NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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REFLECTION OF WEAK SHOCK WAVE FROM A BOUNDARY LAYER ALONG A FLAT PLATE. II - INTERACTION OF OBLIQUE SHOCK WAVE WITH A LAMINAR BOUNDARY LAYER ANALYZED BY DIFFERENTIAL-EQUATION METHOD

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SUMMARY

By analogy with the boundary-layer concept, the flow produced by the interaction between a shock wave and a laminar boundary layer is subdivided into a viscous layer and a potential field. The assumptions that the compressibility effect in the inner layer is negligible and that the original flow in the outer layer is uniform lead to simple analytic solutions for the perturbed flow. The joining conditions at the interface between the layers determine an eigenvalue which gives the rate of decay and the character of the disturbances both upstream and downstream of the point of incidence. The final conclusions are in agreement with experiments.

INTRODUCTION

The present investigation is an independent study of the interaction of an oblique shock with a laminar boundary layer in a compressible supersonic stream. In reference 1, where interaction of weak shock waves with both laminar and turbulent boundary layers was treated, the integrated momentum across the boundary layer was considered, rather than the balance among various dynamic effects at each point. This momentum-integral method is simple and, in certain respects, powerful and capable of yielding useful qualitative information such as the upstream pressure influence, pressure distribution, and the growth of boundary-layer thickness due to the presence of a shock, but it fails in regard to what actually happens inside the boundary layer. In the present report a different approach has been adopted, with the intention of filling the gap left by the previous investigation. The purpose will, on the whole, be complementary, so as to provide a physical picture for the understanding of this complex phenomenon.

Contrary to reference 1, the differential-equation method is employed here. According to available experimental observation, when an oblique shock is incident upon a laminar boundary layer the resultant flow bears no resemblance to the flow predicted by potential theory. For if the viscous flow is absent the flow ahead of the shock will not be affected. Because of the presence of the boundary layer in which there is a subsonic layer, however, a sudden decrease of pressure at a point will immediately be transmitted forward by the inability of the subsonic layer to support an excess pressure rise. When the pressure is transmitted, the flow in the boundary layer will be retarded and the streamlines distorted. Since the outer field is supersonic, this change occurring in the viscous layer will affect the whole potential field. This is actually observed. For stronger shocks, the flow in the boundary layer generally will separate and will have backflow under the influence of an adverse pressure gradient. An adequate theory that is able to account for the observed effects cannot be formulated unless the boundary-layer hypothesis is abandoned entirely. One therefore is faced by a much more difficult mathematical problem.

To restrict the scope and complexity of this study, let it be assumed first of all that the boundary is an insulated flat plate, and, secondly, that the incident shock is weak and its angle of incidence is such that regular reflection would be possible, were the flow frictionless, and, lastly, that the free-stream Mach number is not large. Under these assumptions, the angle of deflection of the flow in passing through the shock wave will be small, and the temperature variation between the free-stream condition and that of the plate will not be large. In fact, the study of a laminar boundary layer indicates that, for moderate Mach number, the temperature as well as the compressibility effects are unimportant (reference 2). This must remain true even if the flow is not boundary-layer flow. Therefore, without loss of generality, it will be assumed that the viscosity and thermal conductivity of the gas will be taken as constant and the Prandtl number is unity.

In order to bring the interaction problem within the scope of practical mathematical analysis, these simplifying hypotheses have to be made in the absence of a proper method of approximation, such as the boundary-layer theory. Broadly speaking, examination of schlieren photographs of the flow produced by the interaction between a shock and a laminar boundary layer will reveal that two characteristically different outer and inner regions exist for sufficiently high Reynolds number. The outer field is characterized by its strong potential character, whereas the region close to the wall is predominantly viscous, which is quite reminiscent of the boundary-layer flow. It appears natural, therefore, to assume a priori that the viscous effect is confined to a thin layer in the vicinity of the boundary and the outer main flow is potential. These two different flows are then in dynamical equilibrium. If one is disturbed, the other will be affected. Since

the outer field is supersonic, any local change will be felt in a much larger region than that in a subsonic field.

After the flow field is separated into two regions, specific assumptions regarding the structure of the viscous layer can be made. It is important to note that in the case of a compression incident wave, the overwhelming effect taking place in the viscous layer is the sudden decrease of the velocity or even reversal of flow (backflow). If backflow sets in, the flow speed in the subsonic region will be very much reduced. As a result of this, the streamlines will be pushed outward and the flow compressed. On account of the displacement of streamlines, the subsonic region will become thicker, and the thickening of the subsonic region is a decisive factor that distinguishes the strong from weak interactions. The importance of this dimension (the thickness of the subsonic layer) has already been established by Tsien and Finston (reference 3) in the case of inviscid theory.

The viscous layer, thus, must have two distinct sublayers, each displaying a different character. In the supersonic layer, the flow is characterized by large velocities and is not unlike that of an unseparated boundary layer. Therefore, for this layer both viscous and inertia forces are important. On the other hand, in the subsonic layer, especially with backflow, the average speed will be very small. In this case, because of slow motion and moderate temperature change, the change of density is always a lower-order effect. In fact, the contribution due to compressibility is proportional to the square of Mach number, and, if the average Mach number in the subsonic region is small, the compressibility effect is, indeed, negligible. Because of this physical fact, the subsonic layer will be taken as incompressible.

With these assumptions, the problem is finally solved by perturbation of weak incident waves. As a test of these assumptions, a simple flow with broken-line velocity profile is taken as the basic flow: In the incompressible layer, the velocity is a linear function of the distance from the plate; in the compressible layer, it is constant. The density in the basic flow is constant in each layer but takes different values. Thus, at the interface where the two flows join, the velocity is continuous but density is discontinuous. For this case, a first-order solution consistent with these assumptions is completely determined.

In the case of weak shock, excellent agreement with experiments has been achieved for the pressure distribution on the plate. It confirms the conjecture that separation of the flow as well as backflow always occur. Because of the occurrence of backflow, transition downstream of the point of incidence is unavoidable in the viscous layer. There are strong experimental evidences but detailed investigations are yet to be conducted. In the outer field, on the other hand, it is predicted that in the place of the regularly reflected shock there is a strong

expansion, and farther downstream a train of strong compression waves must exist, eventually forming an envelope. Therefore, downstream of the point of incidence a second shock must occur. This deduction is also confirmed by experiments.

Lastly, the importance of nonlinear effect is discussed.

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SYMBOLS

A	constant
a'	speed of sound
$C_1, C_2, C_3, C_4, C_5, C_6$	constants
D	constant
d	constant
d/dt	convective derivative
F, G	scalar functions
f_1, g_1	defined by equations (43) and (41), respectively
H	nondimensional enthalpy
$I_{1/3}\left(\frac{2}{3}\eta^{3/2}\right)$	Bessel function of first kind with imaginary argument
$J_{1/3}$	Bessel function of first kind with real argument
K_1, K_2, K_3, K_4	constants

$K_{1/3}\left(\frac{2}{3}\eta^{3/2}\right)$ Bessel function of second kind with imaginary argument

$k = (\gamma - 1)M^2/R^2\left[1 + \left(\frac{4\gamma}{3} - 1\right)\frac{\lambda M^2}{R}\right]$

$\tilde{k} = \left[1 + \frac{4}{3}(\gamma - 1)\frac{M^2\lambda}{R}\right]\Big/\left(1 + \frac{4}{3}\frac{\gamma M^2\lambda}{R}\right)$

L length of plate from leading edge to point of incidence

M Mach number

n dilatation

P pressure

P_1, P_2, P_3 constants

p nondimensional pressure $\left(p'/\rho_\infty U_\infty^2\right)$

p' pressure

R Reynolds number $\left(U_\infty L/\nu_\infty\right)^{1/2}$

$r = \Gamma(1/3)/3^{2/3}$

T nondimensional temperature

T' temperature

s,t defined by equations (B4)

U velocity

u,v nondimensional velocity components

u',v' velocity components

$U^{(1)}$, $V^{(1)}$ constants

$X = -d/\lambda_2$

x,y	nondimensional Cartesian coordinates
x',y'	coordinates
$Z(\tilde{\eta})$	defined by equation (C4)
z	defined by equation (30)

$\alpha = 0.332$

$\tilde{\alpha} = \rho_o(0)_{\alpha R}$

$\beta = \sqrt{M^2 - 1}$

β_1	defined by equation (43)
β_2	defined by equation (C15)
$\Gamma(z)$	gamma function
γ	ratio of specific heats
Δ	Laplacian operator
$\delta\vartheta$	deflection of flow (equation (B6))
ϵ	flow-deflection angle
η	defined by equation (31)
$\tilde{\eta}$	defined by equation (C6)
$\Lambda(\eta)$	defined by equation (C5)
λ	eigenvalue
ν_∞	free-stream kinematic viscosity
ρ	nondimensional density
ρ'	density

ρ_o	density at $y = 0$
$\sigma = \lambda R - \lambda^2$	
$\tau = \lambda(\lambda\tilde{\alpha})^{-1/3}$	
$\tilde{\tau}$	defined by equation (C7)
φ	nondimensional velocity potential
ψ	stream function
ω	angle between velocity and shock

Subscripts:

C	complementary
c	solution in compressible layer
i	solution in incompressible layer
o	value on plate
p	particular
s	due to step wave
sep	at separation point
t	due to transmitted wave
x	partial derivative with respect to x
y	partial derivative with respect to y
1	in region 1, before shock (see fig. 1)
2	in region 2, behind shock (see fig. 1)
3	in region 3 (see fig. 1)
∞	free stream

$\parallel$ parallel to oblique shock wave

$\perp$ perpendicular to oblique shock wave

Superscripts:

(0) zeroth order

(1) first order

(2) second order

(3) third order

STATEMENT OF PROBLEM AND BASIC ASSUMPTIONS

Let there be a laminar boundary layer in a compressible viscous fluid along an insulated flat plate immersed in a steady uniform supersonic stream, and let an oblique shock be incident upon the plate. When the steady condition is established, the flow in the neighborhood exhibits a character which is entirely different from the original flow.

From all indications, this flow does not obey the boundary-layer approximation; nevertheless, for simplicity for future analysis, the concept of boundary layer, or viscous layer, will be retained. Namely, the whole flow field is visualized as consisting of inviscid and viscous flow in equilibrium with each other. The possibility of existence of such a demarcation line will be assumed at this moment. The solutions obtained are consistent with the assumption, as will be seen later, so that the theory is self-consistent at least. Naturally, its further justification rests upon experimental evidence.

The main feature in the viscous layer is that, in the case of a compression wave in the inviscid outer flow, backflow generally exists. For this reason, terms in the equations of motion which are unimportant according to boundary-layer approximation become decisive as the supposed large-order terms vanish. Therefore, the pressure gradients along both directions have to be considered. To simplify the mathematical process, some minor effects, such as the variation of the viscosity coefficient and thermal conductivity with temperature, will be neglected and the Prandtl number will be taken equal to 1. For moderate Mach numbers, less than 3, say, this neglect, according to boundary-layer investigations, will have little effect on the major characteristics of the flow.

The inviscid flow generally is rotational, as it involves shocks. This is particularly true in the case of a local supersonic zone over

a curved surface. For a purely supersonic flow, if the shock is slightly curved, the vorticity generated behind the shock is of high order and can be neglected. The perturbed flow in the outer region can then be regarded as irrotational.

As the region which is influenced by the presence of the shock, according to experimental observations, is confined to an area about the point of incidence, with a dimension only a fraction of the length of the plate, a point which lies at a distance about the length of the plate from the point of incidence will be considered as at infinity. Consequently, the boundary-layer flow will be replaced by a "shear flow" extending to both positive and negative infinities. This approximation is justified if the derivative along the flow is much larger in the perturbed flow than that in the original flow. For large Reynolds number, this condition can always be satisfied.

METHOD OF SOLUTION

The flow is supposed to be two dimensional and steady and the fluid is compressible. If the flow field can be subdivided into viscous and nonviscous regions, the flow in the viscous layer satisfies in dimensionless variables the system:

$$\rho \frac{du}{dt} = -p_x + \frac{1}{R}\left(\Delta u + \frac{1}{3} n_x\right) \tag{1}$$

$$\rho \frac{dv}{dt} = -p_y + \frac{1}{R}\left(\Delta v + \frac{1}{3} n_y\right) \tag{2}$$

$$(\rho u)_x + (\rho v)_y = 0 \tag{3}$$

$$\rho \frac{dH}{dt} = \frac{1}{R} \Delta H + \frac{(\gamma - 1)M^2}{R}\left[\left(\frac{du}{dt}\right)_x + \left(\frac{dv}{dt}\right)_y - \frac{2}{3}(un)_x - \frac{2}{3}(vn)_y\right] \tag{4}$$

and in the case of perfect gas

$$p = \left(\gamma M^2\right)^{-1} \rho T \tag{5}$$

Here the subscripts denote partial differentiations with respect to the Cartesian coordinate x or y indicated; the Laplacian operator Δ, the convective derivative d/dt, the nondimensional enthalpy H, and the dilatation n are defined by

$$\left.\begin{aligned} \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ \frac{d}{dt} &= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \\ H &= T + \frac{\gamma - 1}{2} M^2\left(u^2 + v^2\right) \\ n &= u_x + v_y \end{aligned}\right\} \tag{6}$$

Furthermore, the velocity components u' and v', the pressure p', the density ρ', the temperature T', and the coordinates x' and y' are nondimensionalized as follows:

$$\left.\begin{aligned} u &= \frac{u'}{U_\infty} \\ v &= \frac{v'}{U_\infty} \\ p &= \frac{p'}{\rho_\infty U_\infty^2} \\ \rho &= \frac{\rho'}{\rho_\infty} \\ x &= \frac{x'}{\sqrt{\frac{\nu_\infty L}{U_\infty}}} \\ y &= \frac{y'}{\sqrt{\frac{\nu_\infty L}{U_\infty}}} \end{aligned}\right\} \tag{7}$$

where U_∞, ρ_∞, and ν_∞ are free-stream velocity, density, and kinematic viscosity, and L is the length of the plate from the leading edge to the point of incidence. Finally, the parameters γ, M, and $R = \left(\frac{U_\infty L}{\nu_\infty}\right)^{1/2}$ stand for, respectively, the ratio of specific heats, the free-stream Mach number, and the Reynolds number.

On the other hand, the inviscid flow in the outer region is assumed to be irrotational and, consequently, is determined by a nondimensional velocity potential $\varphi(x,y)$ satisfying the equation

$$\left[(a')^2 - (u')^2\right]\varphi_{xx} - 2u'v'\varphi_{xy} + \left[(a')^2 - (v')^2\right]\varphi_{yy} = 0 \tag{8}$$

where the sound speed a' is given in terms of the velocity components u' and v' by the relation

$$(a')^2 = (a_\infty')^2 - \frac{\gamma - 1}{2}\left[(u')^2 + (v')^2 - U_\infty^2\right] \tag{9}$$

By the assumption that these two different flows are in equilibrium with each other, the flow determined from equation (9) must join smoothly with that given by the system of equations (1) to (5) subject to the boundary conditions

$$u = v = 0$$

when $y = 0$ and

$$u = \varphi_x$$

$$v = \varphi_y$$

when $y = \infty$.

