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TECHNICAL NOTE 2692

ON THE FORM OF THE TURBULENT SKIN-FRICTION LAW AND ITS EXTENSION TO COMPRESSIBLE FLOWS

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SUMMARY

A derivation of the form of the incompressible turbulent skin-friction law for an insulated flat plate is made in such a way that it may be extended to compressible flows. The ratio of compressible to incompressible skin friction is obtained, and the results are shown to be in agreement with existing experimental results.

INTRODUCTION

The magnitude of the skin-friction drag encountered by a flat plate immersed in a fluid at Reynolds numbers large enough to insure turbulent flow has been one of the basic problems of aerodynamics. From the theoretical approaches of Prandtl and von Kármán (for a resumé, see reference 1) and the experimental work of numerous investigators, principally Nikuradse and Ludwieg and Tillmann (see references 2 and 3), much has been learned of the forms of the turbulent boundary-layer velocity profile and the skin friction associated with these forms in incompressible flows, although the exact mechanisms involved are still not completely understood.

Recently, the magnitude of the turbulent skin friction on a flat plate at high Mach numbers has become of great interest, and several papers have been written on this subject presenting both theoretical treatments of the problem and the results of skin-friction measurements at Mach numbers between 1.5 and 3.0. (See references 4 to 8.)

The agreement between these theories and the experimental data that exist is, in general, satisfactory. The status of the problem, however, is such that a simple physical approach to the extension of the incompressible skin-friction laws to the compressible case would seem desirable. The purpose of this paper is to present such a simple physical picture.

SYMBOLS

A	parameter $\left(\left[\frac{n(r-1)}{k^2}\frac{\nu_o}{u_o\delta}\right]^{\frac{1}{m+1}}\right)$
c_f	skin-friction coefficient $\left(\frac{2\tau_w}{\rho_o u_o^2}\right)$
h	roughness height
k	constant relating mixing length to y
l	mixing length
m,n	constants
M	Mach number
r	definitive ratio of total shear stress to laminar shear stress
R_L	Reynolds number $\left(\frac{u_L\delta_L}{\nu_o}\right)$
R_x	Reynolds number $\left(\frac{u_o x}{\nu_o}\right)$
R_δ	Reynolds number $\left(\frac{u_o\delta}{\nu_o}\right)$
T	absolute temperature
u	velocity in x-direction
v	velocity in y-direction
x	distance along surface
y	distance normal to surface
γ	ratio of specific heats
δ	boundary-layer thickness

μ viscosity

ν kinematic viscosity

ρ density

τ total shear stress

Subscripts:

L conditions at edge of laminar sublayer

o free-stream conditions

lam stress produced by laminar action alone

turb stress produced by turbulent action alone

w wall conditions

THE INCOMPRESSIBLE SKIN-FRICTION LAW

If the shearing stress due to turbulence in a region subject to a velocity gradient is assumed to be given by

$$\tau_{turb} = \rho l^2 \frac{\partial u}{\partial y} \left| \frac{\partial u}{\partial y} \right|$$

which is the mixing-length formula of Prandtl (see reference 1, p. 130), the equation of motion to be satisfied by the boundary layer on a flat plate is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y} + \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{1}$$

Many investigators have established that for most purposes

$$\frac{u}{u_o} = \left(\frac{y}{\delta}\right)^{\frac{1}{n}} \tag{2}$$

is an adequate representation of the boundary-layer velocity profile outside a very thin laminar sublayer. For a flat plate in the absence of pressure gradient the value of n is approximately 7. It is also well-known that the value of the mixing length in the boundary layer near the laminar sublayer is given by

$$l = ky \tag{3}$$

where k is approximately 0.4.

When the preceding facts are used, the ratio of the total stress to the laminar stress at points in the boundary layer near the wall may be found in the following manner:

Since

$$\frac{\tau}{\tau_{lam}} = \frac{\mu \frac{\partial u}{\partial y} + \rho l^2\left(\frac{\partial u}{\partial y}\right)^2}{\mu \frac{\partial u}{\partial y}}$$

the insertion of l from equation (3) and $\partial u/\partial y$ obtained from equation (2), namely,

$$\frac{\partial u}{\partial y} = \frac{u_o y^{\frac{1-n}{n}}}{n\delta^{\frac{1}{n}}}$$

yields

$$\frac{\tau}{\tau_{lam}} = 1 + \frac{u_o k^2 y^{\frac{n+1}{n}}}{n\nu\delta^{\frac{1}{n}}} \tag{4}$$

