# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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## CORRECTIONS FOR LIFT, DRAG, AND MOMENT OF AN AIRFOIL IN A SUPERSONIC TUNNEL HAVING A GIVEN STATIC PRESSURE GRADIENT

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## SUMMARY

The corrections for lift, drag, and moment of a two-dimensional airfoil have been analyzed, on the assumption that the airfoil is tested in the working section of a supersonic tunnel in which the pressure field, instead of being uniform, is characterized by gradients in the axial and transverse directions.

The pressure gradients of the tunnel as well as the effect of the airfoil to be tested are regarded as perturbations of the original rectilinear flow field of given Mach number. Therefore the velocity potential of the flow, the nonlinear differential equation of motion, and the boundary conditions are expanded into double series in powers of two parameters, one characterizing the airfoil thickness $\epsilon$ and the other, the inhomogeneity of the field $b$. In this way the nonlinear problem is split into a system of linear boundary-value problems, corresponding to the different powers of $b$ and $\epsilon$.

In each of the resulting problems there appears, besides the differential equation and boundary condition, an additional condition to be stipulated on the characteristics passing through the leading edge. Particular attention has been paid to the correct formulation of this "characteristic condition."

The solution procedure is carried out up to orders $b^2$, $\epsilon^2$, and $\epsilon b$ in the velocity potential. This means that, for example, the drag is computed up to orders $b^2\epsilon$, $\epsilon^3$, and $\epsilon^2 b$. The physical meaning of the results is discussed. The drag term in $\epsilon b$ represents the "horizontal buoyancy" of the airfoil, the term proportional to $b\epsilon^2$ is a consequence of the interaction of the airfoil field and the inhomogeneous pressure field, and the term in $b^2\epsilon$ may be considered as a "second-order buoyancy." The meaning of the various lift and moment terms may be interpreted similarly. The resulting expressions have been derived for arbitrary given pressure gradients and general profile form.

All solutions are obtained in closed, analytic form ready for immediate evaluation. Representative examples with graphs are included.

## INTRODUCTION

Reports from supersonic tunnels indicate that sometimes there exist in the working section undesired static pressure gradients due to design or construction flaws, which are difficult to eliminate. As a consequence, force measurements made upon models have to be corrected to agree with the forces to be expected in rectilinear flow.

For incompressible flow the correction to be applied to the drag, in the case of a longitudinal pressure gradient is well-known (reference 1). Because of the occurrence of a "horizontal buoyancy," the correction turns out to be equal to the product of the pressure gradient and the sum of volume and "apparent volume" of the test body; the apparent volume, like the real volume, depends only on the geometrical shape.

The present analysis is an attempt to solve the analogous problem in supersonic flow under the most general conditions, which require corrections to be applied also to the lift and moment. A two-dimensional pressure gradient in both the longitudinal and transverse directions is considered, which is equivalent to a stream-angle variation along the tunnel axis. In the first approximation the computations must be expected to yield a superposition of well-known results. Hence, it is important to carry the calculations up to the second approximation, which includes a term characterizing the interaction between the airfoil field and that of the given pressure gradient.

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## SYMBOLS

| | |
|---|---|
| $b$ | parameter characterizing inhomogeneity of field |
| $c$ | local sound speed |
| $c_o$ | sound speed at $M_o$ |
| $k(x)$ | dimensionless airfoil profile function measured above flight direction |
| $\overline{k}(\overline{x})$ | profile function measured above chord |
| $l$ | chord length |

| | |
|---|---|
| $M$ | local Mach number |
| $M_o$ | Mach number of uniform field |
| $m(x),n(x)$ | x- and y-components of velocity, on axes, due to inhomogeneity of flow field (dimensionless) |
| $p$ | local pressure |
| $p_o$ | pressure at $M_o$ |
| $q$ | local velocity |
| $x,y$ | dimensionless coordinates, axial and vertical |
| $\beta = \sqrt{M_o^2 - 1}$ | |
| $\gamma$ | adiabatic exponent |
| $\epsilon$ | airfoil thickness parameter |
| $\theta$ | local angle of attack |
| $\overline{\theta}$ | angle of attack of chord |
| $\xi,\eta$ | characteristic coordinates |
| $\rho$ | local density |
| $\rho_o$ | density at $M_o$ |
| $\varphi$ | dimensionless velocity potential |
| $\varphi^B$ | velocity potential of basic flow field |
| $\varphi'$ | velocity potential due to airfoil disturbance |
| $\psi(x)$ | ordinate of leading-edge characteristic |

## STATEMENT OF PROBLEM

### Physical Formulation

Suppose that in the test section of a supersonic tunnel the velocity field, instead of being uniform, is represented by the velocity distribution along the axis $y = 0$, as follows:

At $y = 0$:

$$\left.\begin{aligned} M_x &= M_o + bm(x) \\ M_y &= bn(x) \end{aligned}\right\} \qquad (1)$$

Here $M_x = q_x/c_o$ and $M_y = q_y/c_o$, where $q_x$ and $q_y$ are the x- and y-components of the local velocity of the flow field, $c_o$ is the sound velocity in a region where the Mach number is $M_o$, $x$ and $y$ are dimensionless coordinates measured in units of the length $l$ of the test body, and $b$ is a dimensionless parameter characterizing the magnitude of the inhomogeneity of the flow field and the pressure gradient. Under actual conditions $b$ is a small quantity such that

$$\left.\begin{matrix} bm(x) \\ bn(x) \end{matrix}\right\} \ll M_o \qquad (2)$$

where $m(x)$ and $n(x)$ are given arbitrary functions along the axis, which may be determined by measurement. With conditions on the axis fixed the flow field in the whole xy-plane is automatically prescribed by the equations of supersonic flow. Terms of the order $b^2$ may also be given arbitrarily on the axis but are assumed to be zero in the present case. This does not imply, however, that terms of the order $b^2$ do not appear in the field away from the axis.

In this velocity field a thin airfoil of given profile form $k(x)$ and thickness ratio $\epsilon$ is inserted, which may be regarded as another small perturbation of the rectilinear flow field $M_o$. Two kinds of disturbances then exist, the one produced by the velocity or pressure gradient measured by $b$ and another one produced by the airfoil and measured by $\epsilon$. Therefore, it appears appropriate to expand the velocity potential $\varphi$, as well as the potential equation of supersonic flow, and the boundary condition on the airfoil surface into a power series in powers of both parameters $b$ and $\epsilon$.

In this way, the nonlinear problem is split into a system of linear boundary-value problems corresponding to the different powers of $b$ and $\epsilon$. In each of the resulting problems there appears, besides the differential equation and boundary condition, an additional condition to be stipulated on the characteristics passing through the leading edge: It is to be required that the disturbance produced by the airfoil vanish along these curved lines in order to make the solution unique. It may

be noted that the characteristics of the complete differential equation are curved, while those of the resulting linear equations are straight. It is shown later how the condition on the characteristics can be formulated for each linear boundary-value problem separately.

## Analytic Formulation

Because of the relatively small thickness of the airfoil, the shocks at the leading edge will be weak and the entropy changes in the flow negligible. Hence, the complete flow field will be described by a velocity potential which obeys the well-known, nonlinear differential equation. Introducing a nondimensional quantity $\varphi$, equal to the potential divided by $lc_o$, this equation may be written:

$$\left(\frac{c^2}{c_o^2} - \varphi_x^2\right)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + \left(\frac{c^2}{c_o^2} - \varphi_y^2\right)\varphi_{yy} = 0 \tag{3}$$

where the subscripts $x$ and $y$ indicate differentiation.

The local sound speed $c$ corresponding to the local velocity $q$ is related to the sound speed $c_o$ by means of the energy equation

$$\frac{c^2}{c_o^2} - 1 = \frac{\gamma - 1}{2}\left(M_o^2 - \frac{q^2}{c_o^2}\right) \tag{4}$$

where

$$q^2/c_o^2 = \varphi_x^2 + \varphi_y^2$$

Substituting equation (4), equation (3) can be written:

$$\left[1 + \frac{\gamma - 1}{2}\left(M_o^2 - \varphi_y^2\right) - \frac{\gamma + 1}{2}\varphi_x^2\right]\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} +$$

$$\left[1 + \frac{\gamma - 1}{2}\left(M_o^2 - \varphi_x^2\right) - \frac{\gamma + 1}{2}\varphi_y^2\right]\varphi_{yy} = 0 \tag{5}$$

Now, $\varphi$ may be expanded as a series in powers of the perturbation parameters $b$ and $\epsilon$, as follows:

$$\varphi = \varphi^{00} + b\varphi^{01} + b^2\varphi^{02} + b\varphi^{10} + b^2\varphi^{11} + \epsilon\varphi^{12} + \epsilon b\varphi^{13} + \epsilon^2\varphi^{14} \qquad (6)$$

The potential coefficients

$$\varphi^B = \varphi^{00} + b\varphi^{01} + b^2\varphi^{02} \qquad (7)$$

represent the potential of the basic inhomogeneous velocity field before an airfoil was placed in it.

The terms

$$\varphi' = b\varphi^{10} + b^2\varphi^{11} + \epsilon^2\varphi^{12} + \epsilon b\varphi^{13} + \epsilon^2\varphi^{14} \qquad (8)$$

represent the perturbation potential produced by the airfoil in the basic field.

It is to be noted that the perturbation potential contains the terms $b\varphi^{10} + b^2\varphi^{11}$ which do not vanish for an airfoil of zero thickness ($\epsilon = 0$).

A disturbance for which $\epsilon = 0$ may be thought of as a very thin flat plate placed at a given angle into the basic flow field. In order for the inhomogeneous flow to follow the surface of the plate, vertical disturbance velocities must be produced of such magnitude (up or downward) as to cancel the vertical velocity components of the inhomogeneous flow field, which are proportional to $b$ or $b^2$.

The potential term proportional to $\epsilon b$ is due to the interaction between the basic field and that produced by the airfoil. The $\epsilon$ and $\epsilon^2$ terms represent the first- and second-order potentials of an airfoil in a homogeneous field.

Finally, the analytic procedure to be followed in this investigation, will be outlined briefly:

(1) Differential equation: Substituting series (6) for $\varphi$ into the nonlinear equation (equation (5)) and splitting in powers of $b$ and $\epsilon$, a set of homogeneous or nonhomogeneous equations of the wave type is obtained.

Those equations which are proportional to $b$, $\epsilon$, and $\epsilon^2$ are homogeneous:

$$\left(M_o^2 - 1\right)\varphi_{xx}{}^{ik} - \varphi_{yy}{}^{ik} = 0 \tag{9}$$

The equations which are proportional to $b^2$ and $\epsilon b$ are of the nonhomogeneous type:

$$\left(M_o^2 - 1\right)\varphi_{xx}{}^{ik} - \varphi_{yy}{}^{ik} = F_{ik}\left(\varphi^{lm}\right) \tag{10}$$

where the functions $F_{ik}$ can be shown to be known functions of lower-order $\varphi^{lm}$ which have been computed before.

(2) Boundary condition: Insert series (6) for $\varphi$ into the usual boundary condition stating that the flow is everywhere tangential to the surface of the body:

$$\left(\frac{\varphi_y}{\varphi_x}\right)_{surface} = \epsilon k'(x) \tag{11}$$

and expand equation (11) about $y = 0$ in powers of $\epsilon$. This imposes on each of the potential coefficients $\varphi^{ik}$ a condition of the form

$$\left(\varphi_y{}^{ik}\right)_{y=0} = G_{ik}\left(\varphi^{lm}\right) \tag{12}$$

where, again, the functions $G_{ik}$ can be shown to be known functions of lower-order $\varphi^{lm}$.

(3) "Characteristic condition": As mentioned before, the potential $\varphi'$ describing the various disturbances produced by the airfoil is required to vanish on the leading-edge characteristics which may be written:

$$y = \psi(x,b) = \psi_0(x) + b\psi_1(x) + \ldots \tag{13}$$

since the form of the characteristics is influenced by the inhomogeneity $b$.

It can be shown (see section "Characteristic Condition") that the "characteristic condition" for $\varphi'$ is equivalent to the following relations to be imposed on the individual $\varphi^{ik}$:

For the orders $b$, $\epsilon$, and $\epsilon^2$

$$\varphi^{ik}(x,\psi_0) = 0 \tag{14}$$

For the orders $b^2$ and $b\epsilon$

$$\varphi^{ik}(x,\psi_0) = \psi_1(x)\varphi^{lm}(x,\psi_0) \tag{15}$$

where, again, the coefficients $\varphi^{lm}$ are known lower-order potentials and $\psi_0(x)$ and $\psi_1(x)$ may be computed by means of the theory of characteristics.

At the end of this procedure the solutions $\varphi^{ik}$ will have been derived in closed, analytic form, and the pressures at any point can be computed. Since the objective is to compute the corrections for forces and moments, it is sufficient to determine the velocities and pressures on the airfoil surface only, thus simplifying the considerations.

## BASIC VELOCITY FIELD

### Differential Equation

The analysis can be clarified by considering first the basic inhomogeneous field alone. Then the series expansion for $\varphi$ reduces to:

$$\varphi^B = \varphi^{00} + b\varphi^{01} + b^2\varphi^{02} \tag{16}$$

Substituting equation (1) into equation (5) yields:

$$\left[1 - M_o^2 - b(\gamma + 1)M_o\varphi_x^{01} + \ldots\right]\left(b\varphi_{xx}^{01} + b^2\varphi_{xx}^{02} + \ldots\right) -$$

$$2(M_o + \ldots)\left(b\varphi_y^{01} + \ldots\right)\left(b\varphi_{xy}^{01} + \ldots\right) +$$

$$\left[1 - b(\gamma - 1)M_o\varphi_x^{01} + \ldots\right]\left(b\varphi_{yy}^{01} + b^2\varphi_{yy}^{02} + \ldots\right) = 0 \qquad (17)$$

In deriving equation (17) $\varphi^{00} = M_o x$ has been substituted in equation (5), since it is obviously a solution; furthermore, all those terms have been kept which could possibly contribute to the differential equation in $b$ and $b^2$. Splitting up in powers of $b$, the following equations result:

In $b$:

$$\left(M_o^2 - 1\right)\varphi_{xx}^{01} - \varphi_{yy}^{01} = 0$$

and in $b^2$:

$$\left(M_o^2 - 1\right)\varphi_{xx}^{02} - \varphi_{yy}^{02} = -M_o\left[(\gamma + 1)\varphi_x^{01}\varphi_{xx}^{01} + (\gamma - 1)\varphi_x^{01}\varphi_{yy}^{01} + 2\varphi_y^{01}\varphi_{xy}^{01}\right]$$

## Boundary Condition

The velocity distribution (equation (1)), prescribed on the axis $y = 0$, may be considered as a boundary condition for the potential of the basic flow.

Substituting equation (16) into equation (1), it follows that

$$\left.\begin{aligned} M_O + b\varphi_x^{01}(x,0) + b^2\varphi_x^{02}(x,0) &= M_O + bm(x) \\ \text{and} \\ b\varphi_y^{01}(x,0) + b^2\varphi_y^{02}(x,0) &= bn(x) \end{aligned}\right\} \tag{18}$$

Ordering terms in powers of $b$, a boundary problem in $b$ and one in $b^2$ are obtained.

## Boundary-Value Problem in Order $b$

The boundary-value problem in order $b$ consists of the differential equation

$$\beta^2\varphi_{xx}^{01} - \varphi_{yy}^{01} = 0 \tag{19a}$$

where $\beta^2 = M_O^2 - 1$, and the boundary conditions, at $y = 0$,

$$\left.\begin{aligned} \varphi_x^{01}(x,0) &= m(x) \\ \varphi_y^{01}(x,0) &= n(x) \end{aligned}\right\} \tag{19b}$$

The solution of differential equation (19a) is

$$\varphi^{01}(x,y) = f(x + \beta y) + g(x - \beta y) \tag{20}$$

where $f$ and $g$ are arbitrary functions to be determined by boundary conditions (19b).

Differentiating equation (20) with regard to $x$ and $y$ and then putting $y = 0$ yield:

$$\left.\begin{aligned} \varphi_x^{01}(x,0) &= f'(x) + g'(x) = m(x) \\ \varphi_y^{01}(x,0) &= \beta\left[f'(x) - g'(x)\right] = n(x) \end{aligned}\right\} \tag{21}$$

where $f'$ and $g'$ indicate derivatives with regard to the whole argument.

From equations (21) there follow:

$$\left.\begin{aligned} f'(x) &= \frac{1}{2}\left[m(x) + \frac{1}{\beta}\,n(x)\right] \\ g'(x) &= \frac{1}{2}\left[m(x) - \frac{1}{\beta}\,n(x)\right] \end{aligned}\right\} \tag{22}$$

Since it can be seen from equation (20) that $f$ and $g$ or their derivatives are constant along their characteristics $x \pm \beta y$, it may be concluded from equations (22) that

$$\left.\begin{aligned} f'(x + \beta y) &= \frac{1}{2}\left[m(x + \beta y) + \frac{1}{\beta}\,n(x + \beta y)\right] \\ \text{and} \qquad g'(x - \beta y) &= \frac{1}{2}\left[m(x - \beta y) - \frac{1}{\beta}\,n(x - \beta y)\right] \end{aligned}\right\} \tag{23}$$

Differentiating equation (20) and substituting equations (23), the velocity components of order $b^1$ for the entire field follow:

$$\left.\begin{aligned} \varphi_x^{01}(x,y) &= \frac{1}{2}\left[m(x + \beta y) + \frac{1}{\beta}\,n(x + \beta y) + m(x - \beta y) - \frac{1}{\beta}\,n(x - \beta y)\right] \\ \varphi_y^{01}(x,y) &= \frac{\beta}{2}\left[m(x + \beta y) + \frac{1}{\beta}\,n(x + \beta y) - m(x - \beta y) + \frac{1}{\beta}\,n(x - \beta y)\right] \end{aligned}\right\} \tag{24}$$

From equations (24) the expressions for $\varphi_{xx}^{01}$ and $\varphi_{xy}^{01}$ which appear in the differential equations for $\varphi^{11}$ and $\varphi^{13}$ may be readily derived.

