NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2837

CORRECTIONS FOR DRAG, LIFT, AND MOMENT OF AN AXIALLY SYMMETRICAL BODY PLACED IN A SUPERSONIC TUNNEL HAVING A TWO-DIMENSIONAL PRESSURE GRADIENT

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SUMMARY

The corrections for drag, lift, and moment are derived for an axially symmetrical body placed in the test section of a supersonic tunnel, on the assumption that the test section is characterized by a two-dimensional pressure field originating from construction flaws. Although relatively simple longitudinal and transverse pressure gradients are assumed, the analytical treatment becomes rather difficult because of the difference in symmetry between the body and the basic flow field.

Assuming irrotational conditions, the velocity potential of the flow around the body is expanded in a threefold manner: (1) In powers of the thickness parameter of the body ϵ, (2) in powers of a parameter b characterizing the inhomogeneity of the basic flow field, and (3) as a Fourier series in the azimuth θ around the body axis. Each expansion is taken into account not further than up to the second term.

Upon substitution of this potential series, the nonlinear equation of motion and the boundary condition on the body surface are split into a set of linearized boundary-value problems which can be solved analytically. The mathematical techniques used for the solution are explained in appendixes.

Assuming the two-dimensional pressure field, the drag, lift, and moment corrections for arbitrary body shapes are obtained in closed analytic form. The physical meaning of the results and their validity are discussed.

INTRODUCTION

Consider an axially symmetrical body placed in the test section of a supersonic tunnel. The test section, instead of providing uniform flow, may be characterized by a two-dimensional pressure field. In general, such a pressure field will consist of a longitudinal as well as a transverse pressure gradient, producing a stream-angle variation along the tunnel axis.

The difference in symmetry between the body and the field engenders considerable difficulty in the analytic treatment of the problem, involving a Fourier expansion of the disturbance potential of the body. Therefore, a relatively simple, linear gradient is assumed: On the axis, the horizontal component of velocity may equal the original velocity of the uniform stream U_o, but the transverse gradient may produce a vertical velocity component

$$q_y = bx \tag{1}$$

yielding the desired stream-angle variation. Here b is a small parameter which characterizes the first-order deviation of the actual basic flow from the uniform field.

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SYMBOLS

b parameter characterizing inhomogeneity of pressure field

$\bar{b} = bl/c_o$

c local sound speed

c_o sound speed corrresponding to U_o

D drag

$f(x)$ body profile function

$k(x) = \frac{1}{2}\left[f(x)\right]^2$

L lift

l body length

M moment

$M_o = U_o/c_o$

p	local static pressure
p_o	static pressure at velocity U_o

$q^2 = u^2 + v^2 + w^2$

U_o	original velocity of basic field
u,v,w	axial, radial, and circumferential velocity components
x,r,θ	cylindrical coordinates
x,y,z	rectangular coordinates
α	local angle of attack

$\beta = \sqrt{M_o^2 - 1}$

γ	adiabatic exponent
ϵ	body thickness parameter
ρ	local density
ρ_o	density at velocity U_o
φ	potential function

TWO-DIMENSIONAL BASIC FLOW FIELD

The two-dimensional velocity pontential φ of the basic flow field satisfies the well-known potential equation:

$$\left(c^2 - \varphi_x^2\right)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + \left(c^2 - \varphi_y^2\right)\varphi_{yy} = 0 \qquad (2)$$

where c is the local velocity of sound. The Bernoulli equation yields

$$c^2 = c_o^2 + \frac{\gamma - 1}{2}\left(U_o^2 - q^2\right) \tag{3}$$

where

$$q^2 = \varphi_x^2 + \varphi_y^2$$

c_o is the value of c when $q = U_o$, and γ is the adiabatic exponent.

Now assume that the basic potential may be represented as a formal power series in b:

$$\varphi = \varphi^{00} + b\varphi^{01} + b^2\varphi^{02} + \ldots \tag{4}$$

This procedure will permit determination of the functions φ^{00}, φ^{01}, and so forth. It is immaterial whether this procedure will lead to a convergent power series, since the results may be considered as satisfactory even if the series has only an asymptotic character.

Choose

$$\varphi^{00} = U_o x \tag{5}$$

and require that for $y = 0$ (i.e., on the x-axis)

$$\left.\begin{aligned} \varphi_x &= U_o \\ \varphi_y &= bx \end{aligned}\right\} \tag{6}$$

(see "Introduction"). This implies that, for $y = 0$,

$$\left.\begin{aligned} \varphi_x^{01} &= 0 \\ \varphi_y^{01} &= x \\ \varphi_x^{0n} &= \varphi_y^{0n} = 0 \qquad \text{for } n > 1 \end{aligned}\right\} \tag{7}$$

Inserting equation (4) into equation (2) and requiring that this be an identity in b, it is found that, up to second-order terms in b,

$$\beta^2\varphi_{xx}{}^{01} - \varphi_{yy}{}^{01} = 0 \tag{8}$$

where

$$\beta^2 = \frac{U_o{}^2}{c_o{}^2} - 1 = M_o{}^2 - 1$$

and

$$\beta^2\varphi_{xx}{}^{02} - \varphi_{yy}{}^{02} = -\frac{M_o}{c_o}\left\{\left[(\gamma + 1) + (\gamma - 1)\beta^2\right]\varphi_x{}^{01}\varphi_{xx}{}^{01} + 2\varphi_y{}^{01}\varphi_{xy}{}^{01}\right\} \tag{9}$$

The general solution of equation (8) is

$$\varphi^{01} = f\left(x - \beta_o y\right) + g\left(x + \beta_o y\right)$$

where f and g are arbitrary functions.

In order that conditions (7) be satisfied, one must have

$$f'(x) + g'(x) = 0$$

$$-\beta\left[f'(x) - g'(x)\right] = x$$

This implies that

$$f(x) = -g(x) = -\frac{1}{4\beta}x^2$$

from which it follows that

$$\varphi^{01} = \frac{1}{4\beta}\left[(x + \beta y)^2 - (x - \beta y)^2\right] = xy \tag{10}$$

Inserting this result into equation (9) gives

$$\beta^2\varphi_{xx}{}^{02} - \varphi_{yy}{}^{02} = -2\,\frac{M_o}{c_o}\,x \tag{9a}$$

Now, $-\dfrac{M_o}{3\beta^2 c_o}\,x^3$ is a particular solution of equation (9a). Hence, the general solution is

$$\varphi^{02} = -\frac{M_o}{3\beta^2 c_o}\,x^3 + F(x - \beta y) + G(x + \beta y)$$

where F and G are arbitrary functions.

Proceeding as before to satisfy conditions (7), one easily finds that

$$F(x) = G(x) = \frac{M_o}{6\beta^2 c_o}\,x^3$$

Substitution yields for φ^{02} the simple form

$$\varphi^{02} = \frac{M_o}{c_o}\,xy^2 \tag{11}$$

From equations (4), (10), and (11), the potential of the basic flow becomes:

$$\varphi_{basic} = U_o x + bxy + \frac{M_o}{c_o}\,xy^2\,b^2 + \ldots \tag{12}$$

Hence:

$$\left.\begin{aligned}\varphi_x &= U_o + by + \frac{M_o}{c_o}\,y^2 b^2 + \ldots\\ \varphi_y &= bx + 2\,\frac{M_o}{c_o}\,xyb^2 + \ldots\end{aligned}\right\} \tag{13}$$

The equation of the streamlines, in the present approximation, turns out to be:

$$y - \frac{2M_o^2 - 1}{2M_o c_o} y^2 b + \frac{4M_o^2 - 1}{c_o^2} y^3 b^2 = \frac{1}{2M_o c_o} x^2 b + \text{Constant} \tag{14}$$

From the Bernoulli equation (equation (3)), in combination with the adiabatic relations

$$\left.\begin{aligned} c^2 &= \gamma \frac{p}{\rho} \\ c_o^2 &= \gamma \frac{p_o}{\rho_o} \end{aligned}\right\} \tag{15}$$

where ρ is the density, one obtains for the excess pressure $\Delta p = p - p_o$ the expression:

$$\frac{\Delta p}{\rho_o} - \frac{(\Delta p)^2}{2\rho_o^2 c_o^2} = \frac{1}{2}\left(U_o^2 - q^2\right) \tag{16}$$

Using equations (13), this yields, up to second-order terms in b,

$$\frac{\Delta p}{\rho_o} = -U_o yb - \frac{1}{2}\left[\left(M_o^2 + 1\right)y^2 + x^2\right]b^2 \tag{17a}$$

and on the x-axis,

$$\frac{\Delta p}{\rho_o} = -\frac{1}{2} x^2 b^2 \tag{17b}$$

Therefore the pressure gradient of the basic flow has the two components:

$$\left.\begin{aligned} p_x &= (\Delta p)_x = -\rho_o x b^2 \\ p_y &= (\Delta p)_y = -\rho_o U_o b - \rho_o\left(M_o^{\,2} + 1\right) y b^2 \end{aligned}\right\} \tag{18}$$

In the manner indicated above, all the higher-order coefficients in the expansion for φ could be determined successively. (They turn out, indeed, to be polynomials in x and y.) These coefficients are, however, not needed in the present work. It is understood that expressions (12), (13), and (17) for the basic potential, the corresponding velocity components, and the excess pressure are valid only in a restricted region of the working section around the disturbing body.