This equation is consistent with the idea that the total stress in the boundary layer becomes principally laminar as the surface of the plate is approached. The extent of this laminar sublayer may be reckoned by computing the value of y where the ratio of the total stress to the laminar stress is given by some definitive ratio. When this definitive ratio of total stress to laminar stress is assumed to be r, then from equation (4)

$$1 + \frac{u_o k^2 \delta_L^{\frac{n+1}{n}}}{n\nu\delta^{\frac{1}{n}}} = r \tag{5}$$

and the relative thickness of the laminar sublayer is

$$\frac{\delta_L}{\delta} = \left[\frac{n(r - 1)}{k^2} \frac{\nu}{u_o\delta}\right]^{\frac{n}{n+1}} \tag{6}$$

The velocity in the boundary layer at this point may be computed from equation (2) and is found to be

$$\frac{u_L}{u_o} = \left[\frac{n(r - 1)}{k^2} \frac{\nu}{u_o\delta}\right]^{\frac{1}{n+1}} \tag{7}$$

In the region $0 \leqq y \leqq \delta_L$ the flow is principally laminar and, since $\frac{\partial\tau}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} = 0$ at the wall as may be seen from equation (1), the velocity profile is essentially a straight line in this region. The boundary layer is now assumed, for the purpose of determining the wall stress, to be sharply divided into a turbulent region having a power-law velocity profile and a laminar region having a straight-line velocity profile as shown in figure 1 by the solid lines. The real state of affairs is indicated approximately by the dashed line. The stress in the laminar layer just below the assumed intersection, the point where $\delta = \delta_L$ and $u = u_L$, is now given by $\left(\mu \frac{\partial u}{\partial y}\right)_L$ which will be taken as $\mu u_L/\delta_L$. Since $\frac{\partial\tau}{\partial y} = 0$ at the wall or the stress in the laminar sublayer is approximately constant,

$$\tau_w = \frac{\mu u_L}{\delta_L} = \frac{\mu u_o}{\delta}\left[\frac{n(r - 1)}{k^2} \frac{\nu}{u_o\delta}\right]^{\frac{1-n}{n+1}} \tag{8}$$

The skin-friction coefficient is

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho u_o^2} = 2\left[\frac{n(r - 1)}{k^2}\right]^{\frac{1-n}{n+1}}\left(\frac{\nu}{u_o\delta}\right)^{\frac{2}{n+1}} \tag{9}$$

For $n = 7$, equation (9) becomes

$$c_f = 2\left[\frac{k^2}{7(r - 1)}\right]^{\frac{3}{4}} R_\delta^{-\frac{1}{4}} = \text{Constant } R_\delta^{-\frac{1}{4}} \tag{10}$$

Equation (10) is one form of the skin-friction law that has been shown to be very close to the true state of affairs for the turbulent boundary layer. The value of the constant found experimentally is 0.045. (See reference 1, p. 147.) The value of the ratio $\frac{k^2}{r - 1}$, which will yield this result, is 0.0444.

The assumption of a definitive ratio of stresses is tantamount to assuming that the Reynolds number $u_L\delta_L/\nu$, which is characteristic of the laminar sublayer, is a constant. This Reynolds number formed from the values of δ_L and u_L from equations (6) and (7) is

$$R_L = \frac{u_L\delta_L}{\nu} = \frac{n(r - 1)}{k^2} \tag{11}$$

Putting $n = 7$ and $\frac{k^2}{r - 1} = 0.0444$ into equation (11) gives

$$R_L = 158$$

It is interesting to consider the significance of this number. It may well be close to the critical Reynolds number of the straight-line velocity profile in the laminar sublayer. Below such a Reynolds number all disturbances would be damped; whereas above such a Reynolds number disturbances would feed on energy from the steady motion. This condition would define the laminar sublayer thickness and also a mechanism by which the turbulent boundary layer might receive energy in order to continue its existence.

An estimate of this stability criterion can be made by using the results of von Kármán (reference 9). The size of roughness height h required just to start to change the skin friction was found to be given approximately by

$$\frac{h}{\nu}\sqrt{\frac{\tau_w}{\rho}} \approx 3$$

also

$$\frac{h}{\delta_L} \approx \frac{1}{4}$$

If $\tau_w \approx \frac{\mu u_L}{\delta_L}$,

$$\frac{u_L \delta_L}{\nu} \approx 144$$

The agreement between this value and the value obtained by the present analysis is good.

An idea of the accuracy of the boundary-layer model assumed may be had by referring to figure 2, which is a replot of the data presented in figure 5 of reference 10. The velocity profile shown is that in a fully developed turbulent channel flow where the Reynolds number based on the half width of the channel is 12,200. The sharp transition from laminar profile to turbulent profile is quite evident. If the value for $\frac{k^2}{r - 1}$ found for the condition of $n = 7$ is substituted into equation (6), there results

$$\frac{\delta_L}{\delta} = \left(\frac{158}{R_\delta}\right)^{\frac{7}{8}} \tag{12}$$

For a value of $R_\delta = 12{,}200$, equation (12) yields a value of $\frac{\delta_L}{\delta} = 0.022$, which agrees well with the experimental value of 0.025 determined from the intersection of the straight lines in figure 2.