Since, later on, the values of the velocity and pressure field are required only on the surface of the body, that is, near the axis, expressions (24) may be expanded about the axis:

$$\left.\begin{aligned} \varphi_x^{01}\left[x, \epsilon k(x)\right] &= m(x) + \epsilon k(x) n'(x) \ldots \\ \varphi_y^{01}\left[x, \epsilon k(x)\right] &= n(x) + \epsilon k(x) \beta^2 m'(x) \ldots \end{aligned}\right\} \tag{25}$$

## Boundary-Value Problem in Order $b^2$

In order $b^2$ the boundary-value problem consists of the differential equation

$$\beta^2\varphi_{xx}^{02} - \varphi_{yy}^{02} = -M_o\left[(\gamma + 1)\varphi_x^{01}\varphi_{xx}^{01} + (\gamma - 1)\varphi_x^{01}\varphi_{yy}^{01} + 2\varphi_y^{01}\varphi_{xy}^{01}\right] \tag{26a}$$

and the boundary conditions

$$\left.\begin{aligned} \varphi_x^{02}(x,0) &= 0 \\ \varphi_y^{02}(x,0) &= 0 \end{aligned}\right\} \tag{26b}$$

In order to determine the velocity and pressure field on the airfoil surface, no explicit solution of equations (26) is necessary. This can be seen by the following reasoning.

For $y = \epsilon k(x)$ the derivatives of $\varphi_y^{02}(x,y)$ may be obtained by expanding about the axis:

$$\left.\begin{aligned} \varphi_x^{02}\left[x,\epsilon k(x)\right] &= \varphi_x^{02}(x,0) + \epsilon k(x)\varphi_{xy}^{02}(x,0) + \cdots \\ \varphi_y^{02}\left[x,\epsilon k(x)\right] &= \varphi_y^{02}(x,0) + \epsilon k(x)\varphi_{yy}^{02}(x,0) + \cdots \end{aligned}\right\} \tag{27}$$

Since according to equation (16) $\varphi^{02}$ is of order $b^2$, the second- and higher-order terms in equations (27) are of order $b^2\epsilon$, $b^2\epsilon^2$, . . . and may thus be neglected so that in order $b^2$:

$$\left.\begin{aligned} \varphi_x^{02}\left[x,\epsilon k(x)\right] &\approx 0 \\ \varphi_y^{02}\left[x,\epsilon k(x)\right] &\approx 0 \end{aligned}\right\} \tag{28}$$

Therefore, the total velocity components of the basic field on the airfoil surface are:

$$\left.\begin{aligned} \varphi_x^B &= M_0 + bm(x) + \epsilon bk(x)n'(x) + b^2 0 \ldots \\ \varphi_y^B &= bn(x) + \epsilon bk(x)\beta^2 m'(x) + b^2 0 \ldots \end{aligned}\right\} \tag{29}$$

It is to be noted that the second derivatives $\varphi_{xx}^{02}$ and $\varphi_{yy}^{02}$ appear in the differential equation for $\varphi^{11}$ (see equation (32c)) and hence it may seem that the boundary-value problem (equations (26)) has to be solved after all. However, the combination $\varphi_{yy}^{02} - \beta^2\varphi_{xx}^{02}$, which is required in equation (32c), may be expressed in terms of the known lower-order potential $\varphi^{01}$, as differential equation (26a) indicates. Carrying out the proper differentiations and substitutions yields:

$$\varphi_{yy}^{02} - \beta^2\varphi_{xx}^{02} = \frac{M_0}{4}\left(M_0^2(\gamma+1)\left\{\left[m(\xi) + \frac{n(\xi)}{\beta}\right]\left[m'(\xi) + \frac{n'(\xi)}{\beta}\right] + \left[m(\eta) - \frac{n(\eta)}{\beta}\right]\left[m'(\eta) - \frac{n'(\eta)}{\beta}\right]\right\} + \left[(\gamma+1) + (\gamma-3)\beta^2\right]\left\{\left[m(\eta) - \frac{n(\eta)}{\beta}\right]\left[m'(\xi) + \frac{n'(\xi)}{\beta}\right] + \left[m(\xi) + \frac{n(\xi)}{\beta}\right]\left[m'(\eta) - \frac{n'(\eta)}{\beta}\right]\right\}\right) \tag{30}$$

where $\xi = x + \beta y$ and $\eta = x - \beta y$ are the characteristic coordinates of the problem.

## VELOCITY FIELD OF AIRFOIL

### Differential Equation

The next objective is to determine the perturbation potential produced by an airfoil which is placed into the (inhomogeneous) basic field at an arbitrary small angle of attack, the leading edge coinciding with the coordinate origin. Inserting series (6) for $\varphi$ into equation (5), the quantities $\varphi^{01}$ and $\varphi^{02}$ may now be considered as known and $\varphi^{10} \ldots \varphi^{14}$, as looked for. Keeping only those terms which may possibly contribute to a differential equation of the order $b$, $\epsilon$, $b^2$, $b\epsilon$, or $\epsilon^2$, (equation 5) yields:

$$\left\{1 - M_o^2 - \frac{\gamma + 1}{2}\left[b2M_o\left(\varphi_x^{01} + \varphi_x^{10}\right) + \epsilon 2M_o\varphi_x^{12} \ldots\right]\right\} \times$$

$$\left[b\left(\varphi_{xx}^{01} + \varphi_{xx}^{10}\right) + \epsilon\varphi_{xx}^{12} + b^2\left(\varphi_{xx}^{02} + \varphi_{xx}^{11}\right) + \epsilon b\varphi_{xx}^{13} + \epsilon^2\varphi_{xx}^{14} \ldots\right] -$$

$$2M_o\left[b\left(\varphi_y^{01} + \varphi_y^{10}\right) + \epsilon\varphi_y^{12} \ldots\right]\left[b\left(\varphi_{xy}^{01} + \varphi_{xy}^{10}\right) + \epsilon\varphi_{xy}^{12} \ldots\right] +$$

$$\left\{1 - \frac{\gamma - 1}{2}\left[b2M_o\left(\varphi_x^{01} + \varphi_x^{10}\right) + \epsilon 2M_o\varphi_x^{12} \ldots\right]\right\}\left[b\left(\varphi_{yy}^{01} + \varphi_{yy}^{10}\right) +\right.$$

$$\left.\epsilon\varphi_{yy}^{12} + b^2\left(\varphi_{yy}^{02} + \varphi_{yy}^{11}\right) + \epsilon b\varphi_{yy}^{13} + \epsilon^2\varphi_{yy}^{14} \ldots\right] = 0 \qquad (31)$$

Ordering terms in powers of $b$ and $\epsilon$, equation (31) will split into the following set of linear equations:

In $b$:

$$\beta^2\varphi_{xx}^{10} - \varphi_{yy}^{10} = 0 \qquad (32a)$$

In $\epsilon$:

$$\beta^2\varphi_{xx}^{12} - \varphi_{yy}^{12} = 0 \tag{32b}$$

In $b^2$:

$$\beta^2\varphi_{xx}^{11} - \varphi_{yy}^{11} = -M_o\left\{\left[(\gamma + 1) + (\gamma - 1)\beta^2\right]\left(\varphi_{xx}^{01} + \varphi_{xx}^{10}\right)\left(\varphi_x^{01} + \varphi_x^{10}\right) + 2\left(\varphi_y^{01} + \varphi_y^{10}\right)\left(\varphi_{xy}^{01} + \varphi_{xy}^{10}\right)\right\} + \varphi_{yy}^{02} - \beta^2\varphi_{xx}^{02} \tag{32c}$$

In $b\epsilon$:

$$\beta^2\varphi_{xx}^{13} - \varphi_{yy}^{13} = -M_o\left\{\left[(\gamma + 1) + (\gamma - 1)\beta^2\right]\left[\varphi_x^{12}\left(\varphi_{xx}^{01} + \varphi_{xx}^{10}\right) + \varphi_{xx}^{12}\left(\varphi_x^{01} + \varphi_x^{10}\right)\right] + 2\left[\varphi_y^{12}\left(\varphi_{xy}^{01} + \varphi_{xy}^{10}\right) + \varphi_{xy}^{12}\left(\varphi_y^{01} + \varphi_y^{10}\right)\right]\right\} \tag{32d}$$

In $\epsilon^2$:

$$\beta^2\varphi_{xx}^{14} - \varphi_{yy}^{14} = -M_o\left[(\gamma + 1)\varphi_x^{12}\varphi_{xx}^{12} + (\gamma - 1)\varphi_x^{12}\varphi_{yy}^{12} + 2\varphi_y^{12}\varphi_{xy}^{12}\right] \tag{32e}$$

Whenever the differential equation is inhomogeneous, the right-hand side turns out to be a function of the lower-order solutions which have been computed in an earlier step.

## Boundary Condition

The usual boundary condition that the velocity of the air is everywhere tangential to the airfoil surface may be written as follows:

At $y = \epsilon k(x)$

$$\varphi_y = \epsilon k'(x)\varphi_x \tag{33}$$

As in the basic field, so also here the values of the various potential coefficients $\varphi^{ik}$ on the airfoil surface may be obtained by expansion about $y = 0$:

$$\left.\begin{aligned} \varphi_x{}^{ik}\left[x,\epsilon k(x)\right] &= \varphi_x{}^{ik}(x,0) + \epsilon k(x)\varphi_{xy}{}^{ik}(x,0) + \cdots \\ \varphi_y{}^{ik}\left[x,\epsilon k(x)\right] &= \varphi_y{}^{ik}(x,0) + \epsilon k(x)\varphi_{yy}{}^{ik}(x,0) + \cdots \end{aligned}\right\} \tag{34}$$

Making use of equations (34), substituting equation (6) into equation (33), and ordering in powers of $b$ and $\epsilon$, the following set of boundary conditions results:

In $b$:

$$\varphi_y{}^{10}(x,0) = -\varphi_y{}^{01}(x,0) = -n(x) \tag{35a}$$

In $\epsilon$:

$$\varphi_y{}^{12}(x,0) = k'(x)\varphi_x{}^{00} = k'(x)M_0 \tag{35b}$$

In $b^2$:

$$\varphi_y{}^{11}(x,0) = -\varphi_y{}^{02}(x,0) = 0 \tag{35c}$$

In $\epsilon b$:

$$\varphi_y{}^{13}(x,0) = k'(x)\left[\varphi_x{}^{01}(x,0) + \varphi_x{}^{10}(x,0)\right] - k(x)\left[\varphi_{yy}{}^{01}(x,0) + \varphi_{yy}{}^{10}(x,0)\right]$$

$$= k'(x)\left[m(x) + \varphi_x{}^{10}(x,0)\right] - k(x)\left[\beta^2 m'(x) + \varphi_{yy}{}^{10}(x,0)\right] \tag{35d}$$

In $\epsilon^2$:

$$\varphi_y{}^{14}(x,0) = k'(x)\varphi_x{}^{12}(x,0) - k(x)\varphi_{yy}{}^{12}(x,0) \tag{35e}$$

Also in equations (35) the right-hand sides are known after the lower-order problem has been solved.

## Characteristic Condition

Procedure.- It may be noted that for the computation of the basic flow field both velocity components were prescribed, separately, along the axis, whereas in the present problem only the ratio of the two velocities is fixed by the airfoil condition. Thus, an additional condition is required.

From the fundamental laws of supersonic flow it is evident that no perturbation of the basic field can exist upstream of the two characteristics passing through the leading edge in a downstream direction. Existence theorems for hyperbolic differential equations require that the potential be continuous everywhere. Hence, the perturbation potential must vanish all along the leading-edge characteristics.

To formulate this characteristic condition analytically, the equation of the characteristics has to be known. Suppose the differential equation is given by:

$$A\left(x,y;\ \varphi_x,\varphi_y\right)\varphi_{xx} + 2B\left(x,y;\ \varphi_x,\varphi_y\right)\varphi_{xy} +$$

$$C\left(x,y;\ \varphi_x,\varphi_y\right)\varphi_{yy} + F\left(x,y;\ \varphi_x,\varphi_y\right) = 0 \tag{36}$$

where $A$, $B$, $C$, and $F$ are linear or nonlinear functions of $x$, $y$, and the first derivatives $\varphi_x$ and $\varphi_y$. Then the equation of the characteristics is as follows:

$$A\ dy^2 - 2B\ dx\ dy + C\ dx^2 = 0 \tag{37}$$

so that

$$\frac{dy}{dx} = \frac{1}{A}\left(B \pm \sqrt{B^2 - AC}\right) \tag{38}$$

Thus, the characteristics are seen to depend only on the coefficients of the second-order derivatives in equation (36). In general, these coefficients are functions not only of $x$ and $y$ but also of the desired function $\varphi$. Hence, in a rigorous theory the characteristics are not known beforehand but only after the function $\varphi$ is determined.

By assuming a power-series solution for $\varphi$, this indeterminancy is removed; the nonlinear equation, equation (36), is replaced by a set of linear differential equations whose second-order derivatives have the same constant coefficients for each equation. In this way, the characteristics of each linear equation of the set turn out to be the same straight Mach lines, $x \pm \beta y = \text{Constant}$, determined by the rectilinear component of the basic flow field, whereas the true characteristics are functions of the entire flow field (including the perturbation field $\varphi'$) and hence they are curved. This is an unsatisfactory state of affairs.

It is possible to improve the computation of the characteristics somewhat by considering once more nonlinear equation (5), before it is split up, and by merely distinguishing between the potential of the basic field $\varphi^B$, which is known, and that of the disturbance field $\varphi'$, which is looked for. If the order of magnitude of the various potential coefficients $\varphi^{ik}$ is, for the moment, left out of consideration, curved characteristics can be obtained also for linear differential equations, as the following illustration may show.

Substituting the potential series (6) in the form:

$$\varphi = \varphi^B + \varphi' \tag{39}$$

equation (5) becomes:

$$\left\{1 + \frac{\gamma - 1}{2}\left[M_o^2 - \left(\varphi_y^B + \varphi_y'\right)^2\right] - \frac{\gamma + 1}{2}\left(\varphi_x^B + \varphi_x'\right)^2\right\}\left(\varphi_{xx}^B + \varphi_{xx}'\right) -$$

$$2\left(\varphi_x^B + \varphi_x'\right)\left(\varphi_y^B + \varphi_y'\right)\left(\varphi_{xy}^B + \varphi_{xy}'\right) + \left\{1 + \frac{\gamma - 1}{2}\left[M_o^2 - \left(\varphi_x^B + \varphi_x'\right)^2\right] -$$

$$\frac{\gamma + 1}{2}\left(\varphi_y^B + \varphi_y'\right)^2\right\}\left(\varphi_{yy}^B + \varphi_{yy}'\right) = 0 \tag{40}$$

Since $\varphi^B$ has been determined already as a function of $x$ and $y$, equation (40) may be considered as a differential equation for the unknown $\varphi'$ only and may then be written in the following way:

$$A\varphi_{xx}' + 2B\varphi_{xy}' + C\varphi_{yy}' + \overline{F}\left(x,y;\ \varphi_x', \varphi_y'\right) = 0 \tag{41}$$

Here $\overline{F}$ is, as before, a given function of $x$, $y$, and the first derivatives $\varphi_x'$ and $\varphi_y'$. Each of the coefficients A, B, and C consists of one part that depends only on $\varphi^B$ and a second part which is proportional to $\varphi_x'$ or $\varphi_y'$:

$$\left.\begin{matrix} A \\ B \\ C \end{matrix}\right\} = f_1(\varphi^B) + \left.\begin{matrix} \varphi_x' \\ \varphi_y' \end{matrix}\right\} f_2(\varphi^B, \varphi') \tag{42}$$

Since it is the prime purpose of this analysis to determine the influence of the inhomogeneous field upon the pressure distribution around the airfoil by means of a linear theory, it is sufficient to approximate the equation for the characteristics, by neglecting in the coefficients A, B, and C that part which depends on $\varphi'$, so that now:

$$\left.\begin{matrix} A \\ B \\ C \end{matrix}\right\} \approx f_1(\varphi^B) \approx g_1(x,y)$$

In this way, the characteristics of the total field around the airfoil are approximated by the characteristics of the basic inhomogeneous field. On the other hand, the lines represented by equation (38) which result from such a procedure will be curved. Thus, the equation for the characteristics becomes:

$$\left\{1 + \frac{\gamma - 1}{2}\left[M_o^2 - \left(\varphi_y^B\right)^2\right] - \frac{\gamma + 1}{2}\left(\varphi_x^B\right)^2\right\} dy^2 + 2\left(\varphi_x^B \varphi_y^B\right) dx\, dy +$$

$$\left\{1 + \frac{\gamma - 1}{2}\left[M_o^2 - \left(\varphi_x^B\right)^2\right] - \frac{\gamma + 1}{2}\left(\varphi_y^B\right)^2\right\} dx^2 = 0 \tag{43}$$

where

$$\varphi^B = M_o x + b\varphi^{01} + b^2\varphi^{02} \tag{44}$$

Analytic formulation.- Substituting equation (44) into equation (43) and retaining only terms up to first powers in b, the coefficients of equation (43) become:

$$\left.\begin{aligned} A &= -\left[\beta^2 + (\gamma + 1)M_o b\varphi_x^{01} + \ldots\right] \\ 2B &= -2M_o b\varphi_y^{01} \\ C &= \left[1 - (\gamma - 1)M_o b\varphi_x^{01} \ldots\right] \end{aligned}\right\} \tag{45}$$

where $\varphi_x^{01}$ and $\varphi_y^{01}$ are functions of $m(x,y)$ and $n(x,y)$ given by equations (24).

