[redacted]

TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 615

MOTION OF THE TWO-CONTROL AIRPLANE IN RECTILINEAR FLIGHT AFTER INITIAL DISTURBANCES WITH INTRODUCTION OF CONTROLS FOLLOWING AN EXPONENTIAL LAW

By Alexander Klemin
New York University

Washington
September 1937

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 615

MOTION OF THE TWO-CONTROL AIRPLANE IN RECTILINEAR FLIGHT AFTER INITIAL DISTURBANCES WITH INTRODUCTION OF CONTROLS FOLLOWING AN EXPONENTIAL LAW

By Alexander Klemin

SUMMARY

An airplane in steady rectilinear flight was assumed to experience an initial disturbance in rolling or yawing velocity. The equations of motion were solved to see if it were possible to hasten recovery of a stable airplane or to secure recovery of an unstable airplane by the application of a single lateral control following an exponential law.

The sample computations indicate that, for initial disturbances complex in character, it would be difficult to secure correlation with one type of exponential control. The possibility is visualized that two-control operation may seriously impair the ability to hasten recovery or counteract instability.

INTRODUCTION

An investigation was recently made for the National Advisory Committee for Aeronautics of the motion of the two-control airplane (i.e., one with either aileron or rudder control eliminated) in rectilinear flight following initial disturbances due to gusts or other causes.

The general plan of attack was as follows. An airplane in steady rectilinear flight was assumed to experience an initial disturbance in rolling velocity or yawing velocity. Its motion without application of the controls was found (by the use of operator methods) and appropriately plotted. Then a single control (either rudder or aileron) was assumed as being applied in a simple fashion realizable by a pilot. The equations of motion were again solved, and the results plotted. From the solutions of the equations and from the curves, it was sought to answer the following questions:

1. Under such types of initial lateral disturbances, is it possible to hasten recovery substantially in the case of a stable airplane by the application of only a single lateral control?

2. Under such types of initial lateral disturbances, is it possible to secure recovery in the case of an unstable airplane by the application of only a single lateral control?

The curves obtained as a result of typical calculations and qualitative answers (which appear to be of some degree of general applicability) ~~to~~ the foregoing questions are presented in this note.

The application of the controls is represented by an exponential law, with a negative exponent, so that the control at maximum power when first introduced into the motion can be decreased more or less rapidly with time.

Thus, if the equations of lateral motion are written in the form:

$$D\,(D - Y_v)\,v - (Y_p D + W_o D + g)\varphi + (U_o D - g\theta_o - Y_r D)\psi = 0$$

$$- L_v v + (D - L_p)\,p - L_r r = 0$$

$$- N_v v - N_p p + (D - N_r)\,r = 0$$

The effect of the controls can be introduced by rewriting the equation as:

$$D\,(D - Y_v)\,v - (Y_p D + W_o D + g)\varphi + (U_o D - g\theta_o - Y_r D)\psi = 0$$

$$- L_v v + (D - L_p)\,p - L_r r = L_o e^{lt}$$

$$- N_v v - N_p p + (D - N_r)\,r = N_o e^{nt}$$

The maximum power of the controls can be changed by giving L_o and N_o different values, and the rate at which the controls decrease in power can be changed by giving l and n different negative values.

The effect of initial disturbances p_o, r_o, v_o can be studied by writing:

$$D\,(D - Y_v)\,v - (Y_pD + W_oD + g)\varphi + (U_o - g\theta_o - Y_rD)\psi = 0$$

$$-L_vv + (D - L_p)\,p - L_rr = Dp_o + L_oe^{lt}$$

$$-N_vv - N_pp + (D - N_r)\,r = Dr_o + N_oe^{nt}$$

The operator methods of solution of these equations are now so well understood as to need no explanation in the present note.

The calculations were applied to the following aircraft:

I - The N.A.C.A. "average" airplane of slightly unstable characteristics, in horizontal flight at cruising attitude.

II - The Bristol "Fighter" at an angle of attack of 0°, giving stable characteristics, in gliding flight.

III - The Bristol "Fighter" at an angle of attack of 16° near the stall, giving unstable characteristics, in gliding flight.

Similar investigations were made with initial disturbances in angle of bank which, of course, indirectly produce disturbances or motions in rolling and yawing velocity, as well as in sideslip velocities. Other calculations were also made in regard to the application of constant couples and of constant couples cut off at a point prior to resumption of an even keel by the aircraft. The results of these calculations are on file with the Committee and will not be dealt with in the present note. It may be said, however, that these calculations lead to conclusions in general fundamental agreement with the conclusions drawn from the calculations summarized in the present note.

Acknowledgments and thanks are due to Perry A. Pepper and Harry Goldstein, formerly graduate students of the Daniel Guggenheim School of Aeronautics, New York University, whose assistance was invaluable in the computations, graphical analysis, and general conduct of the investigation. Thanks are also due to the technical staff of the

Committee for help given in discussions prior to the beginning of the investigation and in review of the work prior to its release in the technical note.

I - CALCULATIONS FOR THE N.A.C.A. AVERAGE AIRPLANE

Characteristics of the Average Airplane

The airplane selected for the first set of experimental calculations was the N.A.C.A. average airplane as described in reference 1:

Type: Monoplane, two-passenger; aspect ratio 6; rectangular, rounded-tip, Clark Y wing; dihedral angle, 1°.

Dimensions:

Weight	1,600 lb.
Wing span	32 ft.
Wing area	171 sq. ft.
Area of fin and rudder . .	10.8 sq. ft.
Tail length	14.6 ft.
mk_X^2	1,216 slug-ft.^{2}
mk_Z^2	1,700 slug-ft.^{2}

Stability derivatives: Taken for cruising speed,

U_o = 150 ft./sec.; C_L = 0.35

L_p	L_r	L_β	N_p	N_r	N_β
-5.44	1.11	-2.16	-0.207	-0.913	5.52
		$L_v = L_\beta/U_o = -0.0144$			$N_v = N_\beta/U_o = 0.0368$

One condition for lateral stability is commonly taken to be

$$L_v N_r > L_r N_v$$

Substituting from the foregoing table, it is found that $L_v N_r = 0.01315$ and $L_r N_v = 0.04085$, so that the airplane is spirally unstable as might be inferred from the low value of the dihedral.

The values of the four roots of the determinantal equation are:

$$\lambda_1 = 0.02896$$

$$\lambda_2 = -5.43422$$

$$\lambda_3 = -0.47387 \pm 2.35095\, i$$

An airplane with a slight degree of spiral instability is definitely of interest in the study of two-control operation. Since the effect of the dihedral varies with the attitude of flight and becomes uncertain at high angles of attack, even an airplane with large dihedral may be spirally unstable in certain attitudes.

Free Motion under Initial Disturbances

An obvious preliminary to the study of two-control operation is the study of the free motion under initial disturbances, so as to establish what the controls have to accomplish in securing or hastening recovery. Initial disturbances might be introduced into the motion by considering the effects of certain gusts and imagining these gusts to cease at a given instant. For the purposes of this investigation, it is, however, quite as useful to introduce arbitrary initial disturbances following the practice of British writers on allied topics. Nor is the magnitude of such disturbances particularly important, so long as they are within the power of the controlling moments.

The first initial disturbance studied was that of a rolling velocity $p_o = 0.5$ radian per second, which might be imagined as being introduced by a very powerful and briefly acting gust under one wing tip, with yaw prevented by the rudder; or it might be regarded as the residual motion due to over-energetic application of the ailerons.

The complete solutions for motions in roll and yaw due to the initial disturbance p_o are given by the following equations:

$$p = 0.00254e^{0.02896t} + 0.4967e^{-5.4342t}$$

$$+ 0.0088e^{-0.47387t} \cos (2.35095t + 1.4710) \quad \text{(I-1)}$$

$$r = 0.01873e^{0.02896t} + 0.01495e^{-5.4342t}$$

$$- 0.0371e^{-0.47387t} \cos (2.35095t - 0.4540) \quad \text{(I-2)}$$

These motions are illustrated in the curves of figures 1 and 2.

The resulting motion in p is therefore compounded of an exponential term with a small coefficient that increases slowly; an exponential term with a large coefficient but very rapid damping; and an oscillation of small amplitude that is damped more slowly. As indicated in figure 1 the final effect, if the airplane is left to itself, is a small, practically constant rolling velocity which, of course, could not be left uncorrected in practice.

The resulting motion in r is of similar character to that in p, but it will be noted that the exponential term which increases slowly (i.e., the "spiral dive" term) has a much larger coefficient; that the rapidly damped exponential term has a much smaller coefficient (as would be expected since this term is related to the powerful damping of the wings in roll); that the oscillatory term (frequently called the "Dutch roll" term) has a much greater amplitude. As indicated in figure 2, the final effect, if the airplane is left to itself, is a practically constant yawing velocity much larger than the constant rolling velocity referred to in the preceding paragraph. The constant yawing velocity also could not be left uncorrected in practice.

It is important to note that the initial disturbance in roll p_o, results in a greater disturbance in yaw than in roll after 1 second.

While not greatly pertinent to the problem of two-control operation, it is interesting to discuss the initial free motion, and particularly the reason why, with an initial roll to the right, the initial yaw is to the left.

