NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2609

ESTIMATE OF SLIP EFFECT ON COMPRESSIBLE LAMINAR-BOUNDARY-LAYER SKIN FRICTION

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SUMMARY

Rayleigh's problem for a compressible viscous gas, subject to slip at the wall, is considered. Expressions for slip velocity and skin friction are derived with the gas temperature at the wall and the product of viscosity times density assumed constant, or nearly so. The solution is related to that for the flow past a flat plate by a transformation which, in the continuum regime, results in exact agreement with the expression for laminar skin friction obtained by Chapman and Rubesin. The expressions for local skin friction are considered to define the extent of the continuum regime and to estimate the effect of slip in the border regime between continuum and slip flow. An estimated upper limit for the over-all drag of a flat plate in the slip flow regime is also obtained.

For the entire range from free-molecule flow to continuum flow, the parameter governing skin friction (expressed in the form $\sqrt{\gamma}\, M^* C_f^*$) was found to be $\sqrt{Re^*}/\sqrt{\gamma}\, M^*$ (where C_f^*, γ, Re^*, and M^* are the local skin-friction coefficient, the ratio of specific heats, Reynolds number, and Mach number, respectively, all based on gas properties at the wall). Continuum theory for local skin friction was found to agree with the expressions for skin friction developed herein to within 1 percent for $\sqrt{Re^*}/\sqrt{\gamma}\, M^*$ greater than 5.2 and to within 5 percent for $\sqrt{Re^*}/\sqrt{\gamma}\, M^*$ greater than 2.4.

INTRODUCTION

Fluid mechanics can be classified, as indicated in reference 1, into continuum, slip, intermediate, and free-molecule flow regimes. In the continuum regime the mean free path of the fluid molecules is negligible compared with body dimensions, and the conventional equations of motion apply. When the mean free path is small, but not negligible, compared with body dimensions or boundary-layer thickness, the fluid in contact with the body surface has a nonzero tangential velocity relative to the surface; this flow regime is consequently termed a slip flow. A free-molecule flow is one wherein the mean free path is very large compared with body dimensions so that the chances for collision of molecules among themselves are much smaller than the chances for collision of molecules with the body surface. Intermediate between the free-molecule and slip flow regimes, collisions between molecules and collisions of the molecules with the wall have the same order of probability.

The basic parameter defining these regimes is the ratio of the molecular mean free path l to the smallest significant physical dimension characterizing the flow. In the case of a flat plate, the smallest significant physical dimension may be taken, for sufficiently large Reynolds numbers, as the boundary-layer thickness δ. The ratio l/δ is shown in reference 1 to be of order $M/\sqrt{Re}$, where M is the free-stream Mach number and Re is the Reynolds number based on distance along the plate. When, for simplicity, $M/\sqrt{Re}$ is considered as the basic parameter (regardless of Reynolds number), the flow regimes may be estimated as 2362

$$100 < \frac{\sqrt{Re}}{M} \qquad \text{Continuum flow}$$

$$1 < \frac{\sqrt{Re}}{M} < 100 \qquad \text{Slip flow}$$

$$0.1 < \frac{\sqrt{Re}}{M} < 1 \qquad \text{Intermediate flow}$$

$$\frac{\sqrt{Re}}{M} < 0.1 \qquad \text{Free-molecule flow}$$

(The definitions of the intermediate and free-molecule flow regimes differ, for small Reynolds numbers, from those proposed by Tsien in reference 1, since plate length, rather than δ, was considered in reference 1 to be the significant physical dimension in these two regimes.)

In the continuum regime the conventional momentum and energy equations apply. However, in the slip flow regime additional terms must be incorporated into the equations. Although these terms increase the order of the differential equations, it has been shown by Schamberg, in a California Institute of Technology thesis, that they do not increase the number of boundary conditions required, nor do they radically change the flow from that encountered in the continuum regime. Thus, in the border region between continuum and slip flow, it might be expected that these additional terms are negligible and that a first approximation for the effect of slip may be obtained by solving the conventional equations of motion subject to slip boundary conditions. Neglection of these terms would, at the least, be the first step in an iteration procedure for solving the complete equations of motion for slip flow.

In the present report, Rayleigh's problem[1] for a compressible gas is considered utilizing the conventional equations of motion with slip boundary conditions. The no-slip case has been discussed in references 2 and 3. Rayleigh's problem is of interest because it is a relatively simple boundary-layer-type flow and may be considered as a model indicating the essential features of the slip effect. An estimate of skin friction on a semi-infinite flat plate, under slip flow conditions, is then obtained by relating time in the Rayleigh problem to distance downstream of the plate leading edge. The transformation from time to distance as the independent variable can be so chosen as to yield an expression for skin friction which is in exact agreement, in the continuum regime, with a known solution (reference 4) of the flat-plate boundary-layer equations. The matching of Rayleigh's problem with the flow past a plate is similar, in principle, to the use of "modified" Oseen equations of motion for the flow past the plate, as discussed in reference 5.

In reference 6, the solution of Rayleigh's problem, with slip, for the case of an incompressible fluid is obtained by Schaaf. The effect of compressibility, which is treated herein, is important, since the parameter $\sqrt{Re}/M$ indicates that slip may be introduced by high flight Mach numbers as well as low Reynolds numbers. Moreover, in relating Rayleigh's problem to the flow past a plate, Schaaf did not attempt to match the solution with the flat-plate solution in the continuum regime.[2] This matching, which is imposed herein, is essential, since the use of the conventional equations of motion with slip boundary conditions is considered justifiable chiefly as an asymptotic extrapolation from the continuum regime into the border regime between continuum and slip flow.

ANALYSIS

The slip flow boundary condition is discussed, after which an approximate solution to Rayleigh's problem is obtained. Finally, the solution to Rayleigh's problem is related to compressible slip flow past a plate.