Inserting equations (45) into equation (38) and keeping consistently only terms up to first powers in b, the equation of the characteristics becomes:

$$\frac{dy}{dx} = \frac{M_o b\varphi_y^{01} \pm \sqrt{\left[1 - (\gamma - 1)M_o b\varphi_x^{01}\right]\left[\beta^2 + (\gamma + 1)M_o b\varphi_x^{01}\right]}}{\beta^2 + (\gamma + 1)M_o b\varphi_x^{01}} \tag{46}$$

and after expanding the right-hand side in powers of b:

$$\frac{dy}{dx} = \pm\frac{1}{\beta}\left\{1 - b\frac{M_o}{2\beta^2}\left[\gamma + 1 + (\gamma - 1)\beta^2\right]\varphi_x^{01} \pm b\frac{M_o}{\beta}\varphi_y^{01} \ldots\right\} \tag{47}$$

The upper (or lower) sign is used for the characteristics above (or below) the airfoil.

Equation (47) may be solved by an iteration process, in which the first approximation is obviously

$$\frac{dy}{dx} = \pm\frac{1}{\beta}$$

This leads to the Mach lines:

$$y = \pm x/\beta \tag{48}$$

where the constant of integration has been set equal to zero so as to yield the leading-edge characteristics.

Substituting equation (48) into the right-hand side of equation (47) yields in the second approximation:

$$\frac{dy}{dx} = \pm\frac{1}{\beta}\left\{1 - \frac{M_o b}{2\beta^2}\left[(\gamma + 1) + (\gamma - 1)\beta^2\right]\left(\varphi_x{}^{01}\right)_{y=\pm x/\beta} \pm \frac{M_o b}{\beta}\left(\varphi_y{}^{01}\right)_{y=\pm x/\beta}\right\} \tag{49}$$

From equations (24) substitute for the characteristics above (or below) the airfoil:

$$\left.\begin{aligned}\left(\varphi_x{}^{01}\right)_{y=\pm x/\beta} &= \frac{1}{2}\left[m(2x) \pm \frac{n(2x)}{\beta} + m(0) \mp \frac{n(0)}{\beta}\right] \\ \left(\varphi_y{}^{01}\right)_{y=\pm x/\beta} &= \frac{\beta}{2}\left[\pm m(2x) + \frac{n(2x)}{\beta} \mp m(0) + \frac{n(0)}{\beta}\right]\end{aligned}\right\} \tag{50}$$

Inserting equations (50) into equation (49) and integrating yield for the characteristics:

$$y = \psi_0(x) + b\psi_1(x) \ldots$$

where

$$\left.\begin{aligned}\psi_0(x) &= \pm x/\beta \\ \psi_1(x) &= \mp\frac{M_o}{4\beta^3}\left\{\left[4 + (\gamma - 3)M_o{}^2\right]\int_0^x \left[m(2x) \pm \frac{n(2x)}{\beta}\right] dx + \right. \\ &\qquad \left. M_o{}^2(\gamma + 1)\left[m(0) \mp \frac{n(0)}{\beta}\right]x\right\}\end{aligned}\right\} \tag{51}$$

The upper (or lower) sign refers to the upper (or lower) surface, respectively.

Coming back to the characteristic condition indicated in the section "Procedure," the following requirement has to be fulfilled: For

$$y = \psi_0(x) + b\psi_1(x) + \ldots$$

$$\varphi' = b\varphi^{10} + b^2\varphi^{11} + \epsilon\varphi^{12} + \epsilon b\varphi^{13} + \epsilon^2\varphi^{14} = 0$$

More explicitly:

$$b\varphi^{10}\left[x;\ \psi_0(x) + b\psi_1(x) \ldots\right] + b^2\varphi^{11}\left[x;\ \psi_0(x) + b\psi_1(x) \ldots\right] +$$

$$\epsilon\varphi^{12}\left[x;\ \psi_0(x) + b\psi_1(x) \ldots\right] + \epsilon b\varphi^{13}\left[x;\ \psi_0(x) + b\psi_1(x) \ldots\right] +$$

$$\epsilon^2\varphi^{14}\left[x;\ \psi_0(x) + b\psi_1(x) \ldots\right] = 0 \tag{52}$$

Expanding $\varphi'$ about $y = \psi_0(x)$ in powers of $b$ yields:

$$b\varphi^{10}(x,\psi_0) + b^2\left[\psi_1(x)\varphi_y{}^{10}(x,\psi_0) + \varphi^{11}(x,\psi_0)\right] + \epsilon\varphi^{12}(x,\psi_0) +$$

$$\epsilon b\left[\psi_1(x)\varphi_y{}^{12}(x,\psi_0) + \varphi^{13}(x,\psi_0)\right] + \epsilon^2\varphi^{14}(x,\psi_0) = 0 \tag{53}$$

Splitting equation (53) in powers of $b$ and $\epsilon$ furnishes for each potential coefficient $\varphi^{ik}$ one characteristic condition:

In $b$:

$$\varphi^{10}(x,\psi_0) = 0 \tag{54a}$$

In $\epsilon$:

$$\varphi^{12}(x,\psi_0) = 0 \tag{54b}$$

In $b^2$:

$$\varphi^{11}(x,\psi_0) = -\psi_1(x)\varphi_y{}^{10}(x,\psi_0) \tag{54c}$$

In $\epsilon b$:

$$\varphi^{13}(x,\psi_0) = -\psi_1(x)\varphi_y{}^{12}(x,\psi_0) \tag{54d}$$

In $\epsilon^2$:

$$\varphi^{14}(x,\psi_0) = 0 \tag{54e}$$

Whenever the relation is inhomogeneous, the unknown potential is expressed in terms of known lower-order potentials.

Each of the boundary-value problems consisting of a differential equation, a boundary condition, and a characteristic condition has now been formulated and will be solved in the appropriate order.

## COMPUTATION OF PERTURBATION POTENTIALS

In the following discussion (see fig. 1) $k(x)$ denotes the ordinate of the profile above (or below) the x-axis, which is the direction of the relative wind $M_0$. To distinguish between the upper and lower surfaces, $k_U$ and $k_L$ are used; $k(1)$ is zero or unequal to zero depending on whether the airfoil is at zero or at finite angle of attack. If there are two signs in front of a term, the upper sign refers to the upper, and the lower sign, to the lower surface.

### Computation of $\varphi^{10}$

That part of the disturbance potential which is proportional to $b$, namely, $\varphi^{10}$, obeys the following equations: The differential equation:

$$\left.\begin{aligned} &\beta^2\varphi_{xx}{}^{10} - \varphi_{yy}{}^{10} = 0 \\ &\text{the boundary condition:} \\ &\varphi_y{}^{10}(x,0) = -n(x) \\ &\text{and the characteristic condition:} \\ &\varphi^{10}(x,\pm x/\beta) = 0 \end{aligned}\right\} \tag{55}$$

To obtain a solution, it is convenient to introduce the characteristic coordinates:

$$\xi = x + \beta y$$

$$\eta = x - \beta y$$

Then

$$\left.\begin{aligned} \varphi_x &= \varphi_\xi + \varphi_\eta \\ \varphi_y &= \beta\left(\varphi_\xi - \varphi_\eta\right) \end{aligned}\right\} \tag{56}$$

and

$$\beta^2\varphi_{xx} - \varphi_{yy} = 4\beta^2\varphi_{\xi\eta} \tag{57}$$

As the characteristic condition is different on the upper and lower surfaces, the two surfaces will be treated separately.

Upper surface.- In characteristic coordinates equations (55) may be written as follows for the upper surface: The differential equation becomes

$$4\beta^2\varphi_{\xi\eta}{}^{10} = 0$$

the boundary condition,

$$\beta\left(\varphi_\xi{}^{10} - \varphi_\eta{}^{10}\right)_{\xi=\eta=x} = -n(x)$$

and the characteristic condition,

$$\varphi^{10}(\xi,0) = 0$$

(58)

Integration of the differential equation yields:

$$\varphi^{10}(\xi,\eta) = F(\xi) + G(\eta) \tag{59}$$

where the arbitrary functions F and G are determined by the boundary condition and the characteristic condition.

From the characteristic condition there follows:

$$\varphi_\xi{}^{10}(\xi,0) = F'(\xi) = 0 \quad \text{for any} \quad \xi \tag{60}$$

Then the boundary condition requires that

$$\beta\left[F'(\xi) - G'(\eta)\right]_{\xi=\eta=x} = \beta\left[F'(x) - G'(x)\right] = -n(x) \tag{61}$$

Taking into account equation (60), it follows that:

$$\left.\begin{aligned} G'(x) &= \frac{1}{\beta}\, n(x) \\ G'(\eta) &= \frac{1}{\beta}\, n(\eta) \end{aligned}\right\} \tag{62}$$

Therefore:

$$\left.\begin{aligned} \varphi_x{}^{10}(x,y) &= \frac{1}{\beta}\, n(x - \beta y) \\ \varphi_y{}^{10}(x,y) &= -n(x - \beta y) \end{aligned}\right\} \tag{63}$$

To obtain the pressures on the airfoil surface $y = \epsilon k(x)$, equations (24) may be expanded:

$$\left.\begin{aligned} \varphi_x{}^{10}\left[x,\epsilon k_U(x)\right] &= \varphi_x{}^{10}(x,0) + \epsilon k_U(x)\varphi_{xy}{}^{10}(x,0) \ldots \\ &= \frac{1}{\beta}\, n(x) - \epsilon k_U(x) n'(x) + \ldots \\ \varphi_y{}^{10}\left[x,\epsilon k_U(x)\right] &= -n(x) + \epsilon\beta k_U(x) n'(x) + \ldots \end{aligned}\right\} \tag{64}$$

Lower surface.- On the lower surface, the boundary condition is the same:

$$\beta\left(\varphi_\xi{}^{10} - \varphi_\eta{}^{10}\right)_{\xi=\eta=x} = -n(x)$$

but the characteristic condition differs:

$$\varphi^{10}\left(x, -\frac{x}{\beta}\right) = \varphi^{10}(0,\eta) = 0 \tag{65}$$

From equation (65) there follows: $\varphi_\eta(0,\eta) = G'(\eta) = 0$, for any $\eta$. Then, from the boundary condition there follows:

$$\beta\varphi_\xi = \beta F'(\xi)_{\xi=x} = -n(x)$$

so that

$$\varphi_\xi(\xi) = -\frac{1}{\beta}\, n(\xi) \tag{66}$$

Using equations (42)

and

$$\left.\begin{aligned} \varphi_x{}^{10}(x,y) &= -\frac{1}{\beta}\, n(x + \beta y) \\ \varphi_y{}^{10}(x,y) &= -n(x + \beta y) \end{aligned}\right\} \tag{67}$$

On the airfoil surface:

$$\left.\begin{aligned} \varphi_x{}^{10}\left[x, \epsilon k_L(x)\right] &= -\frac{1}{\beta}\, n(x) - \epsilon k_L(x) n'(x) \ldots \\ \varphi_y{}^{10}\left[x, \epsilon k_L(x)\right] &= -n(x) - \epsilon\beta k_L(x) n'(x) \ldots \end{aligned}\right\} \tag{68}$$

## Computation of $\varphi^{11}$

For the potential $\varphi^{11}$ which is proportional to $b^2$, the boundary-value problem can be formulated by means of equations (32c), (35c), and (54c): The differential equation is

$$\left.\begin{aligned}\beta^2\varphi_{xx}{}^{11} - \varphi_{yy}{}^{11} = -M_o\Big\{\Big[(\gamma+1)+(\gamma-1)\beta^2\Big]\Big(\varphi_{xx}{}^{01}+\varphi_{xx}{}^{10}\Big)\Big(\varphi_x{}^{01}+\varphi_x{}^{10}\Big)+ \\ 2\Big(\varphi_y{}^{01}+\varphi_y{}^{10}\Big)\Big(\varphi_{xy}{}^{01}+\varphi_{yx}{}^{10}\Big)\Big\} + \varphi_{yy}{}^{02} - \beta^2\varphi_{xx}{}^{02}\end{aligned}\right\} \quad (69)$$

the boundary condition,

$$\varphi_y{}^{11}(x,0) = 0$$

and the characteristic condition,

$$\varphi^{11}(x,\pm x/\beta) = -\psi_1(x)\varphi_y{}^{10}(x,\pm x/\beta)$$

Upper surface.- Inserting the values for $\varphi^{01}$, $\varphi^{10}$, and $\varphi_{yy}{}^{02} - \beta^2\varphi_{xx}{}^{02}$ from equations (24), (64), and (30), for the upper surface problem (69), in characteristic coordinates, is as follows: The differential equation becomes

$$\begin{aligned}4\beta^2\varphi_{\xi\eta}{}^{11} = -\frac{M_o}{2}\Bigg(\Big[(\gamma+1)(\beta^2+1)\Big]\left[m(\eta)\frac{n'(\eta)}{\beta}+m'(\eta)\frac{n(\eta)}{\beta}\right]\times \\ \Big[(\gamma+1)+(\gamma-3)\beta^2\Big]\left\{\frac{n'(\eta)}{\beta}\left[m(\xi)+\frac{n(\xi)}{\beta}\right]+ \\ \frac{n(\eta)}{\beta}\left[m'(\xi)+\frac{n'(\xi)}{\beta}\right]\right\}\Bigg)\end{aligned} \quad (70)$$

the boundary condition,

$$\beta\Big(\varphi_\xi{}^{11} - \varphi_\eta{}^{11}\Big)_{\xi=\eta=x} = 0 \quad (71)$$

and the characteristic condition,

$$\varphi^{11}(\xi,0) = \psi_1(\xi,0)n(0) \quad (72)$$

The rest of the procedure may be indicated here briefly, the details being carried out in appendix A.

Upon integrating equation (70) with regard to $\xi$ and $\eta$, respectively:

$$\left.\begin{aligned} \varphi_{\xi}^{11} &= G_1(m,n) + P_1'(\xi) \\ \varphi_{\eta}^{11} &= G_2(m,n) + P_2'(\eta) \end{aligned}\right\} \tag{73}$$

where $G_1$ and $G_2$ are known functions of $m(\xi)$, $m(\eta)$, $n(\xi)$, and $n(\eta)$ and $P_1$ and $P_2$ are unknown integration constants.

Now, characteristic condition (72), upon differentiating with regard to $\xi$:

$$\varphi_{\xi}^{11}(\xi,0) = \psi_{1_{\xi}}(\xi,0)n(0) \tag{74}$$

determines $P_1'(\xi)$ by means of equations (73).

Then boundary condition (71), after insertion of equations (73), fixes $P_2'(\eta)$.

With $P_1$ and $P_2$ determined, the two velocity components $\varphi_x^{11}(x,0)$ and $\varphi_y^{11}(x,0)$ that are necessary for the computation of the pressure on the airfoil surface can be determined by means of equations (73) and (56).

Lower surface.- For the lower surface an analogous procedure can be carried through; note that the characteristic condition is now:

$$\varphi_{\eta}^{11}(0,\eta) = \psi_1(0,\eta)n(0)$$

The resulting velocity components on either airfoil surface are as follows:

$$\left.\begin{aligned}
\varphi_x^{11}(x,0) &= \mp\frac{M_o}{4\beta^2}\left(\left[(\gamma-3)M_o^2+4\right]\left\{\pm\frac{n(x)^2}{\beta^2}+\left[m'(x)\pm\frac{n'(x)}{\beta}\right]\int_0^x\frac{n(\eta')}{\beta}\,d\eta'\right\}+2\left[(\gamma-1)M_o^2+2\right]\frac{m(x)n(x)}{\beta}\mp\left[(\gamma+1)M_o^2\right]\frac{n(0)^2}{\beta^2}\right)\\
\varphi_y^{11}(x,0) &= 0
\end{aligned}\right\} \quad (75)$$

Computation of $\varphi^{12}$

That part of the disturbance potential which is proportional to $\epsilon$, $\varphi^{12}$, must be expected to equal the expression obtained in the Ackeret theory.