Originally when $p = p_o$ and $\phi = v = r = 0$ these are obtained as the sole equations:

$$Dp = L_p p_o$$

$$Dr = N_p p_o$$

With L_p large and negative, the positive motion in p rapidly damps. At the same time with N_p small and negative, r changes immediately from zero to a small negative value. Eventually the positive sideslip following on the positive roll, makes $N_v v$ so much larger than $N_p p$ (with p decreasing) that the swing or yaw is to the right. These considerations are in agreement with the motion shown in figure 2.

Correlation of the Rolling Controlling Moment in the Form $L_o e^{lt}$ with Motion Due to p_o

It is impossible to say what a sentient pilot would do with an airplane of this type when confronted with an initial pure rolling velocity disturbance. To calculate the very "best" possible control effort by mathematical methods seems a difficult task, and the result of such mathematical methods might lead to a complexity of motion of the control stick that would be quite beyond the capacity of a human pilot. Therefore, there is much to be said for the introduction of a rolling control in the form $L_o e^{lt}$. The effects of such a control are comparatively simple to evaluate. By the method of changing L_o and l it is possible to study changes in power and changes in rate of decrease of power quite flexibly. Again from physical considerations, it is not unreasonable to imagine that, faced with a violent roll, the pilot should counteract powerfully with the ailerons and gradually ease off the stick as the roll is checked and the airplane returns to an even keel.

In the original calculations many values of L_o and l were tried, in the endeavor to find the best correlation between the initial disturbance and the counteracting action of the ailerons. It was necessary, of course, not only to secure the best correlation from considerations of the motion in p but also from considerations of the motion in r. (Similarly multiple and varied calculations were carried out in all other cases studied and this multiplicity of calculation and selection of the best combination will in future be taken for granted.)

As a result of these lengthy calculations and the plotting of the corresponding curves, it was found that the most suitable values to insert in the term $L_o e^{lt}$ were:

$$L_o = -1.609$$

$$l = -3$$

The complete equation of motion in p taking both the initial roll $p_o = 0.5$ radian per second and the impressed control into account, then becomes

$$\begin{aligned} p = {} & 0.00254e^{0.02896t} + 0.4967e^{-5.4342t} \\ & + 0.0088e^{-0.47387t} \cos(2.35095t + 1.4710) \\ & - 1.609 \left[0.4025e^{-3t} + 0.0017e^{0.02896t} - 0.4088e^{-5.4342t} \right. \\ & \left. + 0.0049e^{-0.47387t} \cos(2.35095t + 0.7784)\right] \end{aligned} \qquad \text{(I-3)}$$

$$\begin{aligned} r = {} & 0.01873e^{0.02896t} + 0.01495e^{-5.4342t} \\ & - 0.0371e^{-0.47387t} \cos(2.35095t - 0.4540) \\ & - 1.609 \left[- 0.0077e^{-3t} + 0.0125e^{0.02896t} \right. \\ & \left. - 0.0123e^{-5.4342t} + 0.0218e^{-0.47387t} \cos(2.35095t - 1.2035)\right] \end{aligned} \qquad \text{(I-4)}$$

The curves obtained from these equations are shown in figures 3 and 4.

When equation (I-3) in p is examined, it is seen that:

(1) There is a term $0.00254e^{0.02896t}$, which slowly increases with time. This is the spiral dive term.

(2) There is a term due to damping in roll,

$0.4967e^{-5.4342\,t}$, which is so rapidly damped that it has little importance.

(3) There is a Dutch roll term, a damped oscillation, which damps much more slowly than the roll

$$0.0088e^{-0.47387t}\cos(2.35095t + 1.4710)$$

The effect of the impressed control $-1.609e^{-3t}$, as further examination of equation (I-3) indicates, is to provide:

(1) A term, $-1.609(0.0017)e^{0.02896t} =$

$-0.00273e^{0.02896t}$, which is almost precisely equal and opposite to the spiral dive term of the equation in p_o alone.

(2) A term due to damping in roll,

$-1.609(-0.4088)e^{-5.4342t}$, which is so rapidly damped as to be of no importance.

(3) A term in Dutch roll, $-1.609(0.0049)e^{-0.47387t}$

$\cos(2.35095t+0.7784) = -0.0079e^{-0.47387t}\cos(2.35095t+0.7784)$, whose amplitude is almost equal and opposite to the Dutch roll term in p_o alone.

(4) The "impressed-moment" term, $-1.609(0.4025)e^{-3t}$, which is very large at first but decreases rapidly

Reviewing the foregoing statements and comparing figures 1 and 3, it is seen that:

The effect of introducing the controlling moment $-1.609e^{-3t}$ is, first of all, to reduce p_o to an initial zero very much more rapidly, to remove the slowly increasing term, and to reduce the amplitude of oscillation virtually to the vanishing point.

The matching of the effects of p_o and $L_o e^{\lambda t}$ on the motion has been very successful.

In figure 3 there have been plotted the results of introducing an adverse yawing moment (supposed to accompany the rolling moment of the ailerons) represented by $0.288e^{-3t}$ and the results of introducing a similar favorable yawing moment (likewise supposed to accompany the rolling moment of the ailerons). It is seen that neither the adverse nor the favorable yawing moment affects the motion in p very much.

When the equation (I-4) is examined, it is observed that:

(1) There is a term, $0.01873e^{0.02896t}$, which increases very slowly with time, behaving almost as a constant.

(2) There is a term due to damping in roll, $0.01495e^{-5.4342t}$, which damps out so rapidly as to be of negligible importance.

(3) There is a Dutch roll term, $-0.0371e^{-0.47387t}$ $\cos(2.35095t - 0.4540)$, which amounts to a damped oscillation, the damping occurring much more slowly than that of the roll.

The impressed rolling moment $-1.609e^{-3t}$ is seen to provide:

(1) A term, $-1.609(0.0125)e^{0.02896t} = -0.0201e^{0.02896t}$, which is almost equal and opposite to the spiral dive term.

(2) A term due to damping in roll, $-1.609(-0.0123)e^{-5.4342t} = 0.0198e^{-5.4342t}$, which damps out too rapidly to be of any importance.

(3) A Dutch roll term, $-1.609(0.0218)e^{-0.47387t}$ $\cos(2.35095t-1.2035 = -0.0351e^{-0.47387t}\cos(2.35095t-1.2035)$. This term reinforces, to a certain extent, the Dutch roll term due to p_o alone, but the phase difference nullifies this increase to some extent.

(4) An impressed-moment term, $-1.609(-0.0077)e^{-3t} = 0.0124e^{-3t}$, which damps out rapidly and is always relatively small.

It is also interesting to compare the curves of figure 2 and the solid curve of figure 4.

It will be seen that the application of a simple rolling moment $-1.609e^{-3t}$ produces excellent correlation. The maximum amplitude of the motion in r is reduced to about one-fourth the maximum amplitude in r due to p_o alone. The residual term in r disappears, and the amplitude of the oscillation is reduced to negligible proportions. Recovery in r may be considered very satisfactory.

The dashed curves of figure 4 also indicate that the introduction of both favorable and adverse yawing movements is actually detrimental to the motion in r.

The curves of figures 5 and 6 throw further light on the subject. In these charts are shown the effects of an impressed rolling moment of $-1.609e^{-3t}$ acting alone, on the motion in p and r. It will be seen that the impressed rolling moment gives curves for p and r similar in character to the curves in p and r under p_o alone (after the maximum displacements in p and r under the impressed moments have been obtained).

For this particular case of an airplane with some degree of spiral instability, it follows therefore, that with an initial disturbing motion p_o, application of a simple rolling moment of the character $L_o e^{lt}$ will give excellent correlation and bring quick recovery of motions in both p and r. The application of the aileron in exponential fashion rapidly brings into being what may be called a "virtual initial disturbance" in roll. This virtual initial disturbance acts in opposite but similar fashion to the original initial disturbance in roll and thus the two annul each other.

Correlation of a Yawing Controlling Moment in the Form $N_o e^{nt}$ with Motion Due to p_o

A systematic investigation was made of the effects of introducing pure rudder control in an attempt to counteract the effect of an initial disturbance p_o, giving n successively, the values $n = 0$, $n = -1$, $n = -2$, $n = -3$, $n = -4$. It was very quickly found that the only possibility of correlation was when n was made equal to -3 or -4, and that the other values could be discarded. The complete equations of motion become:

For $n = -3$

$$p = 0.00254e^{0.02896t} + 0.4967e^{-5.4342t} + 0.0088e^{-0.47387t}\cos(2.35095t + 1.4710)$$

$$+ N_o\left[-0.0400e^{-3t} + 0.0006e^{0.02896t} + 0.0526e^{-5.4342t} - 0.0693e^{-0.47387t}\cos(2.35095t + 1.3786)\right] \quad \text{(I-5)}$$

$$r = 0.01873e^{0.02896t} + 0.01495e^{-5.4342t} - 0.0371e^{-0.47387t}\cos(2.35095t - 0.4540)$$

$$+ N_o\left[-0.2555e^{-3t} + 0.0044e^{0.02896t} + 0.0016e^{-5.4342t} + 0.2938e^{-0.47387t}\cos(2.35095t - 0.5519)\right] \quad \text{(I-6)}$$

For $n = -4$

$$p = 0.00254e^{0.02896t} + 0.4967e^{-5.4342t} + 0.0088e^{-0.47387t}\cos(2.35095t + 1.4710)$$

$$+ N_o\left[-0.0879e^{-4t} + 0.0005e^{0.02896t} + 0.0893e^{-5.4342t} - 0.0564e^{-0.47387t}\cos(2.35095t + 1.5399)\right] \qquad \text{(I-7)}$$

$$r = 0.01837e^{0.02896t} + 0.01495e^{-5.4342t} - 0.0371e^{-0.47387t}\cos(2.35095t - 0.4540)$$

$$+ N_o\left[-0.2265e^{-4t} + 0.0037e^{0.02896t} + 0.0027e^{-5.4342t} + 0.2392e^{-0.47387t}\cos(2.35095t - 0.3904)\right] \qquad \text{(I-8)}$$

The motion in p under the influence of $N_o e^{-4t}$ alone is shown in figure 7, and the motion in r under the influence of $N_o e^{-4t}$ alone is shown in figure 8.