Slip Flow Boundary Condition

Consider flow along a wall where x is distance along the wall, y is distance normal to the wall, and the fluid velocities u and v are parallel and normal to the wall, respectively. According to Maxwell's

[1]Rayleigh's problem consists in determining the flow field when an infinite plate, immersed in a viscous fluid, is instantaneously set into motion and maintains a constant velocity thereafter.

[2]After completion of this report, it was noted that in reference 7, the solution to Rayleigh's problem for incompressible flow is properly matched to the Blasius solution in the continuum regime.

theory of slip (reference 8), the velocity at the wall in the slip flow regime may be expressed in the form

$$u_w = 2c\left(\frac{2 - f}{f}\right)\left(l\,\frac{\partial u}{\partial y}\right)_w \tag{1}$$

where l is the mean free path, c is a number between 0.491 and 0.499, and f is the fraction of the tangential momentum lost, on impact, by the molecules impinging on the wall. For diffuse reflection (that is, the molecules rebounding from the wall have no preferred direction), f equals 1. Experimentally, f has been found to vary between 0.79 and 1.00 with the majority of the values close to 1.00. Thus, with the assumptions that $f = 1$ and $c = 1/2$, equation (1) becomes 2362

$$u_w = \left(l\,\frac{\partial u}{\partial y}\right)_w \tag{2}$$

It is apparent that slip occurs only when l is not negligible. The mean free path at the wall may be written (reference 1)

$$l = \sqrt{\frac{\pi\gamma}{2}}\,\frac{\nu_w}{a_w} \tag{3}$$

where the additional symbols are defined in appendix A.

Rayleigh's Problem for Compressible Viscous Gas

Rayleigh's problem consists in finding the flow field induced when an infinite flat plate, immersed in a viscous medium, is instantaneously set into motion in its own plane and is maintained at constant velocity thereafter. An equivalent problem is to consider the plate fixed and have the gas impulsively set into motion at constant velocity parallel to the plate. The case of a gas impulsively set into motion is more directly analagous to the flow past a semi-infinite flat plate and will be the one used.

Equations of motion. - As in reference 3, all x-derivatives are zero and the equations of motion are

Momentum:

$$\rho\,\frac{Du}{Dt} = \frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right) \tag{4}$$

$$\rho\,\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{4}{3}\,\frac{\partial}{\partial y}\left(\mu\,\frac{\partial v}{\partial y}\right) \tag{5}$$

Continuity: $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y}(\rho v) = 0 \tag{6}$$

Energy: $$\rho J c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = J \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \frac{4}{3}\left(\frac{\partial v}{\partial y}\right)^2\right] \tag{7}$$

State: $$p = \rho R T \tag{8}$$

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where the operator D/Dt is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial y} \tag{9}$$

Equation (6) implies the existence of a function ψ such that

$$\frac{\rho}{\rho_0} = \frac{\partial \psi}{\partial y} \qquad \frac{\rho v}{\rho_0} = -\frac{\partial \psi}{\partial t} \tag{10}$$

where the subscript 0 designates conditions far from the wall. Transformation of the independent variables from (t,y) to (t,ψ) is accomplished, as in references 2 and 3, by the following relations:

$$\left(\frac{\partial}{\partial t}\right)_y = \left(\frac{\partial}{\partial t}\right)_\psi - \frac{\rho v}{\rho_0}\left(\frac{\partial}{\partial \psi}\right)_t \tag{11}$$

$$\left(\frac{\partial}{\partial y}\right)_t = \frac{\rho}{\rho_0}\left(\frac{\partial}{\partial \psi}\right)_t \tag{12}$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_\psi \tag{13}$$

With these transformations, the equations of motion become

Momentum: $$\frac{\partial u}{\partial t} = \nu_0 \frac{\partial}{\partial \psi}\left[\left(\frac{\rho \mu}{\rho_0 \mu_0}\right)\frac{\partial u}{\partial \psi}\right] \tag{14}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial \psi} + \frac{4}{3}\nu_0 \frac{\partial}{\partial \psi}\left[\left(\frac{\rho \mu}{\rho_0 \mu_0}\right)\frac{\partial v}{\partial \psi}\right] \tag{15}$$

Continuity:

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) - \frac{1}{\rho_0}\frac{\partial v}{\partial \psi} = 0 \tag{16}$$

Energy:

$$Jc_p \frac{\partial T}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{Jc_p}{\sigma} \nu_0 \frac{\partial}{\partial \psi}\left[\left(\frac{\rho\mu}{\rho_0\mu_0}\right)\frac{\partial T}{\partial \psi}\right] + \nu_0 \left(\frac{\rho\mu}{\rho_0\mu_0}\right)\left[\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{4}{3}\left(\frac{\partial v}{\partial \psi}\right)^2\right] \tag{17}$$

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The ratio c_p/σ is assumed constant in equation (17). To return to the (t,y) coordinate system it is necessary to integrate, for constant t,

$$y = \rho_0 \int_0^{\psi} \frac{d\psi}{\rho} \tag{18}$$

Skin friction for constant $\rho\mu/\rho_0\mu_0$ and T_w. - A simplification can be introduced into equations (14), (15), and (17) by considering $\rho\mu/\rho_0\mu_0$ constant. This condition implies that pressure variations are negligible (which is consistent with the Prandtl boundary-layer assumptions) and that μ is proportional to T, which is not strictly correct except for special cases. (A more accurate expression for $\rho\mu/\rho_0\mu_0$ is, from the Sutherland equation,

$$\frac{\rho\mu}{\rho_0\mu_0} = \left(\frac{T}{T_0}\right)^{1/2}\left(\frac{T_0 + S}{T + S}\right) \tag{19}$$

where the pressure is assumed constant and S is a constant for a given gas and equals approximately 216° R for air.) However, by considering the case of constant T_w and assuming $\rho\mu/\rho_0\mu_0$ to be constant throughout the flow field and equal to $\rho_w\mu_w/\rho_0\mu_0$ (so as to be correct in the vicinity of the wall, which is the region of interest) values of compressible boundary-layer skin friction are obtained in reference 4 which agree with the more accurate numerical integrations of reference 9 within 5 percent for flight conditions up to Mach number 5 and within 1 or 2 percent for supersonic wind tunnel conditions up to even higher Mach numbers. The same simplifying assumptions will be made in the present analysis; namely,

$$\begin{aligned}\frac{\rho\mu}{\rho_0\mu_0} &\equiv \frac{\rho_w\mu_w}{\rho_0\mu_0} \\ &\equiv C\end{aligned} \tag{20}$$

and the gas temperature at the wall T_w is constant.