As the disturbances are initiated by the incident shock, it is expected that they are small if the incident wave is weak. From experience, this is at least the case in the field upstream of the point of incidence. If the shock strength is characterized by the flow-deflection angle ϵ, then for small values of ϵ the solutions are expanded in powers of ϵ:

$$\left.\begin{aligned} u &= u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots \\ v &= v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots \\ p &= p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots \\ \rho &= \rho^{(0)} + \epsilon \rho^{(1)} + \epsilon^2 \rho^{(2)} + \cdots \\ T &= T^{(0)} + \epsilon T^{(1)} + \epsilon^2 T^{(2)} + \cdots \end{aligned}\right\} \tag{10}$$

By substituting the quantities defined in equations (10) into the system of equations (1) to (5), there results the following system of equations, according to the powers of ϵ; namely, for the zero order,

$$\left.\begin{aligned} \rho^{(0)} \frac{d^{(0)}u^{(0)}}{dt} &= -p_x^{(0)} + \frac{1}{R}\left[\Delta u^{(0)} + \frac{1}{3} n_x^{(0)}\right] \\ \rho^{(0)} \frac{d^{(0)}v^{(0)}}{dt} &= -p_y^{(0)} + \frac{1}{R}\left[\Delta v^{(0)} + \frac{1}{3} n_y^{(0)}\right] \\ \left(\rho^{(0)}u^{(0)}\right)_x + \left(\rho^{(0)}v^{(0)}\right)_y &= 0 \\ \rho^{(0)} \frac{d^{(0)}H^{(0)}}{dt} &= \frac{1}{R}\Delta H^{(0)} + \frac{(\gamma - 1)M^2}{R}\left[\left(\frac{d^{(0)}u^{(0)}}{dt}\right)_x + \left(\frac{d^{(0)}v^{(0)}}{dt}\right)_y - \right. \\ &\qquad \left. \frac{2}{3}\left(u^{(0)}n^{(0)}\right)_x - \frac{2}{3}\left(v^{(0)}n^{(0)}\right)_y\right] \\ p^{(0)} &= (\gamma M^2)^{-1}\rho^{(0)}T^{(0)} \end{aligned}\right\} \tag{11}$$

where

$$\left.\begin{aligned} \frac{d^{(0)}}{dt} &= u^{(0)}\frac{\partial}{\partial x} + v^{(0)}\frac{\partial}{\partial y} \\ H^{(0)} &= T^{(0)} + \frac{\gamma - 1}{2}M^2\left(u^{(0)2} + v^{(0)2}\right) \\ n^{(0)} &= u_x^{(0)} + v_y^{(0)} \end{aligned}\right\} \tag{12}$$

for the first order,

$$\left.\begin{aligned} &\rho^{(0)}\left(\frac{d^{(0)}u^{(1)}}{dt} + \frac{d^{(1)}u^{(0)}}{dt}\right) + \rho^{(1)}\frac{d^{(0)}u^{(0)}}{dt} = -p_x^{(1)} + \\ &\frac{1}{R}\left(\Delta u^{(1)} + \frac{1}{3}n_x^{(1)}\right) \\ &\rho^{(0)}\left(\frac{d^{(0)}v^{(1)}}{dt} + \frac{d^{(1)}v^{(0)}}{dt}\right) + \rho^{(1)}\frac{d^{(0)}v^{(0)}}{dt} = -p_y^{(1)} + \\ &\frac{1}{R}\left(\Delta v^{(1)} + \frac{1}{3}n_y^{(1)}\right) \\ &\left(\rho^{(0)}u^{(1)} + \rho^{(1)}u^{(0)}\right)_x + \left(\rho^{(0)}v^{(1)} + \rho^{(1)}v^{(0)}\right)_y = 0 \\ &\rho^{(0)}\frac{d^{(0)}H^{(1)}}{dt} + \left(\rho^{(1)}\frac{d^{(0)}}{dt} + \rho^{(0)}\frac{d^{(1)}}{dt}\right)H^{(0)} = \frac{1}{R}\Delta H^{(1)} + \\ &\frac{(\gamma - 1)M^2}{R}\left[\left(\frac{d^{(0)}u^{(1)}}{dt} + \frac{d^{(1)}u^{(0)}}{dt}\right)_x + \left(\frac{d^{(0)}v^{(1)}}{dt} + \frac{d^{(1)}v^{(0)}}{dt}\right)_y - \right. \\ &\left.\frac{2}{3}\left(u^{(0)}n^{(1)} + u^{(1)}n^{(0)}\right)_x - \frac{2}{3}\left(v^{(0)}n^{(1)} + v^{(1)}n^{(0)}\right)_y\right] \\ &p^{(1)} = (\gamma M^2)^{-1}\left(\rho^{(0)}T^{(1)} + \rho^{(1)}T^{(0)}\right) \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned} \frac{d^{(1)}}{dt} &= u^{(1)}\frac{\partial}{\partial x} + v^{(1)}\frac{\partial}{\partial y} \\ H^{(1)} &= T^{(1)} + (\gamma - 1)\,u^{(0)}u^{(1)} + v^{(0)}v^{(1)} \\ n^{(1)} &= u_x^{(1)} + v_y^{(1)} \end{aligned}\right\} \tag{14}$$

and for the second order,

$$\left.\begin{aligned} &\rho^{(0)}\left(\frac{d^{(0)}u^{(2)}}{dt} + \frac{d^{(1)}u^{(1)}}{dt} + \frac{d^{(2)}u^{(1)}}{dt}\right) + \rho^{(1)}\left(\frac{d^{(0)}u^{(1)}}{dt} + \frac{d^{(1)}u^{(0)}}{dt}\right) + \rho^{(2)}\frac{d^{(0)}u^{(0)}}{dt} = \\ &-p_x^{(2)} + \frac{1}{R}\left(\Delta u^{(2)} + \frac{1}{3}n_x^{(2)}\right) \\ &\rho^{(0)}\left(\frac{d^{(0)}v^{(2)}}{dt} + \frac{d^{(1)}v^{(1)}}{dt} + \frac{d^{(2)}v^{(1)}}{dt}\right) + \rho^{(1)}\left(\frac{d^{(0)}v^{(1)}}{dt} + \frac{d^{(1)}v^{(0)}}{dt}\right) + \rho^{(2)}\frac{d^{(0)}v^{(0)}}{dt} = \\ &-p_y^{(2)} + \frac{1}{R}\left(\Delta v^{(2)} + \frac{1}{3}n_y^{(2)}\right) \\ &\left(\rho^{(0)}u^{(2)} + \rho^{(1)}u^{(1)} + \rho^{(2)}u^{(0)}\right)_x + \left(\rho^{(0)}v^{(2)} + \rho^{(1)}v^{(1)} + \rho^{(2)}v^{(0)}\right)_y = 0 \\ &\rho^{(0)}\frac{d^{(0)}H^{(2)}}{dt} + \left(\rho^{(0)}\frac{d^{(1)}}{dt} + \rho^{(1)}\frac{d^{(0)}}{dt}\right)H^{(1)} + \left(\rho^{(0)}\frac{d^{(2)}}{dt} + \rho^{(1)}\frac{d^{(1)}}{dt} + \rho^{(2)}\frac{d^{(0)}}{dt}\right)H^{(0)} = \\ &\frac{1}{R}\Delta H^{(2)} + \frac{(\gamma - 1)M^2}{R}\left[\left(\frac{d^{(0)}u^{(2)}}{dt} + \frac{d^{(1)}u^{(1)}}{dt} + \frac{d^{(2)}u^{(0)}}{dt}\right)_x + \right. \\ &\left.\left(\frac{d^{(0)}v^{(2)}}{dt} + \frac{d^{(1)}v^{(1)}}{dt} + \frac{d^{(2)}v^{(0)}}{dt}\right)_y - \frac{2}{3}\left(u^{(0)}n^{(2)} + u^{(1)}n^{(1)} + u^{(2)}n\right)_x - \right. \\ &\left.\frac{2}{3}\left(v^{(0)}n^{(2)} + v^{(1)}n^{(1)} + v^{(2)}n^{(0)}\right)_y\right] \\ &p^{(2)} = (\gamma M^2)^{-1}\left(\rho^{(0)}T^{(2)} + \rho^{(1)}T^{(1)} + \rho^{(2)}T^{(0)}\right) \end{aligned}\right\} \tag{15}$$

where

$$\left.\begin{aligned} \frac{d^{(2)}}{dt} &= u^{(2)}\frac{\partial}{\partial x} + v^{(2)}\frac{\partial}{\partial y} \\ H^{(2)} &= T^{(2)} + (\gamma - 1)M^2\left(u^{(0)}u^{(2)} + v^{(0)}v^{(2)} + \frac{u^{(1)^2} + v^{(1)^2}}{2}\right) \\ n^{(2)} &= u_x{}^{(2)} + v_y{}^{(2)} \end{aligned}\right\} \tag{16}$$

In the case of the potential flow, if the velocity potential is expanded as

$$\varphi = x + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)} + \cdots \tag{17}$$

there will occur, similarly, equations for the first- and second-order quantities. These are: For the first order

$$\beta^2\varphi_{xx}{}^{(1)} - \varphi_{yy}{}^{(1)} = 0 \tag{18}$$

and for the second order

$$\beta^2\varphi_{xx}{}^{(2)} - \varphi_{yy}{}^{(2)} = -2M^2\left[\left(1 + \frac{\gamma - 1}{2}M^2\right)\varphi_x{}^{(1)}\varphi_{xx}{}^{(1)} + \varphi_y{}^{(1)}\varphi_{yy}{}^{(1)}\right] \tag{19}$$

where $\beta = \sqrt{M^2 - 1}$.

FIRST-ORDER SOLUTION - UPSTREAM, $x < 0$

Let the point of incidence be chosen as the origin of the Cartesian coordinates. Then, negative x will be called upstream, and positive x, downstream, of the point of incidence. The various regions will be numbered by 1, 2, and 3 as shown in figure 1 in which OS indicates shock and OM, the limiting Mach line of region 2.

As it has been assumed that the basic flow is a laminar boundary-layer flow of a Blasius type, such as considered by Von Kármán and Tsien (reference 2) and by Emmons and Brainerd (references 4 and 5), system (11) simply reduces to the well-known Prandtl boundary-layer equations, whose solution has already been found in these references.

Since the basic flow is a boundary-layer flow with constant pressure $p^{(0)}$, system (13) can be simplified. From experiments it has been established that the space variation of the perturbed quantities is much more rapid than that of the unperturbed quantities. By the boundary-layer approximation, the ratio of the partial derivatives $\partial/\partial x : \partial/\partial y$ is $1:R$. For large Reynolds number, the x-derivatives of the basic flow, as well as $v^{(0)}$, which is of the order R^{-1}, can be neglected in comparison with the y-derivatives. That is, $u^{(0)}$ and $\rho^{(0)}$ are functions of y only. This approximation is confirmed by experiments and, as a matter of fact, it is customarily used in the pressure measurements, because the wave is shifted forward and back relative to a fixed pressure orifice to measure the pressure distribution before and behind the wave. In the case of unit Prandtl number, $H^{(0)}$ is a constant. Then, by dropping terms such as $v^{(0)}u_y^{(1)}$ and $u_x^{(0)}u^{(1)}$, system (13) becomes

$$\left.\begin{aligned}
&\rho^{(0)}\left(u^{(0)}u_x^{(1)} + u_y^{(0)}v^{(1)}\right) = -p_x^{(1)} + \frac{1}{R}\left(\Delta u^{(1)} + \frac{1}{3}n_x^{(1)}\right)\\
&\rho^{(0)}u^{(0)}v_x^{(1)} = -p_y^{(1)} + \frac{1}{R}\left(\Delta v^{(1)} + \frac{1}{3}n_y^{(1)}\right)\\
&\rho^{(0)}n^{(1)} + u^{(0)}\rho_x^{(1)} + v^{(1)}\rho_y^{(0)} = 0\\
&\rho^{(0)}u^{(0)}H_x^{(1)} = \frac{1}{R}\Delta H^{(1)} + \frac{(\gamma-1)M^2}{R}\left[\left(u^{(0)}u_x^{(1)} + v^{(1)}u_y^{(0)}\right)_x + \left(u^{(0)}v_x^{(1)}\right)_y - \frac{2}{3}\left(u^{(0)}n^{(1)}\right)_x\right]\\
&p^{(1)} = \left(\gamma M^2\right)^{-1}\left(\rho^{(0)}T^{(1)} + \rho^{(1)}T^{(0)}\right)
\end{aligned}\right\} \qquad (20)$$

Incompressible Layer

It is noted that the coefficients of system (20) depend on the basic velocity profile $u^{(0)}(y)$. Since $u^{(0)}(y)$ cannot be expressed by simple functions, in order to simplify the analytical work further simplifications are necessary.

First of all, in the case of insulated plate, the temperature and, hence, the density in the basic flow will have a vanishing gradient on the plate. This makes the variation of the density in the viscous layer much smaller than that of the velocity. When the flow is subject to an adverse pressure gradient, the flow will be further retarded. When the backflow occurs, the average speed will be very low and, consequently, the representative average Mach number will be small. Under this condition, the change of density is less important than that of the pressure. A layer where this approximation is valid is called an "incompressible layer" and the dominant effects will be pressure and frictional forces.

Of course, it is difficult to define the thickness of this layer beforehand. Generally, it would correspond to the subsonic portion of the viscous layer. In the basic flow, the sonic boundary can be exactly calculated. When the flow is perturbed, it is unknown, but certainly will be thicker, for the flow is subject to an adverse pressure gradient. Owing to this fact and also for mathematical expediency, the thickness will be taken in accordance with the way the velocity profile $u^{(0)}(y)$ is represented. This will become clear below and the thickness defined turns out to be nearly half the original boundary-layer thickness.

Because of the assumption that density is constant and assumes the value $\rho_o^{(0)}$ on the plate, system (20) simplifies to

$$\left.\begin{aligned}
&\rho_o^{(0)}\left(u^{(0)}u_x^{(1)} + u_y^{(0)}v^{(1)}\right) = -p_x^{(1)} + \frac{1}{R}\Delta u^{(1)} \\
&\rho_o^{(0)}u^{(0)}v_x^{(1)} = -p_y^{(1)} + \frac{1}{R}\Delta v^{(1)} \\
&n^{(1)} = 0 \\
&\rho_o^{(0)}u^{(0)}H_x^{(1)} = \frac{1}{R}\Delta H^{(1)} + \frac{2(\gamma - 1)M^2}{R}u_y^{(0)}v_x^{(1)}
\end{aligned}\right\} \qquad (21)$$

Thus, the velocities, pressure, and temperature can be dealt with independently. For the velocities, the first three equations yield by elimination of $p^{(1)}$:

$$\left.\begin{aligned} &\rho_o{}^{(0)}u^{(0)}\left(u_y{}^{(1)} - v_x{}^{(1)}\right)_x + \rho_o{}^{(0)}u_{yy}{}^{(0)}v^{(1)} = \frac{1}{R}\Delta\left(u_y{}^{(1)} - v_x{}^{(1)}\right) \\ &n^{(1)} = 0 \end{aligned}\right\} \tag{22}$$

After the velocities are known, the temperature will be given by the energy equation. In the present problem, however, temperature is not an interesting quantity and will subsequently not be mentioned. To solve equations (22), let a stream function $\psi^{(1)}(x,y)$ be introduced by the relations

$$u^{(1)} = \psi_y{}^{(1)}$$

$$v^{(1)} = -\psi_x{}^{(1)}$$

From equations (22) $\psi^{(1)}$ then satisfies the equation

$$\rho_o{}^{(0)}u^{(0)}\Delta\psi_x{}^{(1)} = \frac{1}{R}\Delta\,\Delta\psi^{(1)} + \rho_o{}^{(0)}u_{yy}{}^{(0)}\psi_x{}^{(1)} \tag{23}$$

By neglecting the compressibility effect in the inner layer, the basic velocity profile $u^{(0)}(y)$ becomes the Blasius profile. The variable coefficients of equation (23), for small values of y, will be power series in y. However, it has been recognized that in the case of Blasius profile the velocity attains the free-stream value fairly rapidly and the initial portion is nearly linear. Consequently, it will not involve serious error to replace the continuous profile by a broken-line one, so that the initial part is proportional to y and the remaining part, constant with the free-stream value unity. If the skin friction agrees with the exact value, the velocity profile can then be defined as

$$\left.\begin{aligned} &u^{(0)}(y) = \alpha y \quad \text{when} \quad 0 \leqq y \leqq y_1 \\ &u^{(0)}(y) = 1 \quad \text{when} \quad y_1 < y \leqq \infty \end{aligned}\right\} \tag{24}$$

If the velocity is continuous at $y = y_1$, then $y_1 = \alpha^{-1}$ which is about half the boundary-layer thickness, namely $y_1 \sim 3$. If the incompressible layer is defined as the interval $0 \leqq y \leqq y_1$, then equation (23) becomes

$$\tilde{\alpha} y \, \Delta\psi_x^{(1)} = \Delta \, \Delta\psi^{(1)} \tag{25}$$

where $\tilde{\alpha} = \rho_o^{(0)} \alpha R$.