When considering this particular correlation of rudder action with an initial disturbance p_o, there arises immediately a difficulty in regard to the orders of magnitude of the two motions in p and r. As can be readily seen from the equations and curves:

(1) In the motion due to p_o alone, the values in p are far greater than the values in r.

(2) In the motion due to $L_o e^{lt}$ alone, the values in p are far greater than the values in r.

(3) In the motion due to $N_o e^{nt}$ alone, the values in p are far less than the values in r.

This difficulty is, of course, a significant and unfortunate circumstance from the point of two-central operation.

When equation (I-3) for the motion in p under p_o and rolling moment $-1.609e^{-3t}$ is studied, it is readily seen that the oscillations are roughly in phase, and that both the terms in $e^{0.02896t}$ and the oscillations are very nicely correlated.

When consideration is given to equations (I-5) and (I-7) and comparison is also made of the curves of figures 1 and 7, a fundamental dissimilarity is seen in the motion due to p_o and that due to $N_o e^{nt}$. The rudder introduces mainly an oscillation and a motion that is not in phase with the motion due to p_o. Also the rudder has too small a coefficient to match the term in $e^{0.02896t}$. Hence, as can be seen from figure 9, the combined motion in p, due to p_o and the rudder, is not much better from recovery considerations than the p motion due to p_o alone.

When equations (I-6) and (I-8) and figures 2, 8, and 10 are all taken into consideration, it is seen that the action of the rudder in removing the disturbance in r due to p_o is quite ineffective. Even when the value of N_o is made fairly small, the rudder introduces undesirable magnitudes of disturbances in yaw.

When the rudder is used in exponential fashion to correct the motion in p due to p_o, it improves this motion very little, if at all, and it actually increases the oscillations in yaw without removing the residual term. There is a fundamental lack of correlation between the motion impressed by the rudder and the motion due to the initial p_o. Practically the only beneficial effect of the rudder, when used in this manner, is that it brings the motion in p to an initial zero more quickly than when the airplane is left to itself.

Correlation of a Yawing Controlling Moment in the Form $N_o e^{nt} = -e^{-4t}$ with Motion in p Due to r_o

The solutions of the equations of motion in p are:

Motion in p due to r_o = 0.25 radian per second

$$p = 0.0005e^{0.02896t} - 0.03204e^{-5.4342t}$$

$$+ 0.0595e^{-0.47387t}\cos(2.35095t - 1.0138) \quad \text{(I-9)}$$

Motion in p due to yawing moment $-e^{-4t}$

$$p = -0.0005e^{0.02896t} - 0.0893e^{-5.4342t}$$

$$+ 0.0564e^{-0.47387t}\cos(2.35095t + 1.5399) + 0.0879e^{-4t} \quad \text{(I-10)}$$

Figure 11 illustrates the motion in p due to r_o = 0.25 radian per second and also the motion due to r_o = 0.25 radian per second combined with the impressed yawing moment $-e^{-4t}$. Figure 7 shows the motion in p under e^{-4t} alone.

Examination of the solution p under r_o = 0.25 radian per second alone shows that it contains:

(1) A spiral dive term, $0.0005e^{0.02896t}$, which increases so slowly in the first 6 seconds that it behaves as a constant.

(2) A term due to damping in roll, $-0.03204e^{-5.4342t}$, which damps out so rapidly that it has no importance.

(3) A Dutch roll term, $0.0595e^{-0.47387t}\cos(2.35095t - 1.0138)$, which represents a damped oscillation, the damping occurring at a medium rate.

The effect of the impressed yawing moment $-e^{-4t}$ is to provide:

(1) A term, $-0.0005e^{0.02896t}$, which exactly nullifies the spiral dive term.

(2) A term, $-0.0893e^{-5.4342t}$, which magnifies the term due to damping in roll, but which damps out very rapidly.

(3) A term, $0.0564e^{-0.47387t}\cos(2.35095t + 1.5399)$, which, to a very great extent, nullifies the Dutch roll because of the difference in phase.

(4) An impressed-moment term, $0.0879e^{-4t}$, which damps out very rapidly.

It is quite clear that the motion in p under r_o is of very similar character to the motion in p under N_oe^{nt}, hence the effectiveness in hastening recovery in p by using rudder control alone when the initial disturbance is in r.

Correlation of a Yawing Controlling Moment in the Form $N_oe^{nt} = -e^{-4t}$ with Motion in r Due to r_o

The solutions of the equations of motion are:

Motion in r due to r_o = 0.25 radian per second

$$r = 0.0037e^{0.02896t} - 0.0010e^{-5.4342t}$$

$$+ 0.2522e^{-0.47387t}\cos(2.35095t + 0.1977) \quad \text{(I-11)}$$

Motion in r due to $-e^{-4t}$

$$r = -0.0037e^{0.02896t} - 0.0027e^{-5.4342t}$$

$$- 0.2392e^{-0.47387t}\cos(2.35095t + 0.3903) + 0.2265e^{-4t} \quad \text{(I-12)}$$

These motions and the composite of the two motions are illustrated in figures 8 and 12.

Examination of the solution for r under $r_o = 0.25$ radian per second, alone, shows that it contains:

(1) A spiral dive term, $0.0037e^{0.02896t}$, which increases so slowly that it behaves as a constant in the first 6 seconds.

(2) A term due to damping in roll, $-0.0010e^{-5.4342t}$, which decreases so rapidly that it has no importance.

(3) A Dutch roll term, $0.2522e^{-0.47387t}\cos(2.35095t + 0.1977)$, which represents a damped oscillation.

The effect of the impressed yawing moment is to provide:

(1) A term, $-0.0037e^{0.02896t}$, which exactly counteracts the spiral dive term.

(2) A term, $-0.0027e^{-5.4342t}$, which reinforces the original term due to damping in roll, but which damps out very rapidly.

(3) A term, $-0.2392e^{-0.47387t}\cos(2.35095t + 0.3903)$, which counteracts the Dutch roll to a large extent (because of the small difference in phase and opposite signs).

(4) An impressed-moment term, $0.2265e^{-4t}$, which damps out very rapidly but still has importance because of the magnitude of the coefficient.

The motion due to the yawing moment $-e^{-4t}$ is similar in character to that due to r_0, and hastens the recovery in r from the initial disturbance r_0. If there is a disturbance r_0, an appropriate rudder control will therefore hasten recovery in <u>both</u> p and r.

Correlation of the Motion Due to an Initial Disturbance $r_0 = 0.25$ radian per second, with That Due to Impressed Rolling Moment, $L_0 e^{lt}$

By a comparison of the solutions for p and r under an initial disturbance r_0 with those due to impressed rolling moments, it was found that apparently no choice of values for L_0 and l in the rolling-moment control $L_0 e^{lt}$ succeeded in reducing the motions in p and r due to r_0 to any appreciable extent. Finally values of L_0 and l were chosen to obtain agreement in phase and to make p vanish at $t = 6$. These values are $L_0 = -1.997$ and $l = -3$.

Figure 13 shows the effect of this control on the motion in p due to the disturbance in r_o. There follows an analysis of the solution, term by term.

Figure 14 shows the effect of this control on the motion in r due to the disturbance in r_o. A term-by-term analysis is given.

Examination of the solution for p under r_o alone shows that it contains the following:

(1) A term, $0.0005e^{0.02896t}$, the spiral dive term, which increases so slowly in the first 6 seconds that it behaves as a constant.

(2) A term due to damping in roll, $-0.0320e^{-5.4342t}$, which damps out too rapidly to be of any importance.

(3) A Dutch roll term, $0.0595e^{-0.47387t}\cos(2.35095t - 1.0138)$, which represents a damped oscillation, the damping occurring at a medium rate.

The effect of the impressed rolling moment, $-1.997e^{-3t}$ is to provide:

(1) A term, $-0.0034e^{0.02896t}$, which reverses the direction of the spiral dive and increases it in magnitude.

(2) A term, $0.8163e^{-5.4342t}$, which reverses the original roll and greatly increases its magnitude. Although it damps out very rapidly, it is of some importance because of its large coefficient.

(3) A term, $-0.0098e^{-0.47387t}\cos(2.35095 + 0.7784)$, which combats the Dutch roll but only to a small extent because of the difference in coefficients and the difference in phase (approximately $\pi/2$).

(4) An impressed-moment term, $-0.8038e^{-3t}$, which damps out at a rather rapid rate, but is of importance because of its large coefficient. This term produces a motion opposite to that produced by the original yawing disturbance.

The general effect of the impressed rolling moment is, therefore, to reverse the motion in p and to increase its magnitude.

Examination of the solution for r under r_o alone shows that it contains:

(1) A term, $0.0037e^{0.02896t}$, the spiral dive term, which increases so slowly as to behave as a constant during the first 6 seconds.