Hence, the velocity components above (or below) the airfoil are:

$$\left.\begin{aligned}
\varphi_x^{12} &= \mp\frac{M_o}{\beta}\,k_{U(L)}'(x\mp\beta y)\\
\varphi_y^{12} &= M_o k_{U(L)}'(x\mp\beta y)
\end{aligned}\right\} \quad (76)$$

By expanding about $y = 0$, the velocities on the upper (or lower) airfoil surface are:

$$\left.\begin{aligned}
\varphi_x^{12}\left[x,\epsilon k(x)\right] &= \mp\frac{M_o}{\beta}\,k_{U(L)}'(x)+M_o\epsilon k_{U(L)}(x)k_{U(L)}''(x)\ldots\\
\varphi_y^{12}\left[x,\epsilon k(x)\right] &= M_o k_{U(L)}'(x)\mp M_o\beta\epsilon k_{U(L)}(x)k_{U(L)}''(x)\ldots
\end{aligned}\right\} \quad (77)$$

## Computation of $\varphi^{13}$

The potential $\varphi^{13}$, which is proportional to $\epsilon b$, may be considered to originate from some kind of interaction between the inhomogeneity of the basic field and the airfoil disturbance. The boundary-value problem for $\varphi^{13}$ can be formulated by means of equations (32d), (35d), and (54d): The differential equation is

$$\beta^2\varphi_{xx}^{13} - \varphi_{yy}^{13} = -M_o\left\{\left[\gamma + 1 + (\gamma - 1)\beta^2\right]\left[\varphi_x^{12}\left(\varphi_{xx}^{01} + \varphi_{xx}^{10}\right) + \varphi_{xx}^{12}\left(\varphi_x^{01} + \varphi_x^{10}\right)\right] + 2\left[\varphi_y^{12}\left(\varphi_{xy}^{01} + \varphi_{xy}^{10}\right) + \varphi_{xy}^{12}\left(\varphi_y^{01} + \varphi_y^{10}\right)\right]\right\}$$

the boundary condition,

$$\varphi_y^{13}(x,0) = k_{U(L)}'(x)\left[m(x) + \varphi_x^{10}(x,0)\right] - k_{U(L)}(x)\left[\beta^2 m'(x) + \varphi_{yy}^{10}(x,0)\right]$$

and the characteristic condition,

$$\varphi^{13}(x,\pm x/\beta) = -\psi_1(x)\varphi_y^{12}(x,\pm x/\beta)$$

(78)

Upper surface.- The treatment of the upper surface is as follows: Inserting for $\varphi^{10}$ and $\varphi^{12}$ expressions (63) and (77), problem (78) written in characteristic coordinates becomes:

The differential equation:

$$4\beta^2\varphi_{\xi\eta}^{13} = \frac{M_o^2}{2\beta}\left(\left[(\gamma + 1) + (\gamma - 3)\beta^2\right]\left\{k_U'(\eta)\left[m'(\xi) + \frac{n'(\xi)}{\beta}\right] + k_U''(\eta)\left[m(\xi) + \frac{n(\xi)}{\beta}\right]\right\} + \left[(\gamma + 1)(\beta^2 + 1)\right]\left\{k_U'(\eta)\left[m'(\eta) + \frac{n'(\eta)}{\beta}\right] + k_U''(\eta)\left[m(\eta) + \frac{n(\eta)}{\beta}\right]\right\}\right) \quad (79)$$

The boundary condition:

$$\beta\left(\varphi_{\xi}^{13} - \varphi_{\eta}^{13}\right)_{\xi=\eta=x} = k_U'(x)\left[m(x) + \frac{n(x)}{\beta}\right] - \beta^2 k_U(x)\left[m'(x) + \frac{n'(x)}{\beta}\right] \quad (80)$$

The characteristic condition:

$$\varphi^{13}(\xi,0) = -\psi_1(\xi,0)M_o k_U'(0) \quad (= 0) \quad (81)$$

The rest of the procedure may be indicated here briefly, the details being carried out in appendix A.

Upon integrating equation (79) with regard to $\xi$ and $\eta$, respectively:

$$\varphi_{\xi}^{13} = F_1(m,n,k) + J_1'(\xi) \quad (82a)$$

and

$$\varphi_{\eta}^{13} = F_2(m,n,k) + J_2'(\eta) \quad (82b)$$

where $F_1$ and $F_2$ are known functions of $k(\xi)$, $m(\xi)$, $n(\xi)$, $k(\eta)$, $m(\eta)$, and $n(\eta)$ and $J_1$ and $J_2$ are unknown integration constants.

Now characteristic condition (81)

$$\varphi_{\xi}^{13}(\xi,0) = -\psi_{1_{\xi}}(\xi,0)M_o k_U'(0)$$

determines $J_1'(\xi)$ by means of equations (82). The function $F_1$ vanishes for $k(0) = 0$ and $k'(0) = 0$, as can be seen from appendix A. Also equation (81) vanishes in that case. Then it follows simply that:

$$J_1'(\xi) = 0$$

This simplification suggests making the assumption $k'(0) = 0$, which is an artifice frequently used in linearized theories. The error introduced

by this deviation from the true contour is known to yield small inaccuracies in the flow field in a very thin strip adjacent to the two characteristics passing through the leading edge. Therefore the simplification is permissible.

Boundary condition (80), after insertion of equations (82), fixes $J_2'(\eta)$. With $J_1$ and $J_2$ fixed, the two velocity components $\varphi_x^{13}(x,0)$ and $\varphi_y^{13}(x,0)$ that are necessary for the computation of the pressure on the airfoil surface can be determined by means of equations (82) and (56).

Lower surface.- For the lower surface an analogous procedure can be carried through. Note that now the characteristic condition is $\varphi_\eta(0,\eta) = 0$ and that different expressions for $k(x)$, $\varphi^{10}$, and $\varphi^{12}$ have to be inserted.

The resulting velocity components on the airfoil surface are:

$$\left.\begin{aligned}\varphi_x^{13}(x,0) &= \pm\frac{M_o^2}{8\beta^2}\left\{\left[4M_o^2(\gamma-1)+8-\frac{8\beta^2}{M_o^2}\right]k_{U(L)}'(x)\left[m(x)\pm\frac{n(x)}{\beta}\right]+\right.\\ &\qquad\left.\left[2M_o^2(\gamma-3)+8+\frac{8\beta^4}{M_o^2}\right]k_{U(L)}(x)\left[m'(x)\pm\frac{n'(x)}{\beta}\right]\right\}\\ \varphi_y^{13}(x,0) &= k_{U(L)}'(x)\left[m(x)\pm\frac{n(x)}{\beta}\right]-\beta^2 k_{U(L)}(x)\left[m'(x)\pm\frac{n'(x)}{\beta}\right]\end{aligned}\right\} \quad (83)$$

## Computation of $\varphi^{14}$

That part of the disturbance potential which is proportional to $\epsilon^2$, that is, $\varphi^{14}$, obeys equations (32e), (35e), and (54e): The differential equation is

$$\left.\begin{aligned}&\beta^2\varphi_{xx}^{14}-\varphi_{yy}^{14}=-M_o\left[(\gamma+1)\varphi_{xx}^{12}\varphi_x^{12}+(\gamma-1)\varphi_x^{12}\varphi_{yy}^{12}+2\varphi_y^{12}\varphi_{xy}^{12}\right]\\ &\text{the boundary condition,}\\ &\varphi_y^{14}(x,0)=\varphi_x^{12}(x,0)k_{U(L)}'(x)-\varphi_{yy}^{12}(x,0)k_{U(L)}(x)\\ &\text{and the characteristic condition,}\\ &\varphi^{14}(x,\pm x/\beta)=0\end{aligned}\right\} \quad (84)$$

Upper surface.- For the upper surface, after inserting for $\varphi^{12}$ expressions (77), problem (84), written in characteristic coordinates, is as follows: The differential equation becomes

$$4\beta^2\varphi_{\xi\eta}{}^{14} = -\frac{M_o{}^5}{\beta^2}\left[(\gamma + 1)k_U'(\eta)k_U''(\eta)\right] \tag{85}$$

the boundary condition,

$$\beta\left(\varphi_\xi{}^{14} - \varphi_\eta{}^{14}\right)_{\xi=\eta=x} = -\frac{M_o}{\beta}k_U'^2(x) + \beta M_o k_U(x)k_U''(x) \tag{86}$$

and the characteristic condition,

$$\varphi^{14}(\xi,0) = 0 \tag{87}$$

Integrating equation (85) with respect to $\xi$ and $\eta$, respectively:

$$\left.\begin{aligned}\varphi_\xi{}^{14} &= -\frac{M_o{}^5}{4\beta^4}\left[\frac{\gamma + 1}{2}k_U'(\eta)^2\right] + M_1'(\xi)\\ \varphi_\eta{}^{14} &= -\frac{M_o{}^5}{4\beta^4}\left[(\gamma + 1)k_U'(\eta)k_U''(\eta)\xi\right] + M_2'(\eta)\end{aligned}\right\} \tag{88}$$

where $M_1$ and $M_2$ have to be determined from the boundary condition and the characteristic condition.

From characteristic condition (87) there follows:

$$\varphi_\xi{}^{14}(\xi,0) = 0 \tag{89}$$

which, by means of equations (88), fixes $M_1'(\xi) = 0$ if again the assumption $k'(0) = 0$ is made, as in the computation of $\varphi^{13}$.

Then boundary condition (86), after insertion of equations (88), fixes $M_2'(\eta)$.

With $M_1$ and $M_2$ fixed the two velocity components on the airfoil surface can be determined by means of equations (88) and (56).

Lower surface.- After analogous considerations for the lower surface have been carried out, the velocity components on the two surfaces are:

$$\left.\begin{aligned}
\varphi_x^{14}(x,0) &= -\frac{M_o}{4\beta^4}\left\{\left[(\gamma+1)M_o^4 - 4\left(M_o^2-1\right)\right]k_{U(L)}'^2(x) + \right. \\
&\qquad \left.\left[4\beta^4 k_{U(L)}(x)k_{U(L)}''(x)\right]\right\} \\
\varphi_y^{14}(x,0) &= \mp\frac{M_o}{\beta}k_{U(L)}'^2(x) \pm \beta M_o k_{U(L)}(x)k_{U(L)}''(x)
\end{aligned}\right\} \quad (90)$$

## FORCES AND MOMENTS ON AIRFOIL

### Complete Velocity Field on Airfoil Surface

To obtain the pressure distribution, the squares of the velocity components of the whole field are needed. By adding all of the components (up to the second order in $b$ and $\epsilon$), there result:

$$\left.\begin{aligned}
\varphi_x^2 &= M_o^2 + b^2\left[\left(\varphi_x^{01}\right)^2 + \left(\varphi_x^{10}\right)^2 + 2\varphi_x^{01}\varphi_x^{10}\right] + \\
&\qquad 2\epsilon b\varphi_x^{12}\left(\varphi_x^{10} + \varphi_x^{01}\right) + \epsilon^2\left(\varphi_x^{12}\right)^2 + 2M_o\left[b\left(\varphi_x^{01} + \varphi_x^{10}\right) + \right. \\
&\qquad \left.\epsilon\varphi_x^{12} + b^2\left(\varphi_x^{02} + \varphi_x^{11}\right) + \epsilon b\varphi_x^{13} + \epsilon^2\varphi_x^{14}\right] \\
\varphi_y^2 &= \epsilon^2\left(\varphi_y^{12}\right)^2 + b^2\left(\varphi_y^{01} + \varphi_y^{10}\right)^2 + 2\epsilon b\varphi_y^{12}\left(\varphi_y^{01} + \varphi_y^{10}\right)
\end{aligned}\right\} \quad (91)$$

Substituting the expressions derived in the preceding sections:

$$\left.\begin{aligned}
\varphi_x^2 &= M_o^2 + bA_1 + \epsilon A_2 + b^2A_3 + \epsilon bA_4 + \epsilon^2A_5 \\
\varphi_y^2 &= \epsilon^2M_o^2k_{U(L)}'^2(x)
\end{aligned}\right\} \quad (92)$$

Here:

$$\left.\begin{aligned}
A_1 &= M_o\left[2m(x) \pm 2\frac{n(x)}{\beta}\right] \\
A_2 &= \mp\left(\frac{2M_o^2}{\beta}\right)k_{U(L)}'(x) \\
A_3 &= \left[\left[m(x) \pm \frac{n(x)}{\beta}\right]^2 \mp \frac{M_o^2}{2\beta^2}\left(\left[(\gamma - 3)M_o^2 + 4\right] \times \right.\right. \\
&\quad \left\{\pm\frac{n^2(x)}{\beta^2} + \left[m'(x) \pm \frac{n'(x)}{\beta}\right]\int_0^x \frac{n(\eta')}{\beta}\,d\eta'\right\} + \\
&\quad \left.\left. 2\left[(\gamma - 1)M_o^2 + 2\right]\left[m(x)\frac{n(x)}{\beta}\right] \mp M_o^2(\gamma + 1)\frac{n^2(0)}{\beta^2}\right)\right] \\
A_4 &= \pm\frac{M_o^3}{4\beta^3}\left\{k_{U(L)}'(x)\left[4M_o^2(\gamma - 1) + 8 - \frac{16\beta^2}{M_o^2}\right]\left[m(x) \pm \frac{n(x)}{\beta}\right] + \right. \\
&\quad \left. k_{U(L)}(x)\left[8 + 2M_o^2(\gamma - 3) + \frac{8\beta^4}{M_o^2}\right]\left[m'(x) \pm \frac{n'(x)}{\beta}\right]\right\} \\
A_5 &= -\frac{M_o^6}{2\beta^4}\,k_{U(L)}'^2(x)\left(\gamma + 1 - \frac{6\beta^2}{M_o^4}\right)
\end{aligned}\right\} \quad (93)$$

Pressure Distribution on Airfoil Surface

The pressure distribution can be obtained from the velocity distribution by means of the energy equation (reference 2) which, in dimensionless form, is as follows:

$$\frac{1}{\gamma - 1}\left[\frac{p}{p_o}\left(\frac{\rho_o}{\rho}\right) - 1\right] = \frac{1}{2}\left(M_o^2 - \frac{q^2}{c^2}\right) \quad (94)$$

Here $p_o$, $\rho_o$, and $M_o$ are taken for the homogeneous flow field and $p$, $\rho$, $q$, and $c$ are local quantities. By use of the isentropic $p - \rho$ relation, equation (94) becomes:

$$\frac{1}{\gamma - 1}\left[\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = \frac{1}{2}\left[M_o^2 - \left(\varphi_x^2 + \varphi_y^2\right)\right] \tag{95}$$

and after substituting $p = p_o + \Delta p$ and expanding in powers of the relative pressure increment $\frac{\Delta p}{p_o}$, which is small compared with 1:

$$\left[\frac{\Delta p}{\gamma p_o} - \frac{1}{2}\left(\frac{\Delta p}{\gamma p_o}\right)^2 \cdots\right] = \left[\frac{\Delta p}{\rho_o c_o^2} - \frac{1}{2}\left(\frac{\Delta p}{\rho_o c_o^2}\right)^2 \cdots\right]$$

$$= \frac{1}{2}\left[M_o^2 - \left(\varphi_x^2 + \varphi_y^2\right)\right] \tag{96}$$

Upon substituting equations (92) for $\varphi_x^2$ and $\varphi_y^2$, equation (96) may be solved in successive approximations. In the first approximation $\left(\frac{\Delta p}{\gamma p_o}\right)^2$ and second-order quantities in the velocities are neglected. Then:

$$\frac{\Delta p}{\rho_o c_o^2} = -bM_o\left[m(x) \pm \frac{n(x)}{\beta}\right] \pm \epsilon \frac{M_o^2}{\beta} k_{U(L)}'(x) \tag{97}$$

Upon substituting from equation (97) for $\left(\frac{\Delta p}{\rho_o c_o^2}\right)$ and keeping second-order quantities in the velocities throughout equation (96), the second approximation yields:

$$\left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right) = \epsilon P_1 + bP_2 + b^2 P_3 + \epsilon b P_4 + \epsilon^2 P_5$$

Here

$$\left.\begin{aligned}
P_1 &= \pm 2 \frac{M_o^2}{\beta} k_{U(L)}'(x) \\
P_2 &= -2M_o\left[m(x) \pm \frac{n(x)}{\beta}\right] \\
P_3 &= \frac{1}{2\beta^2}\left\{\left[(\gamma - 1)M_o^4 + 2\right]\frac{n^2(x)}{\beta^2} - (\gamma + 1)M_o^4 \frac{n^2(0)}{\beta^2} \pm \right. \\
&\quad \left[(\gamma - 3)M_o^4 + 4M_o^2\right]\left[m'(x) \pm \frac{n'(x)}{\beta}\right]\int_0^x \frac{n(x')}{\beta}\,dx' + \\
&\quad \left. 2\left(M_o^2 - 1\right)^2 m^2(x) \pm \left[2M_o^4(\gamma + 1) - 4\left(M_o^2 - 1\right)\right]\frac{m(x)n(x)}{\beta}\right\} \\
P_4 &= \mp\frac{M_o}{2\beta^3}\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right]\left\{\left[m(x) \pm \frac{n(x)}{\beta}\right]2k_{U(L)}'(x) + \right. \\
&\quad \left. \left[m'(x) \pm \frac{n'(x)}{\beta}\right]k_{U(L)}(x)\right\} \\
P_5 &= \frac{M_o^2}{2\beta^4}\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right]k_{U(L)}'^2(x)
\end{aligned}\right\} \quad (98)$$

As shown in appendix B, equations (98) for the pressure distribution on the airfoil surface can be checked in a limiting case, in which complete agreement of equations (98) with an expression derived by Ferri (reference 2) is obtained.