AXIALLY SYMMETRICAL SLENDER BODY IN TWO-DIMENSIONAL BASIC FLOW - THE DISTURBANCE POTENTIAL

Differential Equations

It is assumed that the total velocity potential φ can be approximately written as

$$\varphi = \left(\varphi^{00} + b\varphi^{01} + b^2\varphi^{02}\right) + \left(\epsilon^2\varphi^{10} + b\epsilon^2\varphi^{11}\right) + \ldots \tag{19}$$

This assumption is by no means trivial nor arbitrary. It is justified by the fact that in this way it is possible to satisfy the boundary conditions on the body and the "characteristic condition" (see appendix A) in such a way that the error committed in φ as given by equation (19) is small compared with the terms written down. The method adopted here is such that it can be extended (stepwise) to better and better approximations. It is not obvious, however, what the dependence of the next term on b and ϵ is. The method, if pushed further, leads to a formal series in b, ϵ, and $\log_e \epsilon$, with, at best, an asymptotic character. For further elucidation, see the section "Boundary Conditions."

The terms in the first parentheses in equation (19) represent the basic flow potential as discussed in the section "Two-Dimensional Basic Flow Field," while those in the second parentheses represent the disturbance potential produced by the body, and ϵ is the thickness parameter of the body whose equation, written in cylindrical coordinates (r, θ, x), is assumed to be

$$r = \epsilon f(x) \qquad 0 \leqq x \leqq l$$

$$f(0) = f(l) = 0$$

The first term in the disturbance potential is the well-known term of the linearized theory. The second term represents the influence of the pressure gradient (of the basic flow) on the disturbance potential. In the present work, higher-order approximations will not be considered.

An important and rather restrictive assumption has been tacitly made by using expansion (19) for φ, namely, that

$$\epsilon^2 \log_e \epsilon \ll \bar{b} \ll \epsilon \tag{20}$$

(In order to compare different orders of magnitude, the dimensionless quantity $\bar{b} = bl/c_0$ is introduced. In the equations in the text, the dimensional parameter b is kept, and inequality (20) is taken into consideration in a suitable manner.)

That expression (20) follows from equation (19) can be seen from the following argumentation. If one sets $\epsilon = 0$, the potential φ must reduce to the basic potential, since a needlelike object produces no disturbance in a three-dimensional flow. On the other hand, if one sets $b = 0$ (i.e., assumes that the basic flow is uniform), φ must reduce to the disturbance potential of a body moving in uniform flow. It is well-known (see reference 1) that the second term of this potential is of the order $\epsilon^4 \log_e \epsilon$. This term has been omitted in equation (19) as small compared with a term in $\epsilon^2 b$, and this is permissible only if $\bar{b} \gg \epsilon^2 \log_e \epsilon$. On the other hand, the term in b^3 in the basic potential has been omitted as small compared with all terms written down, in particular, with the smallest term $b\epsilon^2$. This implies that $\bar{b} \ll \epsilon$. Without this restrictive assumption (20) the computational work would become prohibitive.

The cylindrical coordinates x, r, and θ are introduced in such a way that the plane of the basic flow is the plane $\theta = 0$. Then, putting $y = r\cos\theta$ and $z = r\sin\theta$, the potential equation for $\varphi(x,r,\theta)$ becomes

$$\left(c^2 - \varphi_x^{\,2}\right)\varphi_{xx} + \left(c^2 - \varphi_r^{\,2}\right)\varphi_{rr} + \left(c^2 - \frac{1}{r^2}\varphi_\theta^{\,2}\right)\frac{1}{r^2}\varphi_{\theta\theta} - \frac{2}{r^2}\varphi_r\varphi_\theta\varphi_{r\theta} -$$

$$\frac{2}{r^2}\varphi_x\varphi_\theta\varphi_{x\theta} - 2\varphi_x\varphi_r\varphi_{xr} + \left(c^2 + \frac{1}{r^2}\varphi_\theta^{\,2}\right)\frac{1}{r}\varphi_r = 0 \tag{21}$$

Here, c^2 is again given by equation (3), where now $q^2 = \varphi_x^{\,2} + \varphi_r^{\,2} + \frac{1}{r^2}\varphi_\theta^{\,2}$. Inserting expansion (19) into potential equation (21), and ordering terms in powers of b and ϵ, one obtains in orders b and b^2 equations (8) and (9); in order ϵ^2,

$$\beta^2\varphi_{xx}^{\;10} - \varphi_{rr}^{\;10} - \frac{1}{r}\varphi_r^{\;10} = 0 \tag{22}$$

in order $\epsilon^2 b$,

$$\beta^2\varphi_{xx}^{\;11} - \varphi_{rr}^{\;11} - \frac{1}{r}\varphi_r^{\;11} - \frac{1}{r^2}\varphi_{\theta\theta}^{\;11} =$$

$$-\frac{M_o}{c_o}\left\{\left[(\gamma + 1) + (\gamma - 1)\beta^2\right]r\varphi_{xx}^{\;10} + 2x\varphi_{xr}^{\;10} + 2\varphi_r^{\;10}\right\}\cos\theta \tag{23}$$

Equation (22) is the well-known equation for the linearized potential φ^{10}. The term $\frac{1}{r^2}\varphi_{\theta\theta}^{\;10}$ has been omitted since, as is well-known, the linearized potential does not depend on θ. Equation (23) for the "interaction potential" φ^{11} is inhomogeneous and, since its right-hand side involves θ explicitly, φ^{11} certainly depends on θ. Expanding φ^{11} into a Fourier series (since the problem is symmetric in θ, φ^{11} is obviously a cosine Fourier series) there is obtained:

$$\varphi^{11} = \sum_{n=0}^{\infty} \varphi_n{}^{11}(x,r)\cos n\theta \tag{24}$$

Substituting equation (24) in equation (23), it is found that the Fourier coefficients satisfy the differential equations

$$\beta^2\varphi_{n_{xx}}{}^{11} - \varphi_{n_{rr}}{}^{11} - \frac{1}{r}\varphi_{n_r}{}^{11} + \frac{n^2}{r^2}\varphi_n{}^{11} = 0 \qquad n \neq 1 \tag{25}$$

$$\beta^2\varphi_{1_{xx}}{}^{11} - \varphi_{1_{rr}}{}^{11} - \frac{1}{r}\varphi_{1_r}{}^{11} + \frac{1}{r^2}\varphi_1{}^{11} = -\frac{M_o}{c_o}S(x,r) \tag{26}$$

where

$$S(x,r) = \left[(\gamma + 1) + (\gamma - 1)\beta^2\right]r\varphi_{xx}{}^{10} + 2\varphi_r{}^{10} + 2x\varphi_{xr}{}^{10} \tag{27}$$

It will be shown in the next section that the boundary conditions can be satisfied only by making $\varphi_n{}^{11} \equiv 0$ for $n \neq 1$. Hence the interaction potential reduces to

$$\varphi^{11} = \varphi_1{}^{11}(x,r)\cos\theta \tag{28}$$

Boundary Conditions

The flow velocity must be tangential to the body at its boundary, hence

$$\varphi_r = \epsilon f'(x)\varphi_x \tag{29}$$

for

$$r = \epsilon f(x)$$

Inserting expansion (19) in equation (29) and taking account of equations (5), (10), and (11), the boundary condition becomes:

$$bx \cos \theta + b^2\left(2 \frac{M_o}{c_o} xr \cos^2\theta\right) + \epsilon^2 \varphi_r^{10} + b\epsilon^2 \sum_{n=0}^{\infty} \varphi_{n_r}^{11} \cos n\theta + \ldots =$$

$$\epsilon f'(x)\left(U_o + br \cos \theta + b^2 \frac{M_o}{c_o} r^2 \cos^2\theta + \epsilon^2 \varphi_x^{10} + \right.$$

$$\left. b\epsilon^2 \sum_{n=0}^{\infty} \varphi_{n_x}^{11} \cos n\theta + \ldots \right) \tag{29a}$$

for

$$r = \epsilon f'(x)$$

To be able to order the terms of equation (29a) in an appropriate manner, it has to be known how φ_x^{10}, φ_r^{10}, φ_x^{11}, and φ_r^{11} depend on ϵ for $r = \epsilon f(x)$.

This knowledge is gained by considering solutions of the differential equation

$$\beta^2\psi_{xx} - \psi_{rr} - \frac{1}{r} \psi_r + \frac{n^2}{r^2} \psi = 0 \tag{30}$$

which vanish to the left of the cone $r = x/\beta$ and tend to zero as one approaches infinity along a characteristic $r = x/\beta +$ Constant. It is shown in appendix B that such solutions are given by

$$\psi = -\frac{1}{2^n r^n n!} \int_0^{x-\beta r} \frac{\left[(x - \xi) + \sqrt{(x - \xi)^2 - \beta^2 r^2}\right]^n + \left[(x - \xi) - \sqrt{(x - \xi)^2 - \beta^2 r^2}\right]^n}{\sqrt{(x - \xi)^2 - \beta^2 r^2}} g^{(n)}(\xi)\, d\xi \tag{31}$$

where $g(\xi)$ is an arbitrary, n times differentiable function which is identically zero for $\xi < 0$, and that, as $r \longrightarrow 0$,

$$\left.\begin{aligned} \psi_x &= \begin{cases} O\left(\frac{1}{r^n}\right) & n \neq 0 \\ O(\log_e r) & n = 0 \end{cases} \\ \psi_r &= O\left(\frac{1}{r^{n+1}}\right) \end{aligned}\right\} \tag{32}$$