(2) A term, $-0.0010e^{-5.4342t}$, the term due to damping in roll, which decreases so rapidly that it has no importance.

(3) A Dutch roll term, $0.2522e^{-0.47387t}\cos(2.35095t + 0.1977)$, which represents a damped oscillation.

The effect of the impressed rolling moment, $-1.997e^{-3t}$, is to provide:

(1) A term, $-0.0250e^{0.02896t}$, which reverses the direction of the spiral dive and greatly increases its magnitude.

(2) A term, $0.0246e^{-5.4342t}$, which reverses the direction of the term due to damping in roll and increases it greatly in magnitude. However, this term damps out very rapidly.

(3) A term, $0.0436e^{-0.47387t}\cos(2.35095t - 1.2035)$, which increases the Dutch roll but only to a small extent, mainly because of the difference in phase (almost $\pi/2$).

(4) An impressed-moment term, $-0.0154e^{-3t}$, which damps out rather rapidly.

The general effect of the impressed moment, $-1.997e^{-3t}$ on the original motion in r under $r_o = 0.25$ radian per second is to shift it slightly to the opposite direction but to fail completely to reduce its magnitude.

It is possible that the foregoing correlation could be improved upon by further experimental plotting. Careful comparison of the equations, however, does not give much hope of better correlation. The disparity in the character of the two sets of equations is too great.

Thus comparison of the equations in p indicates that, when the $e^{0.02896t}$ terms are of comparable magnitude, the oscillatory term due to r_o is far too large.

Comparison of the equations in r also indicates that, when the $e^{0.0283t}$ terms are of comparable magnitude, the oscillatory term due to r_o is far too large.

The use of a control of the type $L_o e^{\lambda t}$ does not seem to lend itself to recovery from an initial disturbance of the type r_o.

II - CALCULATIONS FOR THE BRISTOL "FIGHTER" AT 0° ANGLE OF ATTACK WITH STABLE CHARACTERISTICS

Characteristics of the Bristol "Fighter" Airplane

The Bristol "Fighter" airplane, as described in reference 2, was selected for these calculations mainly because it is one of the few airplanes in the world whose lateral-stability derivatives are known. Its characteristics at $\alpha = 0°$ are as follows:

Weight	2,850 lb.
Wing area	405 sq. ft.
Semispan	19.7 ft.
Chord	5.5 ft.
mk_X^2	1,700 slug-ft.^{2}
mk_Y^2	1,700 slug-ft.^{2}
mk_Z^2	2,900 slug-ft.^{2}
V	163 ft./sec.
U_o	163 ft./sec.
W_o	0 ft./sec.
θ_o	$-15° = -0.2618$ radian
Y_v	-0.326
$Y_r - U_o$	-162.6

$Y_p + W_0$	−1.70
L_v	−0.091
L_p	−16.80
L_r	2.11
N_v	0.028
N_p	−0.371
N_r	−0.635

Gliding-flight conditions with chord axes were employed. The value of θ_0 is negative for this condition with the chord axis 15° below the horizontal and lying along the glide path.

The roots of the discriminant equation are:

$$\lambda_1 = -0.0474$$

$$\lambda_2 = -16.79$$

$$\lambda_3 = -0.4605 + 2.240i$$

$$\lambda_4 = -0.4605 - 2.240i$$

showing the airplane to be stable under this condition of flight.

Correlation of an Initial Disturbance p_0 with an Impressed Rolling Moment of the Form $L_0 e^{lt}$ and with an Impressed Yawing Moment of the Form $N_0 e^{nt}$

With an initial disturbance p_0 = 1 radian per second, various impressed rolling moments, and various impressed yawing moments were investigated as in section I. The solutions of the equations of motion and the matching of terms in the composite solutions were carried out in the same fashion as in section I. Owing to the similarity of the work, it seems sufficient to reproduce merely the curves of the best results obtained as in figures 15 and 16, without going into the details of the matching of terms.

The solutions of the equations were as follows:

Free motion after p_o

$$p = p_o \Big[0.0001e^{-0.0474t} + 0.9989e^{-16.79t} + 0.0138e^{-0.4605t} \cos(2.240t + 1.4761) \Big] \qquad \text{(II-1)}$$

$$r = p_o \Big[0.0101e^{-0.0474t} + 0.0222e^{-16.79t} - 0.0328e^{-0.4605t} \cos(2.240t - 0.1777) \Big] \qquad \text{(II-2)}$$

Motion due to $L_o e^{lt}$ where $l = -5$

$$p = L_o \Big[0.0832e^{-5t} + 0.00002e^{-0.0474t} - 0.0847e^{-16.79t} + 0.0027e^{-0.4605t} \cos(2.240t + 1.0191) \Big] \qquad \text{(II-3)}$$

$$r = L_o \Big[0.0050e^{-5t} + 0.0021e^{-0.0474t} - 0.0019e^{-16.79t} - 0.0066e^{-0.4605t} \cos(2.240t - 0.6375) \Big] \qquad \text{(II-4)}$$

Motion due to $N_o e^{nt}$ where $n = -5$

$$p = N_o \Big[0.0160e^{-5t} + 0.00009e^{-0.0474t} + 0.0063e^{-16.79t} - 0.0811e^{-0.4605t} \cos(2.240t + 1.2919) \Big] \qquad \text{(II-5)}$$

Motion due to $N_o e^{nt}$ where $n = -5$ (cont.)

$$r = N_o \Big[-0.1868e^{-5t} + 0.0063e^{-0.0474t} + 0.00015e^{-18.79t} + 0.1927e^{-0.4605t} \cos(2.240t - 0.3603) \Big] \qquad \text{(II-6)}$$

From figure 15 it is seen that the free motion in p due to p_o is of the well-damped oscillatory character. When the rolling-moment control $-5e^{-5t}$ is superimposed on this, the motion in p is very much improved and practically disappears at the end of 6 seconds. From figure 16 it is seen also that the motion in r is very much improved by the application of the pure rolling control.

When a yawing-moment control $0.1538e^{-5t}$ was applied (with a positive value of N_o owing to the fact that the signs of the oscillatory term are reversed as the computations indicated), the character of the composite motion in p was very good indeed, with powerful damping, and virtual disappearance of the motion at the end of 6 seconds.

But the motion in r was scarcely improved by the impressed yawing moment. As can be seen, the oscillations were flattened out, but the yawing moment did not counterbalance a practically constant (because so slowly damped) term in spiral dive.

The conclusion is, that for a stable airplane also, an initial disturbance in roll can be very well handled by a pure aileron control applied in exponential fashion but cannot readily be disposed of by exponential application of the rudder.

Correlation of the Motion Due to an Initial Disturbance r_o with an Impressed Rolling Moment $L_o e^{lt}$ and with an Impressed Yawing Moment $N_o e^{nt}$

It is seen from figure 17 that the motion in p is greatly improved by the application of the rudder. On the other hand, the sole effect of the enormous rolling couple (which could not actually be supplied by the ailerons) is to produce a powerful motion in p in the opposite direction at the start and then to give a motion in p which is substantially similar to the free motion in p.

With an initial disturbance $r_o = 1$ radian per second, and an impressed yawing moment $-5e^{-5t}$, there is very appreciable hastening of the recovery. (See fig. 18.) The period of oscillation is shortened and the damping is much greater.

On the other hand, the application of an enormously powerful rolling couple $-52.1667e^{-5t}$ leaves the motion in r substantially the same as the free motion.

For a stable airplane, with an initial disturbance r_o, a pure rudder control applied in exponential fashion will hasten recovery in both p and r. The aileron control so applied has very little effect on recovery in either p or r.

III - CALCULATIONS FOR THE BRISTOL "FIGHTER" AT 16° ANGLE OF ATTACK WITH UNSTABLE CHARACTERISTICS

The stability derivatives at $\alpha = 16^\circ$ are as follows:

V	74.0 ft./sec.
U_o	71.1 ft./sec.
W_o	40.4 ft./sec.
θ_o	4.55° = 0.0791 radian
Y_v	0.141
$Y_r - U_o$	−70.91
$Y_p + W_o$	20.10
L_v	−0.106
L_p	−5.44
L_r	5.00

N_v 0.003

N_p . . . . -0.092

N_r 0.204

The glide path is 11.55° to the horizontal.

The roots of the discriminantal equation are:

$$\lambda_1 = 0.2346$$

$$\lambda_2 = -5.079$$

$$\lambda_3 = -0.2664 + 0.9851i$$

$$\lambda_4 = -0.2664 - 0.9851i$$

so that the airplane at this angle of attack has some degree of spiral instability.

Free Motion with Initial Disturbance $p_o = 1$

Radian per Second

The free motions in p and r are illustrated in figures 19 and 20 and by the following equations:

$$p = p_o \left[0.0069e^{0.2346t} + 1.0418e^{-5.079t} - 0.1732e^{-0.2664t} \cos(0.9851t - 1.2855) \right] \quad \text{(III-1)}$$

$$r = p_o \left[0.0158e^{0.2346t} + 0.0196e^{-5.079t} - 0.0358e^{-0.2664t} \cos(0.9851t - 0.1542) \right] \quad \text{(III-2)}$$

The free motion due to an initial motion p_o may be characterized as follows:

(1) The motion due to damping in roll characterized by the large negative root -5.079 is so heavily damped that it may be practically neglected.