The substitution of equation (20) into equation (14) yields

$$\frac{\partial u}{\partial t} = C\nu_0 \frac{\partial^2 u}{\partial \psi^2} \tag{21}$$

This equation is independent of the energy equation, and a closed-form solution for skin friction can be obtained for zero slip or for nonzero slip at the wall.

(a) Zero slip at wall

For the case of zero slip at the wall, the boundary conditions are

$$\left.\begin{aligned} u &= U \quad \text{for} \quad & t = 0,\ \psi > 0 \\ u &= 0 & t \geq 0,\ \psi = 0 \\ u &\rightarrow \pi & t \geq 0,\ \psi \rightarrow \infty \end{aligned}\right\} \tag{22}$$

Equations (21) and (22) are formally equivalent to the problem of finding the temperature in a semi-infinite solid, originally at constant temperature U, when a constant surface temperature of zero is impressed at time $t = 0$. The solution (reference 10) is

$$\frac{u}{U} = \operatorname{erf}\left(\psi/2\sqrt{C\nu_0 t}\right) \tag{23}$$

where $\operatorname{erf}(\ )$ is the error function (tabulated, for example, in reference 11) and is defined by

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp(-r^2)dr \tag{24}$$

The shear stress at any point in the flow field is then

$$\tau = \mu \frac{\partial u}{\partial y} = \frac{\mu\rho}{\rho_0} \frac{\partial u}{\partial \psi} = \frac{\mu\rho}{\rho_0} \frac{U}{\sqrt{\pi C\nu_0 t}} \exp(-\psi^2/4C\nu_0 t) \tag{25}$$

At the wall,

$$\tau_w = U \sqrt{\mu_w\rho_w/\pi t} \tag{26}$$

(b) Nonzero slip at wall

When there is slip at the wall, equation (21) has the boundary conditions (utilizing equation (2))

$$
\left.
\begin{aligned}
u &= U && \text{for} \quad t = 0,\ \psi > 0 \\
u &= \left(\frac{l\rho_w}{\rho_0}\frac{\partial u}{\partial \psi}\right)_w && \qquad\quad t \geq 0,\ \psi = 0 \\
u &\rightarrow U && \qquad\qquad\quad \psi \rightarrow \infty
\end{aligned}
\right\} \tag{27}
$$

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Equations (21) and (27), as pointed out in reference 6, are equivalent to the problem of finding the temperature distribution in a semi-infinite solid at temperature U which, at t = 0, starts transferring heat to an adjacent medium at zero temperature. The solution is (reference 10)

$$\frac{u}{U} = \operatorname{erf}(\eta) + \operatorname{erfc}(\eta + \zeta)\ \exp\left[\zeta(2\eta + \zeta)\right] \tag{28}$$

where

$$\operatorname{erfc}(\ ) = 1 - \operatorname{erf}(\ )$$

$$\eta = \psi/2\sqrt{C\nu_0 t}$$

$$\zeta = \rho_0\sqrt{C\nu_0 t/l\rho_w}$$

The velocity of slip is

$$\frac{u_w}{U} = \operatorname{erfc}(\zeta)\ \exp(\zeta^2) \tag{29}$$

The shear stress is

$$\tau = \frac{\mu U \rho}{l\rho_w}\left\{\operatorname{erfc}(\eta + \zeta)\ \exp\left[\zeta(2\eta + \zeta)\right]\right\} \tag{30}$$

At the wall,

$$\tau_w = \frac{\mu_w U}{l}\left[\operatorname{erfc}(\zeta)\ \exp(\zeta^2)\right] \tag{31}$$

These equations will be discussed after the analogy with the flow past a flat plate has been established.

Skin Friction on Flat Plate

It is well known that, for the case of no slip, the variation of boundary-layer thickness, displacement thickness, shear stress, and so forth with time in Rayleigh's problem is similar to the variation of these parameters with distance along a flat plate. (See, for example, reference 12.) If in the solution to Rayleigh's problem (without slip) the transformation $t = Kx/U$, where K is a dimensionless constant, is introduced, it is possible, by proper choice of K, to match some of these parameters so that they are identical to the results obtained for a flat plate (without slip). Since skin friction is the major concern herein, K can be chosen so that the skin friction indicated by equation (26) will exactly match the flat-plate continuum solution obtained in reference 4, which may be written

$$\tau_w = 0.332(U^{3/2})\sqrt{\frac{\rho_w \mu_w}{x}} \tag{32}$$

Substituting $t = Kx/U$ into equation (26) yields

$$\tau_w = \frac{1}{\sqrt{K\pi}}(U^{3/2})\sqrt{\frac{\rho_w \mu_w}{x}} \tag{33}$$

Comparison of equations (32) and (33) indicates that

$$K = \frac{1}{\pi(0.332)^2} = 2.888 \tag{34}$$

The appropriate transformation is then

$$t = 2.888\ x/U \tag{35}$$

This transformation matches, for the no-slip case, the skin friction associated with Rayleigh's problem to the expression obtained in reference 4. If substituted into equation (31), the transformation should give a reasonable estimate for skin friction on a flat plate in slip flow, particularly for small slip velocities.

An alternative approach, which yields the same results but which avoids a reference to Rayleigh's problem, is to write the boundary-layer equations for slip flow past a plate and introduce a "modified" Oseen type linearization, as discussed in reference 5.