To solve equation (25), it is natural to assume the form

$$\left.\begin{aligned} \psi^{(1)} &= \psi(y) e^{\lambda x} \\ \lambda &> 0 \end{aligned}\right\} \tag{26}$$

for region 1 where $x < 0$. This assumes a special form of compression waves induced in the potential field; that is, the solution of equation (18)

$$\varphi^{(1)} = \frac{A}{\lambda} e^{\lambda(x - \beta y)} \tag{27}$$

where A is a constant to be determined. This is entirely in agreement with the conclusions reached in reference 1. Substituting equations (26) in equation (25) and simultaneously putting

$$\psi'' + \lambda^2 \psi = z(y) \tag{28}$$

there results

$$z'' + \left(\lambda \tilde{\alpha} y - \lambda^2\right) z = 0 \tag{29}$$

The general solution of this equation is

$$z = C_1 \eta^{1/2} K_{\frac{1}{3}}\left(\frac{2}{3} \eta^{3/2}\right) + C_2 \eta^{1/2} I_{\frac{1}{3}}\left(\frac{2}{3} \eta^{3/2}\right) \tag{30}$$

where

$$\eta = (\lambda\tilde{\alpha})^{1/3}\left(y - \lambda\tilde{\alpha}^{-1}\right) \tag{31}$$

C_1 and C_2 are constants, and $I_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)$ and $K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)$ denote the Bessel functions of the first and second kind with imaginary argument. With z known, the solution of equation (28) is

$$\psi = C_3 \cos \tau\eta + C_4 \sin \tau\eta +$$

$$\frac{\sin \tau\eta}{\tau}\int_{\eta_o}^{\eta} (\cos \tau\eta)z(\eta)\, d\eta - \frac{\cos \tau\eta}{\tau}\int_{\eta_o}^{\eta} (\sin \tau\eta)z(\eta)\, d\eta \tag{28a}$$

with $\tau = \lambda(\lambda\tilde{\alpha})^{-1/3}$. On the plate $y = 0$ and $u^{(1)} = v^{(1)} = 0$; hence $\psi(0) = \psi'(0) = 0$. For large Reynolds number and small values of λ, $y = 0$ corresponds approximately to $\eta = 0$. The boundary conditions thus require $C_3 = C_4 = 0$. Moreover, at $y = y_1$, η will be large, and since $\tau\eta_1 = \lambda y_1 < 1$, the integrals $\int_0^{\eta} \eta^{1/2} \left\{\begin{matrix}\cos \\ \sin\end{matrix}\right\} \tau\eta\, I_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right) d\eta$ will diverge like $\int_0^{\eta} e^{\frac{2}{3}\eta^{3/2}}\, d\eta$ as η approaches infinity. Therefore, $C_2 = 0$. The solution in the incompressible layer can then be read as follows

$$\psi^{(1)} = \frac{C_1}{\tau}\left[(\sin \tau\eta)\int_0^{\eta} \eta^{1/2}(\cos \tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right) d\eta - \right.$$

$$\left. \cos \tau\eta \int_0^{\eta} \eta^{1/2}(\sin \tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right) d\eta\right] e^{\lambda x} \tag{32}$$

Thereby the velocity components are

$$\left.\begin{aligned} u^{(1)} &= C_1(\tilde{\alpha}\lambda)^{1/3}\left[(\cos\tau\eta)\int_0^\eta \eta^{1/2}(\cos\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta + \right.\right. \\ &\qquad \left.(\sin\tau\eta)\int_0^\eta \eta^{1/2}(\sin\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta\right]e^{\lambda x} \\ v^{(1)} &= -C_1(\tilde{\alpha}\lambda)^{1/3}\left[(\sin\tau\eta)\int_0^\eta (\cos\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta - \right. \\ &\qquad \left.(\cos\tau\eta)\int_0^\eta \eta^{1/2}(\sin\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta\right]e^{\lambda x} \end{aligned}\right\} \tag{33}$$

On the other hand, the pressure can be obtained by integration of the first of equations (21) and is, by the form of solution,

$$p^{(1)} = -\rho_o(0)_u(0)_u(1) - \frac{\rho_o(0)_{u_y}(0)_v(1)}{\lambda} + \frac{1}{\lambda R}\left(\psi^{(1)} + \lambda^2\psi'\right)e^{\lambda x}$$

where the constant of integration vanishes by the condition at negative infinity. Substituting $u^{(1)}$ and $v^{(1)}$ from equation (33) and making use of equation (28), a straightforward reduction yields

$$\begin{aligned} p^{(1)} = \frac{\rho_o(0)_{\alpha C_1}}{\tau}\Bigg\{ & -\tau\eta K_{\frac{2}{3}}\left(\frac{2}{3}\eta^{3/2}\right) + (\sin\tau\eta)\int_0^\eta \eta^{1/2}(\cos\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta - \\ & (\cos\tau\eta)\int_0^\eta \eta^{1/2}(\sin\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta - \\ & \tau\eta\left[(\cos\tau\eta)\int_0^\eta \eta^{1/2}(\cos\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta + \right. \\ & \left.\left.(\sin\tau\eta)\int_0^\eta \eta^{1/2}(\sin\tau\eta)K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)d\eta\right]\right\}e^{\lambda x} \end{aligned} \tag{34}$$

On the plate $y = \eta = 0$ this reduces to

$$p^{(1)}(x,0) = -\rho_o{}^{(0)}\alpha C_1\left[\eta K_{\frac{2}{3}}\left(\frac{2}{3}\eta^{3/2}\right)\right]_{\eta=0} e^{\lambda x} \tag{34a}$$

Compressible Layer

In the compressible layer, because of the forward momentum of the potential flow outside, the velocity is everywhere positive and differs from the basic flow by a small amount in order to support the pressure rise. As the velocity is high, both compressibility and viscous effects will become important. Under this condition, the problem is considerably simplified by taking a uniform basic flow, namely $u^{(0)} = \rho^{(0)} = 1$, as discussed above. Accordingly, equations (20) reduce to

$$\left.\begin{aligned} u_x{}^{(1)} &= -p_x{}^{(1)} + \frac{1}{R}\left(\Delta u^{(1)} + \frac{1}{3}n_x\right) \\ v_x{}^{(1)} &= -p_y{}^{(1)} + \frac{1}{R}\left(\Delta v^{(1)} + \frac{1}{3}n_y\right) \\ n^{(1)} &+ \rho_x{}^{(1)} = 0 \\ H_x{}^{(1)} &= \frac{1}{R}\Delta H^{(1)} + \frac{(\gamma - 1)M^2}{R}n_x{}^{(1)} \\ p^{(1)} &= \left(\gamma M^2\right)^{-1}\left(T^{(1)} + \rho^{(1)}\right) \end{aligned}\right\} \tag{35}$$

where

$$H^{(1)} = T^{(1)} + (\gamma - 1)M^2u^{(1)}$$

The elimination of $p^{(1)}$ from the first two equations gives

$$\left(u_y{}^{(1)} - v_x{}^{(1)}\right)_x = \frac{1}{R}\Delta\left(u_y{}^{(1)} - v_x{}^{(1)}\right) \tag{36}$$

After differentiating the first and the second of equations (35) with respect to x and y, respectively, an addition yields

$$\Delta p^{(1)} = -n_x^{(1)} + \frac{4}{3R}\Delta n^{(1)}$$

moreover, by definition

$$H_x^{(1)} = \gamma M^2 p_x^{(1)} + (\gamma - 1)M^2 u_x^{(1)} + n^{(1)}$$

A substitution of $H_{xx}^{(1)}$ and $\Delta H_x^{(1)}$ in the energy equation leads to

$$n_x^{(1)} - M^2 u_{xx}^{(1)} + \frac{\gamma M^2}{R}\left(\Delta u_x^{(1)} + \frac{1}{3} n_{xx}^{(1)}\right) = \frac{1}{R}\left[\Delta n^{(1)} - \frac{4 - \gamma}{3} M^2 n_{xx}^{(1)} + \frac{4\gamma M^2}{3R}\Delta n_x^{(1)} + (\gamma - 1)M^2\left(u_y^{(1)} - v_x^{(1)}\right)_{xy}\right] \tag{37}$$

Equations (36) and (37) form a system of linear partial differential equations for $u^{(1)}$ and $v^{(1)}$ with constant coefficients. The solutions, if carried out, are expressible in terms of trigonometric and exponential functions. According to the arguments of these functions, they fall into two groups: One varies slowly and is "potential-like"; while the other varies rapidly and, therefore, is "boundary-layer-like." The latter group consists of two exponentials whose arguments differ by a term of $O\left(\frac{1}{R}\right)$. Hence, for large Reynolds number, the latter two exponentials will degenerate into one. It was found that this form of solution can be obtained by solving a much simpler problem, namely, by assuming $H^{(1)} = 0$. This assumption appears to be nothing but a method of approximating the solution of equations (36) and (37) in the case of large Reynolds number.

If $H^{(1)}$ is taken to be zero, then instead of equation (37) there is in its place

$$-n^{(1)} + M^2 u_x^{(1)} = \frac{\gamma M^2}{R}\left(\Delta u^{(1)} + \frac{1}{3} n_x^{(1)}\right) \tag{38}$$

by elimination of $p^{(1)}$ and $T^{(1)}$. To solve the system (36) and (38), two scalar functions $F^{(1)}$ and $G^{(1)}$ are introduced through the relations

$$\left.\begin{aligned} u^{(1)} &= F_x{}^{(1)} + G_y{}^{(1)} \\ v^{(1)} &= F_y{}^{(1)} - G_x{}^{(1)} \end{aligned}\right\} \tag{39}$$

The equivalent system, then, is as follows:

$$\left.\begin{aligned} \Delta G_x{}^{(1)} &= \frac{1}{R}\Delta\,\Delta G^{(1)} \\ -\Delta F^{(1)} + M^2 F_{xx}{}^{(1)} &= \frac{4\gamma M^2}{3R}\Delta F_x{}^{(1)} - M^2 G_{xy}{}^{(1)} + \frac{\gamma M^2}{R}\Delta G_y{}^{(1)} \end{aligned}\right\} \tag{40}$$

According to the form of the potential solution, it will again be assumed that

$$G^{(1)} = g_1(y)e^{\lambda x}$$

$$F^{(1)} = f_1(y)e^{\lambda x}$$

By the condition that $G^{(1)}$ is finite at infinity, the solution is shown to be

$$g_1 = C_3 \cos \lambda y + C_4 \sin \lambda y + \frac{C_2}{\lambda R} e^{-\sigma y} \tag{41}$$

where C_2, C_3, and C_4 are constants of integration and

$$\sigma^2 = \lambda R - \lambda^2 \tag{42}$$

By substituting g_1 from equation (41) in the second of equations (40) the solution for f_1 was found to be

$$f_1 = C_3 \sin \lambda y - C_4 \cos \lambda y + C_5 e^{-\beta_1 \lambda y} + C_6 e^{\beta_1 \lambda y} + \frac{(\gamma - 1)M^2}{R^2 + \left(\frac{4\gamma}{3} - 1\right)\lambda M^2 R} \frac{\sigma C_2}{\lambda} e^{-\sigma y} \tag{43}$$

where

$$\beta_1^2 = \left(\beta^2 - \frac{4\gamma M^2 \lambda}{3R}\right) \Bigg/ \left(1 + \frac{4\gamma M^2 \lambda}{3R}\right)$$

The velocity components for $y > y_1$, by equation (39), are given by

$$u(1) = \left\{ \lambda C_5 e^{-\beta_1 \lambda (y - y_1)} + \lambda C_6 e^{\beta_1 \lambda (y - y_1)} + \left[\frac{(\gamma - 1)M^2}{R^2 + \left(\frac{4\gamma}{3} - 1\right)\lambda M^2 R} - \frac{1}{\lambda R} \right] \sigma C_2 e^{-\sigma (y - y_1)} \right\} e^{\lambda x} \tag{44}$$

$$v(1) = \left\{ -\beta_1 \lambda C_5 e^{-\beta_1 \lambda (y - y_1)} + \beta_1 \lambda C_6 e^{\beta_1 \lambda (y - y_1)} - \left[\frac{(\gamma - 1)M^2}{\lambda R^2 + \left(\frac{4\gamma}{3} - 1\right)\lambda^2 M^2 R} + \frac{1}{R} \right] C_2 e^{-\sigma (y - y_1)} \right\} e^{\lambda x}$$

by redefining the constants C. It is noticed that the first group of exponentials varies with both x and y with a slope proportional to λ, which, according to experimental evidence for this type of flow, is a small number; that is, the variation with x is relatively slow. On the other hand, the second exponential varies with y with derivative proportional to σ, which is of $O(\sqrt{\lambda R})$. For large Reynolds number, it appears that σ could be thousands of times larger than λ. Therefore, in the case of large Reynolds number where the viscous layer is thin, the first group of terms changes only slightly while the second exponential drops to zero. Applying the boundary-layer concept, the potential-like terms will be taken as constant and equal to the boundary value of the potential solution. Consequently, the solution can be written as

$$\left.\begin{aligned} u^{(1)} &= \left[A + \left(k - \frac{1}{\lambda R}\right)\sigma C_2 e^{-\sigma(y-y_1)}\right]e^{\lambda x} \\ v^{(1)} &= \left[-\beta A - \left(\frac{k\sigma^2}{\lambda} + \frac{1}{R}\right)C_2 e^{-\sigma(y-y_1)}\right]e^{\lambda x} \end{aligned}\right\} \tag{44a}$$

with

$$k = (\gamma - 1)M^2 \Big/ R^2\left[1 + \left(\frac{4\gamma}{3} - 1\right)\frac{\lambda M^2}{R}\right]$$

It is seen that when $\sigma(y - y_1)$ becomes large, this solution joins the potential solution at $y = 0$ and the constants C_5 and C_6 will be considered as eliminated. By means of this approximate solution the pressure $p^{(1)}$ is shown to be

$$p^{(1)} = -\left[\tilde{k}A - \frac{k\sigma}{3} C_2 e^{-\sigma(y-y_1)}\right]e^{\lambda x} \tag{45}$$

where

$$\tilde{k} = \left[1 + \frac{4}{3}(\gamma - 1)\frac{M^2\lambda}{R}\right] \Big/ \left(1 + \frac{4}{3}\frac{\gamma M^2 \lambda}{R}\right)$$