(2) The increasing term $p_0(0.0158e^{0.2346t})$ in the r motion is larger than the increasing term $p_0(0.0069e^{0.2346t})$ in the p motion.

(3) The oscillatory term in the r motion has smaller amplitude than the oscillatory term in the p motion.

(4) In the first 2 seconds for the motion in p the important term is the oscillatory term, while for the motion in r the important term is that due to the positive real root.

(5) The initial motion p_0 produces an important disturbance in r.

Comparison of the various equations, and plotting of curves not included in the present report led to the selection of $l = -3$, $L_0 = -3.2857$ as giving the best correlation. While there is at first a reversal of the sign of p (as shown in fig. 19), the increasing term containing $e^{0.2346t}$ is evidently powerfully counteracted, and the magnitude of the oscillation is greatly decreased.

Figure 20 indicates that the values selected for L_0 and l also eliminate the motion in r rapidly and completely.

The motion in r is originally a derivative of the initial motion p_0. If L_0e^{lt} counteracts the disturbance in p due to p_0, it should therefore counteract the effect of p_0 in producing r, just as the curves indicate. In other words, an aerodynamic coupling of the same character exists between p_0 and r, as between L_0e^{lt} and r.

The value of $N_0 = -0.0970$ (with $n = -3$) was selected to make the value of $p = 0$ at $t = 6$. The application of yawing moment has little effect on the character of the motion in p as indicated by the curve of figure 19.

It is apparent from figure 20 that, if the motion in r due to p_0 has been satisfactorily opposed by L_0e^{lt}, the introduction of a yawing moment is actually detrimental.

Free Motion with Initial Disturbance $r_0 = 1$

Radian per Second

The equations of motion are:

$$p = r_0 \Big[0.298e^{0.2346t} - 0.678e^{-5.079t}$$

$$+ 1.654e^{-0.2664t} \cos(x-y)\ (0.9851t - 1.3384) \Big] \qquad \text{(III-3)}$$

$$r = r_0 \Big[0.678e^{0.2346t} - 0.0128e^{-5.079t}$$

$$+ 0.3423e^{-0.2664t} \cos(0.9851t - 0.2071) \Big] \qquad \text{(III-4)}$$

The free motion in p due to r_0 indicates decided instability but with the oscillatory term much more powerful in relation to the term produced by the positive real root.

The free motion in r due to r_0 indicates decided instability, with the term due to the positive real root so predominant as to mask the oscillatory term.

The motion in p is well disposed of since the term due to the positive real root is eliminated, and the oscillation is strongly reduced in magnitude as shown in figure 21.

As shown in figure 22, the selection of $N_0 = -4.239$ and $n = -4$ after experimental calculations (based on the idea that the positive real root should be eliminated) was particularly fortunate. The motion in r disappears very rapidly.

As indicated by the equations, the value of L_0 to be employed in order to make an impression on the motion in r due to the positive real term has to be enormous. With L_0 made equal to -85.467, the motion in r is improved by partial counteracting of the constantly increasing term, but the dangerous oscillation persists. (See fig. 22.)

At the same time, when $L_o e^{lt}$ is used to make a partial recovery in the r motion due to r_o, a very undesirable increase in the p motion is evidenced.

It is again evident that for a disturbance p_o, there is ready correlation with a control of the type $L_o e^{lt}$ but <u>not</u> with a control of the type $N_o e^{nt}$. On the other hand, with an initial disturbance r_o, there is ready correlation with a rudder control applied exponentially, but very difficult correlation with a pure rolling control so applied.

DISCUSSION

The investigation presented in this note may perhaps be criticized on the following grounds:

(1) That the introduction of controls following exponential laws is of an arbitrary character and that a skilled pilot might find better methods of introducing pure aileron or pure rudder controls with more flexibility in meeting varying conditions.

(2) That the aircraft employed are not of modern design.

(3) That with more experience with calculations of this sort more appropriate values for the impressed controls might have been found.

(4) That these calculations, while no doubt accurate and starting with permissible assumptions, are of an empirical character and not in the elegant mathematical form from which generalizations may be made.

The rejoinders might be:

(1) That it is practically impossible to cover mathematically all possible manipulations of the controls. That the conception of an impressed control with rapid decrease in power is physically defensible, simple, and close to a manipulation which might well be adopted by a pilot. It is a better conception than one of constant couples adopted by British writers, though perhaps inferior to the idea of constant couples appropriately cut off. Also these expo-

nential controls have been shown as actually hastening recovery, or annulling instability.

(2) That it is not necessary that the airplanes be of modern design. It is sufficient that the derivatives and their ratios be of a reasonable character.

(3) That while better values might have been found with more experience, an enormous range of variations in both the coefficient and the exponent of the impressed controls was covered in the investigation.

(4) That while the calculations are empirical and not in generalized form, it cannot be quite an accident that the same general conclusion emerges from case after case. Also the physical conception of rudder to oppose turn, aileron to oppose roll, is in no way violated.

CONCLUSIONS

The laborious and lengthy computations have led to the following conclusions:

(1) Controls that are exponential functions of time, and decrease in intensity with time, are admissible in the study of two-control operation or of control action in general.

(2) Impressed couples following an exponential law can actually be made to hasten recovery or to counteract instability.

(3) For a simple initial disturbance of the type p_o, the appropriate exponential control is one in roll.

(4) For an initial disturbance of the type p_o, correlation with an impressed exponential control in yaw is difficult.

(5) Where the exponential control appropriate to a disturbance is one in roll, neither adverse nor favorable yawing moments of the ailerons are desirable.

(6) For a simple initial disturbance of the type r_o, the appropriate exponential control is one in yaw.

(7) For an initial disturbance of the type r_0, correlation with an impressed exponential control in roll is difficult.

(8) These deductions are applicable to airplanes not departing greatly from the conventional, whether stable or slightly unstable.

(9) Where initial disturbances are complex in character, as they may well be, it would be difficult to secure correlation with one type of exponential control.

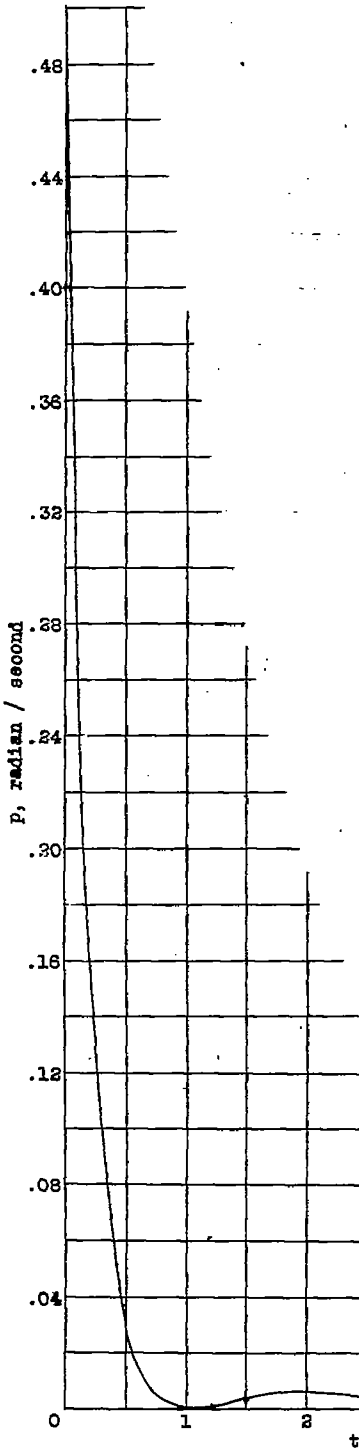
(10) Even if circular flight is achieved by two-control operation, designers of two-control aircraft should guard against the possibility that two-control operation may seriously impair the ability to hasten recovery or counteract instability. This drawback would be particularly serious at or near the stall.

Daniel Guggenheim School of Aeronautics,
New York University,
New York, N. Y., July 1937.

REFERENCES

1. Jones, Robert T.: A Simplified Application of the Method of Operators to the Calculation of Disturbed Motions of an Airplane. T.R. No. 560, N.A.C.A., 1936.

2. Halliday, A. S., and Burge, C. H.: Lateral Stability Calculations for the Bristol Fighter Aeroplane. R. & M. No. 1306, British A.R.C., 1930.