Slip velocity and local skin-friction coefficients. - Substituting equations (3) and (35) into the expression for ζ gives

$$\zeta = \sqrt{\frac{2K}{\gamma\pi}} \sqrt{\frac{\mu_0}{\mu_w}} \frac{\sqrt{Re}}{M} = \frac{1.356}{\sqrt{\gamma}} \sqrt{\frac{\mu_0}{\mu_w}} \frac{\sqrt{Re}}{M} = \frac{1.356}{\sqrt{\gamma}} \frac{\sqrt{Re^*}}{M^*} \tag{36}$$

where the superscript * indicates that the parameter is based on state properties evaluated at the wall; that is,

$$Re^* = \frac{\rho_w U x}{\mu_w} \qquad M^* = \frac{U}{a_w} \qquad q^* = \frac{\rho_w U^2}{2} \tag{37}$$

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Substituting these parameters into equations (29) and (31) gives the following expressions for slip velocity and wall shear:

$$\frac{u_w}{U} \doteq \operatorname{erfc}\left(\frac{1.356}{\sqrt{\gamma}} \frac{\sqrt{Re^*}}{M^*}\right) \exp\left(\frac{1.356}{\sqrt{\gamma}} \frac{\sqrt{Re^*}}{M^*}\right)^2 \tag{38}$$

$$\tau_w = \frac{q^*}{M^*} \sqrt{\frac{8}{\pi\gamma}} \operatorname{erfc}\left(\frac{1.356}{\sqrt{\gamma}} \frac{\sqrt{Re^*}}{M^*}\right) \exp\left(\frac{1.356}{\sqrt{\gamma}} \frac{\sqrt{Re^*}}{M^*}\right)^2 \tag{39}$$

The local skin-friction coefficient may be expressed as

$$\sqrt{\gamma}\, M^* C_f^* = \frac{\sqrt{\gamma}\, M^*}{q^*} \tau_w = \sqrt{\frac{8}{\pi}} \operatorname{erfc}\left(\frac{1.356}{\sqrt{\gamma}} \frac{\sqrt{Re^*}}{M^*}\right) \exp\left(\frac{1.356}{\sqrt{\gamma}} \frac{\sqrt{Re^*}}{M^*}\right)^2 \tag{40}$$

It is seen that $\sqrt{Re^*}/\sqrt{\gamma}\, M^*$ is the basic parameter defining skin friction.

The numerical evaluation of equations (38) to (40) is difficult for very small or very large values of the parameter. In these cases, it is convenient to express the error function and the exponential in series form. For small values of ζ (reference 10),

$$\operatorname{erf}(\zeta) = \frac{2}{\sqrt{\pi}}\left(\zeta - \frac{\zeta^3}{3 \cdot 1!} + \frac{\zeta^5}{5 \cdot 2!} - \frac{\zeta^7}{7 \cdot 3!} + \cdots\right)$$

$$\exp(\zeta^2) = 1 + \zeta^2 + \frac{(\zeta^2)^2}{2!} + \frac{(\zeta^2)^3}{3!} + \cdots$$

For larger values of ζ, the error function can be expressed by the asymptotic series

$$\operatorname{erf}(\zeta) = 1 - \frac{\exp(-\zeta^2)}{\zeta\sqrt{\pi}}\left(1 - \frac{1}{2\zeta^2} + \frac{1 \cdot 3}{(2\zeta^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2\zeta^2)^3} + \cdots\right)$$

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The local skin-friction coefficient can then be written: for $\sqrt{Re^*}/\sqrt{\gamma}\, M^* < 0.4$,

$$\sqrt{\gamma}\, M^* C_f^* = \sqrt{\frac{8}{\pi}}\left[1 - 1.530\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right) + 1.839\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^2 - 1.875\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^3 + 1.691\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^4 - 1.380\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^5 + 1.037\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^6 - \cdots\right] \tag{41}$$

for $\sqrt{Re^*}/\sqrt{\gamma}\, M^* > 2.5$,

$$\sqrt{\gamma}\, M^* C_f^* = 0.664\,\frac{\sqrt{\gamma}\, M^*}{\sqrt{Re^*}}\left[1 - 0.272\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^{-2} + 0.222\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^{-4} - 0.301\left(\frac{\sqrt{Re^*}}{\sqrt{\gamma}\, M^*}\right)^{-6} + \cdots\right] \tag{42}$$

Equation (42) is of primary interest because it indicates the variation of skin friction in the border region between continuum and slip flow. Equations (38) and (40) were evaluated using the expansions indicated in equations (41) and (42) and are plotted in figures 1 and 2 for $\gamma = 1.40$.

In the limiting case of free-molecule flow, $\sqrt{Re^*}/\sqrt{\gamma}\, M^* \doteq 0$, equations (38) and (41) become

$$\frac{u_w}{U} = 1 \tag{43}$$

$$\sqrt{\gamma}\, M^* C_f^* = 2\sqrt{\frac{2}{\pi}} \tag{44}$$

Equation (44) yields twice the result indicated by kinetic theory (reference 13). It should be recalled that continuum equations of motion have been used and that equation (2), which was derived on the basis that the molecular mean free path is small (but not negligible) compared with the boundary-layer thickness, is clearly inapplicable in the free-molecular flow and intermediate flow regimes. Equation (2), which relates shear stress to the local slip velocity, is too large by a factor of 2 in the free-molecule regime, as can be seen by comparing references 8 and 13. For this reason, equation (44) has the proper form but yields twice the correct result.