EXTENSION TO COMPRESSIBLE FLOWS

The extension of the foregoing derivation to compressible flows is straightforward. Equation (4) for the ratio of the total stress to the laminar stress still holds; however, since the value of ν is not constant, it must take on its local value dependent on y. The ratio of stresses at the edge of the laminar layer is therefore

$$\frac{\tau}{\tau_{lam}} = r = 1 + \frac{u_o k^2 \delta_L^{\frac{n+1}{n}}}{n\nu_L \delta^{\frac{1}{n}}}$$

so that

$$\frac{\delta_L}{\delta} = \left[\frac{n(r - 1)}{k^2} \frac{\nu_L}{u_o \delta}\right]^{\frac{n}{n+1}} \tag{13}$$

and

$$\frac{u_L}{u_o} = \left[\frac{n(r - 1)}{k^2} \frac{\nu_L}{u_o \delta}\right]^{\frac{1}{n+1}} \tag{14}$$

When the stress at the wall is evaluated it must be kept in mind that the value of the viscosity is not constant through the laminar sublayer. The velocity-profile shape assumed is shown in figure 3. The stress in the laminar sublayer just below the intersection is again assumed to be given by $\left(\mu \frac{\partial u}{\partial y}\right)_L$, which will be taken as $\mu_L u_L/\delta_L$. Although the viscosity may vary as the wall is approached, the stress must remain constant as before; therefore the resulting wall stress is

$$\tau_w = \frac{\mu_L u_L}{\delta_L} = \frac{\mu_L u_o}{\delta}\left[\frac{n(r - 1)}{k^2} \frac{\nu_L}{u_o \delta}\right]^{\frac{1-n}{n+1}} \tag{15}$$

The skin-friction coefficient becomes

$$c_f = 2\left[\frac{n(r - 1)}{k^2}\right]^{\frac{1-n}{n+1}}\left(\frac{\nu_L}{u_o \delta}\right)^{\frac{2}{n+1}} \frac{\rho_L}{\rho_o} \tag{16}$$

When equation (16) is written in the form

$$c_f = 2\left[\frac{n(r - 1)}{k^2}\right]^{\frac{1-n}{n+1}}\left(\frac{\nu_o}{u_o\delta}\right)^{\frac{2}{n+1}}\frac{\rho_L}{\rho_o}\left(\frac{\nu_L}{\nu_o}\right)^{\frac{2}{n+1}} \tag{17}$$

it may be compared with the value for incompressible flow given by

$$c_{f_{M=0}} = 2\left[\frac{n(r - 1)}{k^2}\right]^{\frac{1-n}{n+1}}\left(\frac{\nu_o}{u_o\delta}\right)^{\frac{2}{n+1}}$$

If the value of k is not changed appreciably in going to fairly high Mach numbers, for example, $0 \leqq M \leqq 4$ or 5, then for the same value of R_δ and n in each case,

$$\frac{c_f}{c_{f_{M=0}}} = \frac{\rho_L}{\rho_o}\left(\frac{\nu_L}{\nu_o}\right)^{\frac{2}{n+1}} \tag{18}$$

The magnitude of the reduction in skin friction with Mach number as given by equation (17) can be found in the following manner. Since the pressure is constant throughout the boundary layer

$$\frac{\rho_L}{\rho_o} = \frac{T_o}{T_L} \tag{19}$$

and when it is assumed that

$$\frac{\mu_L}{\mu_o} = \left(\frac{T_L}{T_o}\right)^m \tag{20}$$

equation (18) becomes

$$\frac{c_f}{c_{f_{M=0}}} = \left(\frac{\rho_L}{\rho_o}\right)^{\frac{n-1}{n+1}}\left(\frac{\mu_L}{\mu_o}\right)^{\frac{2}{n+1}} = \left(\frac{T_o}{T_L}\right)^{\frac{n-1}{n+1}}\left(\frac{T_L}{T_o}\right)^{\frac{2m}{n+1}} = \left(\frac{T_o}{T_L}\right)^{\frac{n-2m-1}{n+1}} \tag{21}$$

Since the discussion is for zero heat transfer and since the assumption of constant total energy in the turbulent layer is not liable to introduce too much error, the temperature ratio in equation (21) may be written

$$\frac{T_o}{T_L} = \left\{1 + \frac{\gamma - 1}{2} M^2\left[1 - \left(\frac{u_L}{u_o}\right)^2\right]\right\}^{-1} \tag{22}$$