It may now be shown how the boundary conditions for φ^{10}, φ^{11}, and so forth can be determined successively in a unique way. First, φ^{10} satisfies equation (30) with $n = 0$; hence $\varphi_r^{10} = O\left(\frac{1}{r}\right)$ for $r \longrightarrow 0$, and $\epsilon^2 \varphi_r^{10}$ is of the order ϵ for $r = \epsilon f(x)$. Matching terms in ϵ in equation (29a), one now gets the condition

$$\lim_{r \longrightarrow 0} r\varphi_r^{10} = U_0 f(x) f'(x) \tag{33}$$

This condition determines φ^{10} uniquely (see the section "Solution of Boundary-Value Problems"). Next consider φ_0^{11} which also satisfies equation (30) with $n = 0$. Using equations (32), the term $b\epsilon^2 {\varphi_0}_r^{11} = O(b\epsilon)$ as $r = \epsilon f(x)$; since there are no other terms of this order in equation (29a), one must set $\varphi_0^{11} \equiv 0$, which of course is a solution of equation (25). Now, the terms $b\epsilon^2 {\varphi_n}_r^{11}$ for $n > 1$ are of the order $b\epsilon^{1-n}$ and become infinite as $\epsilon \longrightarrow 0$, unless $\psi_n^{11} \equiv 0$ for all values of n greater than 1. This proves the statement made at the end of the section "Differential Equations." The potential φ_1^{11} does not satisfy equation (30) but is a solution of an equation with the same left-hand side with a nonvanishing right-hand side. Thus φ_1^{11} is obtained by adding to an expression of the type of equation (31) a particular solution of equation (26). It is shown in the section "Solution of Boundary-Value

Problems" that it is possible to find such a particular solution with the r derivative of higher order than $\frac{1}{r^2}$ for $r \longrightarrow 0$. Hence the singularity of ${\varphi_1}_r^{11}$ is that of ψ for $n = 1$, that is, ${\varphi_1}_r^{11} = 0\left(\frac{1}{r^2}\right)$ as $r \longrightarrow 0$, and the term

$$b\epsilon^2 {\varphi_1}_r^{11} \cos\theta = \frac{b}{\left[f(x)\right]^2} \lim_{r \longrightarrow 0} \left(r^2 {\varphi_1}_r^{11}\right) \cos\theta +$$

Terms vanishing to higher order

The only term of this order in equation (29a) is $bx \cos\theta$ and so one gets the condition:

$$\lim_{r \longrightarrow 0} \left(r^2 {\varphi_1}_r^{11}\right) = -xf^2(x) \tag{34}$$

Grouping the matched and unmatched terms in equation (29a) and substituting $r = \epsilon f(x)$ wherever r appears explicitly, equation (29a) now reads:

$$\frac{1}{f(x)}\left[\lim_{r \longrightarrow 0} \left(r\varphi_r^{10}\right) - U_o f(x)f'(x)\right]\epsilon + \frac{1}{f^2(x)}\left[\lim_{r \longrightarrow 0} \left(r^2 {\varphi_1}_r^{11}\right) + xf^2(x)\right] b\cos\theta + \frac{M_o}{c_o} xf(x)b^2\epsilon + \frac{M_o}{c_o} xf(x)b^2\epsilon \cos 2\theta - f'(x)f(x)b\epsilon^2 \cos\theta -$$

$$\frac{M_o}{2c_o} f'(x)f^2(x)b^2\epsilon^3 - \frac{M_o}{2c_o} f'(x)f^2(x)b^2\epsilon^3 \cos 2\theta + \left[\epsilon^2 \varphi_r^{10} - \frac{\epsilon}{f(x)} \lim_{r \longrightarrow 0} \left(r\varphi_r^{10}\right)\right] + \left[b\epsilon^2 {\varphi_1}_r^{11} - \frac{b}{f^2(x)} \lim_{r \longrightarrow 0} \left(r^2 {\varphi_1}_r^{11}\right)\right]\cos\theta -$$

$$\epsilon^3 \varphi_x^{10} f'(x) - \epsilon^3 b f'(x) {\varphi_1}_x^{11} \cos\theta + \ldots = 0 \tag{35}$$

The first two brackets in equation (35) are zero, by virtue of the imposed boundary conditions (33) and (34). All the unmatched terms are of higher order in ϵ and b and can be disregarded. In particular, the terms in the last two sets of brackets are of order higher than ϵ and b, respectively, whereas the last two terms are of the order $\epsilon^3 \log \epsilon$ and $\epsilon^2 b$, respectively, as may be seen by equations (32). These terms could be matched only by considering higher approximations to φ.

The preceding discussion makes clear how to proceed with higher-order approximations. The various terms in the expansion all satisfy equations of the type of equation (30) with nonhomogeneous terms. At each step only a finite number of Fourier coefficients will be considered. The singular behavior of these Fourier coefficients becomes worse with the order of the coefficients, but this is in turn compensated by an appropriately large exponent of ϵ in the expansion, thus making the coefficients appropriately small at the boundary of the body.

Solution of Boundary-Value Problems

The solution of equation (22) with boundary condition (33) is the well-known expression for the linearized disturbance potential of the Kármán-Moore problem, which is used herein in the form given by Courant and Friedrichs (see reference 2)

$$\varphi^{10} = -U_0 \int_0^{x-\beta r} \frac{k'(\xi)\, d\xi}{\sqrt{(x - \xi)^2 - \beta^2 r^2}} \tag{36}$$

where

$$k(x) = \frac{1}{2} f^2(x)$$

This is exactly the solution ψ of equation (31) with $n = 0$ and $g(x) = \frac{1}{2} U_0 k'(x)$. Clearly, using equation (B6b) of appendix B, $\lim_{r \to 0} \left(r\varphi_r^{10}\right) = 2g(x) = U_0 f(x) f'(x)$. It is assumed that:

$$f(0) = f(l) = 0 \tag{37}$$

This implies that

$$\left.\begin{aligned} k(0) &= k(l) = 0 \\ k'(0) &= k'(l) = 0 \end{aligned}\right\} \qquad (37a)$$

Using formulas (B1) and (B4) of appendix B, upon setting $n = 0$ and $g(x) = \frac{1}{2} U_o k'(x)$, it follows that:

$$\left.\begin{aligned} \varphi_x^{10} &= -U_o \int_0^{x-\beta r} \frac{k''(\xi)\, d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \\ &= -U_o \int_1^{x/\beta r} \frac{k''(x - \beta r\tau)\, d\tau}{\sqrt{\tau^2 - 1}} \\ \varphi_r^{10} &= \frac{U_o}{r} \int_0^{x-\beta r} \frac{(x-\xi)k(\xi)\, d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \\ &= U_o \beta \int_1^{x/\beta r} \frac{\tau k(x - \beta r\tau)\, d\tau}{\sqrt{\tau^2 - 1}} \end{aligned}\right\} \qquad (38)$$

To solve equation (26) for φ_1^{11} subject to boundary condition (34), one must first find a particular solution of equation (26) and add to it an appropriate solution of equation (26) with the right-hand side equal to zero (homogeneous case), such that the sum satisfies the required boundary condition, equation (34). It is verified in appendix C that a particular solution of equation (26) is given by

$$\left(\varphi_1^{11}\right)^* = \frac{M_o}{c_o}\left[r\varphi^{10} + xr\varphi_x^{10} + \frac{M_o^2(\gamma + 1)}{4\beta^2} r^2 \varphi_r^{10}\right] \qquad (39)$$

It is observed that:

$$\lim_{r \to 0} r^2\left(\varphi_1{}^{11}\right)^*_r = 0 \tag{40}$$

This follows, since $\varphi^{10} = O(\log_e r)$, $\varphi_x{}^{10} = O(\log_e r)$, $\varphi_r{}^{10} = O\left(\frac{1}{r}\right)$, $\varphi_{rx}{}^{10} = O\left(\frac{1}{r}\right)$, and $\varphi_{rr}{}^{10} = O\left(\frac{1}{r^2}\right)$, in virtue of equations (32) with $n = 0$. Hence the appropriate solution of the homogeneous equation $\left(\varphi_1{}^{11}\right)^{**}$ must satisfy the same boundary condition as that imposed on $\varphi_1{}^{11}$ itself; $\left(\varphi_1{}^{11}\right)^{**}$ is now immediately found, since it satisfies equation (30) with n= 1, and

$$\lim r^2\left(\varphi_1{}^{11}\right)^{**}_r = -xf^2(x) = -2xk(x)$$

By equation (B6a), appendix B, the desired solution is obtained from formula (31) with $n = 1$ and $g(x) = -2xk(x)$. Hence:

$$\left(\varphi_1{}^{11}\right)^{**} = \frac{2}{r}\int_0^{x-\beta r} \frac{(x - \xi)\left[k(\xi) + \xi k'(\xi)\right] d\xi}{\sqrt{(x - \xi)^2 - \beta^2 r^2}} \tag{41}$$

Using equations (39), (41), (36), and (38), there is finally obtained

$$\begin{aligned}
\varphi_1{}^{11} &= \left(\varphi_1{}^{11}\right)^* + \left(\varphi_1{}^{11}\right)^{**} \\
&= -M_o{}^2 r\int_0^{x-\beta r} \frac{\left[k'(\xi) + xk''(\xi)\right] d\xi}{\sqrt{(x - \xi)^2 - \beta^2 r^2}} + \\
&\quad \frac{1}{r}\int_0^{x-\beta r} \frac{(x - \xi)\left[2k(\xi) + 2\xi k'(\xi) + Pr^2 k''(\xi)\right] d\xi}{\sqrt{(x - \xi)^2 - \beta^2 r^2}} \\
&= -M_o{}^2 r\int_1^{x/\beta r} \frac{\left[k'(x - \beta r\tau) + xk''(x - \beta r\tau)\right] d\tau}{\sqrt{\tau^2 - 1}} + \\
&\quad \beta\int_1^{x/\beta r} \frac{\tau\left[2k(x - \beta r\tau) + 2(x - \beta r\tau)k'(x - \beta r\tau) + Pr^2 k''(x - \beta r\tau)\right] d\tau}{\sqrt{\tau^2 - 1}}
\end{aligned} \tag{42}$$

where

$$P = \frac{M_o^4(\gamma + 1)}{4\beta^2}$$

Differentiating, one gets:

$$\varphi_{1_x}^{11} = -M_o^2 r \int_1^{x/\beta} \frac{\left[2k''(x - \beta r\tau) + xk'''(x - \beta r\tau)\right] d\tau}{\sqrt{\tau^2 - 1}} +$$
$$\beta \int_1^{x/\beta} \frac{\tau\left[4k'(x - \beta r\tau) + 2(x - \beta r\tau)k''(x - \beta r\tau) + Pr^2 k'''(x - \beta r\tau)\right] d\tau}{\sqrt{\tau^2 - 1}} +$$
$$\left(P - M_o^2\right) \frac{xr}{\sqrt{x^2 - \beta^2 r^2}} k''(0) \qquad (43)$$

$$\varphi_{1_r}^{11} = -M_o^2 \int_1^{x/\beta r} \frac{\left\{\left[k'(x - \beta r\tau) + xk''(x - \beta r\tau)\right] - \beta r\tau\left[k''(x - \beta r\tau) + xk'''(x - \beta r\tau)\right]\right\} d\tau}{\sqrt{\tau^2 - 1}} +$$
$$\beta \int_1^{x/\beta r} \frac{\left\{2Pr\tau k''(x - \beta r\tau) - \beta\tau^2\left[4k'(x - \beta r\tau) + 2(x - \beta r\tau)k''(x - \beta r\tau) + Pr^2 k'''(x - \beta r\tau)\right]\right\} d\tau}{\sqrt{\tau^2 - 1}} -$$
$$\left(P - M_o^2\right) \frac{x^2}{\sqrt{x^2 - \beta^2 r^2}} k''(0) \qquad (44)$$

and obviously:

$$\left.\begin{aligned} \varphi_r^{11} &= \varphi_{1_r}^{11} \cos\theta \\ \varphi_x^{11} &= \varphi_{1_x}^{11} \cos\theta \\ \varphi_\theta^{11} &= -\varphi_1^{11} \sin\theta \end{aligned}\right\} \qquad (45)$$

COMPUTATION OF VELOCITIES, PRESSURES, AND FORCES

Velocity Components on Surface of Body

Denote by u, v, and w the axial, radial, and circumferential velocity components, respectively, and split each one into three parts, due to the basic potential, the first-order disturbance potential, and the interaction potential. The first of these are computed immediately from equation (12), whereas the computations involved in evaluating the others are carried out in appendix D.

In the following relations it is important to have all terms arranged in the proper order of magnitude. In view of relation (20)

$$\epsilon^2 \log_e \epsilon \ll \bar{b} \ll \epsilon$$

one has

$$\left.\begin{aligned}
&1 \gg \epsilon \gg \bar{b} \gg \epsilon^2 \log_e \epsilon \gg \epsilon^2 \gg \epsilon\bar{b} \gg \epsilon^3 \log_e \epsilon \gg \bar{b}\epsilon^2 \log_e \epsilon \gg \bar{b}\epsilon^2 \gg \bar{b}^2\epsilon \gg \bar{b}^2\epsilon^2 \\
&\epsilon\bar{b} \gg \bar{b}^2 \gg \bar{b}\epsilon^2 \log_e \epsilon \gg \epsilon^4 \log_e^2 \epsilon \gg \epsilon^4 \log_e \epsilon \\
&\epsilon^3 \log_e \epsilon \gg \epsilon^3 \gg \bar{b}\epsilon^2
\end{aligned}\right\} \tag{46}$$

Up to terms in $b\epsilon^2$, one now has, by differentiating equation (12),

$$\left.\begin{aligned}
u_{basic} &= U_o + b\epsilon\sqrt{2k(x)} \cos\theta + \ldots \\
v_{basic} &= bx \cos\theta + \ldots \\
w_{basic} &= -bx \sin\theta + \ldots
\end{aligned}\right\} \tag{47}$$

Using formulas (D8) and (D9) of appendix D:

$$\left.\begin{aligned}
\epsilon^2\varphi_x{}^{10} &= \epsilon^2 \log_e \epsilon U_o k''(x) - \epsilon^2 U_o\left\{\int_0^x \frac{\left[k''(x-\sigma) - k''(x)\right] d\sigma}{\sigma} + \right. \\
&\qquad \left. k''(x) \log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}}\right\} + \text{Terms of order } \epsilon^4 \log_e \epsilon \\
\epsilon^2\varphi_r{}^{10} &= \epsilon \frac{U_o k'(x)}{\sqrt{2k(x)}} + \epsilon^3 \log_e \epsilon U_o \beta^2\sqrt{\frac{k(x)}{2}}\, k'''(x) - \\
&\qquad \epsilon^3 U_o \beta^2\sqrt{\frac{k(x)}{2}}\left\{\int_0^x \frac{\left[k'''(x-\sigma) - k'''(x)\right] d\sigma}{\sigma} + \frac{k''(0)}{x} + \right. \\
&\qquad \left. k'''(x)\left[\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} + \frac{1}{2}\right]\right\} + \ldots \\
\epsilon^2\varphi_\theta{}^{10} &= 0
\end{aligned}\right\} \quad (48)$$

Using formulas (D10) to (D12) of appendix D:

$$\left.\begin{aligned}
b\epsilon^2\varphi_{1_x}{}^{11} \cos\theta &= b\epsilon\left[\sqrt{2k(x)} + \frac{2xk'(x)}{\sqrt{2k(x)}}\right] \cos\theta + \ldots \\
b\epsilon^2\varphi_{1_r}{}^{11} \cos\theta &= -bx\cos\theta + b\epsilon^2 \log_e \epsilon\left[\left(3M_o{}^2 - 2\right)k'(x) + \right. \\
&\qquad \left. \left(2M_o{}^2 - 1\right)xk''(x)\right] \cos\theta + b\epsilon^2 Q(x)\cos\theta + \ldots \\
b\epsilon^2\left(\frac{1}{r}\varphi_\theta{}^{11}\right) &= -b\epsilon^2\left(\frac{1}{r}\varphi_1{}^{11}\right)\sin\theta \\
&= -bx\sin\theta - b\epsilon^2 \log_e \epsilon\left[\left(3M_o{}^2 - 2\right)k'(x) + \right. \\
&\qquad \left. \left(2M_o{}^2 - 1\right)xk''(x)\right] \sin\theta - b\epsilon^2 Q^*(x)\sin\theta + \ldots
\end{aligned}\right\} \quad (49)$$

From equations (47), (48), and (49) one obtains:

$$u = u_{basic} + u_{disturbance}$$

$$= U_o + \epsilon^2 \log_e \epsilon U_o k''(x) - \epsilon^2 U_o \left[\int_0^x \frac{k''(x - \sigma) - k''(x)}{\sigma} d\sigma + \right.$$

$$\left. k''(x) \log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} \right] + 2b\epsilon \left[\sqrt{2k(x)} + \frac{xk'(x)}{\sqrt{2k(x)}} \right] \cos\theta +$$

$$\text{Terms of order } \epsilon^4 \log_e \epsilon \tag{50}$$

At present all one needs to know about w and v is that

$$w = -2bx \sin\theta + \left\{ \quad \right\} b\epsilon^2 \log_e \epsilon + \text{Terms of higher order} \tag{51}$$

and that

$$v = \epsilon \frac{U_o k'(x)}{\sqrt{2k(x)}} + \left\{ \quad \right\} \epsilon^3 \log_e \epsilon + \text{Terms of higher order} \tag{52}$$

It is immaterial whether one knows what the above braces actually represent.

Now consider again boundary condition (29a)

$$\varphi_r = \epsilon f'(x)\varphi_x \quad \text{for} \quad r = \epsilon f(x)$$

or

$$v = \epsilon f'(x)u = \epsilon \frac{k'(x)}{\sqrt{2k(x)}} u$$

Substituting u and v obtained in equations (50) and (52), it is now observed that the unmatched term of lowest order is proportional to $\epsilon^3 \log_e \epsilon$. It may be easily verified by use of the method of the section "Boundary Conditions" that introducing a new term in the potential, to cancel this term in the boundary condition, will result in the addition to u of a term of order $\epsilon^4 \log_e^2\epsilon$; higher-order corrections will introduce in u terms of still higher order. Consequently, formula (50) for u is certainly correct to the order $b\epsilon^2 \log_e \epsilon$ (see relations (46)). The same cannot be said about formula (52) for v and formula (51) for w. In these equations, higher-order potentials will introduce corrections of order $\epsilon^3 \log_e \epsilon$, so that the method yields expressions for v and w correct, on the body, only to order $\epsilon^3 \log_e \epsilon$. This is already sufficient to carry out the computation of the pressure field near the axis, correct to the order $b\epsilon^2 \log_e \epsilon$ (see next section). On the other hand, one can easily find a better formula for v, on the body, than the previous one. Indeed, using the boundary condition and the present knowledge about u, the expression

$$v = \epsilon \frac{k'(x)}{\sqrt{2k(x)}} u \tag{53}$$

is correct to the order $b\epsilon^3 \log_e \epsilon$, hence definitely to the order $b\epsilon^2$.

The quantity

$$S = \frac{1}{2}\left(U_o^2 - q^2\right)$$

where

$$q^2 = u^2 + v^2 + w^2 \tag{54}$$

is of importance in the evaluation of pressure on the body, drag, and lift. Since the expansion for u begins with a term of order unity, this quantity may be computed at best to an order not higher than or equal to $\epsilon^4 \log_e^2\epsilon$. Using equations (50), (53), and (51) one easily finds:

$$S = -\epsilon^2 \log_e \epsilon U_o{}^2 k''(x) + \epsilon^2 U_o{}^2 \left\{ \int_0^x \frac{k''(x - \sigma) - k''(x)}{\sigma} d\sigma + k''(x) \log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} - \frac{\left[k'(x)\right]^2}{4k(x)} \right\} - b\epsilon \left\{ 2U_o \cos\theta \left[\sqrt{2k(x)} + \frac{xk'(x)}{\sqrt{2k(x)}} \right] \right\} - b^2\left(2x^2 \sin^2\theta\right) + \ldots \tag{55}$$

correct to order $b\epsilon^2 \log_e \epsilon$.