Eigenvalues λ

In the last two sections, the velocity and pressure have been calculated by entirely different methods of approximation in two different layers. For the incompressible layer, the nonslip conditions are satisfied on the plate, whereas in the compressible layer, the velocity agrees with the potential flow at infinity. The complete solution is then left with three undetermined constants C_1, C_2, and A and with an arbitrary parameter λ. To determine these constants, it is assumed that the two solutions (33) and (34) and (44) and (45) must join at the interface $y = y_1$. Now, because of the simplification made in connection with equation (44), the conditions at the interface are to be stated as follows for $y = y_1$:

$$\left.\begin{aligned} u_c^{(1)} &= u_i^{(1)} \\ v_c^{(1)} &= v_i^{(1)} \\ p_c^{(1)} &= p_i^{(1)} \end{aligned}\right\} \tag{46}$$

where the subscripts c and i indicate, respectively, solutions of compressible and incompressible layers. The u velocity profile thus will have a discontinuity in slope. This could be improved by dropping the assumption $H^{(1)} = 0$. However, as pressure depends only on the velocity and its second derivatives, an error in shear can produce only minor contributions and can be ignored. By substituting the solutions in equations (46), there results the following system of equations:

$$\left.\begin{aligned} A + \left(k - \frac{1}{\lambda R}\right)\sigma C_2 - U^{(1)}(\eta_1)C_1 &= 0 \\ -\beta A - \left(\frac{k\sigma^2}{\lambda} + \frac{1}{R}\right)C_2 + V^{(1)}(\eta_1)C_1 &= 0 \\ -\tilde{k}A + \frac{k\sigma}{3}C_2 - P^{(1)}(\eta_1)C_1 &= 0 \end{aligned}\right\} \tag{47}$$

where $U^{(1)}$, $V^{(1)}$, and $P^{(1)}$ are defined by

$$U^{(1)} = (\tilde{\alpha}\lambda)^{1/3}\left[(\cos \tau\eta_1)\int_0^{\eta_1} \eta^{1/2}(\cos \tau\eta)K_{\frac{1}{3}}\, d\eta + (\sin \tau\eta_1)\int_0^{\eta_1} \eta^{1/2}(\sin \tau\eta)K_{\frac{1}{3}}\, d\eta\right]$$

$$V^{(1)} = (\tilde{\alpha}\lambda)^{1/3}\left[(\sin \tau\eta_1)\int_0^{\eta_1} \eta^{1/2}(\cos \tau\eta)K_{\frac{1}{3}}\, d\eta - (\cos \tau\eta_1)\int_0^{\eta_1} \eta^{1/2}(\sin \tau\eta)K_{\frac{1}{3}}\, d\eta\right]$$

$$P^{(1)} = \frac{\rho_o(0)_\alpha}{\tau}\left[-\tau\eta_1 K_{\frac{2}{3}}\left(\frac{2}{3}\eta_1^{3/2}\right) + (\tilde{\alpha}\lambda)^{-1/3}V^{(1)} - \tau\eta_1(\tilde{\alpha}\lambda)^{-1/3}U^{(1)}\right]$$

In order that linear system (47) will admit a nontrivial solution for C_1, C_2, and A, it is both necessary and sufficient that the determinant

$$\begin{vmatrix} 1 & \left(k - \frac{1}{\lambda R}\right)\sigma & U^{(1)} \\ \beta & \left(\frac{k\sigma^2}{\lambda} + \frac{1}{R}\right) & V^{(1)} \\ -\tilde{k} & \frac{1}{3}k\sigma & P^{(1)} \end{vmatrix} = 0 \qquad (48)$$

vanish. This equation serves to determine all proper values of λ. After λ has been determined, the constants C_1 and C_2 corresponding to this λ can be solved for in terms of A, namely

$$\left.\begin{aligned} C_1 &= A\,\frac{\sigma\left(k - \frac{1}{\lambda R}\right)\beta - \left(\frac{k\sigma^2}{\lambda} + \frac{1}{R}\right)}{\sigma\left(k - \frac{1}{\lambda R}\right)V^{(1)} - \left(\frac{k\sigma^2}{\lambda} + \frac{1}{R}\right)U^{(1)}} \\ C_2 &= A\,\frac{\beta U^{(1)} - V^{(1)}}{\sigma\left(k - \frac{1}{R\lambda}\right)V^{(1)} - \left(\frac{k\sigma^2}{\lambda} + \frac{1}{R}\right)U^{(1)}} \end{aligned}\right\} \tag{49}$$

where the constant A remains to be determined.

To solve determinantal equation (48) analytically even for this simplest case does not seem to be possible. The procedure from here on is essentially numerical. For the present purpose, the numerical solution will be carried out, based on the approximation that η_1 will be taken mathematically as infinity and $\tau\eta_1$, small. Accordingly, the integrals in $U^{(1)}$, $V^{(1)}$, and $P^{(1)}$ will have an infinite upper limit and $\cos \tau\eta_1 \approx 1$ and $\sin \tau\eta_1 \approx \lambda y_1$. Furthermore, for large values of η_1 the Bessel function $K_{\frac{1}{3}}\left(\frac{2}{3}\eta_1^{3/2}\right) \approx \eta_1^{-3/4} e^{-\frac{2}{3}\eta_1^{3/2}}$ will be very small and can, therefore, be neglected. Equation (48) is finally expressed explicitly in terms of the parameters R, β, and α. To retain terms up to the order $O\left(R^{-5/6}\right)$, the determinantal equation becomes

$$\beta y_1 \lambda^2 - \tilde{\alpha}^{1/3} r y_1 \lambda^{4/3} + \lambda + \rho_o^{(0)} \alpha\beta = 0 \tag{50}$$

where $r = \Gamma\left(\frac{1}{3}\right)\Big/3^{2/3}$.

For $M = 2$, $R = 774$, and $\alpha = 0.332$, there are two pairs of positive and negative roots. The negative roots would make the perturbation infinite at negative infinity, which is contrary to assumption. Therefore, negative roots are not admitted. For the pair of positive roots, one is roughly 20 times the other which is 0.0467. For large Reynolds number, these would correspond to $\lambda_1 \approx \beta^{-3/2} R^{1/2}$ and $\lambda_2 \approx \beta^{3/4} R^{-1/4}$.

In region 1, x is negative. The disturbance with a logarithmic rate of decay λ_1 will quickly disappear. The observed disturbance must have a decay rate equal to λ_2.

According to this solution, the dependence of the distance influenced by pressure disturbance on the Mach number and Reynolds number can be discussed. By the form of the first-order solution, the pressure decays exponentially from the point of incidence. For a given value of ϵ, when $-\lambda_2 X = d$, d being a constant, the pressure disturbance would have dropped to a certain fraction of its initial value. Thus, $-X = d/\lambda_2$ would serve as a measure of the distance reached by the pressure. Therefore, by varying M and R, the distance X'/L will change according to the law $\beta^{-3/4}R^{-3/4}$. Namely, by increasing both Mach number and Reynolds number, the distance reached by pressure disturbances will decrease. This result is quantitatively different from that arrived at in reference 1, which is $\beta^{-1/2}R^{-1/2}$. By comparison there is an increase of both compressibility and viscous effects in the new result. This shows how gross an error can be made if the boundary-layer approximation is applied. It is surprising also that the upstream pressure propagation depends only on M and R but, to the first order at least, is independent of the shock strength. This seems to be in agreement with experiment (reference 6).

It can be shown from equations (33) and (34) that for arbitrary values of A the pressure $p^{(1)}\left(x_{sep},0\right)$ at the point of separation where $u_y = 0$ is proportional to $\beta^{-1/2}R^{-1/2}$.

FIRST-ORDER SOLUTION - DOWNSTREAM $x > 0$

As in the upstream first-order solution, one must start with the potential solution. In order to determine the flow in region 2, the interaction between the incoming wave ${}_1\varphi^{(1)}$ and the shock must be considered.

Potential Solution

In equation (27) the incoming wave is given by

$${}_1\varphi^{(1)} = \frac{A}{\lambda} e^{\lambda(x-\beta y)}$$

When this train of waves hits the shock, the shock will be slightly modified according to Rankine-Hugoniot conditions. The velocity potential in region 2 with these conditions satisfied is shown to be (appendixes A and B)

$$ {}_2\varphi^{(1)} = \frac{1}{\beta}(x + \beta y) + \frac{A}{\lambda} e^{\lambda(x-\beta y)} \tag{51} $$

That is, to the first order, the flow in region 2 is simply a superposition of a step wave upon the transmitted wave. Since the conditions are prescribed on the shock OS, they are uniquely defined in region 2, terminated only by the extreme Mach line OM.

To continue this solution into region 3, two possibilities present themselves. It may be either continuous or discontinuous there. If it were continuous on OM, there would be a discontinuity in pressure at the point of incidence. In the case of inviscid flow, this, of course, would be admissible. But, by the condition that the viscous layer will join smoothly with the potential field, this would make the velocity $u^{(1)}(x,y)$ discontinuous at the origin $x = 0$. Hence, the first possibility must be discarded. If the solution is discontinuous at OM, and the pressure as well as the velocity $u^{(1)}$ are continuous at the origin, then the discontinuity must correspond to an expansion. This, of course, is what has been observed.

Assuming that the pressure returns to the value just in front of the shock, a simple calculation shows that the turn of the flow through a Prandtl-Meyer expansion has to be

$$ -\epsilon\beta\left({}_1u^{(1)}(0) - {}_2u^{(1)}(0)\right) $$

where ${}_1u^{(1)}(0)$ and ${}_2u^{(1)}(0)$ are, respectively, the velocity $u^{(1)}$ just before and after the shock. By means of solutions (27) and (51), the turn required is ϵ. Now the direction of the flow before the expansion is $\epsilon(1 - \beta A)$; therefore the total inclination of the velocity vector at OM is

$$ \epsilon(2 - \beta A) $$

A solution in region 3 subject to these conditions is found to be (appendix B):

$$ {}_3\varphi^{(1)} = \frac{1}{\beta}(x + \beta y) - \frac{1}{\beta}(x - \beta y) + \frac{A}{\lambda} e^{\lambda(x-\beta y)} \tag{52} $$

The velocity in region 3 can then be given by

$$_3u^{(1)} = Ae^{\lambda(x-\beta y)}$$

$$_3v^{(1)} = 2 - \beta Ae^{\lambda(x-\beta y)}$$

Were the perturbation velocity to remain finite at infinity, λ would have to be negative in region 3, as $x - \beta y \geqq 0$.

Viscous Solutions

As potential solution (52) in region 3 is a superposition of two different types of waves, namely, step waves and transmitted waves, it is expected that the viscous solution will also be composed of two different parts, each associated with a special wave in the external field. Since the origin and character of the disturbances are different, they can best be discussed separately as the following:

In view of the fact that the step waves are opposite in sign, they do not change the pressure but give rise to a uniform vertical velocity in the potential field. The flow in the viscous layer due to this uniform deflection may vary rapidly in the y-direction but certainly not in the x-direction because of the constancy of pressure. The main effect of the step waves, first of all, then, is to produce a constant deflection along the edge of the viscous layer. If the velocities due to step waves are $u_s^{(1)}$ and $v_s^{(1)}$, the problem should be solved subject to the boundary conditions:

$$\left.\begin{aligned} &u_s^{(1)} = v_s^{(1)} = 0 && \text{when } y = 0 \\ &u_s^{(1)} = 0, \quad v_s^{(1)} = 2 && \text{at outer edge of viscous layer} \\ &u_s^{(1)} = 0 && \text{when } x = 0, \quad y \geqq 0 \end{aligned}\right\} \qquad (53)$$

It must be borne in mind that generally the flow in region 1 has been separated, with a considerable region of backflow in the neighborhood of the point of incidence; the resultant flow in the incompressible layer must be small. If $u_s^{(1)}$ and $v_s^{(1)}$ are considered as additional

perturbations due to the step waves, the problem can then be simplified by neglecting the inertia forces in the neighborhood of the plate. The perturbation stream function $\psi_s^{(1)}$ then satisfies

$$\Delta \Delta \psi_s^{(1)} = 0 \tag{54}$$

in the incompressible layer $0 \leqq y \leqq y_1$.

As the objective here is to demonstrate approximately the effect of the step waves on the viscous layer, only simple solutions will be considered. A simple solution that satisfies both the first and third of conditions (53) is clearly

$$\psi_s^{(1)} = D\left(\frac{y}{y_1}\right)^2\left(1 - \frac{2}{3}\frac{y}{y_1}\right)x \tag{55}$$

for y in $0 \leqq y \leqq y_1$, where D is a constant. The velocities are thereby

$$\left.\begin{aligned} u_s^{(1)} &= 2D\,\frac{y}{y_1}\left(1 - \frac{y}{y_1}\right)\frac{x}{y_1} \\ v_s^{(1)} &= -D\left(\frac{y}{y_1}\right)^2\left(1 - \frac{2}{3}\frac{y}{y_1}\right) \end{aligned}\right\} \tag{56}$$

Accordingly, the pressure, or the pressure gradients, in the incompressible layer due to the step waves is of the order R^{-1} and for the present approximation will be neglected.

In the compressible layer where the viscous effect is not as important as the compressibility effect, a flow that does not associate with a pressure rise can either be a uniform field or the one with a vertical velocity varying linearly as y. According to equation (37), the following, with the second of equations (53) satisfied, is a solution for $y_1 \leqq y \leqq y_2$, y_2 being defined as the outer edge of the compressible layer:

$$\left.\begin{aligned} u_s^{(1)} &= 0 \\ v_s^{(1)} &= 2\left(\frac{y}{y_1}\right) \end{aligned}\right\} \qquad (57)$$

The fact that equation (57) is a solution shows the importance of both the compressibility and the viscosity in this layer, which allow the independent variation of $\rho^{(1)}$ and $T^{(1)}$. The joining condition at the interface gives the constant D a value $-6y_1/y_2$.

Thus, it is seen that the deflection of the flow due to step-waves, though it contributes no pressure, induces a forward velocity in the incompressible layer. Since it increases linearly with x, the back-flow should be expected to be reduced in the downstream direction. Of course, the possibility exists that an additional pressure might also enter if the inertia forces, though small, were not neglected. Nevertheless, it can be safely stated that the pressure is always of the secondary importance in this case and hence the above conclusions will remain valid.

On the other hand, the perturbation velocities $u_t^{(1)}$ and $v_t^{(1)}$ due to the transmitted wave, according to equation (52), are subject to boundary conditions

$$\left.\begin{aligned} u_t^{(1)} = v_t^{(1)} &= 0 && \text{when } y = 0 \\ u_t^{(1)} = Ae^{\lambda x} \qquad v_t^{(1)} &= -\beta Ae^{\lambda x} && \text{when } y = \infty \\ u_t^{(1)} &= u^{(1)} && \text{when } x = 0,\ y \geqq 0 \end{aligned}\right\} \qquad (58)$$

To solve this problem, the viscous layer is again subdivided into incompressible and compressible layers. Since the boundary conditions and the differential equations are the same, $u_t^{(1)}$ and $v_t^{(1)}$ will have the same forms as those given in equations (33) and (44), with eigenvalue λ satisfying equation (48). Now if the perturbation is required to vanish at positive infinity, λ should be negative. But were λ negative, solutions (33) and (44) would be highly oscillatory, as the

arguments of both Bessel and the exponential functions involve a factor $\sqrt{\lambda R}$. That is, the solution of the incompressible layer would involve $J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{\rho_o^{(0)}\alpha\lambda R}\cdot y^{3/2}\right)$ and that of the compressible layer would have $\cos\sqrt{\lambda R}\,y$ and $\sin\sqrt{\lambda R}\,y$. For large Reynolds number, these types of solutions for a steady laminar flow must be rejected as impossible. By the same reason, complex roots are excluded. The only alternative is to accept the positive λ.