In the limiting case of continuum flow, $\sqrt{Re^*}/\sqrt{\gamma}\ M^* \rightarrow \infty$, equation (42) becomes

$$\sqrt{\gamma}\ M^* C_f^* = 0.664\ \frac{\sqrt{\gamma}\ M^*}{\sqrt{Re^*}} \tag{45}$$

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which, because of the matching procedure, is in exact agreement with reference 4. Equation (45) agrees with equation (40) to within 1 percent for $\sqrt{Re^*}/\sqrt{\gamma}\ M^* > 5.2$ and within 5 percent for $\sqrt{Re^*}/\sqrt{\gamma}\ M^* > 2.4$. Equation (42) thus indicates that the effect of slip first appears from a term proportional to the square of the parameter $\sqrt{\gamma}\ M^*/Re^*$.

Equation (40) is plotted in figure 2 for $\gamma = 1.40$. The four flow regimes, based on the criteria mentioned in the Introduction, are also shown. For a semi-infinite flat plate in a rarefied gas stream, it is seen that the flow starts as a "free-molecule" type flow and ultimately develops into a "continuum" type flow; the indicated estimate for the extent of the continuum regime is conservative in regard to local skin friction. Equation (40) indicates that continuum theory for local skin friction is applicable for $\sqrt{Re^*}/M^* > 5$. The plot of equation (40) is dashed for $\sqrt{Re^*}/M^* < 2$ since it is of questionable validity beyond the border regime between continuum and slip flow. Since equation (40) asymptotes to twice the correct value for free-molecule flow, it appears that this equation underestimates the reduction (in skin friction) associated with slip. In fact, it may be argued, from a kinetic theory viewpoint, that the local skin friction must be a monotonic decreasing function of $\sqrt{Re^*}/M^*$, starting with the free-molecule flow value. According to kinetic theory, local skin friction is proportional to the axial momentum (parallel to the wall) of the molecules striking a unit area of the wall per unit time (see, for example, reference 8). As the fluid moves downstream, the molecules in the vicinity of the wall have less and less axial momentum due to previous collisions with the wall and thus bring successively less and less momentum to the wall at the downstream stations. Barring an instability in the flow (such as the transition to turbulent flow in the continuum regime), the local skin friction must then decrease monotonically with $\sqrt{Re^*}/M^*$. On the basis of this reasoning, equation (40) is definitely invalid for $\sqrt{Re^*}/M^*$ less than approximately 0.7. An interpolation function can be devised which will decrease monotonically from the free-molecule value and fair into equation (40) in the vicinity of the continuum-slip flow border regime. For example, if equation (40) is multiplied by

$$\left[1 - \frac{1}{2}\exp\left(-\frac{\sqrt{Re^*}}{\sqrt{\gamma}\ M^*}\right)\right]$$

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the resulting expression for local skin friction will decrease monotonically with $\sqrt{Re^*}/\sqrt{\gamma}\, M^*$ and will have the proper end points. However, insufficient information is available, concerning skin friction in the intermediate regime, to justify a particular choice for the interpolation function.

Integrated drag. - The drag coefficient for the plate is written

$$C_D = \frac{1}{L}\int_0^L C_f dx \tag{47}$$

Since equation (40) overestimates the skin friction in the free-molecule and intermediate flow regimes its use in equation (47) will result in an overestimate of the flat-plate drag coefficient. Such a procedure will provide some useful information, however, and will therefore be presented. Following this integration, an estimate of the upper limit for the flat-plate drag coefficient will be indicated.

If equation (40) is substituted into equation (47) the resulting expression for drag coefficient may be written

$$\sqrt{\gamma}\, M^* C_D^* = \sqrt{\frac{8}{\pi}}\frac{1}{\zeta_L^2}\left[\frac{2\zeta_L}{\sqrt{\pi}} - 1 + \operatorname{erfc}(\zeta_L)\exp(\zeta_L^2)\right] \tag{48}$$

where

$$\zeta_L = 1.356\sqrt{Re^*_L}/\sqrt{\gamma}\, M^*$$

In expanded form: for $\sqrt{Re_L^*}/\sqrt{\gamma}\, M^* < 0.4$,

$$\sqrt{\gamma}\, M^* C_D^* = \sqrt{\frac{8}{\pi}}\left[1 - 1.020\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\, M^*} + 0.920\left(\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\, M^*}\right)^2 - 0.750\left(\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\, M^*}\right)^3 + 0.564\left(\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\, M^*}\right)^4 - \cdots\right] \tag{49}$$

for $\sqrt{Re_L^*}/\sqrt{\gamma}\, M^* > 2.5$,

$$\sqrt{\gamma}\, M^* C_D^* = 1.328\frac{\sqrt{\gamma}\, M^*}{\sqrt{Re^*_L}}\left[1 - 0.654\left(\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\, M^*}\right)^{-1} + 0.272\left(\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\, M^*}\right)^{-2} - 0.0739\left(\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\, M^*}\right)^{-4} + \cdots\right] \tag{50}$$

Equation (48) is plotted as a dashed curve in figure 3.

It was previously argued that the local skin friction must decrease monotonically from the free-molecule flow value. Therefore, equation (40) tends to overestimate the local skin friction upstream of its region of validity. If it is assumed that the free-molecule flow value $\sqrt{\gamma}\ M^*C_f^* = \sqrt{2/\pi}$ applies for $\sqrt{Re^*}/\sqrt{\gamma}\ M^* \leq 0.55$ and that equation (40) applies for $\sqrt{Re^*}/\sqrt{\gamma}\ M^* > 0.55$, then this distribution of skin friction should, at all points, equal or exceed the correct local values. (Note that at $\sqrt{Re^*}/\sqrt{\gamma}\ M^* = 0.55$ equation (40) yields the free-molecule flow value for skin friction.) Then integrating this distribution of skin friction should yield an upper limit for the over-all drag of a flat plate. Integration yields (for $\sqrt{Re^*_L}/\sqrt{\gamma}\ M^* > 0.55$)

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$$\sqrt{\gamma}\ M^*C_D^* = \sqrt{\frac{8}{\pi}}\frac{1}{\zeta_L^2}\left[\frac{2\zeta_L}{\sqrt{\pi}} - 1 + \operatorname{erfc}(\zeta_L)\ \exp(\zeta_L^2)\right] - 0.0623\left(\frac{\sqrt{Re^*_L}}{\sqrt{\gamma}\ M^*}\right)^{-2} \quad (51)$$