Thus,

$$\frac{c_f}{c_{f_{M=0}}} = \left\{1 + \frac{\gamma - 1}{2} M^2\left[1 - \left(\frac{u_L}{u_o}\right)^2\right]\right\}^{\frac{1+2m-n}{n+1}} \tag{23}$$

For air where $\gamma = 1.4$, $m = 0.76$, and $n = 7$, equation (23) becomes

$$\frac{c_f}{c_{f_{M=0}}} = \left\{1 + \frac{M^2}{5}\left[1 - \left(\frac{u_L}{u_o}\right)^2\right]\right\}^{-0.56} \tag{24}$$

Equation (24) shows that the ratio of skin-friction coefficient for the compressible case to the incompressible skin-friction coefficient depends principally on Mach number and only to a very limited extent on Reynolds number since u_L/u_o is not particularly sensitive to Reynolds number.

The value of $\left(\frac{u_L}{u_o}\right)^2$ to be put into equation (24) is found from equation (13) as follows:

$$\frac{u_L}{u_o} = \left[\frac{n(r - 1)}{k^2} \frac{\nu_L}{u_o\delta}\right]^{\frac{1}{n+1}} = \left[\frac{n(r - 1)}{k^2} \frac{\nu_o}{u_o\delta}\right]^{\frac{1}{n+1}}\left(\frac{\nu_L}{\nu_o}\right)^{\frac{1}{n+1}}$$

then

$$\left(\frac{u_L}{u_o}\right)^2 = \left[\frac{n(r - 1)}{k^2} \frac{\nu_o}{u_o\delta}\right]^{\frac{2}{n+1}} \left(\frac{T_L}{T_o}\right)^{\frac{2(m+1)}{n+1}}$$

or

$$\left(\frac{u_L}{u_o}\right)^2 = \left[\frac{n(r - 1)}{k^2} \frac{\nu_o}{u_o\delta}\right]^{\frac{2}{n+1}} \left\{1 + \frac{M^2}{5}\left[1 - \left(\frac{u_L}{u_o}\right)^2\right]\right\}^{\frac{2(m+1)}{n+1}}$$

Thus,

$$\left[\left(\frac{u_L}{u_o}\right)^2\right]^{\frac{n+1}{2(m+1)}} + \frac{AM^2}{5}\left(\frac{u_L}{u_o}\right)^2 - \frac{A}{5}\left(5 + M^2\right) = 0 \tag{25}$$

where

$$A = \left[\frac{n(r - 1)}{k^2} \frac{\nu_o}{u_o\delta}\right]^{\frac{1}{m+1}}$$

Equation (25) may be solved for $\left(\frac{u_L}{u_o}\right)^2$ for any Mach number and Reynolds number R_δ. These results may then be inserted into equation (23) for $c_f/c_{f_{M=0}}$.

Figure 4 is a plot of $\left(\frac{u_L}{u_o}\right)^2$ and $c_f/c_{f_{M=0}}$ against Mach number obtained from equations (24) and (25) for several typical values of R_δ for a one-seventh-power velocity profile. The insensitivity of skin-friction-coefficient ratio to Reynolds number is quite evident.

It is always interesting for the sake of argument to see the effect on the skin-friction ratio as the Mach number approaches infinity, although in this case such an extension is not justifiable. As the Mach number approaches infinity, the term $\left(\frac{u_L}{u_o}\right)^2$ approaches unity (see equation (25)), but the expression inside the brace in equation (24) approaches $1/A$, so that

$$\lim_{M\to\infty}\left(\frac{c_f}{c_{f_{M=0}}}\right) = \left[\frac{n(r - 1)}{k^2}\frac{\nu_o}{u_o\delta}\right]^{\frac{n-1-2m}{(n+1)(m+1)}}$$

COMPARISON WITH EXPERIMENT AND OTHER METHODS

In figure 5 the results of this analysis are compared with the experimental results presented by Wilson (reference 7) and with the experimental and theoretical results of Rubesin, Maydew, and Varga (reference 6) as well as with the original suggestion by von Kármán (reference 11). The result obtained theoretically by Van Driest (reference 8) for a Reynolds number R_x equal to 7×10^6 is also compared. This result may be compared since the values of R_δ used in figure 5 cover a range sufficient to encompass the value of δ that might exist at x in the analysis of Van Driest. The agreement between this method and the experimental results to date which are described very well by the curves of Rubesin and Wilson is good. The curve obtained by the method suggested by Monaghan (reference 4) almost coincides with the experimental curve of Wilson and therefore is in good agreement with that of the present method.