Pressure, Drag, Lift, and Pitching Moment

Using formula (16) for the excess pressure, one finds

$$\left. \begin{aligned} \Delta p &= \rho_o S \\ p &= p_o + \rho_o S \end{aligned} \right\} \tag{56}$$

correct to the order $b\epsilon^2 \log_e \epsilon$.

The drag is the x component of the pressure force integrated over the body surface. The surface element of the body is $\frac{r\,dx\,d\theta}{\cos\alpha}$, where θ is the azimuth angle and α, the local angle of attack. Then the element of the drag force is

$$dD = pr \tan\alpha \,dx\, d\theta$$

Since

$$pr \tan\alpha = p\epsilon^2 f(x)f'(x) = \epsilon^2 pk'(x)$$

one gets:

$$D = \epsilon^2 \int_0^l dx \int_0^{2\pi} pk'(x)\, d\theta \tag{57}$$

As

$$\int_0^x k'(x)\ dx = 0$$

$$\int_0^x k'(x)k''(x)\ dx = 0$$

$$\int_0^{2\pi} \cos\theta\ d\theta = 0$$

only terms in ϵ^2 and in b^2 of S, equation (55), contribute to the drag. Thus, one obtains

$$D = D_1 + D_2 \tag{58}$$

Here

$$D_1 = -2\pi\rho_o U_o{}^2\epsilon^4 \int_0^l dx \int_0^x k''(x)k''(\sigma)\ \log_e (x - \sigma)\ d\sigma \tag{59}$$

is the well-known formula for the drag in the linearized theory (see reference 2, section 153; for a simpler computation, see appendix E) and

$$D_2 = -\rho_o\epsilon^2 b^2 \int_0^l dx \int_0^{2\pi} 2x^2 k'(x)\ \sin^2\theta\ d\theta$$

$$= 4\pi\rho_o\epsilon^2 b^2 \int_0^l xk(x)\ dx \tag{60}$$

where D_2 represents the effect of the inhomogeneity of the flow. Observing that

$$\pi\left[\epsilon f(x)\right]^2 dx = \epsilon^2\left[2\pi k(x)\right]\ dx = dV$$

is the volume element of the body, one may write:

$$D_2 = 2\rho_o \bar{x} V b^2 \tag{61}$$

where $\bar{x}$ is the distance of the nose of the body from its center of mass and $V = 2\pi\epsilon^2 \int_0^l k(x)\,dx$ is the volume of the body. Formula (58) for the drag is correct to the order $b\epsilon^4 \log_e \epsilon$.

The lift is the y component of the pressure forces integrated over the body surface. The element of lift is

$$\begin{aligned} dL &= -pr \cos\theta \, dx \, d\theta \\ &= -\epsilon p\sqrt{2k(x)} \cos\theta \, dx \, d\theta \end{aligned}$$

All terms of p contribute nothing to the lift, except for the term in ϵb (see equations (55) and (56)). Thus

$$\begin{aligned} L &= 4\rho_o U_o b\epsilon^2 \int_0^l dx \int_0^{2\pi} \left[k(x) + \frac{1}{2} xk'(x)\right] \cos^2\theta \, d\theta \\ &= \rho_o U_o V b \end{aligned} \tag{62}$$

an expression which is correct to the order $b\epsilon^3 \log_e \epsilon$.

The element of the pitching moment about the nose is:

$$\begin{aligned} dM &= p \cos\theta \, rx \, dx \, d\theta \\ &= \epsilon p\sqrt{2k(x)}\, x \cos\theta \, dx \, d\theta \end{aligned}$$

Again, all terms of p contribute nothing to the moment, except for the term ϵb (see equations (55) and (56)). Hence

$$M = -4\rho_o U_o b\epsilon^2 \int_0^l dx \int_0^{2\pi} \left[xk(x) + \frac{1}{2} x^2 k'(x)\right] \cos^2\theta \, d\theta = 0 \tag{63}$$

an expression which is correct to the order $b\epsilon^3 \log_e \epsilon$.

Validity of Results - Physical Interpretation

As was stated at various points in the text, the derivations do not yield results which are all correct to the same order of magnitude. For clarity, the main conclusions are listed again.

The main assumption on orders of magnitude is that

$$\epsilon^2 \log_e \epsilon \ll \bar{b} \ll \epsilon$$

The potential and velocity components away from the body are determined to the order $b\epsilon^2$. They contain terms of orders

$$1 \gg b \gg \epsilon^2 \gg b^2 \gg b\epsilon^2$$

On the body they are determined as follows:

(1) The u component of velocity is determined to the order $b\epsilon^2 \log_e \epsilon$ and contains terms of orders

$$1 \gg \epsilon^2 \log_e \epsilon \gg \epsilon^2 \gg b\epsilon$$

(see formula (50)).

(2) The v component of velocity is determined to the order $b\epsilon^3 \log_e \epsilon$ and contains terms of orders

$$\epsilon \gg \epsilon^3 \log_e \epsilon \gg \epsilon^3 \gg b\epsilon^2$$

(see formula.(53)).

(3) The w component of velocity is determined to the order b and contains only a term in b (see formula (51)).

(4) The pressure is determined to the order $b\epsilon^2 \log_e \epsilon$ and contains terms of orders

$$1 \gg \epsilon^2 \log_e \epsilon \gg \epsilon^2 \gg b\epsilon \gg b^2$$

(see formulas (56) and (55)).

The drag is determined to the order $b\epsilon^4 \log_e \epsilon$ and contains terms of orders $\epsilon^4 \gg b^2\epsilon^2$ (see formula (58)). Both the lift and the pitching moment are determined to the order $b\epsilon^3 \log_e \epsilon$, the former containing the only term of order $b\epsilon^2$ and the latter being zero to this order.

Of the two drag terms, the one proportional to ϵ^4 is the wave drag of the body, in the representation by Courant and Friedrichs.

One might anticipate from general physical considerations that a "horizontal-buoyancy" term will occur which should be equal to the product of volume and pressure gradient in the x-direction. From equations (18) it can be seen that for the assumed basic field, the x gradient of the pressure, up to order b, is zero. Therefore no drag term of the order $b\epsilon^2$ may be expected. Note that in order to compute the drag, the expression for the pressure has to be multiplied by ϵ^2, since the surface element is proportional ϵ^2.

The second drag term occurring in equation (58), of order $b^2\epsilon^2$, can be seen to originate from two different terms in the velocity potential. One contribution comes from the so-called interaction potential φ^{11}, having as a factor $b\epsilon^2$; φ^{11} itself is shown to be proportional to $1/r$. The corresponding pressure term turns out to be proportional to $\left(\varphi_r^{11}\right)^2$ and is thus proportional to b^2 near the axis, so that the drag term becomes of order $b^2\epsilon^2$. The other contribution originates from the potential coefficients φ^{01} and φ^{02} of the basic field, which are proportional to b and b^2. Since the basic field has no singularity on the axis, the corresponding velocities and pressures do not change their order, as the body surface is approached. Therefore, also, these terms will make contributions to the drag of order $b^2\epsilon^2$.

A drag term of order $b\epsilon^3$ which could be anticipated from contributions of φ^{01} (proportional to b) and of φ^{11} (proportional to $b\epsilon^2$) can be shown to vanish when the pressure distribution is integrated over the body surface.

As seen from equation (62), the lift consists only of one term which is proportional to $b\epsilon^2$. The origin of this term can be traced back to contributions from φ^{01} as well as φ^{11}; φ^{01} leads to a pressure term which is proportional to y so that the whole expression for the pressure on the surface becomes proportional to $b\epsilon$ and the lift expression becomes proportional to $b\epsilon^2$; φ^{11} leads to a pressure term which is proportional to $\varphi_x{}^{11}$ and is thus altogether of order $b\epsilon$, hence again a lift term of order $b\epsilon^2$ results.

One might expect that lift terms of order $b\epsilon$, $b\epsilon^2$, and ϵ^3 occur, in analogy to the corresponding result for two-dimensional bodies. But it can be shown that the respective expressions vanish when the corresponding pressure terms are integrated over the body surface.

The moment about the nose of the body has been shown to vanish, up to order $b\epsilon^3 \log_e \epsilon$. This peculiar result, which holds for an arbitrary body contour, must be attributed to the very special basic field that has been chosen.

New York University
New York, N. Y., November 5, 1951

APPENDIX A

CHARACTERISTIC CONDITION

It must be postulated that the total disturbance potential

$$\epsilon^2\varphi^{10} + b\epsilon^2\varphi^{11} \tag{A1}$$

vanishes at the "characteristic surface," the equation of which may be written in the form

$$r = \Omega(x) = \Omega^{(0)}(x) + b\Omega^{(1)}(x) + \ldots \tag{A2}$$

where

$$\Omega^{(0)}(x) = x/\beta_o \tag{A3}$$

and $\beta_o^{\,2} = M_o^{\,2} - 1$. Inserting equation (A2) into disturbance potential (A1) and using a Taylor expansion, there is obtained, neglecting higher-order terms,

$$\epsilon^2\varphi^{10}\left[x;\Omega^{(0)}(x)\right] + b\epsilon^2\left\{\varphi_r^{10}\left[x;\Omega^{(0)}(x)\right]\Omega^{(1)}(x) + \varphi^{11}\left[x;\Omega^{(0)}(x)\right]\right\} = 0$$

Hence:

$$\varphi^{10}\left[x;\Omega^{(0)}(x)\right] = 0 \tag{A4}$$

and

$$\varphi_r^{10}\left[x;\Omega^{(0)}(x)\right]\Omega^{(1)}(x) + \varphi^{11}\left[x;\Omega^{(0)}(x)\right] = 0 \tag{A5}$$