Equation (51) is plotted in figure 3 and is considered to define an upper limit for flat-plate drag coefficient. Equations (48) and (51) agree within 1 percent for $\sqrt{Re_L^*}/\sqrt{\gamma}\ M^* > 5$, indicating that for these values of the parameter the free-molecule and intermediate flow regimes are confined to a relatively small region close to the leading edge, and a more correct estimate of the local skin friction in these regimes may be unnecessary when estimating over-all drag. Continuum theory agrees with both equations (48) and (51) to within 1 percent for $\sqrt{Re_L^*}/\sqrt{\gamma}\ M^* > 65$ and to within 5 percent for $\sqrt{Re_L^*}/\sqrt{\gamma}\ M^* > 13$.

Surface temperature. - In the previous developments the gas temperature at the wall has been assumed constant. However, for zero heat transfer, the gas temperature at the surface of a plate in slip flow varies from the free-stream value at the leading edge to the continuum value for recovery temperature at downstream points. In order to evaluate skin friction from the previous equations, it is necessary to base equation (20) on an appropriate average value of T_W, as was done in reference 4. An approximate method for estimating the variation of T_W along a plate with zero heat transfer is indicated in appendix B.

SUGGESTIONS FOR ADDITIONAL STUDY

The method of solution used herein is attractive because of its relative simplicity. However, several aspects of the development require additional study. Use of continuum equations, for example, means that terms of order $(M^*/\sqrt{Re^*})^2$ are neglected in the equations of motion (reference 1). But, equation (42) indicates that the correction to continuum theory, in the border region between continuum and slip flow, is of the order $\left(M^*/\sqrt{Re^*}\right)^2$. Thus the correction is of the same order as the terms neglected in the differential equation. The use of continuum equations with slip boundary conditions, for estimating skin-friction coefficients in the slip flow regime, is justifiable if the effect of slip is introduced primarily through the boundary conditions rather than through the differential equations. The region of validity of this approach can be established by solving Rayleigh's problem using equations of motion which include the higher order terms neglected herein and comparing the solution with the results obtained by the present analysis. Moreover, the transformation from Rayleigh's problem to a flat plate, although exact in regard to skin friction in the continuum regime, may be only approximate for the slip flow regime. An evaluation of this effect can be obtained by solving directly for skin friction on a flat plate, using continuum equations with slip boundary conditions, and comparing the solution with the results of the present report. Finally, more accurate solutions for local skin friction in the intermediate regime would be of interest.

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CONCLUDING REMARKS

A solution to Rayleigh's problem for a compressible fluid, with slip boundary conditions, has been obtained and related to the flow past a semi-infinite flat plate. The results have been interpreted to define the border region between continuum and slip flow and to estimate the effect of slip.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, September 13, 1951

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APPENDIX A

SYMBOLS

a		speed of sound
C	=	$\rho_w\mu_w/\rho_0\mu_0$
C_D		flat-plate drag coefficient, $\text{drag}\Big/\frac{1}{2}\rho U^2 L$
C_f		local skin friction coefficient, $\tau\Big/\frac{1}{2}\rho U^2$
c_p		specific heat at constant pressure
erf(x)	=	$\frac{2}{\sqrt{\pi}}\int_0^x \exp(-r^2)\,dr$
erfc(x)	=	$1 - \text{erf}(x)$
J		mechanical equivalent of heat
K		constant relating Rayleigh's solution to flow past plate
k		thermal conductivity
L		length of plate
l		molecular mean free path
M		Mach number of free stream
p		static pressure
q		dynamic pressure, $\frac{1}{2}\rho U^2$
R		gas constant
Re		Reynolds number, $\rho_0 U x/\mu_0$
S		constant in Sutherland formula
T		gas temperature
t		time
U		velocity in free stream
u		velocity parallel to x-axis

v		velocity parallel to y-axis
x,y x,ψ		coordinate systems
γ		ratio of specific heats
δ		boundary-layer thickness
ζ	=	$\rho_0\sqrt{C\nu_0 t}/l\rho_w$
η	=	$\psi/2\sqrt{C\nu_0 t}$
μ		viscosity
ν		kinematic viscosity, μ/ρ
ρ		density
σ		Prandtl number, $\mu c_p/k$
τ		shear stress

Subscripts:

w	wall value
L	based on length of plate
0	free-stream value

Superscript:

*	state properties of gas evaluated at wall

APPENDIX B

EVALUATION OF T_w FOR ZERO HEAT TRANSFER

In the continuum regime, the wall temperature is given by (references 4 and 14)

$$\frac{T_w}{T_0} = 1 + \frac{\gamma - 1}{2} M^2(\sigma)^{1/2} \tag{B1}$$

where $\sigma = 0.72$ for air. A reasonable approximation for wall temperature, in the case of slip, might be

$$\frac{T_w}{T_0} = 1 + \frac{\gamma - 1}{2} M^2(\sigma)^{1/2} \left[1 - \left(\frac{u_w}{U}\right)^2\right] \tag{B2}$$

This expression is correct in the limiting cases $u_w = U$ and $u_w = 0$ and the dependency of T_w/T_0 on the square of u_w/U would seem correct from energy considerations. In the border region between continuum and slip flow, u_w/U is small (fig. 1); $(u_w/U)^2$ is then negligible in equation (B2) and equation (B1) applies.

The procedure for estimating the variation of T_w along an insulated plate would then be as follows: The wall temperature is first assumed constant at the value indicated by equation (B1). The corresponding variation of u_w/U along the plate can then be found from figure 1. These values for u_w/U can then be substituted into equation (B2) to yield the estimate of the local T_w.