If λ is positive, it follows that the velocity $u_t^{(1)}$ and pressure $p_t^{(1)}$ will continue to increase until the process of linearization breaks down. This would imply that in the potential field the train of compression waves, immediately following the expansion, will grow exponentially with x. The flow direction as well as the curvature of the streamlines will also increase sharply. For such a flow, it is well-known that the Mach waves will converge and form an envelope. Therefore, in region 3 a shock eventually must be developed. On the other hand, in the viscous field, as $u_t^{(1)} + u_s^{(1)}$ for any constant y will increase with x, backflow, though slightly reduced by the step waves, will become stronger in the downstream direction. Now, it has been well-established that the laminar velocity profile with a point of inflection is highly unstable. As the pressure continues to grow at a Reynolds number generally above the critical value, transition must occur after a critical pressure gradient is reached. The flow from there on cannot be theoretically studied without considering unsteady flow.

It is therefore concluded that in the case of an incident compression shock, laminar flow is not possible for the whole viscous layer and transition always occurs. This appears to be in complete agreement with the present available experimental observations.

Character of Flow After Transition

Although the flow from a certain point on is unknown, the interesting fact is, however, that the flow up to the point of transition is very insensitive to what happens beyond the point of transition. In the previous sections, the problem has been reduced to depend only on one constant A on which the quantitative behavior, but not the character of the flow, depends. A quite similar conclusion has also been reached in reference 1 and seems to be an experimental fact (reference 7). In one of his experiments, Liepmann introduced an expansion wave immediately after the incident shock, and the observed upstream flow field was practically unchanged. Therefore, in order to account

for the observed pressure distribution, say it is sufficient just to take approximately the effect of the flow downstream of the point of transition.

Assume that the second shock starts at the edge of the viscous layer and, by linear theory, has the same strength ϵ. Inasmuch as the direction of the flow on OM is constant and equal to $\epsilon(2 - \beta A)$, if the flow after the shock TR (fig. 2) is parallel to the plate, the direction in front of the shock at large distance must be ϵ. This condition gives $A = \beta^{-1}$. Moreover, as the velocity vector is turning away from the plate, the pressure continues to rise and will be stopped only by transition. In that event, the sudden increase of shear of the viscous layer will thicken the viscous layer. This thickening will make the flow expand and consequently a drop of pressure will ensue. After this, the streamlines will gradually level off and the reattachment of the partially turbulent layer, if not yet accomplished prior to the transition point, will be certain to follow. If the point of transition is chosen to coincide with the point where $v = 0$ along the boundary streamline, the location is then determined by $\lambda x_1 = \log_e 2$.

For values of x greater than x_1, the flow in the immediate neighborhood of transition would have a greater influence than that far downstream where the flow is more uniform. Since the flow in the transition region is of a boundary-layer type, namely, the backflow ceases to be a factor, it can again be approximately represented by the integrated effects, such as the momentum and pressure. If the flows before and after the point of transition have the same pressure and total momentum, the dynamical equilibrium can then be maintained. According to the solution given in reference 1, the pressure distribution in the transition region is approximately

$$\frac{2}{\beta} + K_1 e^{\lambda_1 x} + K_2 e^{\lambda_2 x}$$

where λ_1 and λ_2 are negative constants, being functions of M and R and K_1 and K_2 are integration constants. By the conditions that at $x = x_1$ the pressure and its derivative are continuous, K_1 and K_2 can then be determined.

It must be emphasized again that the conditions stated in this section are tentative. No obvious reasons beyond the ones outlined at the beginning of the section can be given at this moment.

NUMERICAL EXAMPLES

As a numerical example, the case of $M = 2$ and $R = 774$ is presented. First, the determinantal equation (50) is solved numerically for only one Mach number 2 but at different Reynolds numbers. The results are shown in figure 3. At this Reynolds number, λ is 0.0467. The velocity $u(x,y)$ and the surface pressure $p(x,0)$ for this λ are calculated. It is seen that for the shock strength of $\epsilon = -1^{\circ}$, the boundary layer first separates at $x_{sep} = 1.106$ (fig. 4(a)) and, subsequently, backflow sets in. When the shock strength is increased to $\epsilon = -3^{\circ}$, say, the separation occurs at a much earlier station, namely at $x_{sep} = -22.4$. The region of backflow is proportionally wider (fig. 4(b)). However, if the step waves are taken into account, then in the case of $\epsilon = -1^{\circ}$ there would be no separation (dashed curve in fig. 4(a) for $y_1 = 3$ and $y_2 = 9$); whereas in the case of $\epsilon = -3^{\circ}$ the flow at the same location remains separated but the backflow has been very much reduced (dashed curve in fig. 4(b)). These results show quite the same characteristics as the experimental measurements by Ackeret, Feldman, and Rott (reference 8) of the velocity distribution over a curved plate.

The pressure distributions over the plate for these cases are shown in figure 5. For the weaker shock $\epsilon = -1^{\circ}$, surprisingly good agreement with experiment (reference 6) is obtained. This very fact seems to justify the present assumptions regarding the structure of the viscous layer. In the case of stronger shock, for example, $\epsilon = -3^{\circ}$, there is, however, a distinct difference between theory and experiment. Theoretically, the pressure would still decay exponentially upstream but would begin with a larger amplitude. Experimentally, it was found, strangely enough, that, over a considerable range of the upstream disturbed region, the pressure first decreases very slowly and then decays more or less exponentially. This "pressure bump" seems to be characteristic of the pressure distribution in the case of interaction between stronger shocks and the laminar boundary layer. This bumpy character in the pressure distribution, by all evidences, must be attributed to the nonlinear effect of the flow. This will be exhibited in the following section.

APPRAISAL OF HIGHER-ORDER EFFECTS

It has been shown that, if there is no backflow in region 1, the pressure disturbances will decay exponentially. When the shock strength increases, however, the pressure in the disturbed region becomes much higher and drops much more slowly than that predicted by the theory. This appears to be due to the fact that when backflow develops, there

will be an underestimate of the perturbed velocity and, consequently, a much lower pressure.

To estimate the effect due to higher-order terms, a second-order solution is found and is given in appendix C. According to the second-order solution, the correction terms for wall pressure would be

$$P_1 e^{\lambda x} + P_2 e^{2\lambda x}$$

In view of the fact that there is an overestimation of the first-order pressure by the condition $A = \beta^{-1}$, the correction term is expected to be negative for small values of x and positive far upstream. Then $P_1 > 0$ and $P_2 < 0$. Since $e^{2\lambda x}$ increases faster than x^2 and $e^{\lambda x}$, when x is positive, the correction would be negatively large far downstream. It has been pointed out previously that the iteration process fails for positive values of x; this solution shows further that it diverges oscillatorily. In order that the solution will behave properly, it will have to be carried to an odd order, such as one or three. To third order, the correction for wall pressure would be

$$P_1 e^{\lambda x} + P_2 e^{2\lambda x} + P_3 e^{3\lambda x}$$

If $P_1 = 0.75$, $P_2 = -1.83$, and $P_3 = 1$, the wall-pressure distribution for $\epsilon = -3^\circ$ would resemble the curve as shown in figure 6, which does exhibit the same character as measured by experiments. This, of course, cannot be considered as conclusive evidence, but at least it shows that higher-order terms such as given above do have the possibility of accounting for the observed behavior of the wall-pressure distribution. It is very important that this step be carried out.

DISCUSSION

This study being intended as an exploratory study, the numerical results obtained are not expected to be exact, but only accurate enough to insure the conclusions. As far as the first-order solution is concerned, the entire problem depends on the determination of one parameter λ, which was calculated on the basis of two main approximations:

(a) the contribution of the integral $\int_{\eta_1}^{\infty} \eta^{1/2} K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right) d\eta$ is negligible

and (b) λy_1 is small such that terms of order $(\lambda y_1)^2$ and higher can be neglected. The subsequent solution for λ satisfies this criterion. The question remains, however, whether $\lambda y_1 = n(1)$ is also a solution or not. If $\lambda y_1 = n(1)$, the integrals $\int_0^{\eta_1} \eta^{1/2} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \tau\eta \, K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right) d\eta$ would require a more elaborate evaluation. The possibility exists and is worth testing.

As from the outset the boundary-layer approximation is rejected, it is of interest to examine the character of the perturbed flow in the light of boundary-layer theory. According to the boundary-layer approximation, if the velocity $u' = O(U_\infty)$ and $\partial/\partial x' = O(L^{-1})$, then the velocity $v' = O(U_\infty R^{-1})$ and $\partial/\partial y' = O(L^{-1}R)$. Now, from the character of the solution for perturbed flow in the present theory, $\partial/\partial x' = O(L^{-1}R^{3/4})$ and $\partial/\partial y' = O(L^{-1}R^{5/4})$. By comparison, the derivatives for the perturbed flow are much larger than the corresponding derivatives of boundary-layer theory. Moreover, from the equation of continuity the ratio $v^{(1)}/u^{(1)}$ of the perturbed velocities is $O(R^{-1/2})$. As in the vicinity of the wall $u^{(1)} \approx u^{(0)}$, $v^{(1)}$ is much larger than the vertical component in the boundary layer. Since in this case $\partial/\partial x' < \partial/\partial y'$ and $v^{(1)} < u^{(1)}$, a set of equations analogous to those for the boundary-layer flow can be deduced as long as backflow does not take place. Therefore, for expansion and even weak compression incident waves, a much simpler problem would be feasible.

Finally, it might also be noted that, owing to the assumption that the flow far away from the plate is inviscid and irrotational, there is introduced a sharp discontinuity in higher derivatives at the demarcation line between the two flow fields. Because of the presence of shock, however, a first-order discontinuity is also expected, because, by the assumption of frictionless flow at large distance, a smooth transition from small shock thickness at large distance to a larger one at the vicinity of the wall is precluded. The viscous-layer concept then idealizes the situation by taking the shock as a discontinuity in the outer field but continuous in the viscous flow. The effect of this is exhibited in a discontinuity in slope of the streamlines. This picture is entirely in agreement with the observed flow patterns.

According to the numerical example, the step waves tend to weaken the backflow downstream of the point of incidence but are unable to make

the flow reattach if the separation has occurred upstream of the point of incidence. This may very well be due to the poor approximation inherent in the Stokes flow which may underestimate the rate of change in the y-direction. If a more exact solution is given, reattachment of the separated flow might be accomplished in the laminar regime. This point should be considered as open.

CONCLUSIONS

The following conclusions are drawn from an analysis by the differential-equation method of the interaction of an oblique shock wave with a laminar boundary layer along a flat plate:

1. The pressure perturbation decays exponentially forward of the point of incidence and the distance of pressure propagation varies with different Mach and Reynolds numbers as $\beta^{-3/4}R^{-3/4}$ (where β is $\sqrt{M^2 - 1}$, M is Mach number, and R is Reynolds number) and very slowly with the shock strength.

2. If the shock strength is strong enough, separation of the flow always occurs. For given Mach and Reynolds numbers, the separation point depends strongly on the shock strength.

3. The pressure at the point of separation varies with Mach and Reynolds numbers as $\beta^{-1/2}R^{-1/2}$.

4. In the viscous layer, laminar flow is not possible everywhere, no matter whether the incident shock is strong or weak. In the distance of about two or three boundary-layer thicknesses, transition would occur.

5. The curvature of the streamlines after the shock is positive and the Mach waves in the potential field must coalesce to form a shock which approaches asymptotically the regularly reflected shock in the inviscid fluid. As its position depends on the point of transition, the exact location cannot be predicted by the present theory.

6. From the calculated pressure distribution over the wall, it is definitely proved that the observed overcompression of the wall pressure is a consequence of the positive curvature of the streamlines and the expansion is associated with transition.

7. The observed "bump" in the pressure distribution in region 1 for strong shocks is definitely a nonlinear effect and is an indirect consequence of separation.

Cornell University
Ithaca, N. Y., January 11, 1952

APPENDIX A

PERTURBATION OF AN OBLIQUE SHOCK

Let ${}_1u_{\parallel}$, ${}_1u_{\perp}$, p_1, and ρ_1 denote, respectively, the parallel and normal velocities with respect to an oblique shock, the pressure and density in front of the shock, and, by replacing 1 by 2, the corresponding quantities behind the shock. These variables are related to each other by the Rankine-Hugoniot conditions:

$$\left.\begin{aligned} &\rho_1\,{}_1u_{\perp} = \rho_2\,{}_2u_{\perp} \\ &{}_1u_{\parallel} = {}_2u_{\parallel} \\ &p_1 + \rho_1\,{}_1u_{\perp}^2 = p_2 + \rho_2\,{}_2u_{\perp}^2 \\ &\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}\,{}_1u_{\perp}^2 = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}\,{}_2u_{\perp}^2 \end{aligned}\right\} \tag{A1}$$

If the flow in front of shock is given, equations (A1) yield the following solutions:

$$\left.\begin{aligned} p_2 &= p_1 + \frac{2}{\gamma + 1}\rho_1\left({}_1u_{\perp}^2 - a_1^2\right) \\ \rho_2 &= \rho_1\frac{\gamma + 1}{\gamma - 1 + 2\left(\frac{a_1}{{}_1u_{\perp}}\right)^2} \\ {}_2u_{\perp} &= {}_1u_{\perp}\left(\frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\frac{a_1^2}{{}_1u_1^2}\right) \end{aligned}\right\} \tag{A2}$$

where a_1 is the speed of sound in front of the shock. If a_∞ and U_∞ are the speed of sound and velocity at infinity, then a_1 can be expressed in terms of velocities by the relation:

$$a_1^2 + \frac{\gamma - 1}{2}\left(u_1^2 + v_1^2\right) = a_\infty^2 + \frac{\gamma - 1}{2} U_\infty^2$$

Furthermore, if ω is the angle between the velocity (u_1, v_1) and the shock, it is easy to show that the Cartesian components before and after the shock are related by

$$\left.\begin{aligned} u_2 &= u_1 + \frac{2}{\gamma + 1}\left(u_1 \sin^2\omega + V_1 \sin\omega \cos\omega\right)\left(\frac{a_1^2}{q_1^2 \sin^2\omega} - 1\right) \\ V_2 &= V_1 - \frac{2}{\gamma + 1}\left(u_1 \cos\omega \sin\omega - V_1 \sin^2\omega\right)\left(\frac{a_1^2}{q_1^2 \sin^2\omega} - 1\right) \end{aligned}\right\} \quad (A3)$$

with $q_1^2 = u_1^2 + V_1^2$.