Using the notation of appendix D,

$$\left.\begin{aligned} \varphi^{10} &= -U_oI_0(k') \\ \varphi_r^{10} &= U_o\beta_oI_1(k'') \end{aligned}\right\} \tag{A6}$$

$$\varphi^{11} = \cos\theta\varphi_1^{\,11}$$

or

$$\varphi^{11} = \cos\theta\Big[-M_o{}^2 rI_0(k') - xrM_o{}^2 I_0(k'') + 2\beta_o I_1(k) + 2\beta_o xI_1(k') + P\beta_o r^2 I_1(k'') - 2\beta_o{}^2 rI_2(k')\Big] \tag{A7}$$

(See equation (42).) It is easy to prove that all the integrals $I_n(g)$ approach zero when x approaches $\beta_o r$. In the present case, only the special values $n = 0$, 1, and 2 are needed. Now

$$\left.\begin{aligned} I_0(g) &= \int_1^{x/\beta_o r} \frac{g(x - \beta_o r\tau)}{\sqrt{\tau^2 - 1}}\, d\tau \\ I_1(g) &= \int_1^{x/\beta_o r} \frac{\tau g(x - \beta_o r\tau)}{\sqrt{\tau^2 - 1}}\, d\tau \\ I_2(g) &= \int_1^{x/\beta_o r} \frac{\tau^2 g(x - \beta_o r\tau)}{\sqrt{\tau^2 - 1}}\, d\tau \end{aligned}\right\} \tag{A8}$$

Put $x/\beta_o r = 1 + \delta$. Then

$$\left|I_0(g)\right| = \left|\int_1^{x/\beta_o r} \frac{g(x - \beta_o r\tau)}{\sqrt{\tau^2 - 1}}\, d\tau\right| < G\int_1^{1+\delta} \frac{d\tau}{\sqrt{\tau^2 - 1}}$$

$$= \log_e\left[1 + \delta + \sqrt{(1 + \delta)^2 - 1}\right]$$

where $G = \left|g(x - \beta_o r\tau)\right|_{max}$ is always finite. The above expression goes to zero if $\delta \longrightarrow 0$. Similarly,

$$\left|I_1(g)\right| < G\int_1^{1+\delta} \frac{\tau\, d\tau}{\sqrt{\tau^2 - 1}} = G\sqrt{(1 + \delta)^2 - 1}$$

approaches zero if $\delta \longrightarrow 0$ and

$$\left|I_2(g)\right| < G_1 \int_1^{1+\delta} \frac{\tau \, d\tau}{\sqrt{\tau^2 - 1}} = G_1\sqrt{(1 + \delta)^2 - 1}$$

where $G_1 = \left|\tau g(x - \beta_o r\tau)\right|_{max}$ is finite. Also this integral approaches zero if $\delta \longrightarrow 0$. Hence, all the integrals $I_n(g)$ occurring in equations (A4) and (A5) vanish for $r = x/\beta_o = \Omega^{(0)}(x)$, meaning that the characteristic condition is fulfilled.

APPENDIX B

PROPERTIES OF THE FUNCTION ψ DEFINED BY EQUATION (31)

It is shown first that equation (31) satisfies equation (30). The differentiation of ψ is carried out easily after introducing $\tau = \frac{x - \xi}{\beta r}$ as a new integration variable. One obtains then

$$\psi = -\frac{\beta^n}{2^n n!}\int_1^{x/\beta r} \frac{\left(\tau + \sqrt{\tau^2 - 1}\right)^n + \left(\tau - \sqrt{\tau^2 - 1}\right)^n}{\sqrt{\tau^2 - 1}} g^{(n)}(x - \beta r\tau)\, d\tau \quad \text{(B1)}$$

and it follows that

$$\beta^2\psi_{xx} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{n^2}{r^2}\psi \equiv$$

$$-\frac{\beta^{n+2}}{2^n n!}\left\{\int_1^{x/\beta r} \frac{\left(\tau + \sqrt{\tau^2 - 1}\right)^n + \left(\tau - \sqrt{\tau^2 - 1}\right)^n}{\sqrt{\tau^2 - 1}}\left[-\left(\tau^2 - 1\right)g^{(n+2)}(x - \beta r\tau) + \right.\right.$$

$$\left.\frac{\tau}{r\beta}g^{(n+1)}(x - \beta r\tau) + \frac{n^2}{\beta^2 r^2}g^{(n)}(x - \beta r\tau)\right] d\tau -$$

$$\frac{\left(x + \sqrt{x^2 - \beta^2 r^2}\right)^n + \left(x - \sqrt{x^2 - \beta^2 r^2}\right)^n}{\beta^{n+2} r^{n+2}}\sqrt{x^2 - \beta^2 r^2}\; g^{(n+1)}(0) -$$

$$\left. n\,\frac{\left(x + \sqrt{x^2 - \beta^2 r^2}\right)^n - \left(x - \sqrt{x^2 - \beta^2 r^2}\right)^n}{\beta^{n+2} r^{n+2}}\, g^{(n)}(0)\right\} \quad \text{(B2)}$$

Now, one may verify that the integrand on the preceding page,

$$\frac{\left(\tau + \sqrt{\tau^2 - 1}\right)^n + \left(\tau - \sqrt{\tau^2 - 1}\right)^n}{\sqrt{\tau^2 - 1}}\left[-\left(\tau^2 - 1\right)g^{(n+2)}(x - \beta r\tau) + \frac{\tau}{r\beta}g^{(n+1)}(x - \beta r\tau) + \frac{n^2}{\beta^2 r^2}g^{(n)}(x - \beta r\tau)\right] =$$

$$\frac{d}{d\tau}\left\{\frac{1}{\beta r}\left[\left(\tau + \sqrt{\tau^2 - 1}\right)^n + \left(\tau - \sqrt{\tau^2 - 1}\right)^n\right]\sqrt{\tau^2 - 1}\ g^{(n+1)}(x - \beta r\tau) + \frac{n}{\beta^2 r^2}\left[\left(\tau + \sqrt{\tau^2 - 1}\right)^n - \left(\tau - \sqrt{\tau^2 - 1}\right)^n\right]g^{(n)}(x - \beta r\tau)\right\}$$

so that the right-hand side is identically zero and equation (30) is satisfied by ψ. Differentiating equation (B1) with respect to x and r and transforming back to the original variable of integration ξ, one obtains

$$\psi_x = -\frac{1}{2^n r^n n!}\int_0^{x-\beta r}\frac{\left[(x - \xi) + \sqrt{(x - \xi)^2 - \beta^2 r^2}\right]^n + \left[(x - \xi) - \sqrt{(x - \xi)^2 - \beta^2 r^2}\right]^n}{\sqrt{(x - \xi)^2 - \beta^2 r^2}}g^{(n+1)}(\xi)\ d\xi - \frac{1}{2^n r^n n!}\frac{\left(x + \sqrt{x^2 - \beta^2 r^2}\right)^n + \left(x - \sqrt{x^2 - \beta^2 r^2}\right)^n}{\sqrt{x^2 - \beta^2 r^2}}g^{(n)}(0) \tag{B3}$$

$$\psi_r = \frac{1}{2^n r^{n+1} n!}\int_0^{x-\beta r}\left\{\frac{\left[(x - \xi) + \sqrt{(x - \xi)^2 - \beta^2 r^2}\right]^n}{\sqrt{(x - \xi)^2 - \beta^2 r^2}} + \frac{\left[(x - \xi) - \sqrt{(x - \xi)^2 - \beta^2 r^2}\right]^n}{\sqrt{(x - \xi)^2 - \beta^2 r^2}}\right\}(x - \xi)g^{(n+1)}(\xi)\ d\xi + \frac{x}{2^n r^{n+1} n!}\frac{\left(x + \sqrt{x^2 - \beta^2 r^2}\right)^n + \left(x - \sqrt{x^2 - \beta^2 r^2}\right)^n}{\sqrt{x^2 - \beta^2 r^2}}g^{(n)}(0) \tag{B4}$$

As $r \longrightarrow 0$, the integrands in equations (B3) and (B4) remain finite and continuous over the full range $0 < \xi < x - \beta r$, except in the case of ψ_x for $n = 0$. Hence

$$\lim_{r \longrightarrow 0} \left(r^n \psi_x\right) = -\frac{1}{n!}\int_0^x (x - \xi)^{n-1} g^{(n+1)}(\xi)\, d\xi - \frac{x^{n-1}}{n!} g^{(n)}(0)$$

$$= -\frac{n - 1}{n!}\int_0^x (x - \xi)^{n-2} g^{(n)}(\xi)\, d\xi$$

$$= -\frac{1}{n} g'(x) \qquad n > 0 \tag{B5}$$

$$\lim_{r \longrightarrow 0} \left(r^{n+1} \psi_r\right) = \frac{1}{n!}\int_0^x (x - \xi)^{n} g^{(n+1)}(\xi)\, d\xi + \frac{x^n}{n!} g^{(n)}(0)$$

$$= \frac{1}{(n - 1)!}\int_0^x (x - \xi)^{n-1} g^{(n)}(\xi)\, d\xi$$

$$= g(x) \qquad n > 0 \tag{B6a}$$

$$\lim_{r \longrightarrow 0} \left(r \psi_r\right) = 2\int_0^x g'(\xi)\, d\xi + 2g(0) = 2g(x) \qquad n = 0 \tag{B6b}$$

provided that $g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0$. It follows then that

$$\left.\begin{aligned} \psi_x &= -\frac{1}{n} g'(x)\frac{1}{r^n} + \ldots \\ \psi_r &= g(x)\frac{1}{r^{n+1}} + \ldots \end{aligned}\right\} \quad n > 0 \tag{B7}$$

where the dots indicate terms of higher order in r as $r \longrightarrow 0$. It is shown in appendix D that, for $n = 0$, ψ_x is of the order of $\log_e r$.