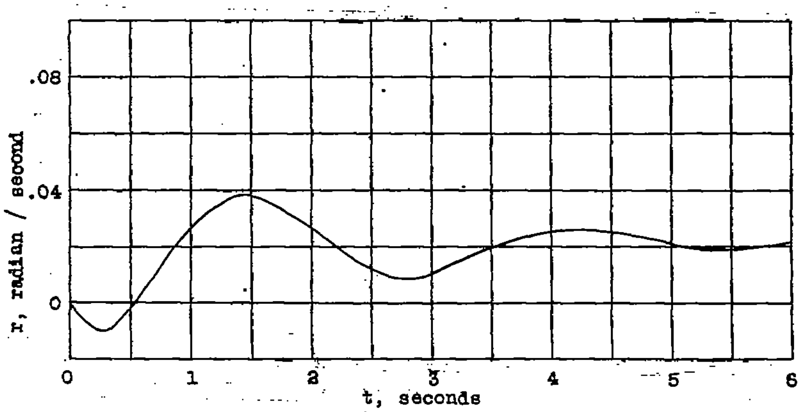


Figure 2.- Solution for r under an initial disturbance of p_0 = 0.5 radian per second. The N.A.C.A. average airplane. $r = 0.01873\ e^{0.02896t} + 0.01495\ e^{-5.4342t} - 0.0371\ e^{-0.47387t} \cos (2.35095t - 0.4540)$

Figure 1.- Solution for p under an initial disturbance of p_0 = 0.5 radian per second. The N.A.C.A. average airplane. $p = 0.00254\ e^{0.02896t} + 0.4967\ e^{-5.4342t} + 0.0088\ e^{-0.47387t} \cos (2.35095t + 1.4710)$

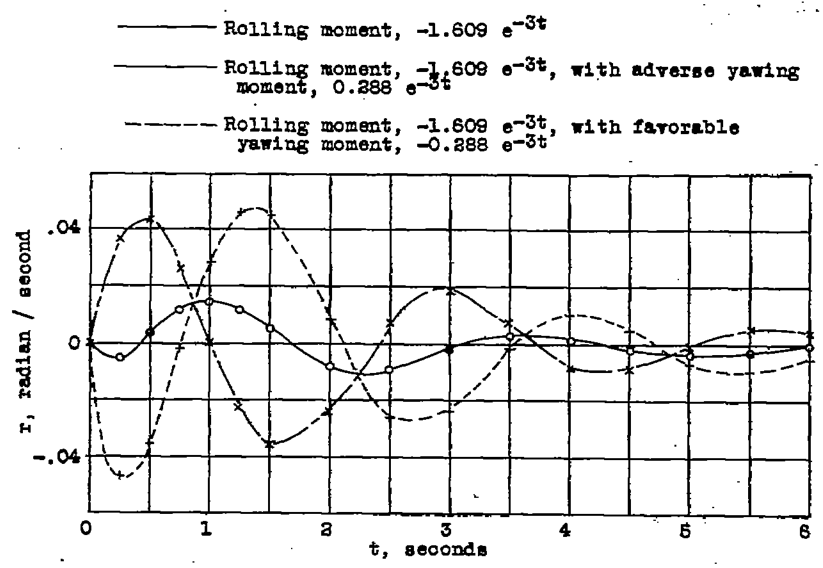


Figure 4.- Solution for the resultant motion in r with the initial disturbance of p_o = 0.5 radian per second for three amounts of yawing moment. The N.A.C.A. average airplane.

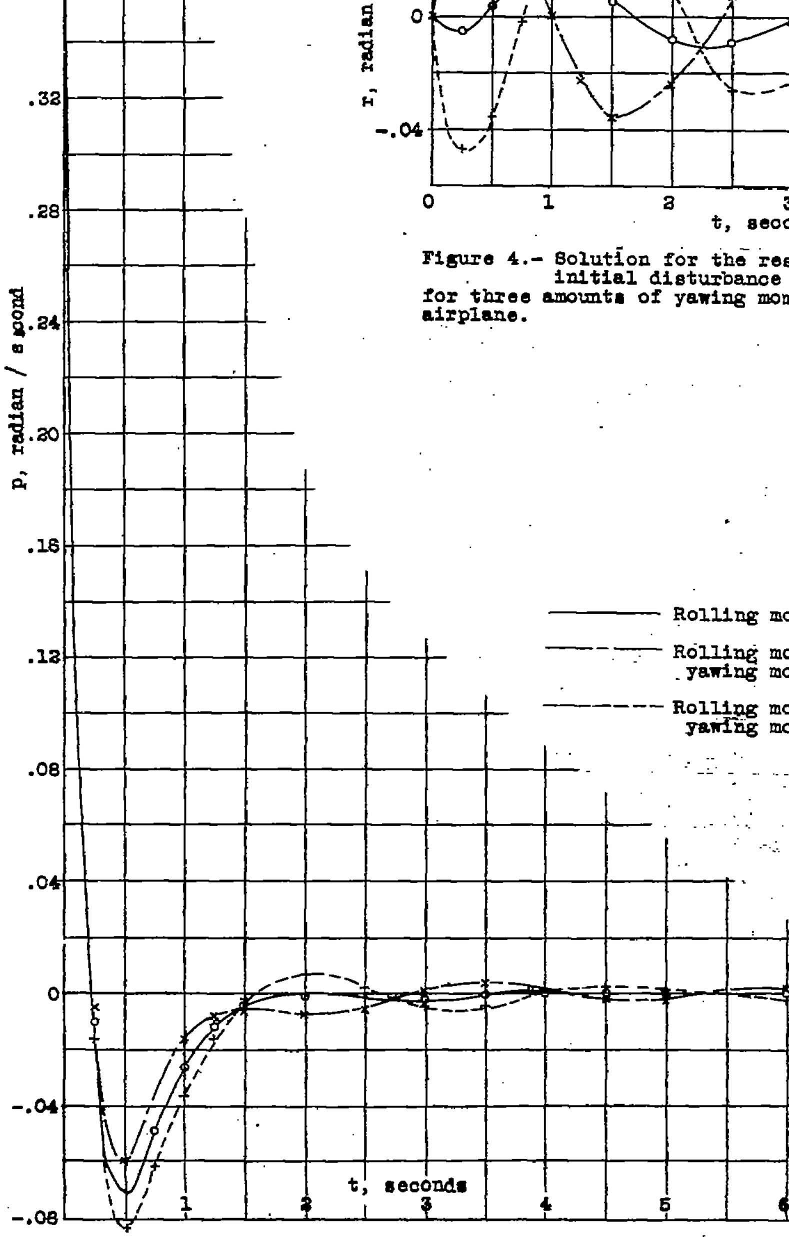


Figure 3.- Solution for the resultant motion in p with the initial disturbance of p_o = 0.5 radian per second for three amounts of yawing moment. The N.A.C.A. average airplane.

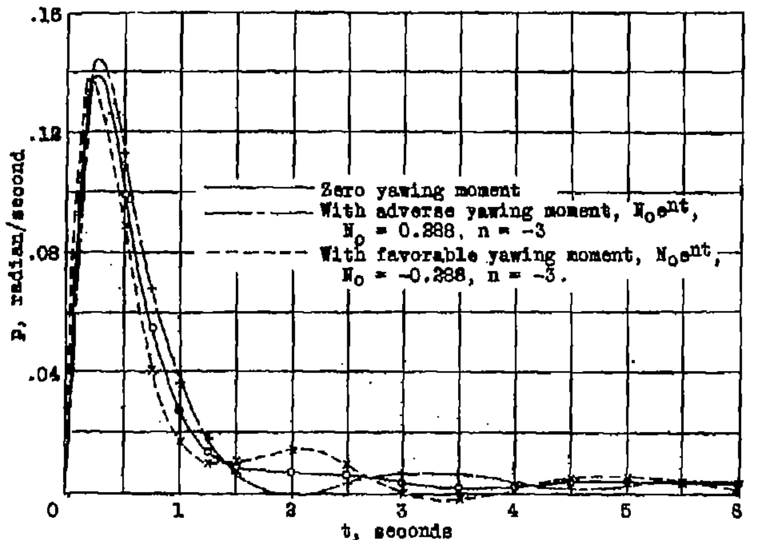


Figure 5.- Solution for the motion in p with an impressed rolling moment of L_0e^{lt}, L_0 = -1.609, l = -3 for three amounts of yawing moment. The N.A.C.A. average airplane.

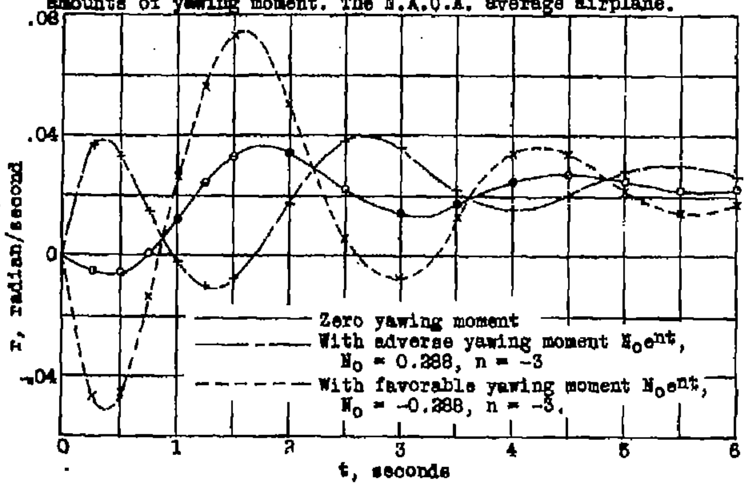


Figure 6.- Solution for the motion in r with an impressed rolling moment of L_0e^{lt}, L_0 = -1.609, l = -3 for three amounts of yawing moment. The N.A.C.A. average airplane.

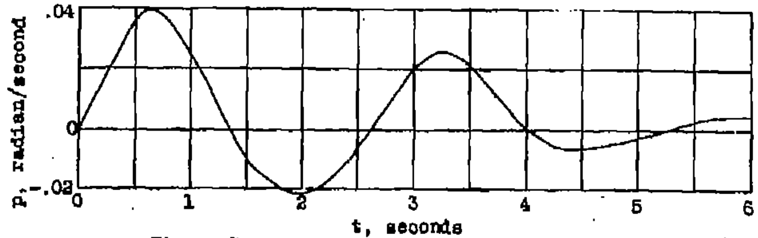


Figure 7.- Solution for p under yawing moment, N_0e^{nt}, N_0 = 1, n = -4.