In the case of weak shock, the shock angle ω can be expanded in terms of the deflection angle ϵ. Then $\cos\omega$ and $\sin\omega$ are shown to be

$$\left.\begin{aligned} \cos\omega &= \frac{\beta}{M}\left\{1 + \frac{\gamma + 1}{4\beta^3} M^2\epsilon - \right. \\ &\qquad \left. \frac{(\gamma + 1)M^2}{4\beta^4}\left[\frac{(\gamma + 1)M^2}{8\beta^2}\left(2M^2 + 1\right) - 1\right]\epsilon^2 + \cdots\right\} \\ \sin\omega &= \frac{1}{M}\left\{1 - \frac{\gamma + 1}{4\beta} M^2\epsilon + \right. \\ &\qquad \left. \frac{(\gamma + 1)}{4\beta^2} M^2\left[\frac{(\gamma + 1)M^2}{8\beta^2}\left(M^2 + 1\right) - 1\right]\epsilon^2 + \cdots\right\} \end{aligned}\right\} \quad (A4)$$

Now since the velocity components u_1 and v_1 can be expanded:

$$u_1 = 1 + \epsilon u_1^{(1)} + \epsilon^2 u_1^{(2)} + \cdots$$

$$v_1 = \epsilon v_1^{(1)} + \epsilon^2 v_1^{(2)} + \cdots$$

If ω has the expansion:

$$\omega = \omega^{(0)} + \epsilon\omega^{(1)} + \epsilon^2\omega^{(2)} + \cdots$$

and $\omega^{(0)}$ is defined by equations (A4), then a straightforward reduction gives

$$u_2 = 1 + \epsilon u_2^{(1)} + \epsilon^2 u_2^{(2)} + \cdots$$

$$v_2 = \epsilon v_2^{(1)} + \epsilon^2 v_2^{(2)} + \cdots \tag{A5}$$

where

$$\left.\begin{aligned}
u_2^{(1)} &= \left[1 - \frac{4}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M}\right)\right]u_1^{(1)} + \frac{1}{\beta}\left[1 + \frac{4\beta^2\omega^{(1)}}{(\gamma+1)M^2}\right] \\
v_2^{(1)} &= v_1^{(1)} - \frac{4\beta}{\gamma+1}u_1^{(1)}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right) + \left[1 + \frac{4\beta^2\omega^{(1)}}{(\gamma+1)M^2}\right] \\
u_2^{(2)} &= \left[1 - \frac{4}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right)\right]u_1^{(2)} + \frac{1}{\beta}\left[1 + \frac{4\beta^2\omega^{(1)}}{(\gamma+1)M^2}\right]\left(u_1^{(1)} - \frac{v_1^{(1)}}{\beta}\right) + \\
&\quad \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right)\left[\left(u_1^{(1)}\right)^2 - \left(v_1^{(1)}\right)^2\right] + \frac{4\beta}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right)u_1^{(1)}v_1^{(1)} + \\
&\quad \left[-\frac{1}{\beta^2}\left(\frac{\gamma+1}{4}\frac{M^4}{\beta^2} - 1\right) - \frac{(M^2-2)}{\beta^2}\omega^{(1)} + \frac{4\beta\omega^{(2)}}{(\gamma+1)M^2} - \frac{2(M^2-2)}{(\gamma+1)M^2}\left(\omega^{(1)}\right)^2\right] \\
v_2^{(2)} &= -\frac{4\beta}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right)u_1^{(2)} + v_1^{(2)} + \frac{2\beta}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right)\left[\left(u_1^{(1)}\right)^2 - \left(v_1^{(1)}\right)^2\right] - \\
&\quad \frac{1}{\beta}\left(1 + \frac{4\beta^2\omega^{(1)}}{(\gamma+1)M^2}\right)\left(1 + \frac{\gamma-1}{2}M^2\right)u_1^{(1)} + \frac{1}{\beta}\left[1 + \frac{4\beta^2\omega^{(1)}}{(\gamma+1)M^2}\right]v_1^{(1)} - \\
&\quad \frac{4}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right)u_1^{(1)}v_1^{(1)} + \left[\frac{1}{\beta} + \frac{M^2+2}{\beta}\omega^{(1)} + \frac{4\beta^2\omega^{(2)}}{(\gamma+1)M^2} + \frac{2\beta(1+M^2)}{(\gamma+1)M^2}\left(\omega^{(1)}\right)^2\right]
\end{aligned}\right\} \tag{A6}$$

APPENDIX B

POTENTIAL FLOW IN REGION 2

If the flow in region 2 is irrotational, there exists a potential ${}_2\varphi(x,y)$ and, if the shock conditions are satisfied, it must possess the expansion:

$$ {}_2\varphi = x + \epsilon\, {}_2\varphi^{(1)} + \epsilon^2\, {}_2\varphi^{(2)} + \cdots \tag{B1} $$

For the first-order solution of the shock conditions, namely, on a line $x + \beta y = 0$,

$$ \left.\begin{aligned} {}_2\varphi_x^{(1)} &= \left[1 - \frac{4}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right)\right] u_1^{(1)} + \frac{1}{\beta}\left(1 + \frac{4\beta^2\omega^{(1)}}{(\gamma+1)M^2}\right) \\ {}_2\varphi_y^{(1)} &= v_1^{(1)} - \frac{4\beta}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M^2}\right) u_1^{(1)} + 1 + \frac{4\beta^2\omega^{(1)}}{(\gamma+1)M^2} \end{aligned}\right\} \tag{B2} $$

Now on the shock

$$ u_1^{(1)} = Ae^{2\lambda x} $$

$$ v_1^{(1)} = -\beta Ae^{2\lambda x} $$

If the perturbed shock angle is

$$ \omega^{(1)} = se^{\lambda(x-\beta y)} $$

the solution ${}_2\varphi^{(1)}$ must be

$$ {}_2\varphi^{(1)} = \frac{1}{\beta}(x + \beta y) + te^{\lambda(x-\beta y)} \tag{B3} $$

Substituting $\omega^{(1)}$ and ${}_2\varphi^{(1)}$ in equation (B2), the two equations yield uniquely

$$\left.\begin{aligned} s &= \frac{A}{\lambda} \\ t &= \frac{M^2}{\beta}\left(\frac{\gamma - 1}{2} + \frac{1}{M^2}\right)A \end{aligned}\right\} \tag{B4}$$

The solution in region 2 is thus a superposition of the transmitted waves $e^{\lambda(x-\beta y)}$ and a uniform step wave $x + \beta y$ introduced by external agency.

Since this solution will terminate on the Mach line OM, the solution in region 3 will be found by the condition that on $x - \beta y = 0$,

$${}_3u^{(1)} = {}_1u^{(1)} \tag{B5}$$

By this condition, the Prandtl-Meyer flow requires a deflection

$$\delta\vartheta = -\epsilon\beta\ {}_1u^{(1)}(0) - {}_2u^{(1)}(0) \tag{B6}$$

Solution (A3) gives $\delta\vartheta = \epsilon$. Consequently, the initial direction of the flow in region 3, to the first order, is

$${}_2v^{(1)}(0) + \delta\vartheta = (2 - \beta A)\epsilon \tag{B7}$$

APPENDIX C

SECOND-ORDER SOLUTION

Potential Field

If the first-order potential is of the form:

$$\varphi^{(1)} = \frac{A}{\lambda} e^{\lambda(x-\beta y)}$$

it can be shown that the second-order equation (19) admits a solution which reduces, at the edge of viscous layer $y = 0$, to

$$\left.\begin{aligned} u^{(2)} &= Be^{\lambda x} + Ce^{2\lambda x} \\ v^{(2)} &= -\beta Be^{\lambda x} - \beta\left[C + \frac{(\gamma+1)M^4A^2}{4\beta^2}\right]e^{2\lambda x} \end{aligned}\right\} \qquad (C1)$$

where B and C are arbitrary constants and the term $-\frac{(\gamma+1)M^4A^2}{4\beta} e^{2\lambda x}$ is derived from the particular integral $-\frac{(\gamma+1)M^2A^2}{4\beta} ye^{2\lambda(x-\beta y)}$. The reason that both $e^{2\lambda x}$ and $e^{\lambda x}$ are included in the complementary part of the solution is that the solution in region 2, according to equation (A6), contains both solutions.

Incompressible Layer

Assuming that, to the second order, the density in the inner layer remains constant, the system of equations (15) simplifies to

$$\Delta\Delta\psi^{(2)} - \tilde{\alpha} y \Delta\psi_x^{(2)} = R\left(\psi_y^{(1)}\Delta\psi_x^{(1)} - \psi_x^{(1)}\Delta\psi_y^{(1)}\right) \qquad (C2)$$

where the second-order stream function $\psi^{(2)}$ is defined as

$$u^{(2)} = \psi_y^{(2)}$$

$$v^{(2)} = -\psi_x^{(2)}$$

By substituting $\psi^{(1)}$ into the right-hand side of equation (C2), the particular integral can be found; this is

$$\psi_p^{(2)} = \left[\frac{\sin \tilde{\tau}\tilde{\eta}}{\tilde{\tau}(2\tilde{\alpha}\lambda)^{2/3}} \int_0^{\tilde{\eta}} (\cos \tilde{\tau}\tilde{\eta}) Z(\tilde{\eta})\, d\tilde{\eta} - \frac{\cos \tilde{\tau}\tilde{\eta}}{\tilde{\tau}(2\tilde{\alpha}\lambda)^{2/3}} \int_0^{\tilde{\eta}} (\sin \tilde{\tau}\tilde{\eta}) Z(\tilde{\eta})\, d\tilde{\eta} \right] e^{2\lambda x} \tag{C3}$$

where the function $Z(\tilde{\eta})$ stands for

$$Z(\tilde{\eta}) = \frac{2}{3}\tilde{\eta}^{1/2} I_{\frac{1}{3}}\left(\frac{2}{3}\tilde{\eta}^{3/2}\right) \left[\int_\infty^{\tilde{\eta}} \tilde{\eta}^{1/2} K_{\frac{1}{3}}\left(\frac{2}{3}\tilde{\eta}^{3/2}\right) \Lambda(\eta)\, d\tilde{\eta} \right] - \frac{2}{3}\tilde{\eta}^{1/2} K_{\frac{1}{3}}\left(\frac{2}{3}\tilde{\eta}^{3/2}\right) \left[\int_0^{\tilde{\eta}} \tilde{\eta}^{1/2} I_{\frac{1}{3}}\left(\frac{2}{3}\tilde{\eta}^{3/2}\right) \Lambda(\eta)\, d\tilde{\eta} \right] \tag{C4}$$

with

$$\Lambda(\eta) = \frac{1}{4}(2\tilde{\alpha}\lambda)^{4/3}\frac{C_1^2}{\rho_o(0)_\alpha}\left\{\eta^{1/2}K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right)\left[(\cos\tau\eta)\int_0^\eta \eta^{1/2}(\cos\tau\eta)K_{\frac{1}{3}}\,d\eta + (\sin\tau\eta)\int_0^\eta \eta^{1/2}(\sin\tau\eta)K_{\frac{1}{3}}\,d\eta\right] - \frac{1}{\tau}\eta K_{\frac{2}{3}}\left(\frac{2}{3}\eta^{3/2}\right)\left[(\sin\tau\eta)\int_0^\eta \eta^{1/2}(\cos\tau\eta)K_{\frac{1}{3}}\,d\eta - (\cos\tau\eta)\int_0^\eta \eta^{1/2}(\sin\tau\eta)K_{\frac{1}{3}}\,d\eta\right]\right\} \tag{C5}$$

and the variables:

$$\tilde{\eta} = (2\tilde{\alpha}\lambda)^{1/3}\left(y - \frac{2\gamma}{\tilde{\alpha}}\right) \tag{C6}$$

$$\tilde{\tau} = 2\lambda/(2\tilde{\alpha}\lambda)^{1/3} \tag{C7}$$

The general solution, consequently, can be written as

$$\psi^{(2)} = \psi_c^{(2)} + \psi_p^{(2)} \tag{C8}$$

where the complementary integral $\psi_c^{(2)}$, by the first-order solution, will be given by

$$\psi_c^{(2)} = C_3\psi^{(1)}(\eta)e^{\lambda x} + C_4\psi^{(1)}(\tilde{\eta})e^{2\lambda x} \tag{C9}$$

Since the form of $\psi^{(1)}(\eta)$ is known, and, furthermore, $\psi^{(1)}(0) = \psi^{(1)'}(0) = 0$, the solution $\psi^{(2)}$ is determined with two arbitrary constants C_3 and C_4.

With $\psi^{(2)}$ determined, the second-order pressure $p^{(2)}(x,0)$ on the plate can be written, similarly, as

$$p^{(2)}(x,0) = P_1 e^{\lambda x} + P_2 e^{2\lambda x} + P_p e^{2\lambda x} \tag{C10}$$

where

$$P_p = -\frac{(2\tilde{\alpha}\lambda)^{1/3} 3^{2/3}}{3\Gamma\left(\frac{1}{3}\right)(\lambda R)} \int_0^\infty \eta^{1/2} K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{3/2}\right) \Lambda(\eta)\, d\tilde{\eta}$$

and P_1 and P_2 take the same form as the symbol P for the first-order pressure $p^{(1)}$.

Compressible Layer

In the compressible layer, $\rho^{(0)} = u^{(0)} = T^{(0)} = 1$. Again, if $H^{(2)} = 0$, equations (15) become

$$\left.\begin{aligned}
&\left(u_y^{(2)} - v_x^{(2)}\right)_x - \frac{1}{R}\Delta\left(u_y^{(2)} - v_x^{(2)}\right) = \lambda\left(u^{(1)}\Delta G^{(1)} + v^{(1)}\Delta F^{(1)}\right) + \\
&\qquad\qquad v^{(1)}\Delta G_y^{(1)} - u^{(1)}\Delta F_y^{(1)} \\
&M^2 u_x^{(2)} - n^{(2)} - \frac{\gamma M^2}{R}\left(\Delta u^{(2)} + \frac{1}{3} n_x^{(2)}\right) = -\left(1 + \gamma M^2\right) v^{(1)} u_y^{(1)} + \\
&\left[2\beta^2 - \left(1 + \gamma M^2\right)\right] u^{(1)} v_y^{(1)} - 2\lambda\left(1 + \frac{\gamma - 1}{2} M^2\right)\left(u^{(1)}\right)^2 + \\
&\lambda M^2(\gamma - 1)\left(v^{(1)}\right)^2 - \frac{1}{\lambda}\left(v_y^{(1)}\right)^2 - \frac{1}{\lambda} v^{(1)} v_{yy}^{(1)}
\end{aligned}\right\} \tag{C11}$$

Similarly, by introducing $F^{(2)}$ and $G^{(2)}$ through the relations

$$u^{(2)} = F_x^{(2)} + G_y^{(2)}$$

$$v^{(2)} = F_y^{(2)} - G_x^{(2)}$$

there result the following equations:

$$\left.\begin{aligned}
&\Delta G_x^{(2)} - \frac{1}{R}\Delta\,\Delta G^{(2)} = \lambda\left(u^{(1)}\Delta G^{(1)} + v^{(1)}\Delta F^{(1)}\right) + \\
&\qquad v^{(1)}\Delta G_y^{(1)} - u^{(1)}\Delta F_y^{(1)} \\
&M^2\left(F_{xx}^{(2)} + G_{xy}^{(2)}\right) - \Delta F^{(2)} - \frac{\gamma M^2}{R}\left[\Delta\left(F_x^{(2)} + G_y^{(2)}\right) + \frac{1}{3}\Delta\,\Delta F^{(2)}\right] = \\
&-\left(1 + \gamma M^2\right)v^{(1)}u_y^{(1)} + \left[2\beta^2 - \left(1 + \gamma M^2\right)\right]u^{(1)}v_y^{(1)} - \\
&2\lambda\left(1 + \frac{\gamma - 1}{2}M^2\right)u^{(1)2} + \lambda(\gamma - 1)M^2v^{(1)2} - \frac{1}{\lambda}v_y^{(1)2} - \frac{1}{\lambda}v^{(1)}v_{yy}^{(1)}
\end{aligned}\right\} \quad (C12)$$

The solutions $F^{(2)}$ and $G^{(2)}$ can be written in an analogous manner. That is:

$$\left.\begin{aligned}
F^{(2)} &= F_C^{(2)} + F_p^{(2)} \\
G^{(2)} &= G_C^{(2)} + G_p^{(2)}
\end{aligned}\right\} \quad (C13)$$