CONCLUSIONS

A derivation of the form of the incompressible turbulent skin-friction law for an insulated flat plate is made in such a way that it may be extended to compressible flows. The following conclusions may be drawn:

(1) The incompressible argument demonstrates that a Reynolds number characteristic of the laminar sublayer $u_L\delta_L/\mu$ is a constant approximately equal to 158. This value may be close to the critical Reynolds number of the straight-line velocity profile in the laminar sublayer below which all disturbances are damped and above which disturbances may feed on the energy of the steady motion.

(2) An expression for the ratio of compressible skin-friction coefficient to incompressible skin-friction coefficient at the same value of Reynolds number based on the boundary-layer thickness on an insulated flat plate is derived. The agreement between the compressible and incompressible skin-friction coefficients found experimentally and those predicted by the present method is good.

(3) The ratio of compressible to incompressible skin friction is shown to be rather insensitive to Reynolds number based on the boundary-layer thickness and to be principally a function of Mach number.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
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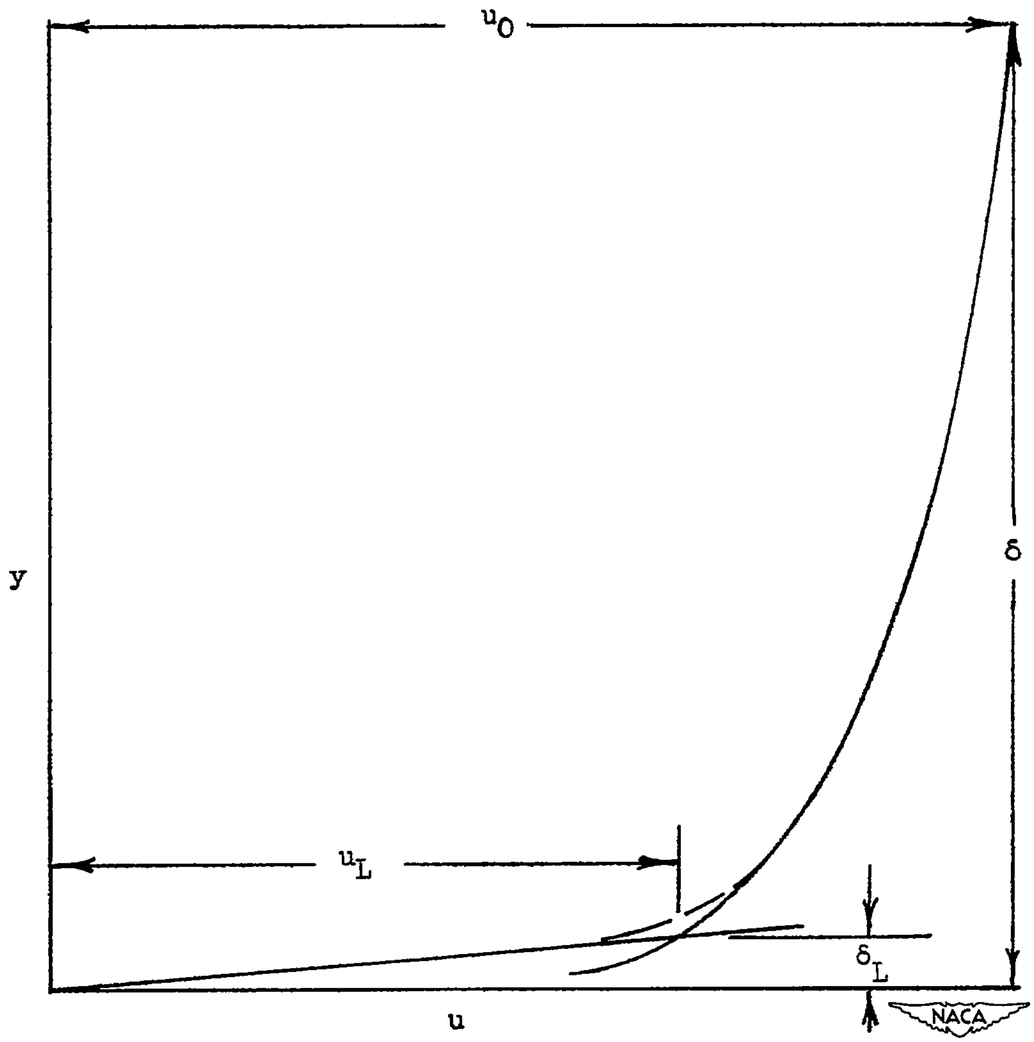

Figure 1.- Velocity profile assumed for incompressible analysis.