$p = -0.0879\, e^{-4t} + 0.0005\, e^{0.02898t} + 0.0893\, e^{-5.4342t} - 0.0584\, e^{-0.47387t} \cos(2.35095\, t + 1.5399)$

The N.A.C.A. average airplane.

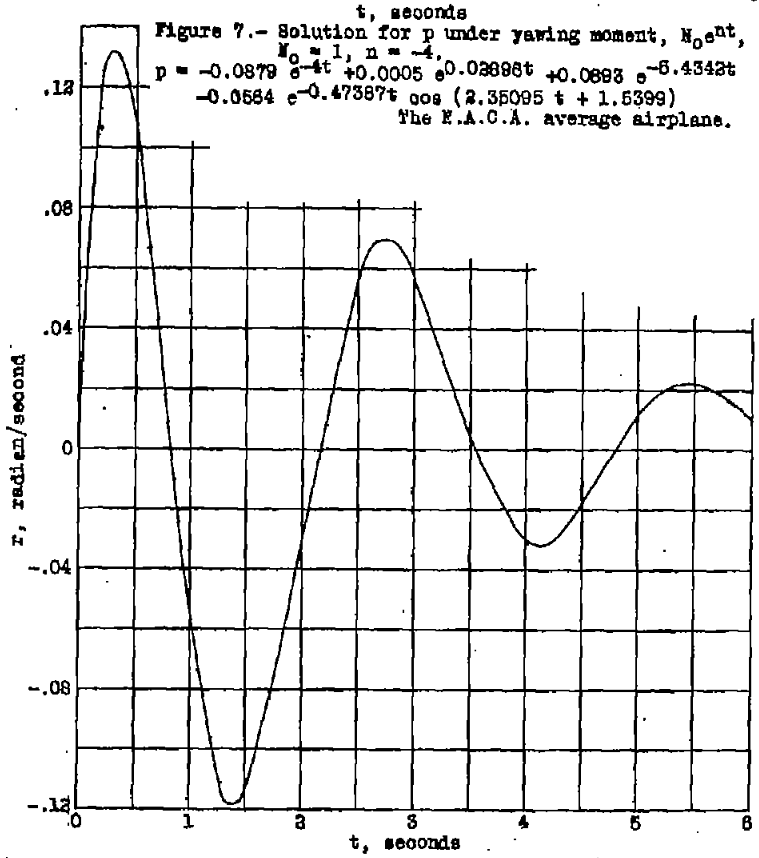


Figure 8.- Solution for r under yawing moment, N_0e^{nt}, N_0 = 1, n = -4

$r = -0.2265\, e^{-4t} + 0.0037\, e^{0.02898t} + 0.0027\, e^{-5.4342t} + 0.2392\, e^{-0.47387t} \cos(2.35095t - 0.3904)$

The N.A.C.A. average airplane.

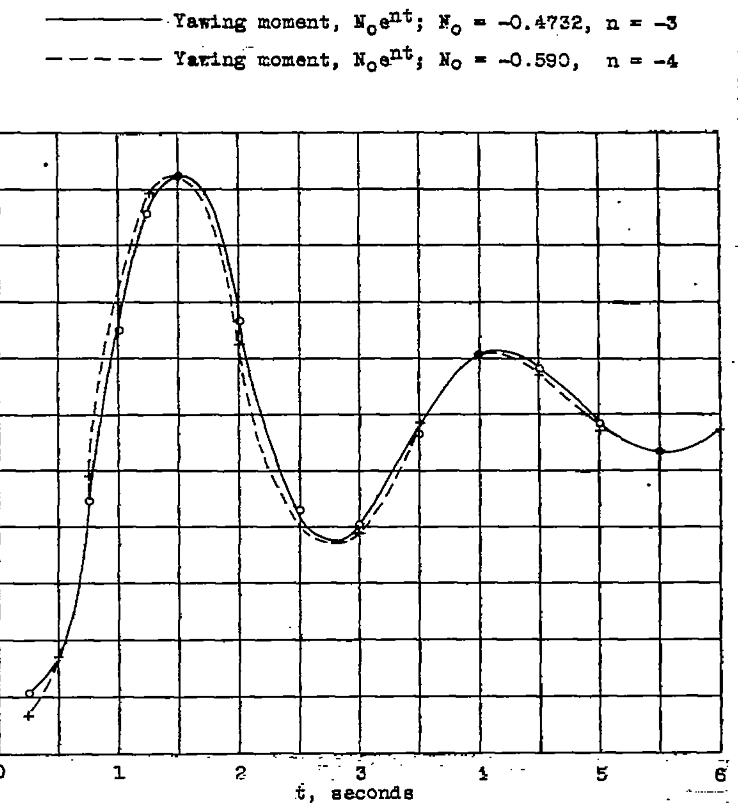


Figure 10.- Solution for the resultant motion in r for an initial disturbance of p_o = 0.5 radian per second with different amounts of impressed yawing moment. The N.A.C.A. average airplane.

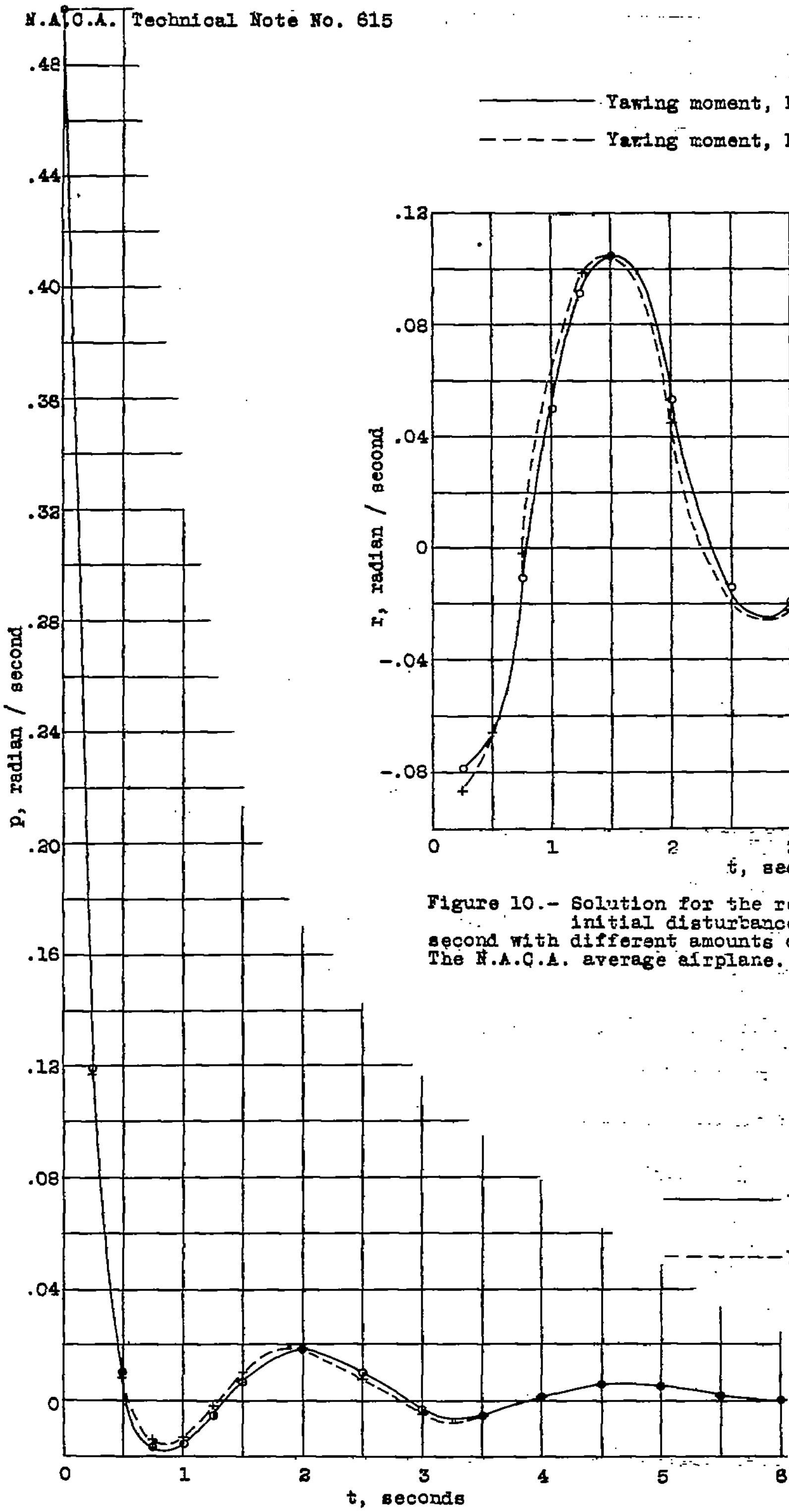


Figure 9.- Solution for the resultant motion in p for an initial disturbance of p_o = 0.5 radian per second with different amounts of impressed yawing moment. The N.A.C.A. average airplane.

