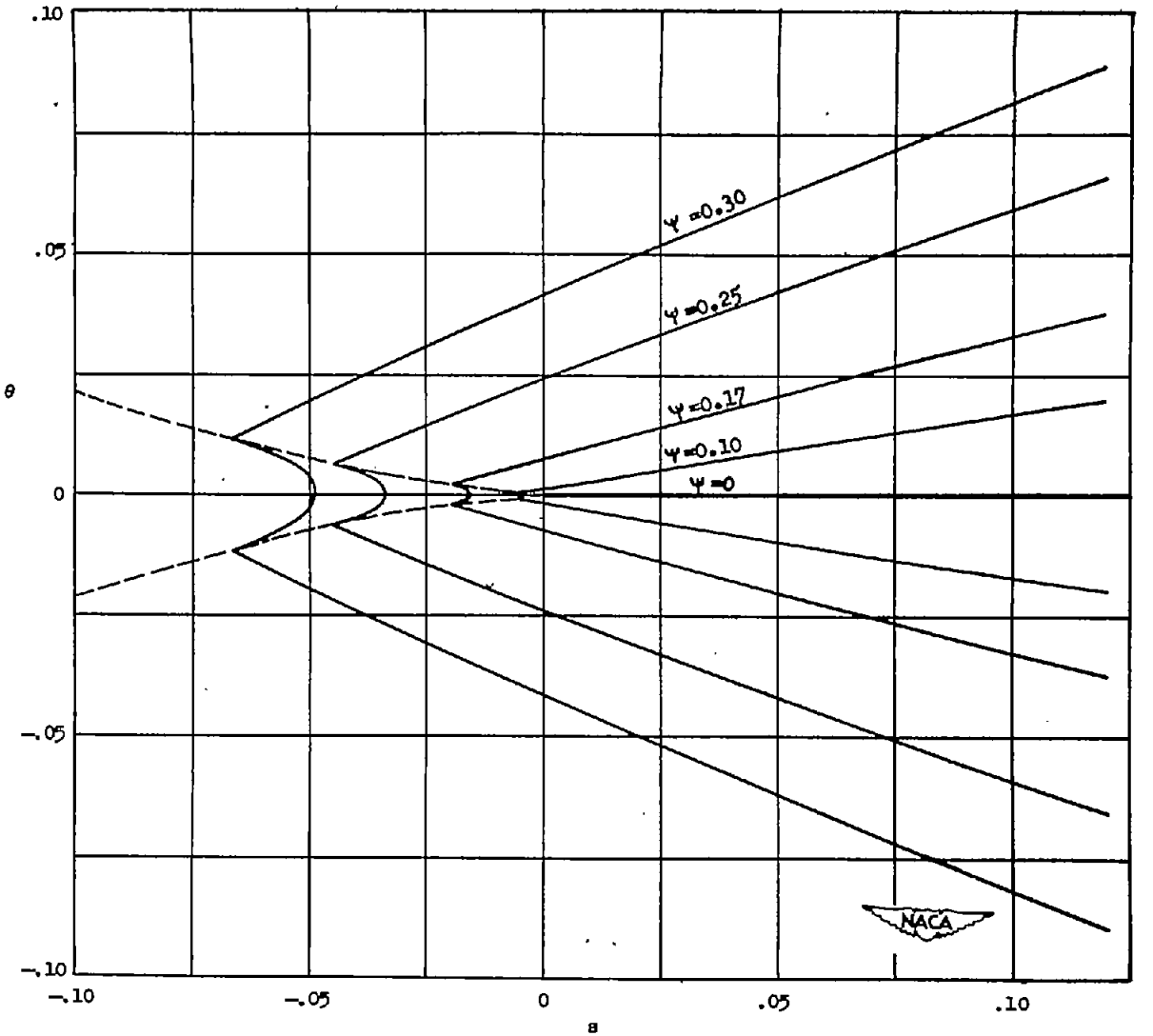
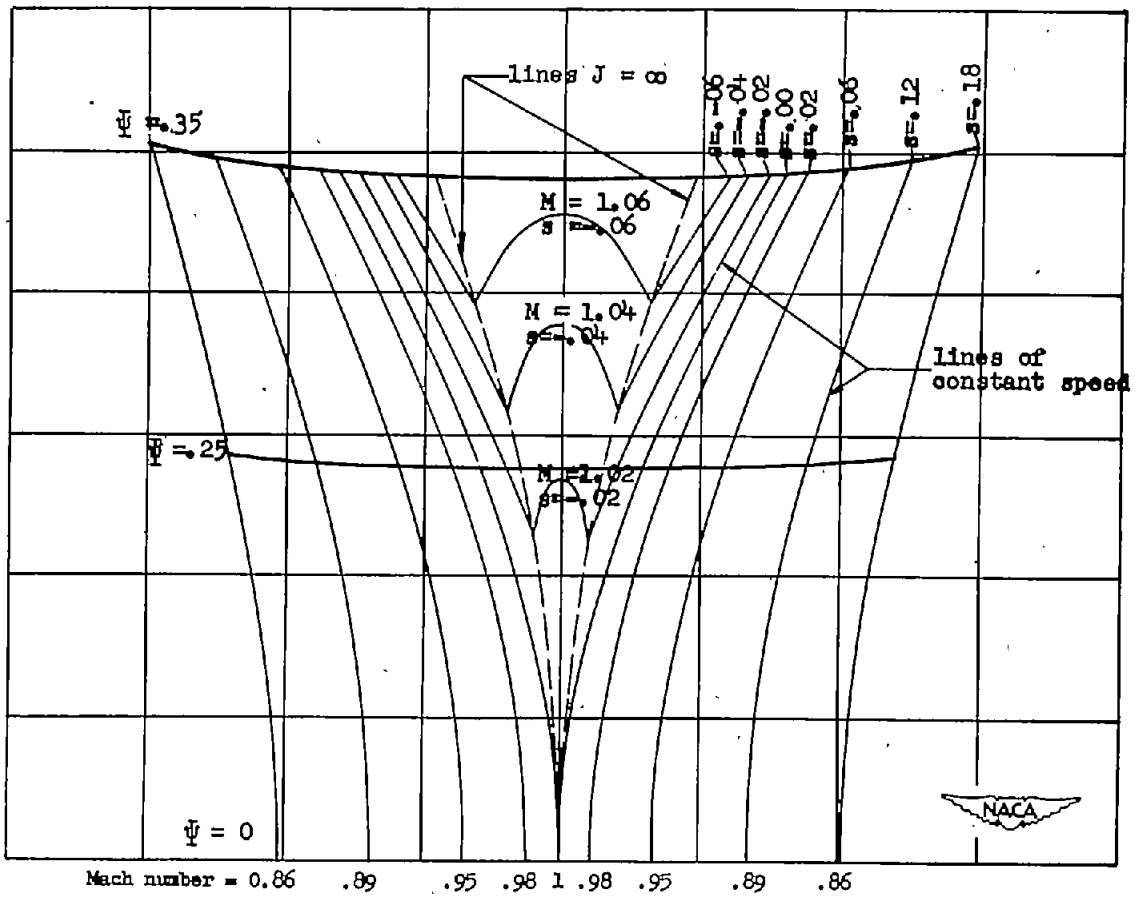


Figure 10.- Hodograph of flow through a nozzle computed by Tomotika and Tamada.



(a) Hodograph plane.

Figure 11.- Symmetrical flow through a nozzle for $\lambda = 0$.



(b) Physical plane.

Figure 11.- Concluded.