That ψ, ψ_x, and ψ_r vanish for $x < \beta r$ is obvious, since the integrands then become identically zero. It may also be verified that both ψ_x and ψ_r are proportional to $\frac{1}{\sqrt{r}}$ as one lets r tend to infinity along a characteristic $x = \beta r +$ Constant.

APPENDIX C

VERIFICATION OF EXPANSION (39) AS A PARTICULAR SOLUTION OF EQUATION (26)

It is to be verified that expansion (39)

$$\left(\varphi_1{}^{11}\right)^* = \frac{M_o}{c_o}\left[r\varphi^{10} + xr\varphi_x{}^{10} + \frac{M_o{}^2(\gamma + 1)}{4\beta^2} r^2\varphi_r{}^{10}\right] \tag{C1}$$

is a solution of the nonhomogenous equation (26)

$$L(\varphi) = \beta^2\varphi_{xx} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi$$

$$= -\frac{M_o}{c_o}\left\{\left[(\gamma + 1) + (\gamma - 1)\beta^2\right]r\varphi_{xx}{}^{10} + 2\varphi_r{}^{10} + 2x\varphi_{xr}{}^{10}\right\} \tag{C2}$$

One has

$$L\left(r^2\varphi_r{}^{10}\right) = \beta^2 r^2\varphi_{rxx}{}^{10} - r^2\varphi_{rrr}{}^{10} - 5r\varphi_{rr}{}^{10} - 3\varphi_r{}^{10}$$

$$= r^2\left(\beta^2\varphi_{xx}{}^{10} - \varphi_{rr}{}^{10} - \frac{1}{r}\varphi_r{}^{10}\right)_r + 4r\left(\beta^2\varphi_{xx}{}^{10} - \varphi_{rr}{}^{10} - \frac{1}{r}\varphi_r{}^{10}\right) - 4r\beta^2\varphi_{xx}{}^{10}$$

$$= -4\beta^2 r\varphi_{xx}{}^{10} \tag{C3}$$

$$L\left(r\varphi^{10} + xr\varphi_x^{\ 10} + \frac{1}{2} r^2\varphi_r^{\ 10}\right) =$$

$$\beta^2 r\varphi_{xx}^{\ 10} - r\varphi_{rr}^{\ 10} - 3\varphi_r^{\ 10} + \beta^2 xr\varphi_{xxx}^{\ 10} + 2\beta^2 r\varphi_{xx}^{\ 10} -$$

$$xr\varphi_{xrr}^{\ 10} - 3x\varphi_{xr}^{\ 10} - 2\beta^2 r\varphi_{xx}^{\ 10} =$$

$$r\left(\beta^2\varphi_{xx}^{\ 10} - \varphi_{rr}^{\ 10} - \frac{1}{r}\varphi_r^{\ 10}\right) + xr\left(\beta^2\varphi_{xx}^{\ 10} - \varphi_{rr}^{\ 10} -\right.$$

$$\left.\frac{1}{r}\varphi_r^{\ 10}\right)_x - 2\varphi_r^{\ 10} - 2x\varphi_{xr}^{\ 10} =$$

$$-2\varphi_r^{\ 10} - 2x\varphi_{xr}^{\ 10} \tag{C4}$$

since $\beta^2\varphi_{xx}^{\ 10} - \varphi_{rr}^{\ 10} - \frac{1}{r}\varphi_r^{\ 10} = 0$ (cf. equation (22)). Since $M_o^{\ 2} = \beta^2 + 1$,

$$r\varphi^{10} + xr\varphi_x^{\ 10} + \frac{M_o^{\ 2}(\gamma + 1)}{4\beta^2} r^2\varphi_r^{\ 10} =$$

$$r\varphi^{10} + xr\varphi_x^{\ 10} + \frac{1}{2} r^2\varphi_r^{\ 10} + \frac{(\gamma + 1) + \beta^2(\gamma - 1)}{4\beta^2} r^2\varphi_r^{\ 10}$$

From equations (C3) and (C4) it follows then that

$$L\left(\varphi_1^{\ 11}\right)^* = -\frac{M_o}{c_o}\left\{\left[(\gamma + 1) + \beta^2(\gamma - 1)\right] r\varphi_{xx}^{\ 10} + 2\varphi_r^{\ 10} + 2x\varphi_{xr}^{\ 10}\right\}$$

completing the verification.

Expression (C1) for $\left(\varphi_1^{\ 11}\right)^*$ was found by guessing that equation (C2) has a solution of the form

$$\Phi = Ar\varphi^{10} + Br^2\varphi_r^{\ 10} + Cxr\varphi_x^{\ 10}$$

and determining the constants A, B, and C.

APPENDIX D

COMPUTATION OF VELOCITY COMPONENTS ON SURFACE OF BODY

In order to evaluate the velocity components at the body, one must find the dependence on ϵ of integrals of the type

$$I_n(g) = \int_1^{x/r\beta} \frac{\tau^n g(x - \beta r\tau)}{\sqrt{\tau^2 - 1}} d\tau = \int_{\beta r}^{x} \frac{1}{(\beta r)^n} \frac{\sigma^n g(x - \sigma)}{\sqrt{\sigma^2 - \beta^2 r^2}} d\sigma \qquad \text{(D1)}$$

for $n = 0$, 1, and 2, as $r = \epsilon f(x)$ and $\epsilon \longrightarrow 0$. Using the identities

$$\tau = -\frac{1}{\beta r}\Big[(x - \beta r\tau) - x\Big]$$

$$\tau^2 = \frac{1}{(\beta r)^2}\Big[(x - \beta r\tau)^2 - 2x(x - \beta r\tau) + x^2\Big]$$

it is seen immediately that

$$\left.\begin{aligned} I_1(g) &= \frac{1}{\beta r}\Big[xI_0(g) - I_0(xg)\Big] \\ I_2(g) &= \frac{1}{(\beta r)^2}\Big[x^2 I_0(g) - 2xI_0(xg) + I_0\left(x^2 g\right)\Big] \end{aligned}\right\} \qquad \text{(D2)}$$

In order to get that part which does not vanish as $\epsilon \longrightarrow 0$, the expression for I_0 is needed up to the terms in r^2.

Write $R = \beta r$ and integrate

$$I_0(g) = \int_R^x \frac{g(x - \sigma)\, d\sigma}{\sqrt{\sigma^2 - R^2}} \qquad \text{(D3)}$$

three times by parts. Since the successive integrals of $\frac{1}{\sqrt{\sigma^2 - R^2}}$ are

$$\log_e \left(\sigma + \sqrt{\sigma^2 - R^2}\right)$$

$$\sigma \log_e \left(\sigma + \sqrt{\sigma^2 - R^2}\right) - \sqrt{\sigma^2 - R^2}$$

$$\left(\frac{\sigma^2}{2} + \frac{R^2}{4}\right) \log_e \left(\sigma + \sqrt{\sigma^2 - R^2}\right) - \frac{3}{4} \sigma\sqrt{\sigma^2 - R^2}$$

one easily finds that

$$\begin{aligned} I_0(g) = {} & g(0) \log_e \left(x + \sqrt{x^2 - R^2}\right) - g(x - R) \log_e R + \\ & g'(0)\left[x \log_e \left(x + \sqrt{x^2 - R^2}\right) - \sqrt{x^2 - R^2}\right] - g'(x - R)R \log_e R + \\ & g''(0)\left[\left(\frac{x^2}{2} + \frac{R^2}{4}\right) \log_e \left(x + \sqrt{x^2 - R^2}\right) - \frac{3}{4} x\sqrt{x^2 - R^2}\right] - \frac{3}{4} g''(x - R)R^2 \log_e R + \\ & R^3 \int_1^{x/R} g'''(x - R\tau)\left[\left(\frac{\tau^2}{2} + \frac{1}{4}\right) \log_e R\left(\tau + \sqrt{\tau^2 - 1}\right) - \frac{3}{4} \tau\sqrt{\tau^2 - 1}\right] d\tau \end{aligned}$$

in the last part the original variable of integration τ has been used. Expanding the brackets in the above integrand in inverse powers of τ ($1 < \tau < \infty$ as $R \longrightarrow 0$), one gets

$$\left(\frac{\tau^2}{2} + \frac{1}{4}\right) \log_e 2R\tau - \frac{3}{4} \tau^2 + \frac{1}{4} + \text{Terms of order } \frac{1}{\tau^2} \text{ or higher}$$

The contribution of the unwritten terms of the integral is of order R^3 and may be neglected. Keeping terms up to R^2, one now gets, after simplifying the integral by successive integrations by parts,

$$\begin{aligned} I_0(g) = {} & \left[g(x) + \frac{R^2}{4} g''(x)\right] \log_e 2 - \frac{1}{4}\left[g(0) - xg'(0)\right]\left(\frac{R}{x}\right)^2 - \\ & g'(x)R + \frac{1}{2} g''(x)R^2 + \int_R^x \left[g(x - \sigma) + \frac{R^2}{4} g''(x - \sigma)\right] \frac{d\sigma}{\sigma} \end{aligned}$$

Now

$$\int_R^x \frac{g(x - \sigma)}{\sigma} d\sigma = \int_0^x \frac{g(x - \sigma) - g(x)}{\sigma} d\sigma + g(x) \log_e \frac{x}{R} - \int_0^R \frac{g(x - \sigma) - g(x)}{\sigma} d\sigma$$

The first integral is finite; the second integral yields

$$g'(x)R - \frac{R^2}{4} g''(x) + . . .$$

Similarly,

$$\frac{R^2}{4}\int_R^x \frac{g''(x - \sigma)}{\sigma} d\sigma = \frac{R^2}{4}\int_0^x \frac{g''(x - \sigma) - g''(x)}{\sigma} d\sigma + \frac{1}{4} g''(x)R^2 \log_e \frac{x}{R} + . . .$$