## Drag Force

In order to obtain the drag force, the component of the pressure which is parallel to the flight direction $x$ has to be integrated over

the upper and lower surfaces. Denoting, as in figure 1, by $\epsilon\theta_{U(L)}$ the angle subtended by the local tangent of the surface with the flight direction, the drag becomes:

$$D = \int \Delta p_U \sin \epsilon\, \theta_U \, ds_U - \int \Delta p_L \sin \epsilon\, \theta_L \, ds_L \tag{99}$$

where $ds_U$ and $ds_L$ are the line elements of the airfoil surface.

Keeping only terms up to third orders in $\epsilon$, equation (99) becomes:

$$D \approx l \int_0^l \Delta p_U \epsilon k_U'(x)\, dx - l \int_0^l \Delta p_L \epsilon k_L'(x)\, dx$$

Hence, the drag coefficient is:

$$C_D = \frac{D/{c_o}^2}{\frac{1}{2}\rho_o {M_o}^2 l} = \epsilon \frac{1}{{M_o}^2} \int_0^l \left[ \left( \frac{\Delta p}{\frac{1}{2}\rho_o {c_o}^2} \right)_U k_U' - \left( \frac{\Delta p}{\frac{1}{2}\rho_o {c_o}^2} \right)_L k_L' \right] dx \tag{100}$$

For $(\Delta p)_U$ and $(\Delta p)_L$ expressions (98) have to be substituted in equation (100).

In the case of a symmetrical airfoil at zero angle of attack $k_U(x) = -k_L(x) = k(x)$. Then:

$$(C_D)_{\bar{\theta}=0} = \epsilon^2 D_1 + b\epsilon D_2 + \epsilon b^2 D_3 + \epsilon^2 b D_4 + \epsilon^3 D_5$$

Here:

$$\left.\begin{aligned}
D_1 &= \frac{4}{\beta}\int_0^1 \left[k'(x)\right]^2 dx \\
D_2 &= \frac{4}{M_o}\int_0^1 m'(x)k(x)\,dx \\
D_3 &= \frac{1}{M_o^2\beta^2}\left\{\left[(\gamma-1)M_o^4+2\right]\int_0^1 \frac{n^2(x)}{\beta^2}k'(x)\,dx + \right. \\
&\quad \left[(\gamma-3)M_o^4+4M_o^2\right]\int_0^1 \frac{n'(x)}{\beta^2}k'(x)\left[\int_0^x n(x')\,dx'\right]dx + \\
&\quad \left. 2\left(M_o^2-1\right)^2\int_0^1 m^2(x)k'(x)\,dx\right\} \\
D_4 &= -\frac{1}{M_o\beta^3}\left\{\left[(\gamma+1)M_o^4-4\left(M_o^2-1\right)\right]\left[2\int_0^1 m(x)k'^2(x)\,dx + \right.\right. \\
&\quad \left.\left.\int_0^1 m'(x)k(x)k'(x)\,dx\right]\right\} \\
D_5 &= \frac{1}{\beta^4}\left\{\left[(\gamma+1)M_o^4-4\left(M_o^2-1\right)\right]\int_0^1 k'^3(x)\,dx\right\}
\end{aligned}\right\} \quad (101)$$

In the case of a symmetrical airfoil at a finite angle of attack $\epsilon\bar{\theta}$ the decomposition of $k_U'(x)$ and $k_L'(x)$, suggested in the section on airfoil geometry (appendix C), has to be carried out. In other words: Replace $k_U'$ in $(\Delta p)_U$ and $k_L'$ in $(\Delta p)_L$ by:

and

$$\left.\begin{aligned}
k_U'(x) &= \bar{k}(x) - \bar{\theta} \\
k_L'(x) &= -\bar{k}(x) - \bar{\theta}
\end{aligned}\right\} \quad (102)$$

where now $\bar{k}(\bar{x}) \approx \bar{k}(x)$ is the customary profile function, for which $\bar{k}'(1) = 0$.

Then, upon substituting the modified expressions (98) into equation (100), the following generalizations of the coefficients of $C_D$ have to be noted:

$$\left.\begin{aligned}
D_1' &= D_1 + \frac{4\bar{\theta}^2}{\beta} \\
D_2' &= D_2 + \frac{4\theta}{M_o\beta}\int_0^1 n(x)\,dx \\
D_3' &= D_3 - \frac{\bar{\theta}}{M_o^2\beta^2}\left\{\left[(\gamma-3)M_o^4 \quad 4M_o^2\right]\int_0^1 m'(x)\left[\int_0^x \frac{n(x')}{\beta}\,dx'\right]dx + \right. \\
&\qquad \left.\left[2M_o^4(\gamma+1) - 4\left(M_o^2-1\right)\right]\left[\int_0^1 m(x)\,\frac{n(x)}{\beta}\,dx\right]\right\} \\
D_4' &= D_4 + \frac{\theta}{M_o\beta^3}\left[(\gamma+1)M_o^4 - 4\left(M_o^2-1\right)\right]\left\{\left[4\int_0^1 \frac{n(x)}{\beta}\,\bar{k}'(x)\,dx + \right.\right. \\
&\qquad \left.\int_0^1 \frac{n'(x)}{\beta}\,\bar{k}'(x)x\,dx + \int_0^1 \frac{n'(x)}{\beta}\,\bar{k}(x)\,dx\right] + \\
&\qquad \left.\bar{\theta}\left[\int_0^1 m'(x)x\,dx - 2\int_0^1 m(x)\,dx\right]\right\} \\
D_5' &= D_5
\end{aligned}\right\} \quad (103)$$

The physical meaning of the various terms in equations (101) or (103) may be readily interpreted: $D_1$ (or $D_1'$) is the wave drag of the airfoil well-known from the Ackeret theory; $D_5$ (or $D_5'$) is the second-order drag known from the Busemann approximation (reference 3); $D_2$ represents the horizontal buoyancy acting on the airfoil; $D_3$ may be denoted "second-order buoyancy"; and $D_4$ is the term characterizing the interaction between the airfoil field and the given pressure gradient of the wind tunnel.

Insofar as $b\epsilon D_2$ is concerned, it is to be noted that the integral is taken over the product of the local volume element $\epsilon k(x)\,dx$ and the local pressure-gradient coefficient $b\frac{4}{M_0}m'(x)$. Since the pressure gradient is varying from point to point, this expression represents, indeed, the generalization of the classical result which is thus seen to hold not only for incompressible but also for supersonic flow.

However, the other correction term for the drag, which can be derived for incompressible flow (see "Introduction"), does no longer appear in the supersonic range: The simple concept of the "apparent mass" is to be replaced by more complicated expressions.

## Lift Force

In order to obtain the lift, the component of the pressure which is perpendicular to the flight direction has to be integrated over the upper and lower surfaces of the airfoil. Thus:

$$L = -\int(\Delta p)_U \cos \epsilon\,\theta_U\,ds_U + \int(\Delta p)_L \cos \epsilon\,\theta_L\,ds_L$$

which up to and including second orders in $\epsilon$ becomes:

$$L \approx -l\int_0^1 \left[(\Delta p)_U - (\Delta p)_L\right] dx \qquad (104)$$

The lift coefficient is:

$$C_{L_\cdot} = -\frac{1}{M_o^2}\int_0^1 \left[\left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right)_U - \left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right)_L\right] dx \tag{105}$$

For $(\Delta p)_U$ and $(\Delta p)_L$ expressions (98) have to be substituted in equation (105).

In the case of a symmetrical airfoil at zero angle of attack $k_U(x) = -k_L(x) = k(x)$. Then:

$$(C_L)_{\bar{\theta}=0} = bL_1 + b^2L_2 + \epsilon bL_3$$

Here:

$$\left.\begin{aligned}
L_1 &= \frac{4}{M_o}\int_0^1 \frac{n(x)}{\beta}\,dx \\
L_2 &= -\frac{1}{M_o^2\beta^2}\left\{\left[(\gamma - 3)M_o^4 + 4M_o^2\right]\int_0^1 m'(x)\left[\int_0^x \frac{n(x')}{\beta}\,dx'\right]dx + \right. \\
&\qquad \left. 2\left[(\gamma + 1)M_o^4 - 2\left(M_o^2 - 1\right)\right]\int_0^1 m(x)\frac{n(x)}{\beta}\,dx\right\}. \\
L_3 &= \frac{1}{M_o\beta^3}\left\{\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right]\left[2\int_0^1 \frac{n(x)}{\beta}\,k'(x)\,dx + \right.\right. \\
&\qquad \left.\left. \int_0^1 \frac{n'(x)}{\beta}\,k(x)\,dx\right]\right\}
\end{aligned}\right\} \tag{106}$$

Note that equations (106) vanish for $m = n = 0$, as they should.

In the case of a symmetrical airfoil at a finite angle of attack $\bar{\theta}$, after making the same modifications as in the case of the drag, the following changes in the coefficients $L_i$ have to be made:

$$L_1' = L_1$$

$$L_2' = L_2$$

$$L_3' = L_3 - \frac{\bar{\theta}}{M_o\beta^3}\left\{\left[(\gamma+1)M_o^4 - 4\left(M_o^2-1\right)\right]\left[2\int_0^1 m(x)\,dx + \int_0^1 xm'(x)\,dx\right]\right\} \quad (107)$$

Furthermore, a coefficient

$$\epsilon L_4' = \epsilon\frac{4\bar{\theta}}{\beta}$$

appears.

The coefficients $L_4'$ and $L_5'$ represent the lift expressions known from the work of Ackeret and Busemann (see reference 3), $L_1$ and $L_2$ are vertical forces due mainly to the vertical pressure gradient, and $L_3$ is the interaction term.

## Pitching Moment

The moment about the leading edge is obtained by integrating over the moments of the differential lift forces dL acting at a distance x. Therefore the moment coefficient becomes:

$$C_M = \frac{M/c_o^2}{\frac{1}{2}\rho_o M_o^2 l} = \frac{1}{M_o^2}\int_0^1 \left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right)x\,dx$$

The integral is to be taken over the upper as well as the lower surface.

Since an upward lift which is to be considered positive produces a negative (diving) moment, the contribution of the upper surface is:

$$\left(C_M\right)_U = \frac{1}{M_o^2}\int_0^1 \left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right)_U x\,dx$$

The contribution of the lower surface is:

$$(C_M)_L = -\frac{1}{M_o^2}\int_0^1 \left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right)_L x\,dx$$

Hence:

$$C_M = \frac{1}{M_o^2}\int_0^1 \left[\left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right)_U - \left(\frac{\Delta p}{\frac{1}{2}\rho_o c_o^2}\right)_L\right] x\,dx \tag{108}$$

In the case of a symmetrical airfoil at zero angle of attack $k_U(x) = -k_L(x) = k(x)$. Then:

$$(C_M)_{\bar{\theta}=0} = bM_1 + b^2M_2 + \epsilon bM_3$$

Here:

$$\left.\begin{aligned}
M_1 &= -\frac{4}{M_o}\int_0^1 x\,\frac{n(x)}{\beta}\,dx \\
M_2 &= \frac{1}{M_o^2\beta^2}\left\{\left[(\gamma-3)M_o^4 + 4M_o^2\right]\int_0^1 m'(x)x\left[\int_0^x \frac{n(x')}{\beta}\,dx'\right]dx + \right. \\
&\qquad \left. 2\left[M_o^4(\gamma+1) - 2\left(M_o^2-1\right)\right]\int_0^1 m(x)\frac{n(x)}{\beta}\,x\,dx\right\} \\
M_3 &= -\frac{1}{M_o\beta^3}\left\{\left[(\gamma+1)M_o^4 - 4\left(M_o^2-1\right)\right]\left[2\int_0^1 \frac{n(x)}{\beta}\,xk'(x)\,dx + \right.\right. \\
&\qquad \left.\left. \int_0^1 \frac{n'(x)}{\beta}\,xk(x)\,dx\right]\right\}
\end{aligned}\right\} \tag{109}$$

In the case of a symmetrical airfoil at a finite angle of attack $\bar{\theta}$ after making the same replacements as before:

$$\left.\begin{aligned} M_1' &= M_1 \\ M_2' &= M_2 \\ M_3' &= M_3 + \frac{\bar{\theta}}{M_o\beta^3}\left\{\left[(\gamma+1)M_o^4 - 4\left(M_o^2-1\right)\right]\left[\int_0^1 xm(x)\,dx + \int_0^1 x^2 m'(x)\,dx\right]\right\} \end{aligned}\right\} \quad (110)$$

Furthermore, two additional coefficients appear:

$$\epsilon M_4' = -\epsilon\frac{2\bar{\theta}}{\beta}$$

and

$$\epsilon^2 M_5' = \epsilon^2\frac{2\bar{\theta}}{\beta^4}\left\{\left[(\gamma+1)M_o^4 - 4\left(M_o^2-1\right)\right]\int_0^1 x\bar{k}'(x)\,dx\right\}$$

The coefficients $M_4'$ and $M_5'$ represent the moments known from the Ackeret and Busemann calculations, $M_1$ and $M_2$ are the moment corrections due to the pressure gradients of the wind tunnel, and $M_3$ is the interaction term.

## Application of Theory

As a simple example of the foregoing general considerations, the force corrections will be given for a symmetrical airfoil placed at zero angle of attack into a tunnel field, for which

$$m(x) = n(x) = x$$

(i.e., for which the pressure gradient $b\rho U m'(x)\frac{c_o}{l}$ is linear in $x$):

$$\left.\begin{aligned}
C_D = {} & \epsilon^2 \frac{4}{\beta} \int_0^1 \left[k'(x)\right]^2 dx + \epsilon b \frac{4}{M_o} \int_0^1 k(x)\, dx + \\
& \epsilon^3 \frac{1}{\beta^4}\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right] \int_0^1 \left[k'(x)\right]^3 dx - \\
& \epsilon b^2 \frac{1}{\beta^4}\left[M_o^2(3\gamma - 17) + 4\left(M_o^4 + 4\right)\right] \int_0^1 xk(x)\, dx - \\
& \epsilon^2 b \frac{2}{M_o\beta^3}\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right] \int_0^1 xk'^2(x)\, dx \\
C_L = {} & b \frac{2}{M_o\beta} - b^2 \frac{1}{6M_o^2\beta^3}\left[M_o^4(5\gamma + 1) - 4\left(M_o^2 - 2\right)\right] - \\
& \epsilon b \frac{1}{M_o\beta^4}\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right] \int_0^1 k(x)\, dx \\
C_M = {} & -b \frac{4}{3M_o\beta} + b^2 \frac{1}{8M_o^2\beta^3}\left[M_o^4(5\gamma + 1) - 4\left(M_o^2 - 2\right)\right] + \\
& \epsilon b \frac{3}{M_o\beta^4}\left[M_o^4(\gamma + 1) - 4\left(M_o^2 - 1\right)\right] \int_0^1 xk(x)\, dx
\end{aligned}\right\} \quad (111)$$

For a doubly symmetric diamond profile of vertex angle $\epsilon$ these formulas reduce to:

$$\left.\begin{aligned}
C_D &= \epsilon^2 \frac{4}{\beta} + \epsilon b \frac{1}{M_o} - \epsilon b^2 \frac{1}{8\beta^4}\left[M_o^2(3\gamma - 17) + 4\left(M_o^4 + 4\right)\right] - \\
&\quad \epsilon^2 b \frac{1}{M_o\beta^3}\left[M_o^4(\gamma + 1) - 4\left(M_o^2 - 1\right)\right] \\
C_L &= b \frac{2}{M_o\beta} - b^2 \frac{1}{6M_o^2\beta^3}\left[M_o^4(5\gamma + 1) - 4\left(M_o^2 - 2\right)\right] - \\
&\quad \epsilon b \frac{1}{4M_o\beta^4}\left[M_o^4(\gamma + 1) - 4\left(M_o^2 - 1\right)\right] \\
C_M &= -b \frac{4}{3M_o\beta} + b^2 \frac{1}{8M_o^2\beta^3}\left[M_o^4(5\gamma + 1) - 4\left(M_o^2 - 2\right)\right] + \\
&\quad \epsilon b \frac{3}{8M_o\beta^4}\left[M_o^4(\gamma + 1) - 4\left(M_o^2 - 1\right)\right]
\end{aligned}\right\} \quad (112)$$

Numerical results obtained by means of equations (112) are plotted in figures 2 to 9 for representative values of $b$ and $\epsilon$. In figure 2 the components of $C_D$ which are proportional to $\epsilon^2$, $\epsilon b$, $\epsilon^2 b$, and $\epsilon b^2$ are graphed separately, and in figure 3 the total $C_D$ is shown as a function of the Mach number $M_o$. Similarly, the components of $C_L$ and $C_M$ which are proportional to $b$, $\epsilon b$, and $b^2$ are plotted separately in figures 4 and 6, respectively, while the total $C_L$ and $C_M$ are shown as functions of $M_o$ in figures 5 and 7, respectively. For the case of a finite, small angle of attack $\epsilon\overline{\theta}$ the behavior of the lift coefficient and its components is demonstrated in figures 8 and 9.

The relative importance of the various terms may be seen from the plots; for example, for the drag the first-order horizontal buoyancy is the major correction to be applied and is on the order of 25 percent of the true drag. Next in importance is the interaction term which amounts to about 10 percent of the wave drag. The second-order buoyancy amounts to about 1 percent.

The same relative magnitude of the correction terms exists in the lift coefficient, if the case of an angle of attack is considered as in figure 8.

For the lift and moment at zero angle of attack the term proportional to $b$ represents again the largest correction, while the remainder are small compared with it.