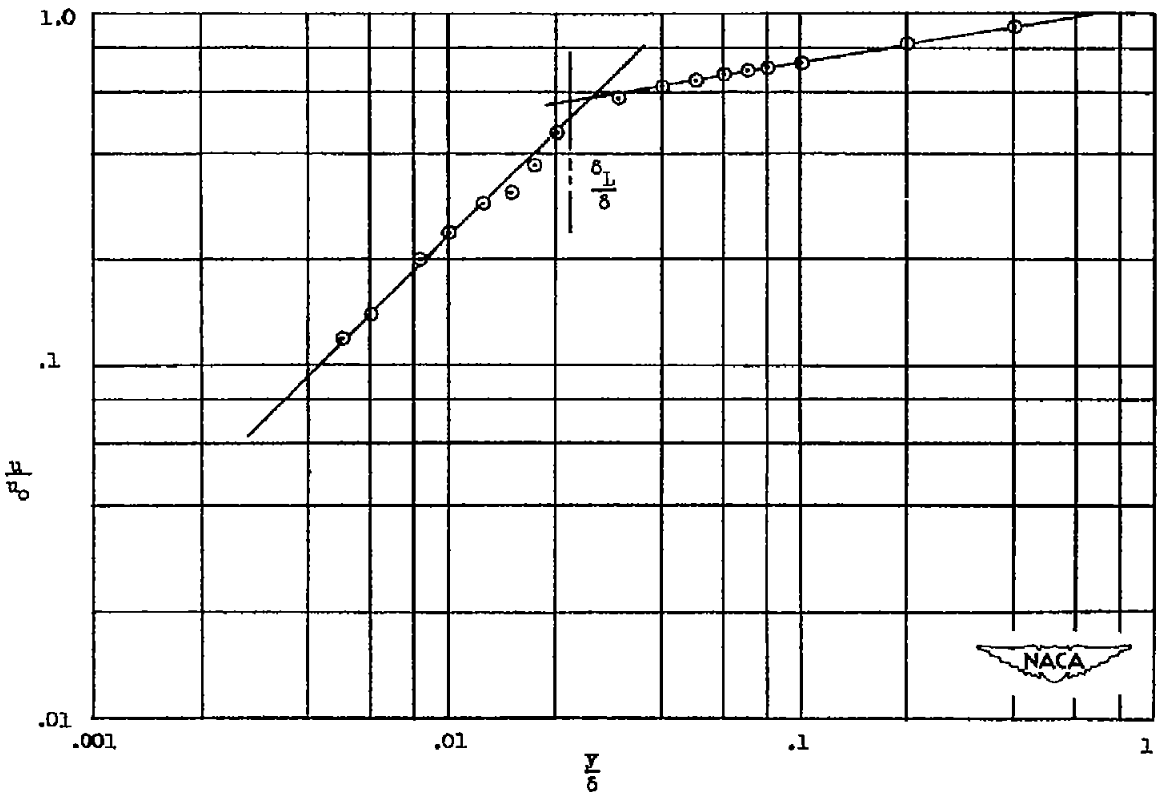

Figure 2.- Agreement between assumed and experimental profiles from reference 10. $R_\delta = 12,200$.