Here $F_C^{(2)}$ and $G_C^{(2)}$ are known from the first-order solution and $F_p^{(2)}$ and $G_p^{(2)}$ are given as follows:

$$\left.\begin{aligned}
F_p^{(2)} &= \left(K_1 + K_2e^{-\sigma y} + K_3e^{-2\sigma y}\right)e^{2\lambda x} \\
G_p^{(2)} &= K_4e^{-\sigma y + 2\lambda x}
\end{aligned}\right\} \quad (C14)$$

where the constants K are defined by

$$K_1 = \frac{1 + \left(\beta^2/\beta_1^2\right)}{8\lambda\left(\beta_1^2 - \beta_2^2\right)} \frac{\left[4 + (\gamma - 1)M^2\right]\beta_1^2 + (\gamma - 1)M^2 + 2 + 2\beta_1^4}{1 + \frac{8}{3}\frac{\gamma M^2 \lambda}{R}} A^2 +$$

$$\frac{1 - \left(\beta^2/\beta_1^2\right)}{8\lambda\beta_2^2} \frac{\left[2\beta^2 - (\gamma - 1)M^2\right]\beta_1^2 + 2 + (\gamma - 1)M^2}{1 + \frac{8}{3}\frac{\gamma M^2 \lambda}{R}} A^2$$

$$K_2 = \frac{1 - (\beta/\beta_1)}{2\left(1 + \frac{8}{3}\frac{\gamma M^2 \lambda}{R}\right)}\Bigg[\gamma(\lambda\beta_1 - \sigma)M^2\left[\frac{2}{\gamma} - \frac{4\lambda^2 + (\lambda\beta_1 - \sigma)^2}{\lambda R}\right] \times$$

$$\frac{\left(\lambda^2\beta_1^2 - \sigma^2\right)\left[\frac{K\sigma^2}{\lambda^2} - R^{-1} + \beta_1\sigma(K + R^{-1})\lambda^{-1}\right]}{8\lambda^2(1 - 2\lambda R^{-1}) + (\lambda\beta_1 - \sigma)^2 + 2(1 - 4\lambda R^{-1}) - (\lambda R)^{-1}(\lambda\beta_1 - \sigma)^4} +$$

$$\beta_1(1 + \gamma M^2) - \frac{\sigma K R}{\lambda}\left\{4 + 2(\gamma - 1)M^2 + \beta_1^2\left[2 + (\gamma - 1)M^2 - \beta^2\right]\right\} +$$

$$\left(\frac{\sigma^2 K}{\lambda^2} - R^{-1}\right)\left(\lambda\beta_1\left(\beta_1^2 + \frac{\sigma^2}{\lambda^2} + 2M^2 + 2\right) + \sigma\left\{1 + \gamma M^2 + \beta_1^2\left[(\gamma - 1)M^2 - \beta^2\right]\right\}\right)\Bigg] \times$$

$$\frac{C_2 A}{(\lambda\beta_1 + \sigma)^2 - 4\lambda^2\beta_2^2} +$$

$$\frac{1 + (\beta/\beta_1)}{2\left(1 + \frac{8}{3}\frac{\gamma M^2 \lambda}{R}\right)}\Bigg(\frac{\gamma(\lambda\beta_1 + \sigma)M^2\left\{-\frac{2}{\gamma} + (\lambda R)^{-1}\left[4\lambda^2 + (\lambda\beta_1 - \sigma)^2\right]\right\}\left(\lambda^2\beta_1^2 - \sigma^2\right)\left[K\frac{\sigma^2}{\lambda^2} - \frac{1}{R} - \frac{\beta_1\sigma}{\lambda}(K + R^{-1})\right]}{8\lambda^2\left(1 - \frac{2\lambda}{R}\right) + 2(\lambda\beta_1 + \sigma)^2\left(1 - \frac{4\lambda}{R}\right) - (\lambda R)^{-1}(\lambda\beta_1 + \sigma)^4} -$$

$$\beta_1(1 + \gamma M^2) - \frac{\sigma K R}{\lambda}\left\{\beta_1^2\left[2 + (\gamma - 1)M^2 - \beta^2\right] + 4 + 2(\gamma - 1)M^2\right\} +$$

$$\left(K\frac{\sigma^2}{\lambda^2} - R^{-1}\right)\left\{-\lambda\beta_1\left(\beta_1^2 + \sigma^2 + 2M^2 + 2\right) + \frac{\sigma}{\lambda}\left\{1 + \gamma M^2 + \beta_1^2\lambda\left[(\gamma - 1)M^2 - \beta^2\right]\right\}\right) \times$$

$$\frac{C_2 A}{(\lambda\beta_1 + \sigma)^2 - 4\lambda^2\beta_2^2}$$

$$K_3 = -\left(1 + \frac{8}{3}\frac{\gamma M^2 \lambda}{R}\right)^{-1}\left\{(\gamma - 1)M^2 R\left(\frac{K\sigma^2}{\lambda^2} - R^{-1}\right) - \left[2 + (\gamma - 1)M^2\right]\frac{\sigma^2 K^2 R^2}{\lambda^3}\right\}C_2{}^2$$

$$K_4 = \frac{\left(\lambda\beta_1{}^2 - \sigma^2\right)(1 + \beta/\beta_1)\left[K\frac{\sigma^2}{\lambda^2} - R^{-1} - \frac{\beta_1\sigma}{\lambda}\left(K + \frac{1}{R}\right)\right]AC_2}{2\lambda\left[8\lambda^2(1 - 2\lambda R^{-1}) + 2(\lambda\beta_1 + \sigma)^2(1 - 4\lambda R^{-1}) - (\lambda R)^{-1}(\lambda\beta_1 + \sigma)^4\right]} +$$

$$\frac{\left(\lambda^2\beta_1{}^2 - \sigma^2\right)(1 - \beta/\beta_1)\left[K\frac{\sigma^2}{\lambda^2} - R^{-1} + \frac{\beta_1\sigma}{\lambda}\left(K + \frac{1}{R}\right)\right]AC_2}{2\lambda\left[8\lambda^2(1 - 2\lambda R^{-1}) + 2(\lambda\beta_1 - \sigma)^2(1 - 4\lambda R^{-1}) - (\lambda R)^{-1}(\lambda\beta_1 - \sigma)^4\right]}$$

and the constants K and β_2 stand for

$$K = -\frac{(\gamma - 1)\frac{M^2}{R}\lambda}{R + \left(\frac{4}{3}\gamma - 1\right)M^2\lambda}$$

$$\beta_2{}^2 = \frac{\beta^2 - \frac{8}{3}\frac{\gamma M^2 \lambda}{R}}{1 + \frac{8}{3}\frac{\gamma M^2 \lambda}{R}} \tag{C15}$$

The second-order solution now involves six arbitrary constants and, by the conditions at the interface, all but one can be determined.

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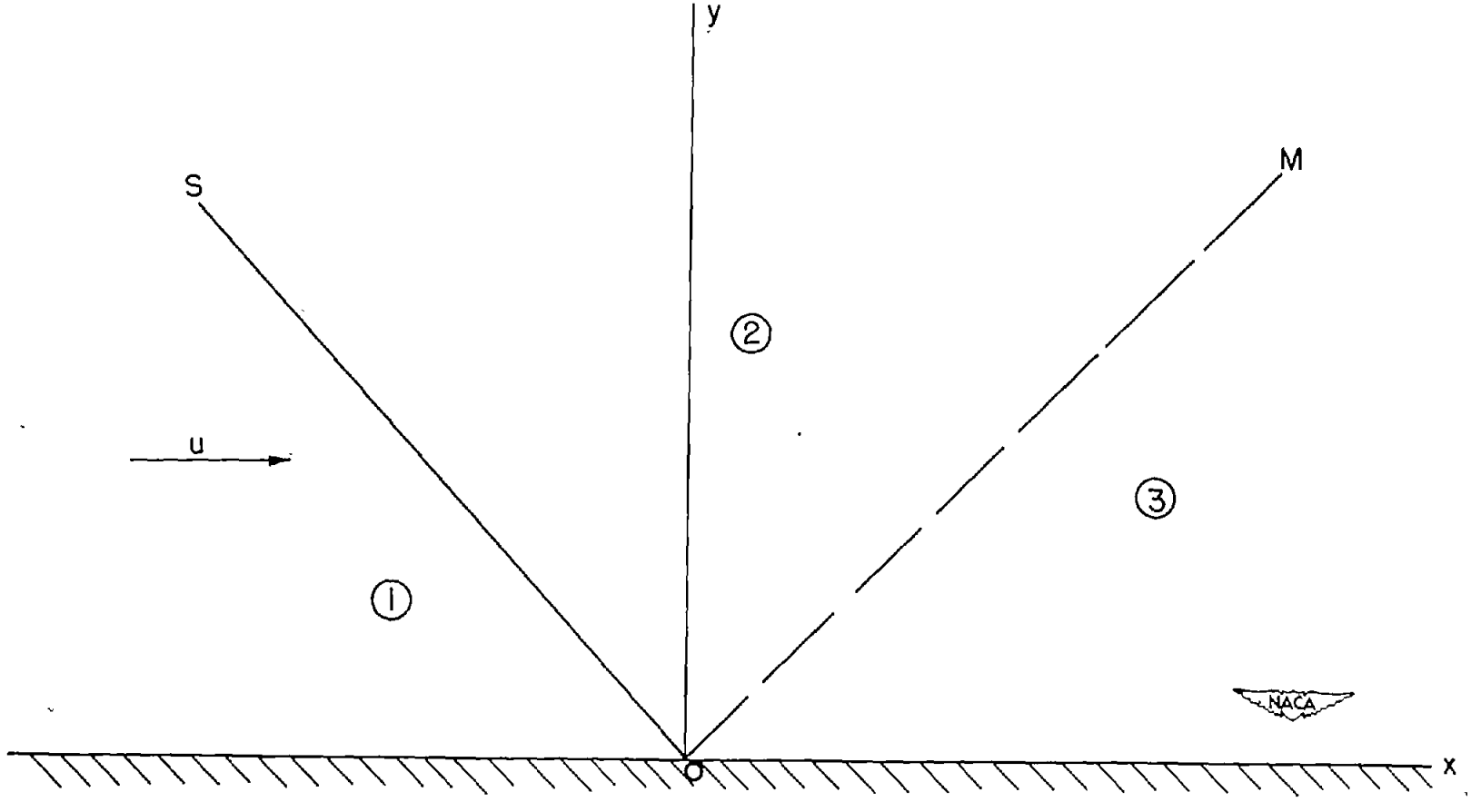


Figure 1.- Diagram of flow field.

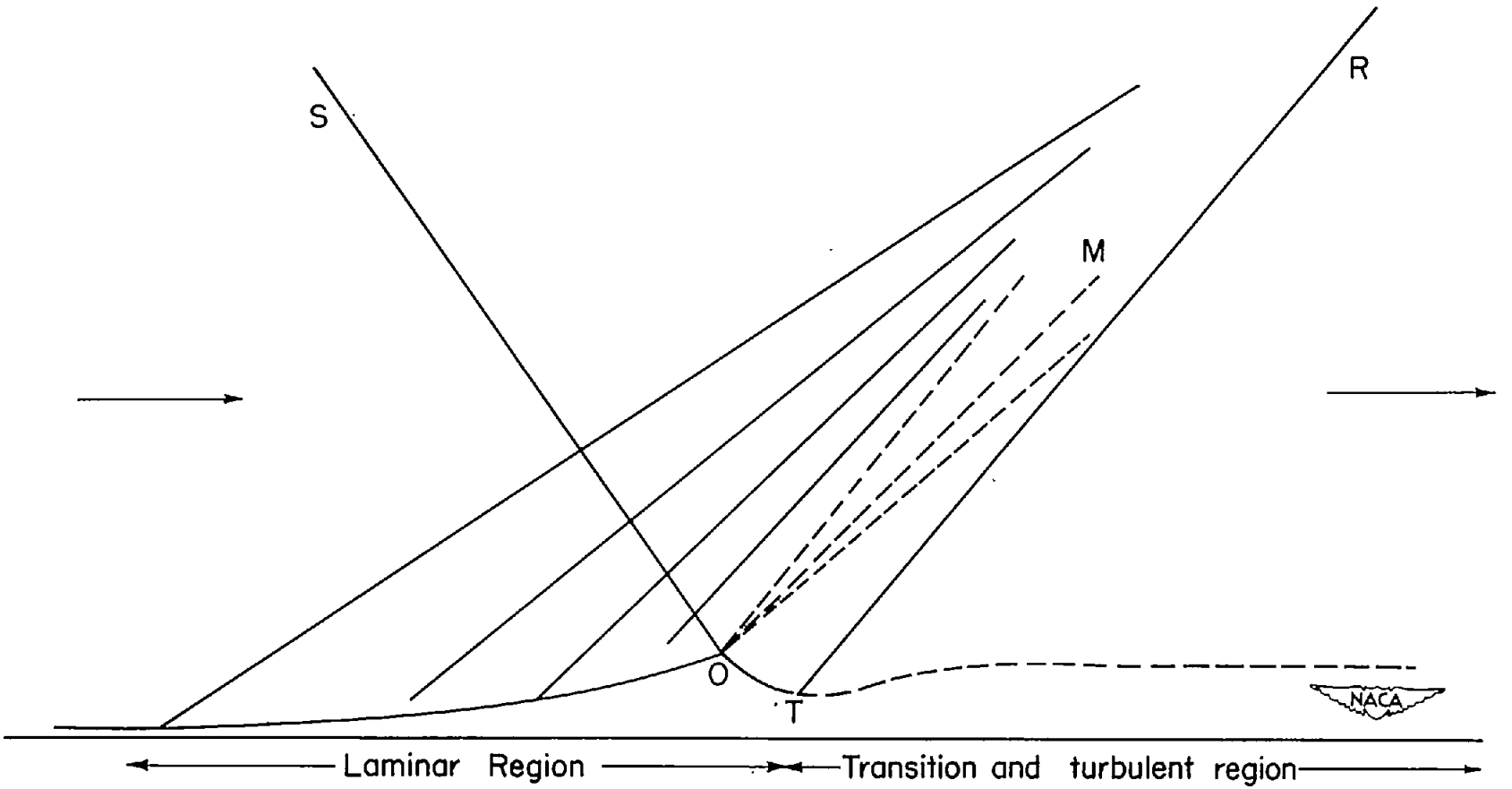


Figure 2.- Sketch of flow field according to first-order theory.