New York University
New York, N. Y., October 17, 1951

# APPENDIX A

## COMPUTATION OF PERTURBATION POTENTIALS $\varphi^{11}$, $\varphi^{13}$, AND $\varphi^{14}$

### Computation of $\varphi^{11}$

The details of the computation of the potential $\varphi^{11}$ are given as follows:

Upper surface.- For the upper surface the derivatives in equations (73) are given by:

$$\varphi_{\xi}^{11} = -\frac{M_0}{8\beta^2}\left((\gamma + 1)(1 + \beta^2)m(\eta)\frac{n(\eta)}{\beta} + \right.$$

$$\left[(\gamma + 1) + (\gamma - 3)\beta^2\right]\left\{\left[m'(\xi) + \frac{n'(\xi)}{\beta}\right]\int^{\eta} \frac{n(\eta')}{\beta}\, d\eta' + \right.$$

$$\left.\left.\left[m(\xi) + \frac{n(\xi)}{\beta}\right]\frac{n(\eta)}{\beta}\right\}\right) + P_1'(\xi) \qquad \text{(A1a)}$$

$$\varphi_{\eta}^{11} = -\frac{M_0}{8\beta^2}\left((\gamma + 1)(1 + \beta^2)\left[m(\eta)\frac{n'(\eta)}{\beta} + m'(\eta)\frac{n(\eta)}{\beta}\right]\xi + \right.$$

$$\left[\gamma + 1 + (\gamma - 3)\beta^2\right]\left\{\left[m(\xi) + \frac{n(\xi)}{\beta}\right]\frac{n(\eta)}{\beta} + \right.$$

$$\left.\left.\frac{n'(\eta)}{\beta}\int^{\xi}\left[m(\xi') + \frac{n(\xi')}{\beta}\right] d\xi'\right\}\right) + P_2'(\eta) \qquad \text{(A1b)}$$

Equation (74) in explicit form is as follows:

$$\varphi_{\xi}^{11}(\xi,0) = -\frac{M_o}{8\beta^3}\left\{(\gamma+1)(1+\beta^2)\left[m(0)-\frac{n(0)}{\beta}\right]+\left[\gamma+1+(\gamma-3)\beta^2\right]\left[m(\xi)+\frac{n(\xi)}{\beta}\right]\right\}n(0) \tag{A2}$$

The constants of integration are found to be:

$$P_1'(\xi) = \frac{M_o}{8\beta^2}\left\{(\gamma+1)(1+\beta^2)\frac{n^2(0)}{\beta^2}+\left[\gamma+1+(\gamma-3)\beta^2\right]\left[m'(\xi)+\frac{n'(\xi)}{\beta}\right]\int^{0}\frac{n(\eta')}{\beta}\,d\eta'\right\} \tag{A3}$$

$$P_2'(\eta) = \frac{M_o}{8\beta^2}\left((\gamma+1)(1+\beta^2)\left\{\frac{n^2(0)}{\beta^2}-\frac{m(\eta)n(\eta)}{\beta}+\eta\left[m(\eta)\frac{n'(\eta)}{\beta}+m'(\eta)\frac{n(\eta)}{\beta}\right]\right\}+\left[\gamma+1+(\gamma-3)\beta^2\right]\left\{\frac{n'(\eta)}{\beta}\int^{\eta}\left[m(\eta')+\frac{n(\eta')}{\beta}\right]d\eta'-\left[m'(\eta)+\frac{n'(\eta)}{\beta}\right]\int_0^{\eta}\frac{n(\eta')}{\beta}\,d\eta'\right\}\right) \tag{A4}$$

Lower surface.- For the lower surface the derivatives corresponding to those in equations (73) are:

$$\varphi_\xi{}^{ll} = \frac{M_o}{8\beta^2}\left((\gamma + 1)(1 + \beta^2)\left[m(\xi)\frac{n'(\xi)}{\beta} + m'(\xi)\frac{n(\xi)}{\beta}\right]\eta + \right.$$

$$\left[\gamma + 1 + (\gamma - 3)\beta^2\right]\left\{\left[m(\eta) - \frac{n(\eta)}{\beta}\right]\frac{n(\xi)}{\beta} + \right.$$

$$\left.\left.\frac{n'(\xi)}{\beta}\int^{\eta}\left[m(\eta') - \frac{n(\eta')}{\beta}\right]d\eta'\right\}\right) + P_1'(\xi) \qquad \text{(A5a)}$$

$$\varphi_\eta{}^{ll} = \frac{M_o}{8\beta^2}\left((\gamma + 1)(1 + \beta^2)m(\xi)\frac{n(\xi)}{\beta} + \right.$$

$$\left[\gamma + 1 + (\gamma - 3)\beta^2\right]\left\{\left[m(\eta) - \frac{n(\eta)}{\beta}\right]\frac{n(\xi)}{\beta} + \right.$$

$$\left.\left.\left[m'(\eta) - \frac{n'(\eta)}{\beta}\right]\int^{\xi}\frac{n(\xi')}{\beta}\,d\xi'\right\}\right) \qquad \text{(A5b)}$$

The characteristic condition in explicit form is as follows:

$$\varphi_\eta{}^{ll}(0,\eta) = \frac{M_o}{8\beta^3}\left\{(\gamma + 1)(1 + \beta^2)\left[m(0) + \frac{n(0)}{\beta}\right] + \right.$$

$$\left.\left[\gamma + 1 + (\gamma - 3)\beta^2\right]\left[m(\eta) - \frac{n(\eta)}{\beta}\right]\right\}n(0) \qquad \text{(A6)}$$

In this case the integration constants come out to be:

$$P_1'(\xi) = -\frac{M_o}{8\beta^2}\left((\gamma+1)(1+\beta^2)\left\{\left[m(\xi)\frac{n'(\xi)}{\beta}+m'(\xi)\frac{n(\xi)}{\beta}\right]\xi - \left[\frac{n^2(0)}{\beta^2}+m(\xi)\frac{n(\xi)}{\beta}\right]\right\}+ \left[\gamma+1+(\gamma-3)\beta^2\right]\left\{\frac{n'(\xi)}{\beta}\int^{\xi}\left[m(\eta')-\frac{n(\eta')}{\beta}\right]d\eta' - \left[m'(\xi)-\frac{n'(\xi)}{\beta}\right]\int_0^{\xi}\frac{n(\xi')}{\beta}\,d\xi'\right\}\right) \tag{A7}$$

$$P_2'(\eta) = -\frac{M_o}{8\beta^2}\left\{(\gamma+1)(1+\beta^2)\left[-\frac{n^2(0)}{\beta^2}\right]+ \left[\gamma+1+(\gamma-3)\beta^2\right]\left[m'(\eta)-\frac{n'(\eta)}{\beta}\right]\int^{0}\frac{n(\xi')}{\beta}\,d\xi'\right\} \tag{A8}$$

## Computation of $\varphi^{13}$

The procedure for computing the potential $\varphi^{13}$ is as follows. For the upper surface the derivatives in equations (82) are given by:

$$\varphi_\xi{}^{13} = \frac{M_o{}^2}{8\beta^3}\left((\gamma+1)(1+\beta^2)\left[m(\eta)+\frac{n(\eta)}{\beta}\right]k_U'(\eta)+ \left[\gamma+1+(\gamma-3)\beta^2\right]\left\{\left[m'(\xi)+\frac{n'(\xi)}{\beta}\right]k_U(\eta)+ \left[m(\xi)+\frac{n(\xi)}{\beta}\right]k_U'(\eta)\right\}\right)+J_1'(\xi) \tag{A9a}$$

$$\varphi_\eta{}^{13} = \frac{M_o{}^2}{8\beta^3}\Bigg((\gamma + 1)(1 + \beta^2)\left\{\left[m'(\eta) + \frac{n'(\eta)}{\beta}\right]k_U'(\eta) + \left[m(\eta) + \frac{n(\eta)}{\beta}\right]k_U''(\eta)\right\}\xi + \left[\gamma + 1 + (\gamma - 3)\beta^2\right]\left\{\left[m(\xi) + \frac{n(\xi)}{\beta}\right]k_U'(\eta) + k_U''(\eta)\int^{\xi}\left[m(\xi') + \frac{n(\xi')}{\beta}\right]d\xi'\right\}\Bigg) + J_2'(\eta) \qquad \text{(A9b)}$$

Putting $k(0) = k'(0) = 0$, equation (A9a), together with equation (81), yields $J_1'(\xi) = 0$. On the other hand, substituting equations (A9a) and (A9b) into equation (80) yields:

$$J_2'(\eta) = \frac{M_o{}^2}{2\beta}\Bigg(-(\gamma + 1)(1 + \beta^2)\left\{\left[m'(\eta) + \frac{n'(\eta)}{\beta}\right]\eta k_U'(\eta) + \left[m(\eta) + \frac{n(\eta)}{\beta}\right]\eta k_U''(\eta)\right\} + \left[(\gamma + 1)(1 + \beta^2) - \frac{8\beta^2}{M_o{}^2}\right]\left[m(\eta) + \frac{n(\eta)}{\beta}\right]k_U'(\eta) - \left[\gamma + 1 + (\gamma - 3)\beta^2\right]k_U''(\eta)\int^{\eta}\left[m(\xi') + \frac{n(\xi')}{\beta}\right]d\xi' + \left[\gamma + 1 + (\gamma - 3)\beta^2 + \frac{8\beta^4}{M_o{}^2}\right]k_U(\eta)\left[m'(\eta) + \frac{n'(\eta)}{\beta}\right]\Bigg) \qquad \text{(A10)}$$

The lower surface can be treated analogously.

## Computation of $\varphi^{14}$

In computing the potential $\varphi^{14}$ the upper surface is treated as follows. Putting in the first of equations (88) $k(0) = k'(0) = 0$, this equation, together with differentiated equation (87), yields $M_1'(\xi) = 0$. On the other hand, substituting equations (88) into equation (86) yields:

$$M_2'(\eta) = -\frac{M_o^5}{\beta^2}\left[\left(\frac{\gamma+1}{2} - \frac{4\beta^2}{M_o^4}\right)k_U'^2(\eta) - (\gamma+1)\eta k_U'(\eta)k_U''(\eta) + \frac{4\beta^4}{M_o^4}k_U(\eta)k_U''(\eta)\right] \qquad \text{(All)}$$

The lower surface can be treated analogously.

## APPENDIX B

## CHECK OF EQUATIONS (98) FOR PRESSURE DISTRIBUTION IN A LIMITING CASE

Ferri (reference 2) represents the pressure distribution over an arbitrary surface, which is exposed to a uniform stream of Mach number $M_1$, as a power series in terms of the local angle of attack $\eta$. Thus,

$$\frac{p - p_1}{\rho_1 c_1^2} = \mp a_1 \eta + a_2 \eta^2 + \ldots \tag{B1}$$

where $a_1$ and $a_2$ are given functions of $M_1$, $p$ is the local pressure, and $p_1$, the free-stream pressure.[1]

One may note that Ferri's result corresponds to a limiting case of the general nonuniform field, namely, to the case in which the basic flow is characterized along the x-axis by an x-component $M_o + bm$ and a y-component $bn$, where both $m$ and $n$ are constant.

The agreement of the present result with Ferri's can be seen as follows. Expressions (24) for $\varphi_x^{01}$ and $\varphi_y^{01}$ indicate that if $m(x)$ and $n(x)$ are constant along the axis, they are also constant in the whole xy-plane. The same is no longer true in order $b^2$, because the differential equation for $\varphi^{02}$ is inhomogeneous. However, up to order $b$, the basic field may be considered uniform in this limiting case.

The Mach number of this uniform field is:

$$M_1 = \frac{c_o}{c_1} \sqrt{(M_o + bm)^2 + (bn)^2} \tag{B2}$$

The angle of the flow direction with the local surface is

$$\eta = \epsilon\theta' + \frac{bn}{M_o + bm} \tag{B3}$$

---

[1]Note that Ferri's coefficient is here multiplied with $M_1^2$.

where $\epsilon\theta'$ and $\frac{bn}{M_o + bm}$ are the angles subtended by the x-axis with the local surface and with the stream direction, respectively.

In applying equation (B1) it should be noted that Ferri's pressure coefficient is taken with respect to $M_1$, while the coefficient $\frac{p - p_o}{\rho_o c_o^2}$ in this report is based upon $M_o$, which is related to $M_1$ by equation (B2). The connection between the two pressure coefficients is given by the relation

$$\frac{p - p_1}{\rho_1 c_1^2} = \frac{c_o^2}{c_1^2}\left(\frac{p - p_o}{\rho_o c_o^2}\frac{\rho_o}{\rho_1} + \frac{p_o - p_1}{\rho_1 c_1^2}\frac{c_1^2}{c_o^2}\right) \tag{B4}$$

where $\frac{p_o - p_1}{\rho_1 c_1^2}$, representing the pressure difference between points where the Mach numbers are $M_o$ and $M_1$, as well as $\rho_o/\rho_1$ and $c_o/c_1$ may be obtained by the isentropic energy relation.

Substituting Ferri's result for $p - p_1$, with $M_1$ and $\eta$ taken from equations (B2) and (B3) and the expression for $p_o - p_1$ obtained from the energy relation, equation (B4) may be expanded in powers of $b$ and yields:

$$\frac{p - p_o}{\rho_o c_o^2} = \mp\epsilon\frac{\theta' M_o^2}{\beta} - bM_o\left(m \pm \frac{n}{\beta}\right) +$$

$$\epsilon^2\frac{(\theta')^2 M_o^2}{4\beta^2}\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right] -$$

$$\epsilon b\frac{\theta' M_o}{2\beta^2}\left[(\gamma + 1)M_o^4 - 4\left(M_o^2 - 1\right)\right]\left(m \pm \frac{n}{\beta}\right) \tag{B5}$$

which includes terms of order $\epsilon$, $b$, $\epsilon b$, and $\epsilon^2$ but not terms of order $b^2$.

On the other hand, equation (B5) can be obtained from equation (98), if $m(x) = m$, $n(x) = n$, and $k'(x) = -\theta'$ are substituted and the $b^2$ term is disregarded. Thus, in the limiting case considered, Ferri's and the present results are identical.

## APPENDIX C

## AIRFOIL GEOMETRY

In order to show that the considerations in the text pertain to airfoils at both zero and finite angles of attack, the geometry of the airfoil will be considered in detail.

If the chord of the airfoil is at zero angle of attack, the local slope is given by

$$\frac{dy}{dx} = \epsilon k'(x) \approx \epsilon\theta \tag{C1}$$

where $y$ and $k(x)$ are positive on the upper, and negative on the lower surface (see fig. 1(a)). If the airfoil is symmetrical, then:

$$\left.\begin{aligned} \left(\frac{dy}{dx}\right)_U &= \epsilon k'(x) \approx \epsilon\theta_U \\ \left(\frac{dy}{dx}\right)_L &= -\epsilon k'(x) \approx \epsilon\theta_L \end{aligned}\right\} \tag{C2}$$

Figure 1(b) may illustrate how these relations have to be modified in case the chord of the airfoil is at a finite angle $\epsilon\overline{\theta}$ relative to the flight direction $x$.

Two coordinate systems may be used, one whose x-axis coincides with the flight direction and another one whose $\overline{x}$-axis coincides with the chord. Then the slope of the profile above the flight direction $x$ is:

$$\begin{aligned} \frac{dy}{dx} &= \epsilon k'(x) \\ &= \tan \epsilon(\theta - \overline{\theta}) \\ &\approx \epsilon(\theta - \overline{\theta}) \end{aligned} \tag{C3}$$

where $\tan \epsilon\theta$ is the slope of the profile above the chord. If the profile function above (or below) the chord, $\bar{y} = \epsilon\bar{k}(\bar{x})$, is introduced, then

$$\epsilon\theta \approx \tan \epsilon\theta = \bar{k}'(\bar{x})$$

and equation (C3) can be rewritten:

$$\frac{dy}{dx} \approx \epsilon\left[\bar{k}'(\bar{x}) - \bar{\theta}\right] \tag{C4}$$

Note that $\bar{y} > 0$ on the upper surface and $\bar{y} < 0$ on the lower surface. Then, at P in figure 1(b), $\epsilon\theta$ is positive because $\bar{y}$ increases with increasing $\bar{x}$. Fixing the sign of $\bar{\theta}$ analogously to that of $\theta$, $\bar{\theta}$ in the sketch is positive and larger than $\theta$, so that in equation (C3) $\frac{dy}{dx}$ comes out negative, as it should according to the figure.