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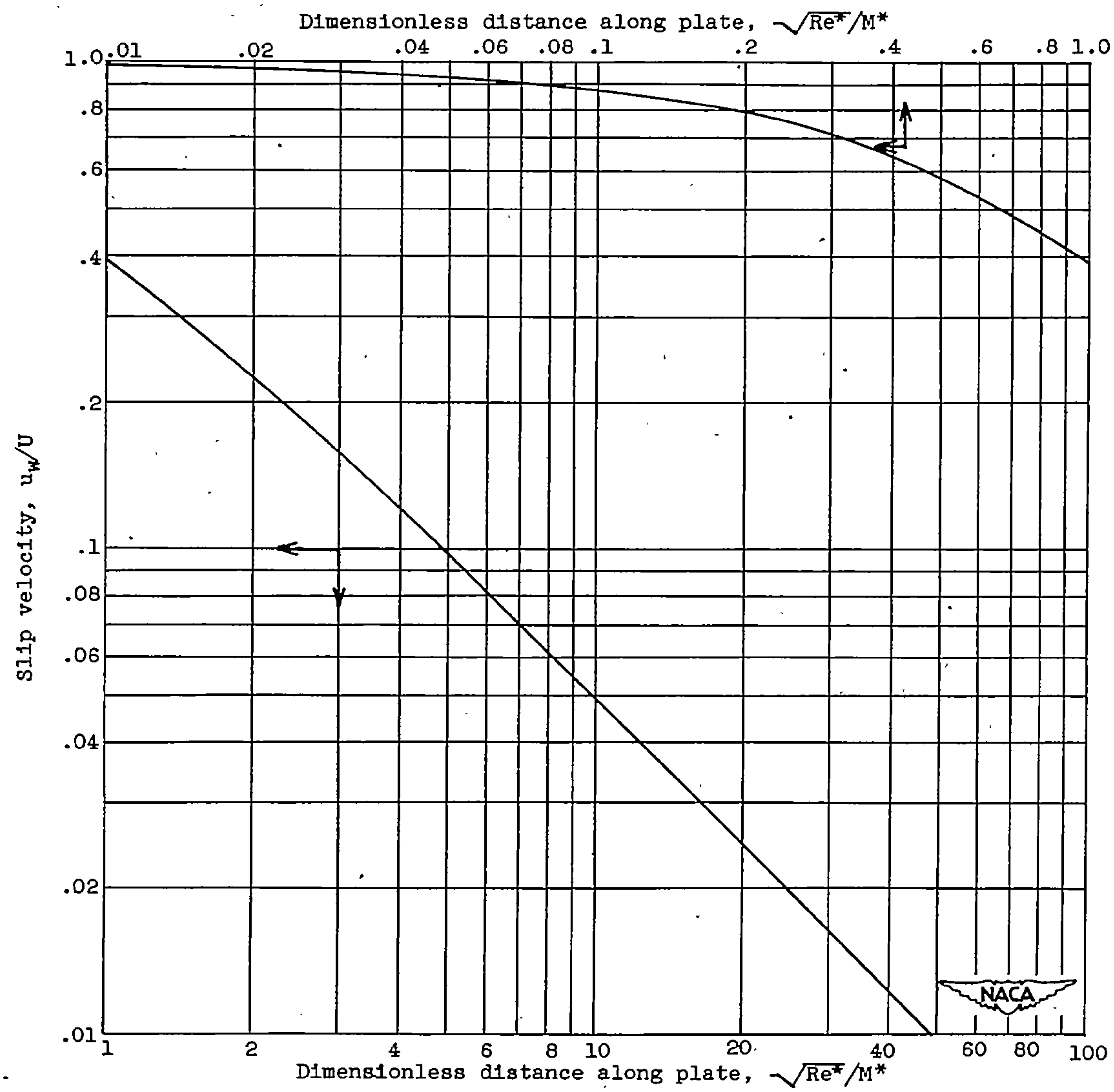

Figure 1. - Variation of slip velocity along flat plate. γ, 1.40.