Hence, finally,

$$I_0(g) = \int_0^x \frac{g(x - \sigma) - g(x)}{\sigma} d\sigma + \frac{R^2}{4}\int_0^x \frac{g''(x - \sigma) - g''(x)}{\sigma} d\sigma + \left[g(x) + \frac{R^2}{4} g''(x)\right] \log_e \frac{2x}{R} - \frac{1}{4}\left[g(0) - xg'(0)\right]\frac{R^2}{x^2} + \frac{1}{4} g''(x)R^2 \tag{D4}$$

Replacing $R = \beta r = \beta\epsilon f(x) = \epsilon\beta\sqrt{2k(x)}$, one now gets, using equations (D4) and (D2),

$$I_0(g) = -g(x) \log_e \epsilon + \int_0^x \frac{g(x - \sigma) - g(x)}{\sigma} d\sigma + g(x) \log_e \left(\frac{x}{\beta}\sqrt{\frac{2}{k(x)}}\right) + \text{Terms of order } \epsilon^2 \log_e \epsilon + . . . \tag{D5}$$

$$I_1(g) = \frac{1}{\epsilon\beta\sqrt{2k(x)}}\int_0^x g(\sigma)\ d\sigma + (\epsilon \log_e \epsilon)\beta\sqrt{\frac{k(x)}{2}}\ g'(x) -$$

$$\epsilon\beta\sqrt{\frac{k(x)}{2}}\left[\int_0^x \frac{g'(x - \sigma) - g'(x)}{\sigma}\ d\sigma + \frac{g(0)}{x} +\right.$$

$$\left. g'(x)\left(\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} + \frac{1}{2}\right)\right] + \ldots \tag{D6}$$

$$I_2(g) = \frac{1}{2\beta^2 k\epsilon^2}\int_0^x \sigma g(x - \sigma)\ d\sigma - \frac{1}{2}\ g(x) \log_e \epsilon +$$

$$\frac{1}{2}\left[\int_0^x \frac{g(x - \sigma) - g(x)}{\sigma}\ d\sigma +\right.$$

$$\left. g(x)\left(\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} - \frac{1}{2}\right)\right] + \ldots \tag{D7}$$

Using formulas (D5), (D6), and (D7), the integrals occurring in the velocity components, equations (38) and (42) to (45), are evaluated by mere substitution. There are obtained, as $r = \epsilon\sqrt{2k(x)}$,

$$\varphi_x^{10} = -U_o I_0\left[k''(x)\right]$$

$$= U_o k''(x) \log_e \epsilon - U_o\left[\int_0^x \frac{k''(x - \sigma) - k''(x)}{\sigma}\ d\sigma +\right.$$

$$\left. k''(x) \log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}}\right] + \text{Terms of order } \epsilon^2 \log_e \epsilon + \ldots \tag{D8}$$

$$\varphi_r{}^{10} = U_o\beta I_1\left[k''(x)\right] = \frac{1}{\epsilon}\frac{U_o k'(x)}{\sqrt{2k(x)}} + \epsilon \log_e \epsilon\; U_o\beta^2\sqrt{\frac{k(x)}{2}}\, k'''(x) -$$

$$\epsilon U_o\beta^2\sqrt{\frac{k(x)}{2}}\left\{\int_0^x \frac{k'''(x - \sigma) - k'''(x)}{\sigma}\, d\sigma + \frac{k''(0)}{x} + \right.$$

$$\left. k'''(x)\left[\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} + \frac{1}{2}\right]\right\} + \ldots \tag{D9}$$

$$\varphi_{1_x}{}^{11} = -M_o{}^2\sqrt{2k(x)}\;\epsilon\left\{2I_0\left[k''(x)\right] + xI_0\left[k'''(x)\right]\right\} +$$

$$4\beta I_1\left[k'(x)\right] + 2\beta I_1\left[xk''(x)\right] + 2P\beta k(x)\epsilon^2 I_1\left[k'''(x)\right] +$$

$$\left(P - M_o{}^2\right)\frac{x\epsilon\sqrt{2k(x)}}{\sqrt{x^2 - \beta^2\epsilon^2 2k(x)}}\, k''(0)$$

$$= \frac{1}{\epsilon}\left[\sqrt{2k(x)} + \frac{2xk'(x)}{\sqrt{2k(x)}}\right] + \epsilon \log_e \epsilon \tag{D10}$$

$$\varphi_{1_r}{}^{11} = -M_o{}^2 I_0\left[k'(x)\right] - M_o{}^2 xI_0\left[k''(x)\right] +$$

$$M_o{}^2\beta\sqrt{2k(x)}\;\epsilon I_1\left[k''(x)\right] +$$

$$M_o{}^2\beta\sqrt{2k(x)}\; x\epsilon I_1\left[k'''(x)\right] + 2P\beta\sqrt{2k(x)}\;\epsilon I_1\left[k''(x)\right] -$$

$$4\beta^2 I_2\left[k'(x)\right] - 2\beta^2 I_2\left[xk''(x)\right] - 2P\beta^2 k(x)\epsilon^2 I_2\left[k'''(x)\right] -$$

$$\left(P - M_o{}^2\right)xk''(0) + \ldots$$

$$= -\frac{x}{\epsilon^2} + \left[\left(3M_o{}^2 - 2\right)k'(x) + \left(2M_o{}^2 - 1\right)xk''(x)\right]\log_e \epsilon + Q(x) + \ldots \tag{D11}$$

where

$$Q(x) = -\left(3M_o^2 - 2\right)\left\{\int_0^x \frac{k'(x - \sigma) - k'(x)}{\sigma}\,d\sigma + k'(x)\left[\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} - \frac{1}{2}\right]\right\} -$$

$$\left(2M_o^2 - 1\right)\left\{\int_0^x \frac{(x - \sigma)k''(x - \sigma) - xk''(x)}{\sigma}\,d\sigma + xk''(x)\left[\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} - \frac{1}{2}\right]\right\} +$$

$$\left(P - \frac{1}{2}M_o^2\right)k'(x) + \frac{1}{2}M_o^2 xk''(x)$$

$$\left.\begin{aligned}\frac{1}{r}\varphi_1^{11} = &-M_o^2 I_0\left[k'(x)\right] - M_o^2 x I_0\left[k''(x)\right] + \\ &\frac{2\beta}{\epsilon\sqrt{2k(x)}} I_1\left[k(x)\right] + \frac{2\beta}{\epsilon\sqrt{2k(x)}} I_1\left[xk'(x)\right] + \\ &P\beta\sqrt{2k(x)}\ \epsilon I_1\left[k''(x)\right] \\ = &\frac{x}{\epsilon^2} + \left[\left(3M_o^2 - 2\right)k'(x) + \left(2M_o^2 - 1\right)xk''(x)\right]\log_e \epsilon + Q^*(x) + \ldots\end{aligned}\right\} \quad \text{(D12)}$$

where

$$Q^*(x) = -\left(3M_o^2 - 2\right)\left\{\int_0^x \frac{k'(x - \sigma) - k'(x)}{\sigma}\,d\sigma +\right.$$

$$\left. k'(x)\left[\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} + \frac{1}{2}\right]\right\} -$$

$$\left(2M_o^2 - 1\right)\left\{\int_0^x \frac{(x - \sigma)k''(x - \sigma) - xk''(x)}{\sigma}\,d\sigma +\right.$$

$$\left. xk''(x)\left[\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} + \frac{1}{2}\right]\right\} + \left(P - \frac{1}{2}M_o^2\right)k'(x) + \frac{1}{2}M_o^2 xk''(x)$$

Dots indicate terms which either vanish as $\epsilon \longrightarrow 0$ or tend to zero faster than the last term considered. In the above five formulas the fact that $k(0) = k'(0) = 0$ was used (see equation (37a)).

APPENDIX E

DERIVATION OF FORMULA FOR DRAG IN LINEARIZED THEORY

Formula (59) for D_1 is the well-known expression for the drag of the linearized theory. For the sake of completeness, a simple derivation of this formula is given. From equations (57), (56), and (55),

$$\frac{D_1}{2\pi\rho_0 U_0{}^2\epsilon^4} = \int_0^l k'(x)\left\{\int_0^x \frac{k''(x-\sigma) - k''(x)}{\sigma}\,d\sigma + k''(x)\log_e \frac{x}{\beta}\sqrt{\frac{2}{k(x)}} - \frac{\left[k'(x)\right]^2}{4k(x)}\right\}dx$$

Since

$$k'(x)k''(x)\log_e \frac{1}{\beta}\sqrt{\frac{2}{k(x)}} - \frac{1}{4}\frac{\left[k'(x)\right]^3}{k(x)} = \frac{d}{dx}\left\{\frac{1}{2}\left[k'(x)\right]^2 \log_e \frac{1}{\beta}\sqrt{\frac{2}{k(x)}}\right\}$$

and $k'(0) = k'(l) = 0$, the above integral reduces to

$$I = \int_0^l k'(x)\left[\int_0^x \frac{k''(x-\sigma) - k''(x)}{\sigma}\,d\sigma + k''(x)\log_e x\right]dx$$

Now, the bracket above

$$\int_0^x \frac{k''(x-\sigma) - k''(x)}{\sigma}\,d\sigma + k''(x)\log_e x =$$

$$\left[k''(x-\sigma) - k''(x)\right]\log_e \sigma\Big|_0^x + k''(x)\log_e x +$$

$$\int_0^x k'''(x-\sigma)\log_e \sigma = k''(0)\log_e x + \int_0^x k'''(x-\sigma)\log_e \sigma\,d\sigma =$$

$$\frac{d}{dx}\int_0^x k''(x-\sigma)\log_e \sigma\,d\sigma$$

so, after integrating by parts,

$$I = -\int_0^l dxk''(x)\int_0^x k''(x - \sigma) \log_e \sigma \, d\sigma$$

$$= -\int_0^l dx\int_0^x k''(x)k''(\sigma) \log_e (x - \sigma) \, d\sigma$$

which is the desired result.

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