Now, the barred and unbarred coordinate systems are related (neglecting second orders in $\epsilon$) as follows:

$$\bar{x} \approx x - y\bar{\theta}$$

$$\bar{y} \approx y + x\bar{\theta}$$

Then

$$\epsilon\bar{k}'(\bar{x}) = \epsilon\bar{k}'\left[x\left(1 + \epsilon^2\bar{\theta}^2\right) - \bar{\theta}\,\bar{k}(\bar{x})\epsilon^2\right]$$

$$\approx \epsilon\bar{k}'(x)$$

Therefore, equation (C4) may be written:

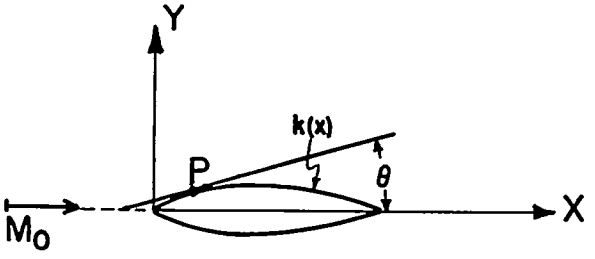
$$\frac{dy}{dx} = \epsilon k'(x) = \epsilon\left[\bar{k}'(x) - \bar{\theta}\right] \tag{C5}$$

## REFERENCES

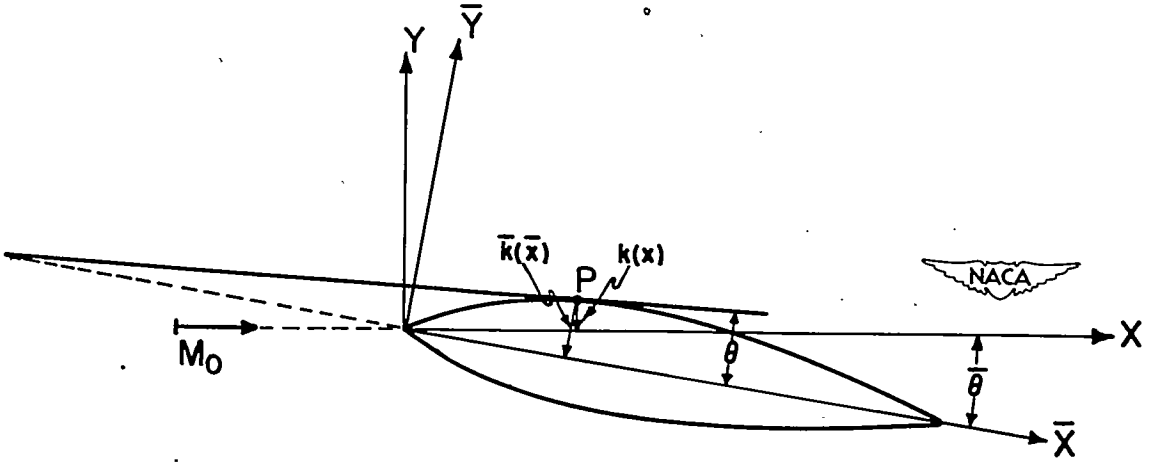
1. Glauert, H.: Wind Tunnel Interference on Wings, Bodies and Airscrews. R. & M. No. 1566, British A.R.C., 1933.

2. Ferri, Antonio: Elements of Aerodynamics of Supersonic Flows. The Macmillan Co., 1949, p. 127.

3. Busemann, A.: Aerodynamischer Auftrieb bei Überschallgeschwindigkeit. Atti dei Convegni 5, R. Accad. d'Italia, 1936, pp. 328-360.




(a) $\bar{\theta} = 0$.




(b) $\bar{\theta} \neq 0$.

Figure 1.- Airfoil geometry.

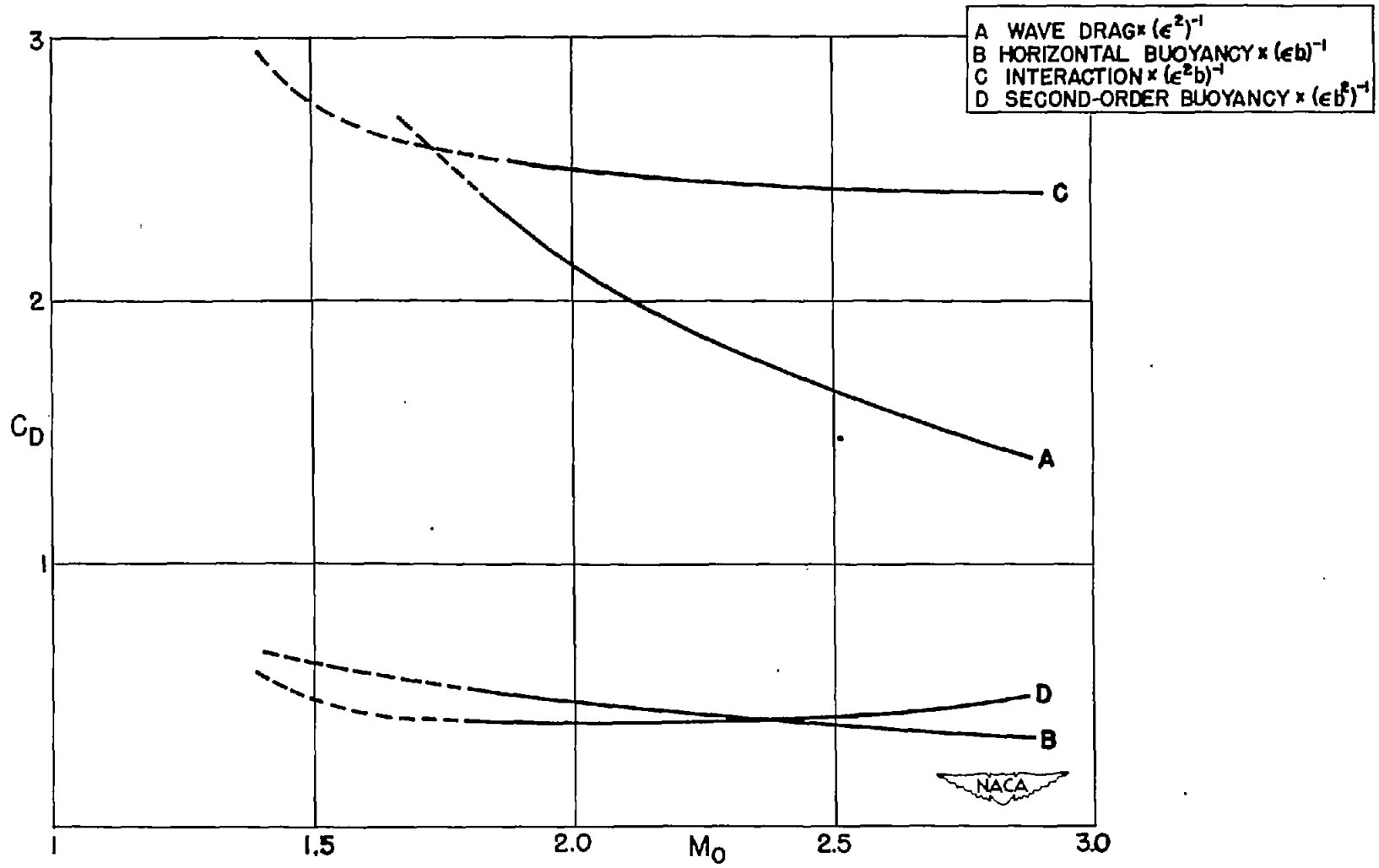



Figure 2.- Components of drag coefficient $C_D$ plotted against Mach number $M_0$ for diamond profile. $\epsilon = 0.05$; $b = 0.05$.

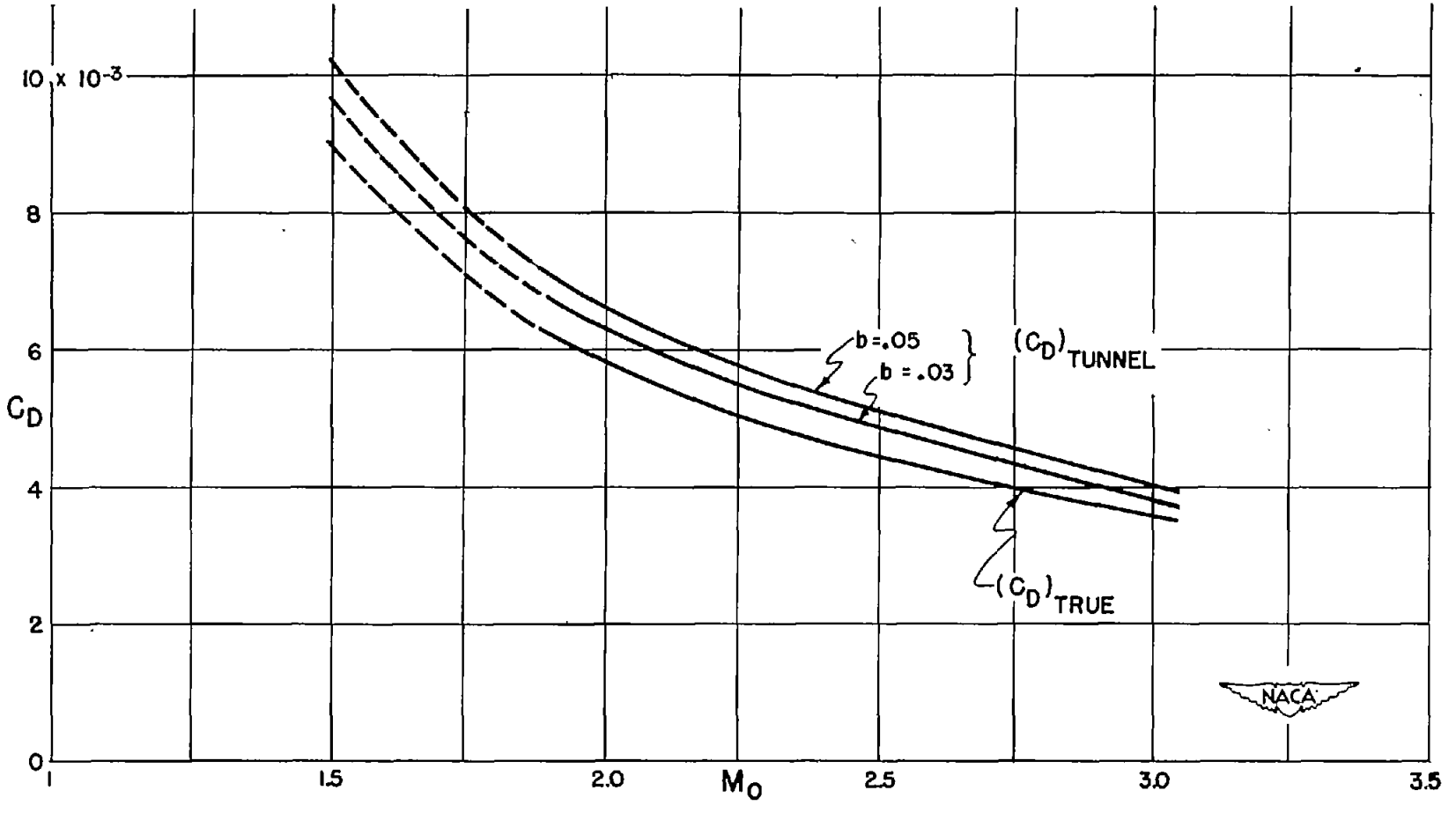



Figure 3.- Total drag coefficient $C_D$ as function of Mach number $M_0$ for diamond profile. $\epsilon = 0.05$.

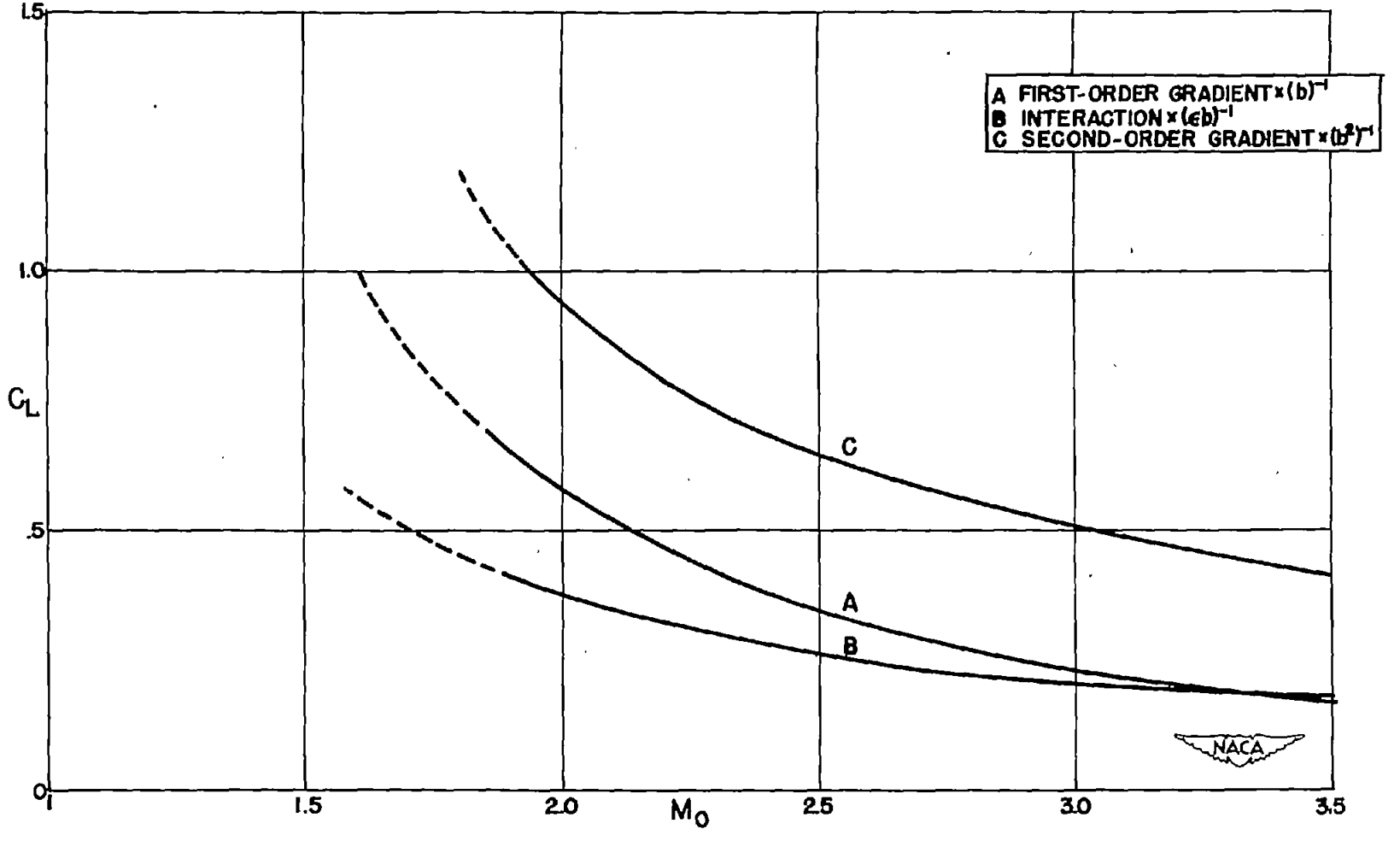



Figure 4.- Components of lift coefficient $C_L$ plotted against Mach number $M_0$ for diamond profile. $\epsilon = 0.05$; $b = 0.05$.

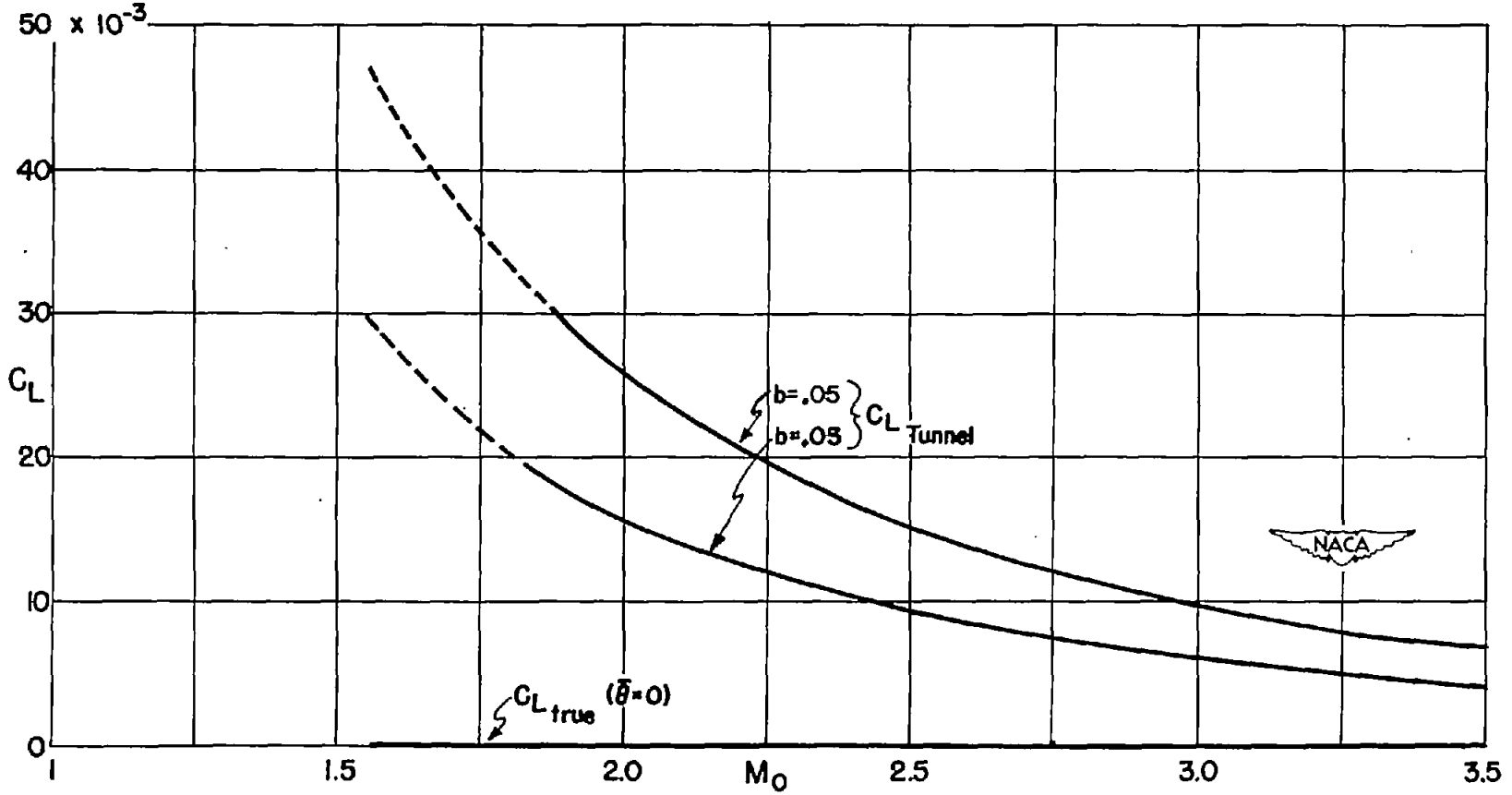



Figure 5.- Total lift coefficient $C_L$ as function of Mach number $M_o$ for diamond profile. $\epsilon = 0.05$.