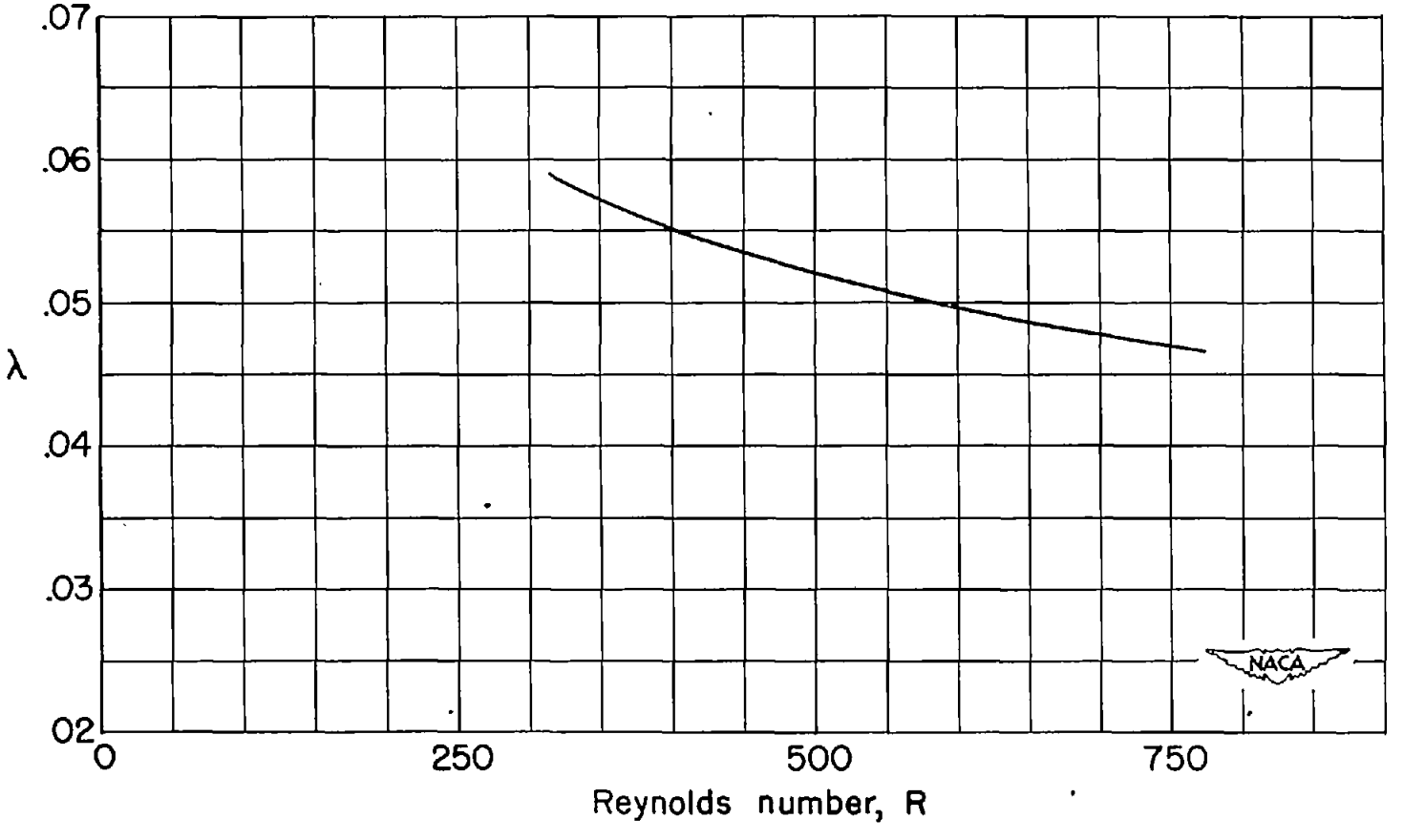
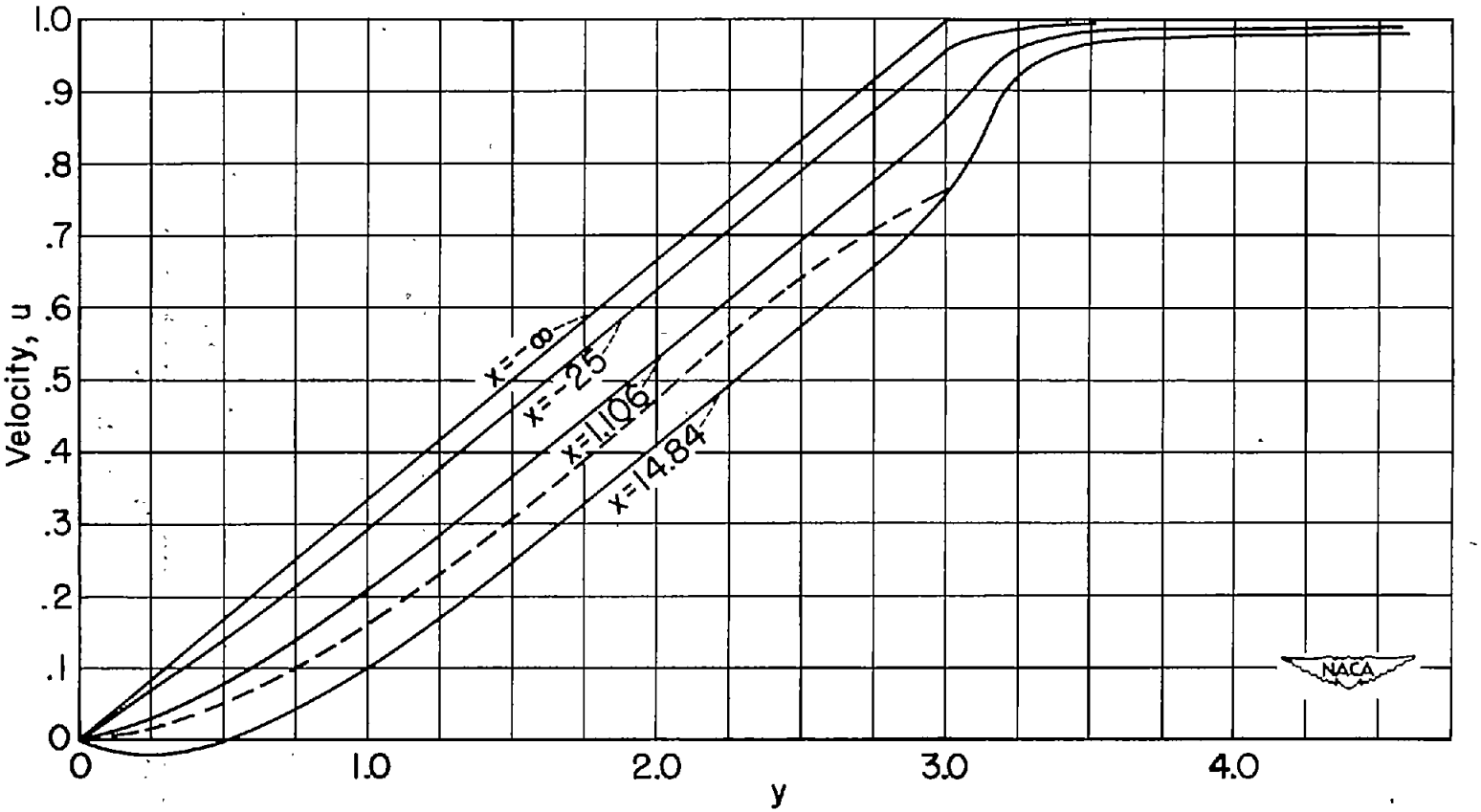
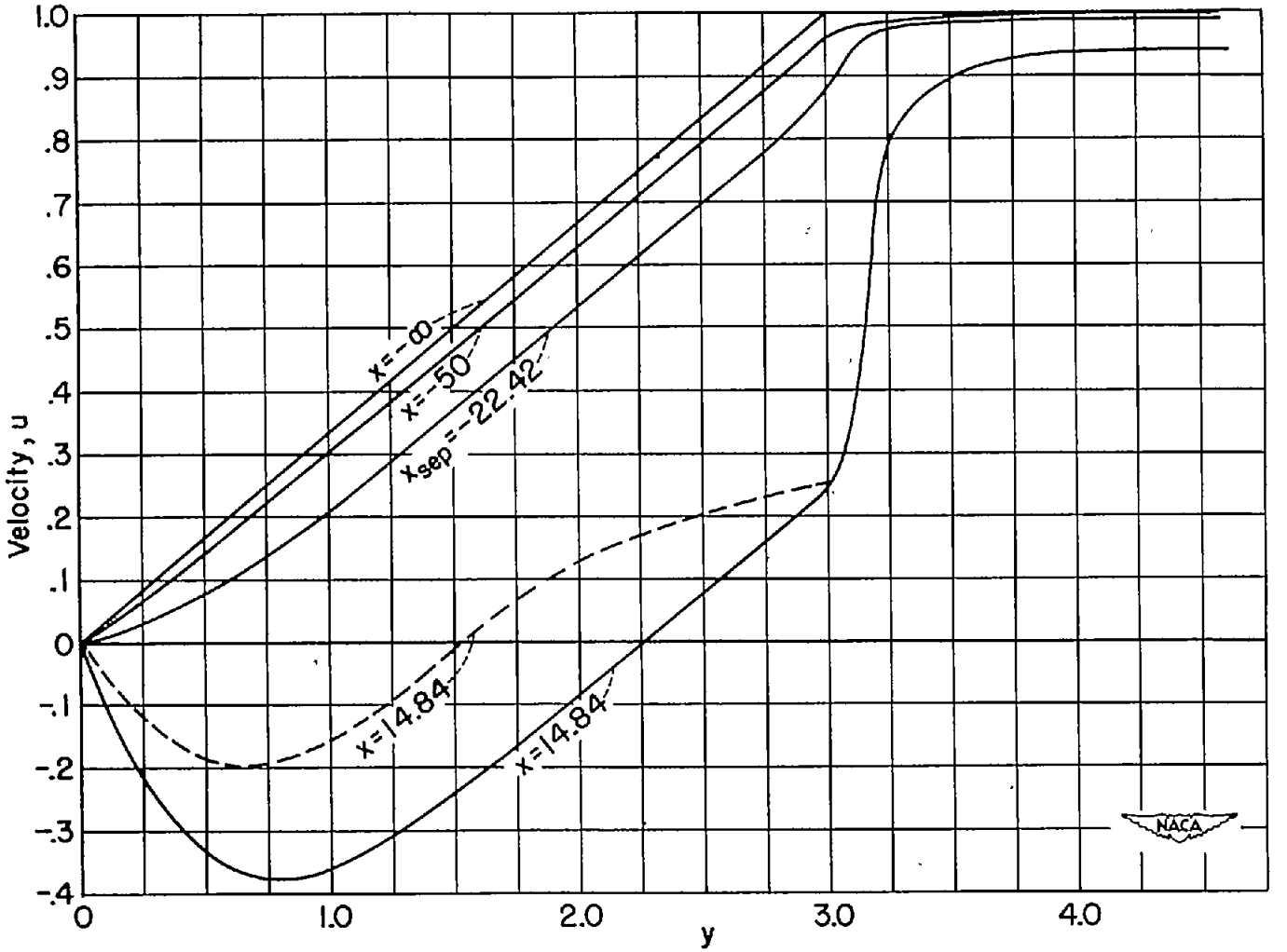


Figure 3.- Curve of λ against R for M = 2.



(a) $\epsilon = -1^{\circ}$; M = 2; R = 774.

Figure 4.- Velocity distribution.



(b) $\epsilon = -3^{\circ}$; M = 2; R = 774.

Figure 4.- Concluded.

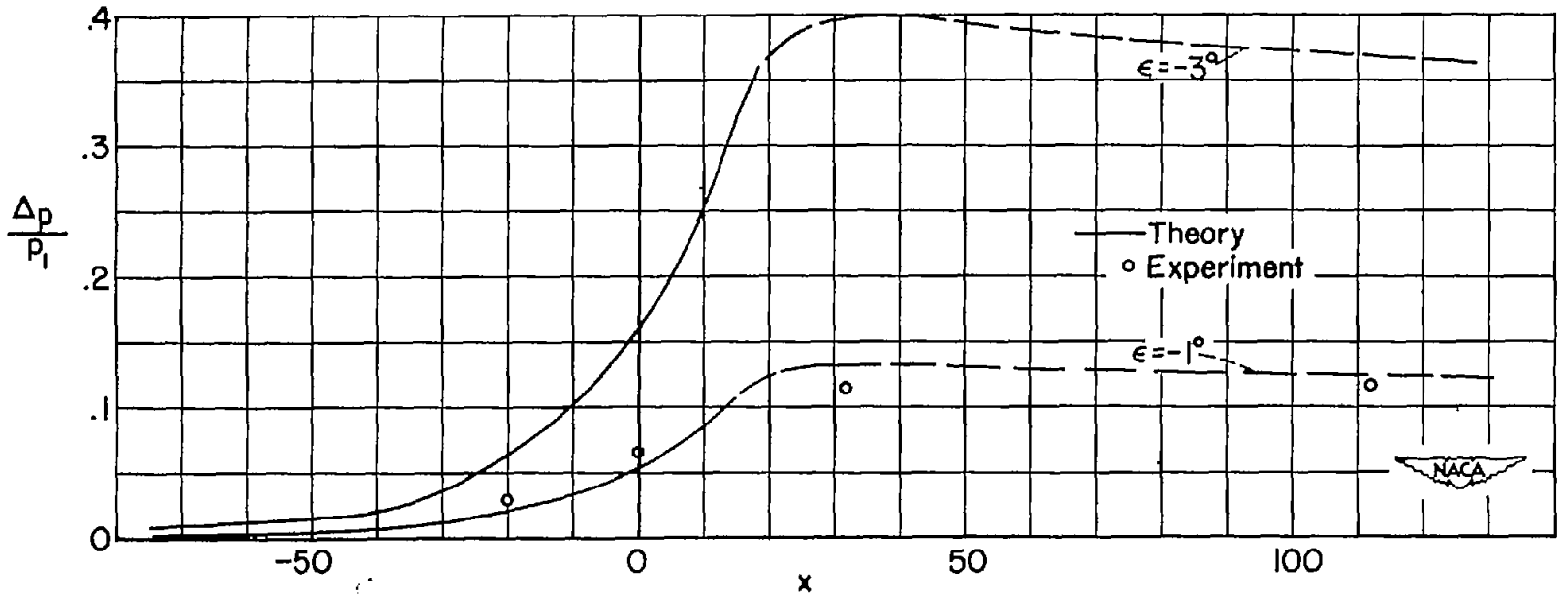


Figure 5.- Pressure distribution on plate for $\epsilon = -3°$ and $\epsilon = -1°$. M = 2; R = 774. Experimental values were taken from reference 6.

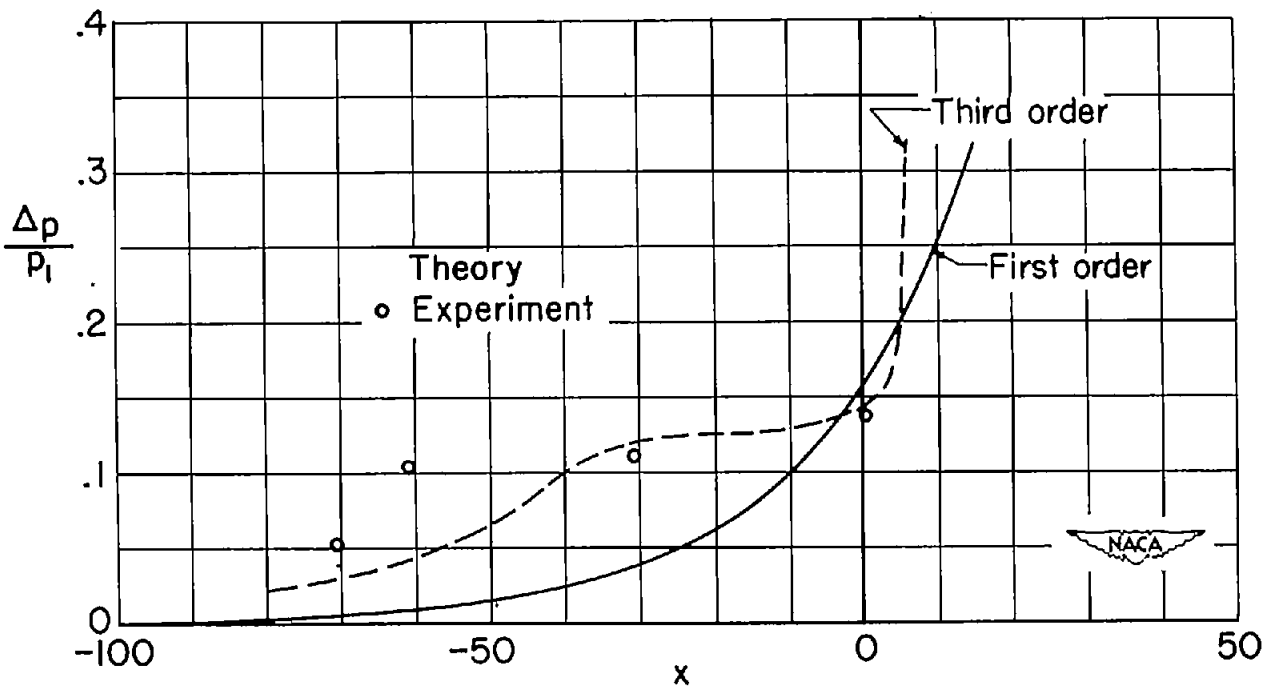


Figure 6.- Pressure distribution on plate for third- and first-order terms for $\epsilon = -3^{\circ}$. M = 2; R = 774.

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