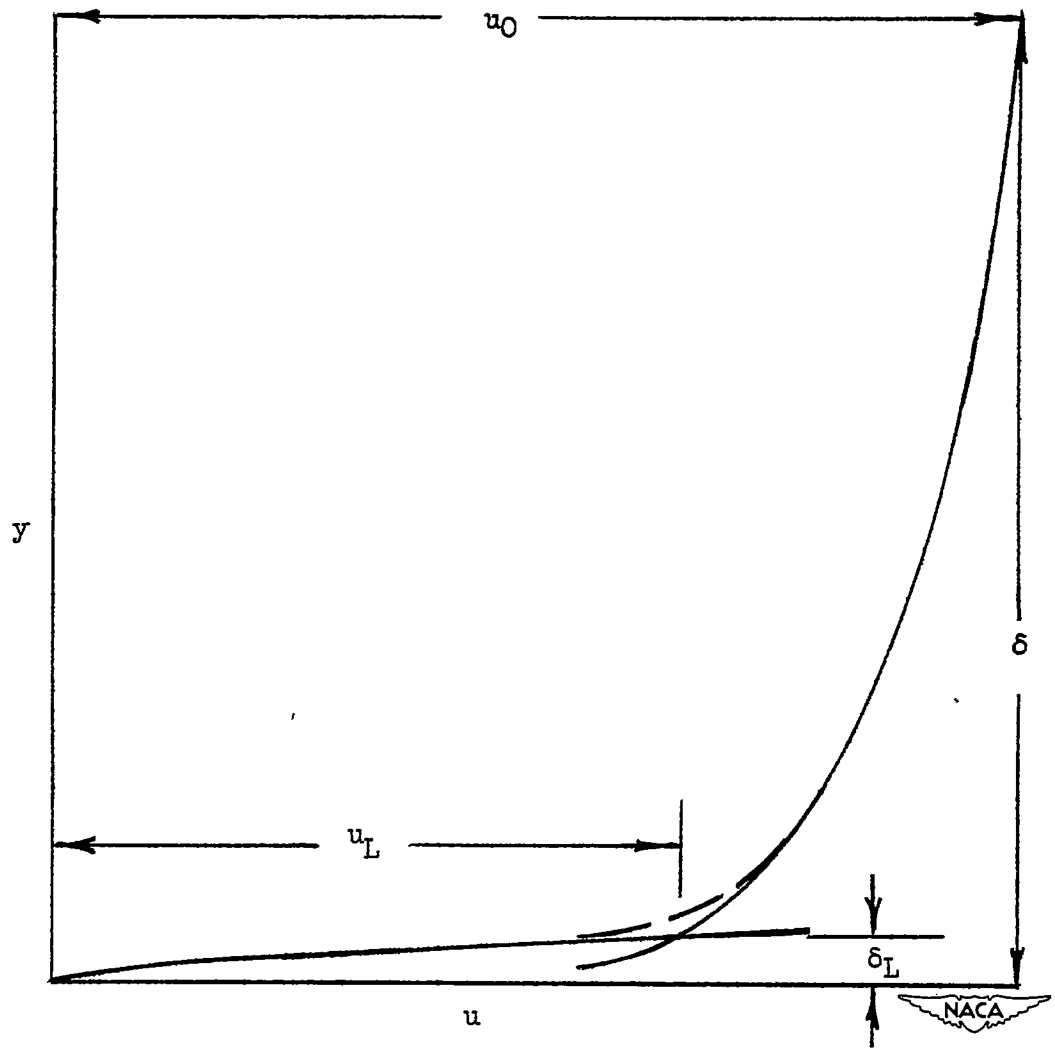

Figure 3.- Velocity profile assumed for compressible analysis.