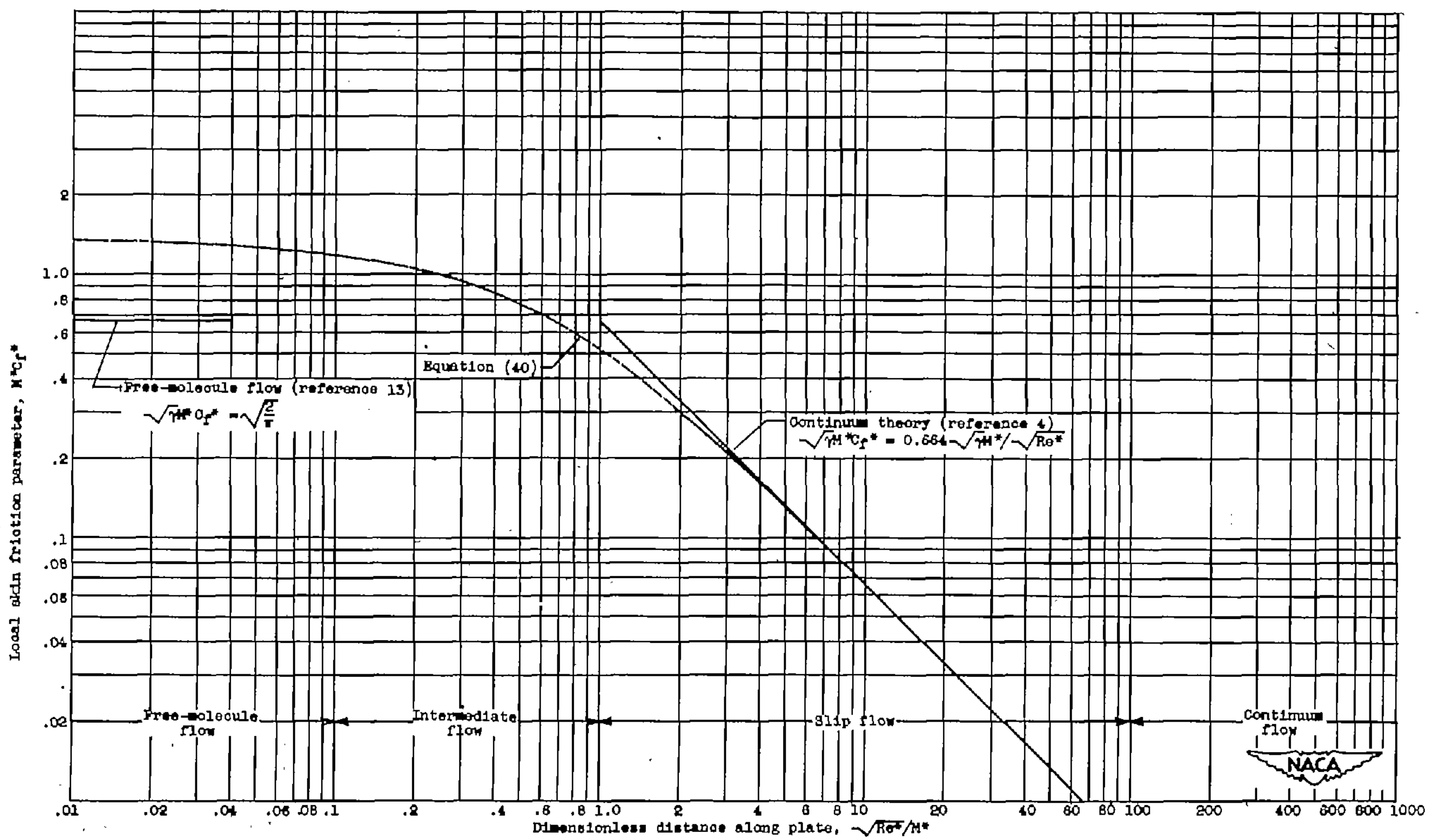

Figure 2. - Variation of local skin friction along flat plate. γ, 1.40.

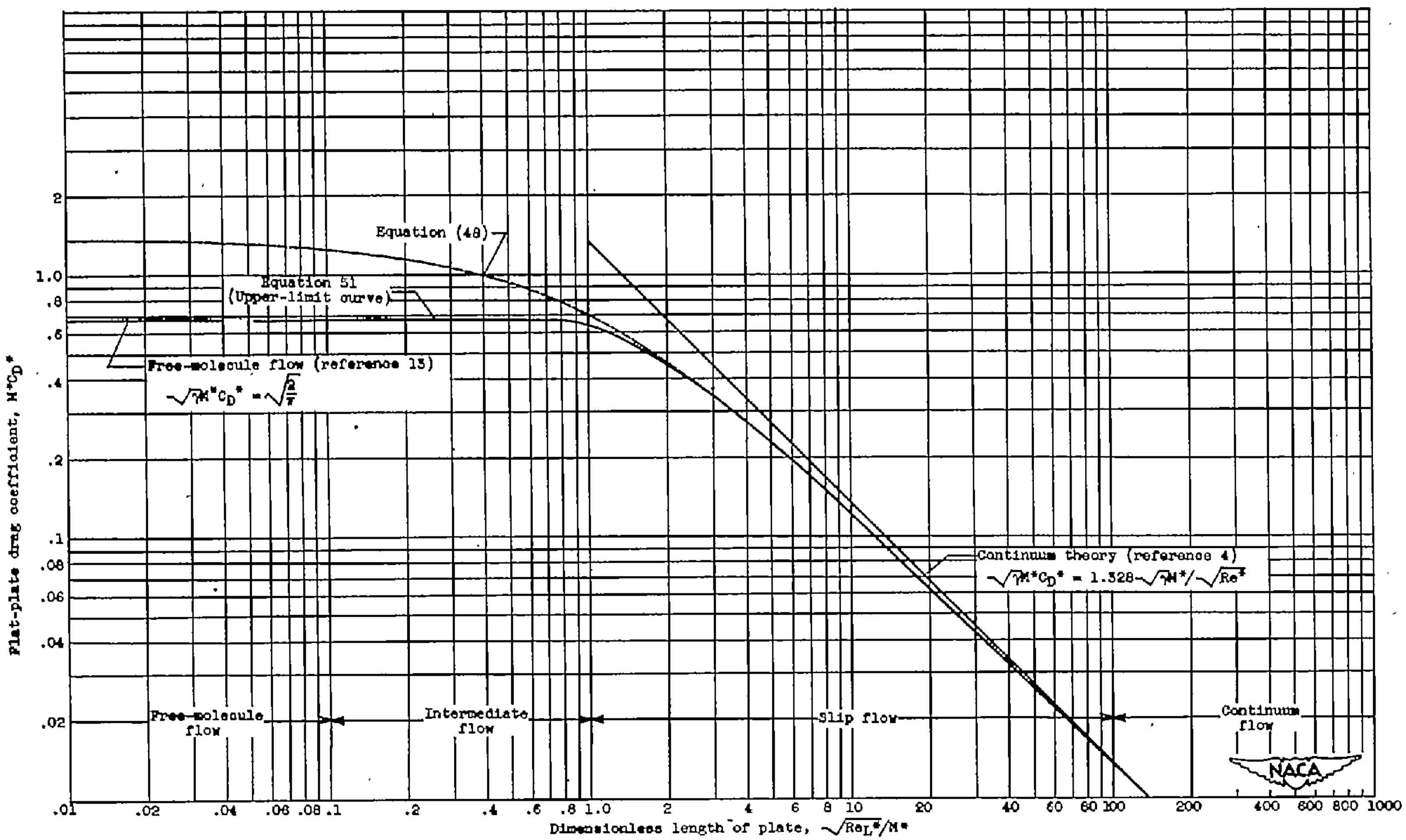

Figure 3. - Drag coefficient of flat plate. γ, 1.40.