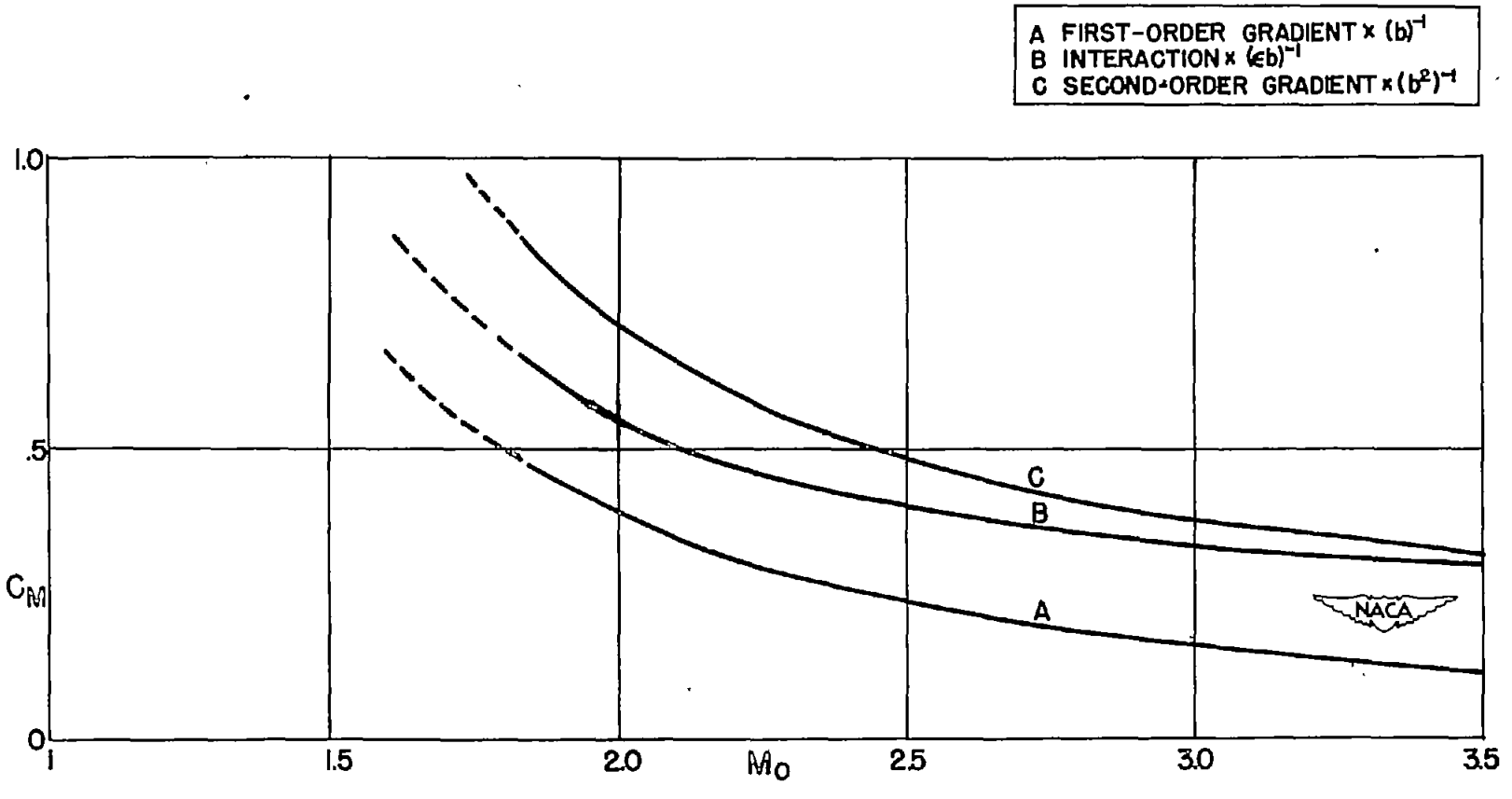



Figure 6.- Components of pitching-moment coefficient $C_M$ plotted against Mach number $M_0$ for diamond profile. $\epsilon = 0.05$; $b = 0.05$.

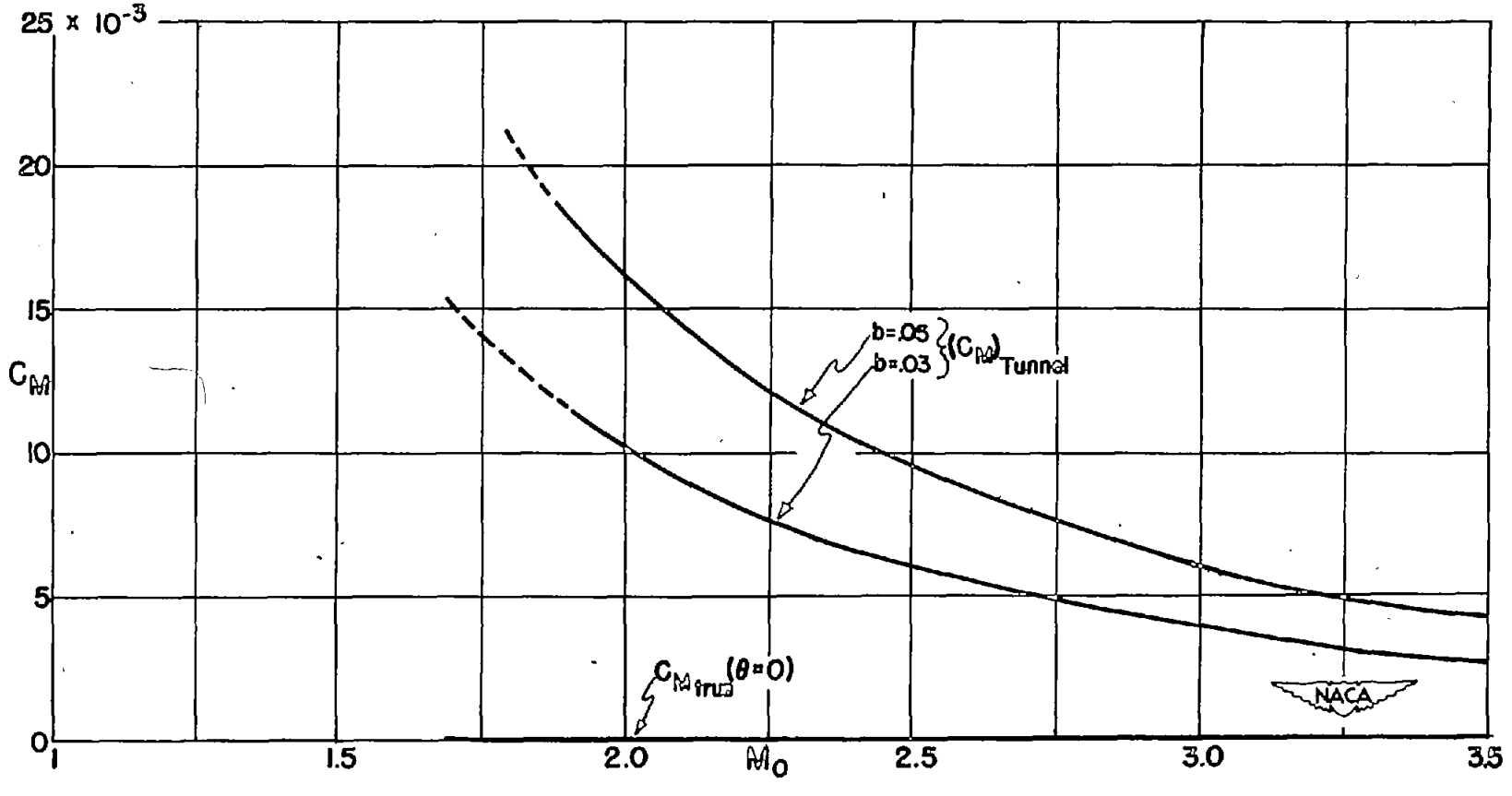



Figure 7.- Total pitching-moment coefficient $C_M$ as function of Mach number $M_0$ for diamond profile. $\epsilon = 0.05$.

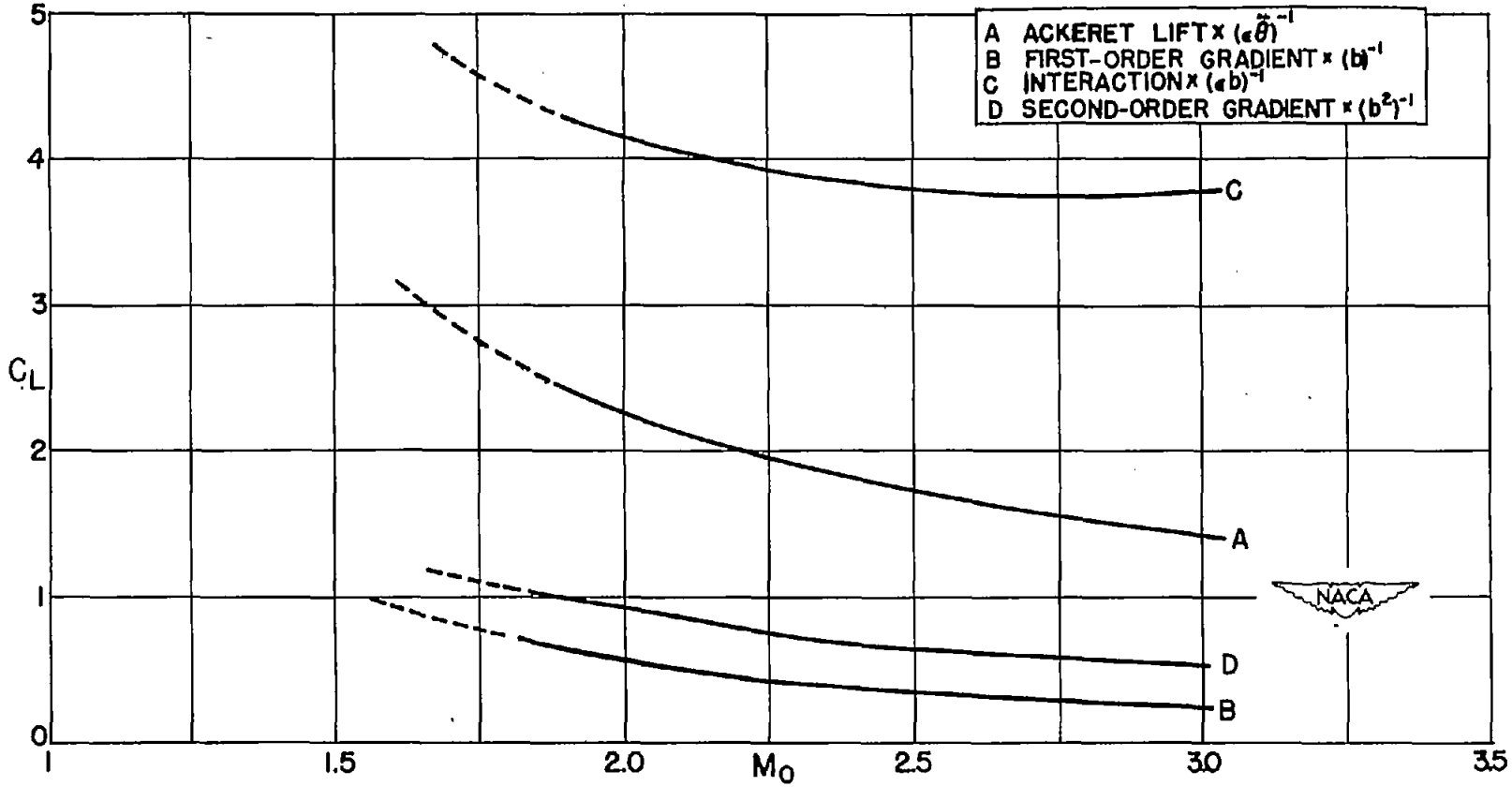



Figure 8.- Component of lift coefficient $C_L$ plotted against Mach number $M_0$ for diamond profile at finite, small angle of attack $\epsilon\bar{\theta}$. $\epsilon\bar{\theta} = 0.05$; $\epsilon = 0.05$; $b = 0.05$.

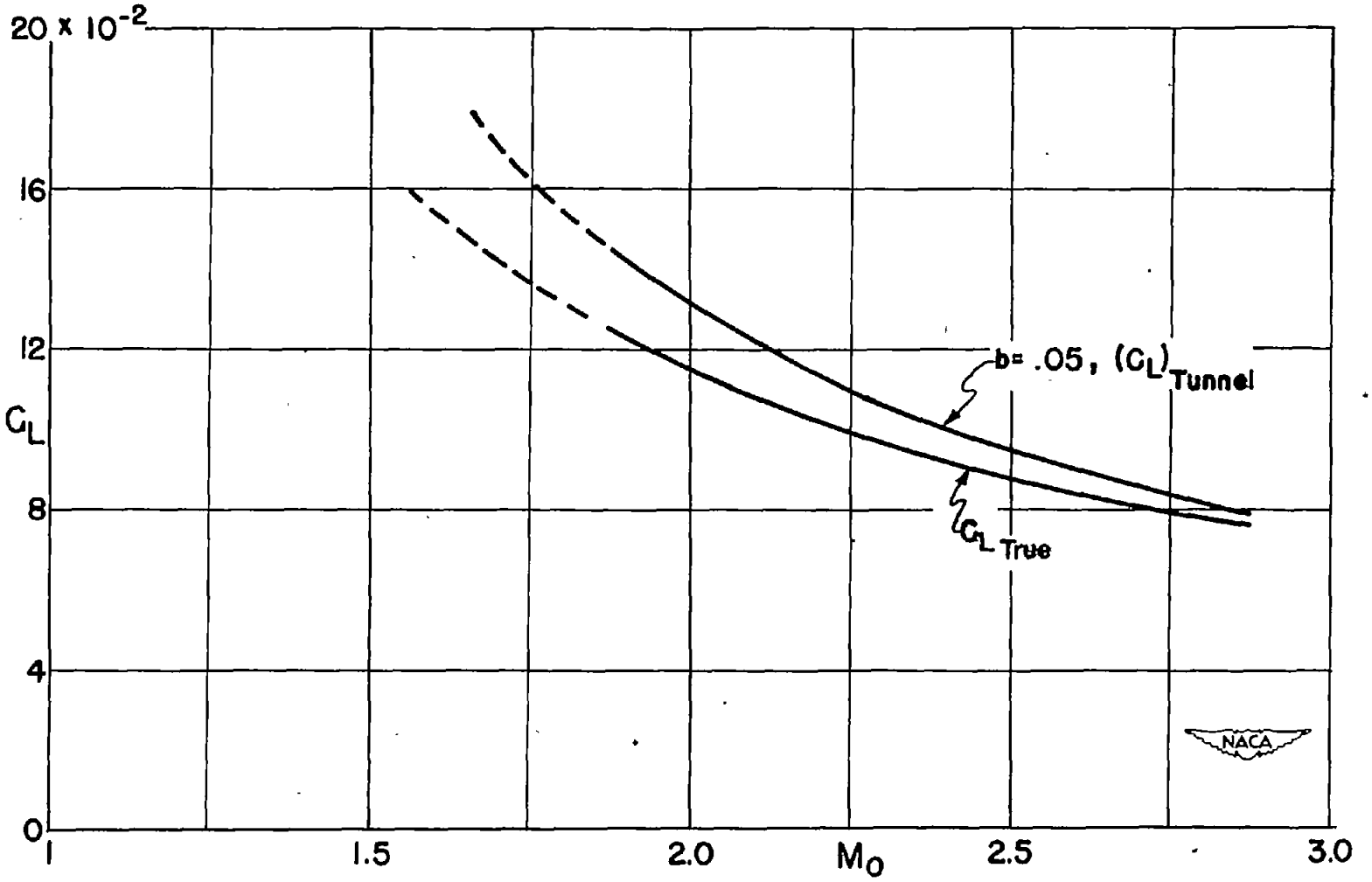



Figure 9.- Total lift coefficient $C_L$ as function of Mach number $M_o$ for diamond profile at finite, small angle of attack $\epsilon\bar{\theta}$. $\epsilon\bar{\theta} = 0.05$; $\epsilon = 0.05$.