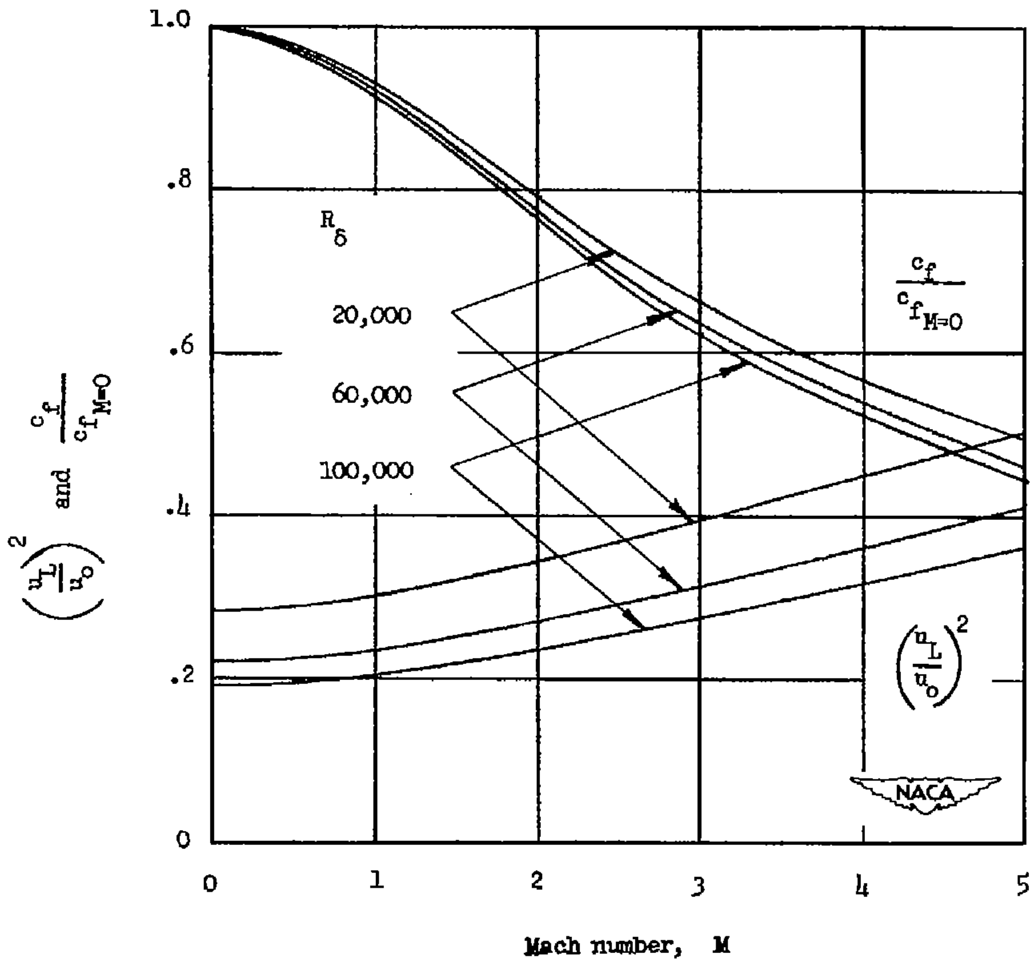

Figure 4.- Plot of $\frac{c_f}{c_{f_{M=0}}}$ and $\left(\frac{u_L}{u_o}\right)^2$ against Mach number as determined from equations (24) and (25).

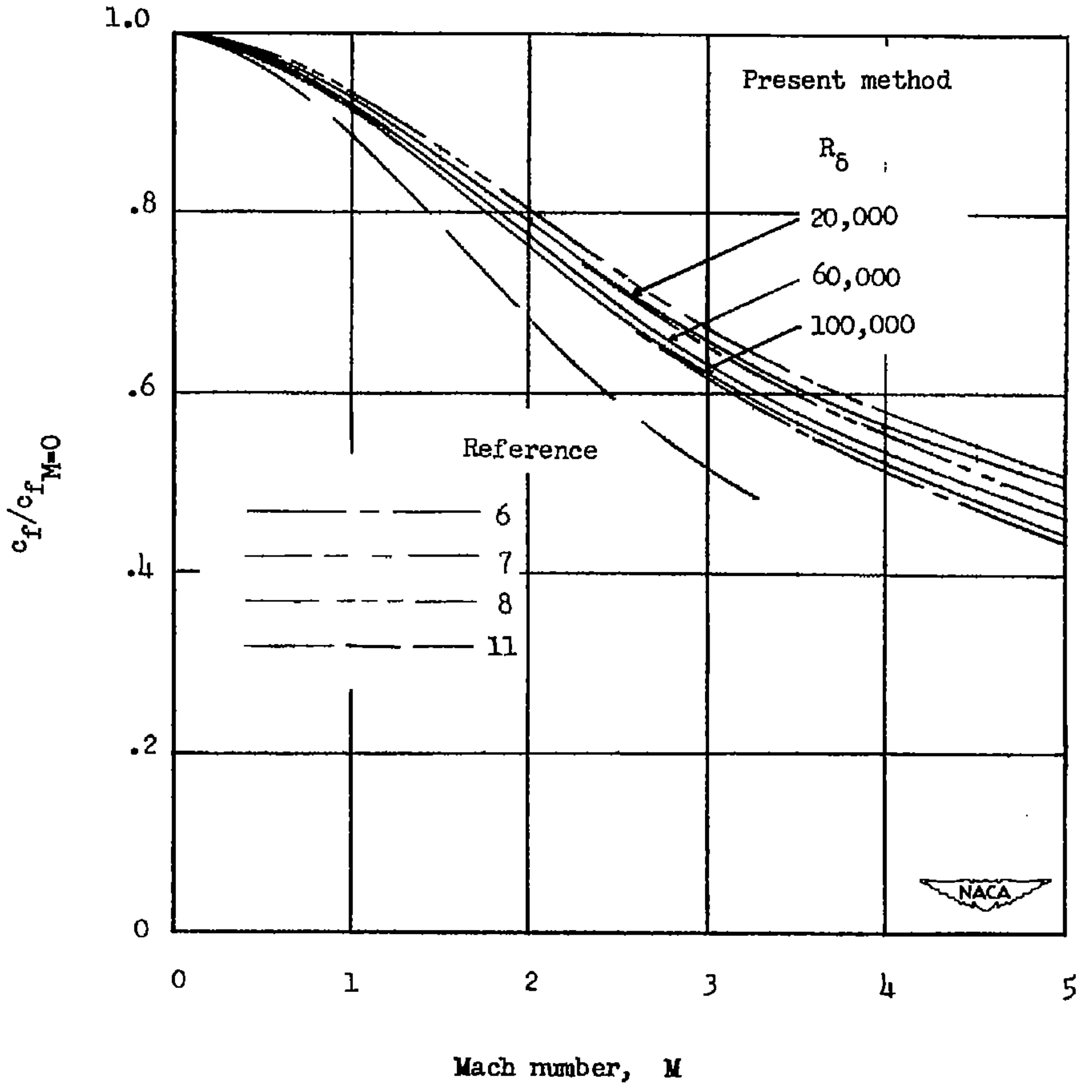

Figure 5.- Comparison of experiment with present method.

NACA-Langley - 7-21-54 - 75