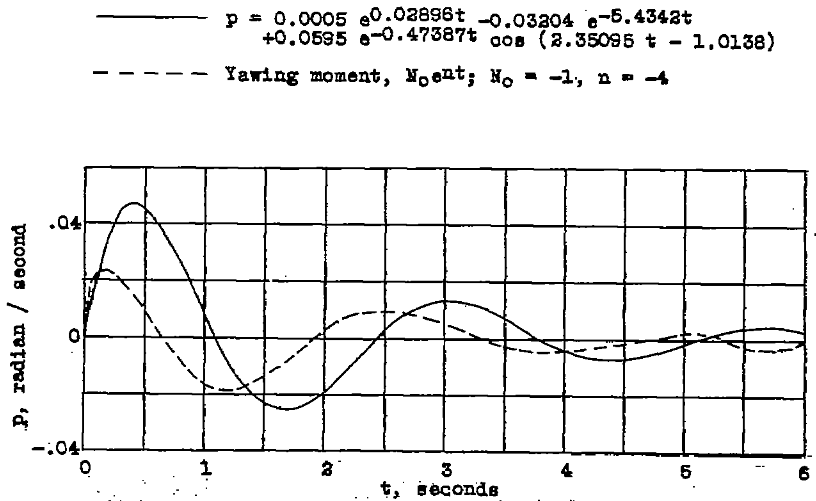


Figure 11.- Solution for p with an initial disturbance of r_o = 0.25 radian per second with and without an impressed yawing moment. The N.A.C.A. average airplane.

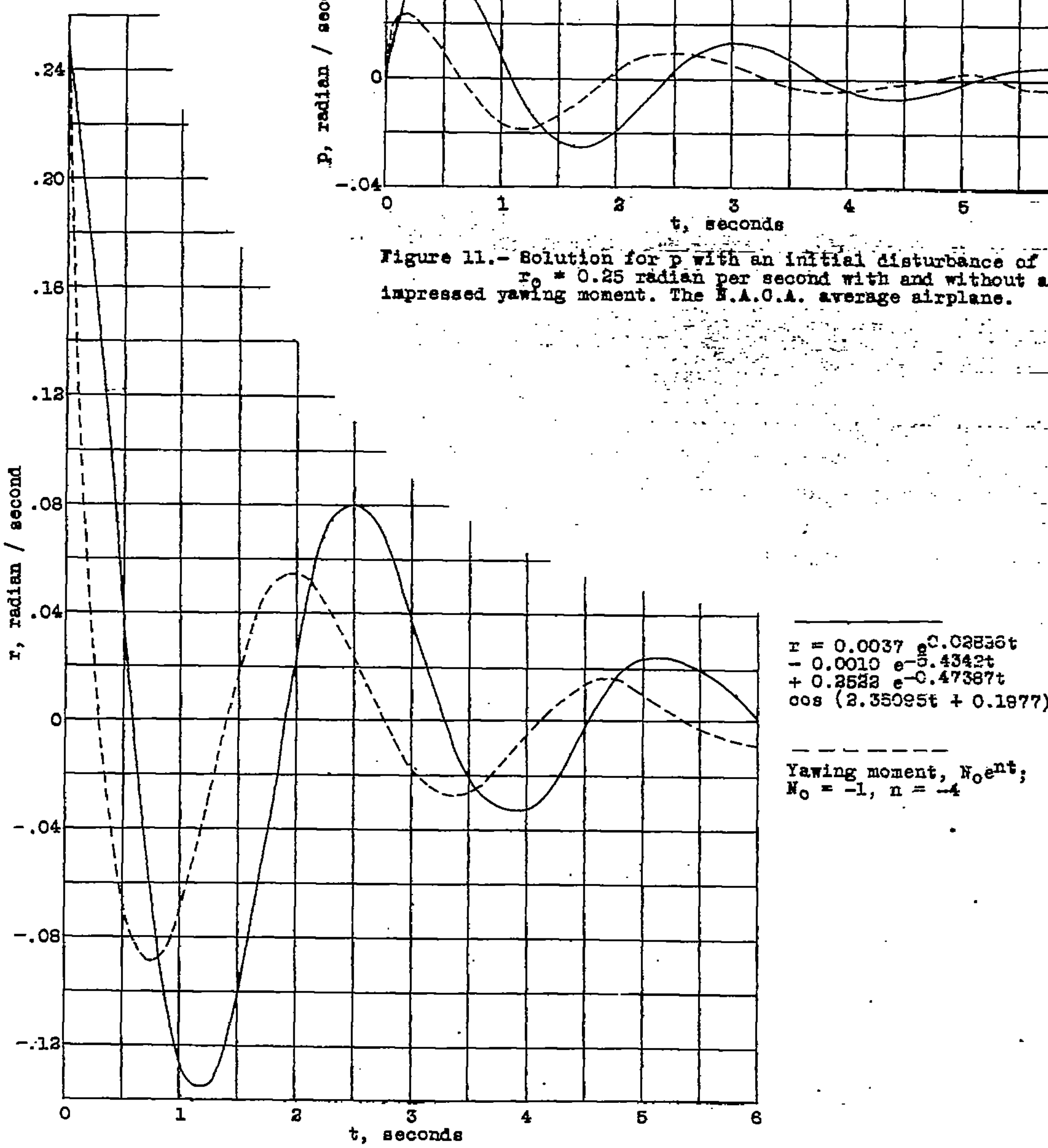


Figure 12.- Solution for r with an initial disturbance of r_o = 0.25 radian per second with and without an impressed yawing moment. The N.A.C.A. average airplane.

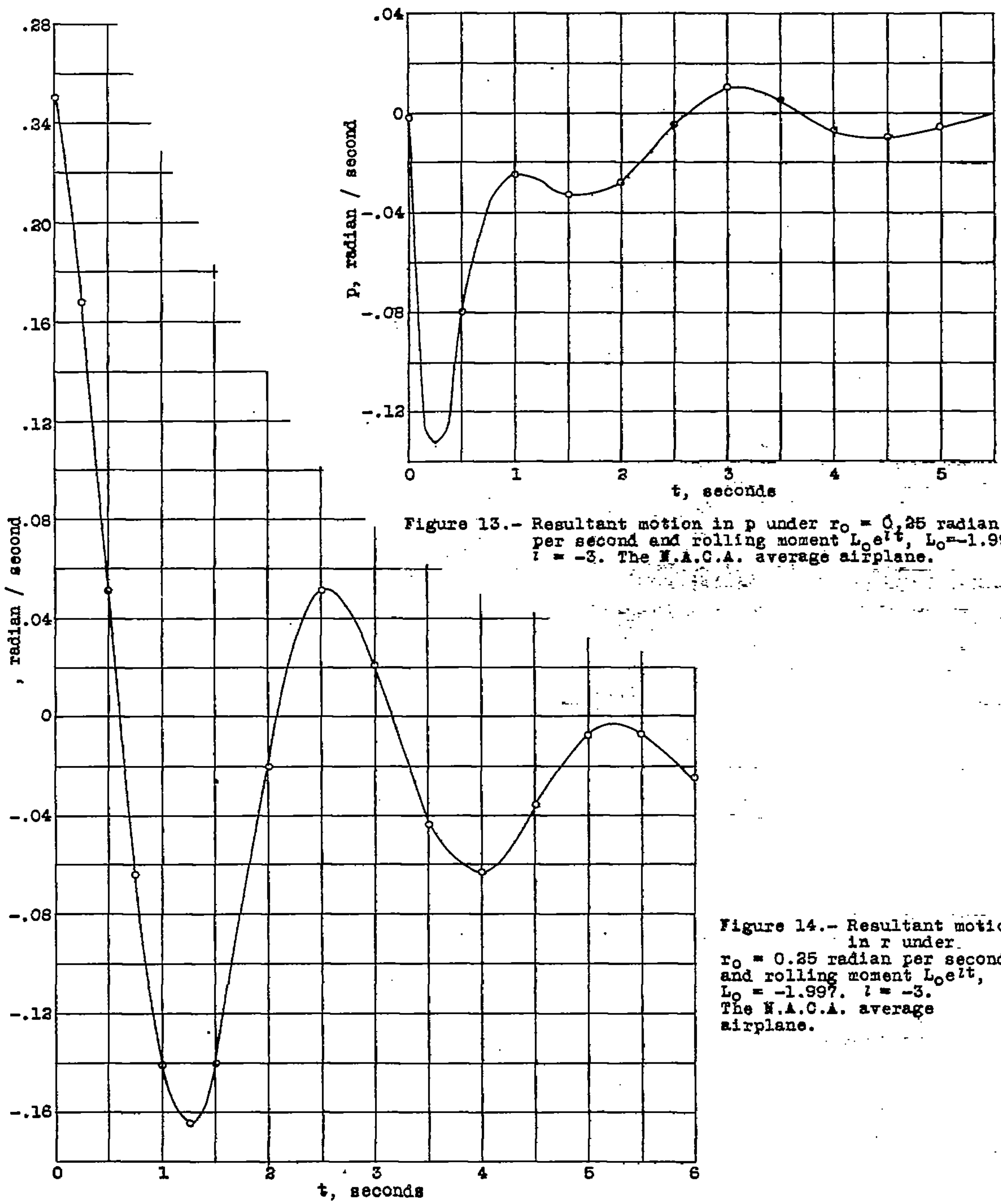


Figure 13.- Resultant motion in p under r_0 = 0.25 radian per second and rolling moment $L_o e^{lt}$, L_o=-1.997, l = -3. The N.A.C.A. average airplane.

Figure 14.- Resultant motion in r under r_0 = 0.25 radian per second and rolling moment $L_o e^{lt}$, L_o = -1.997. l = -3. The N.A.C.A. average airplane.

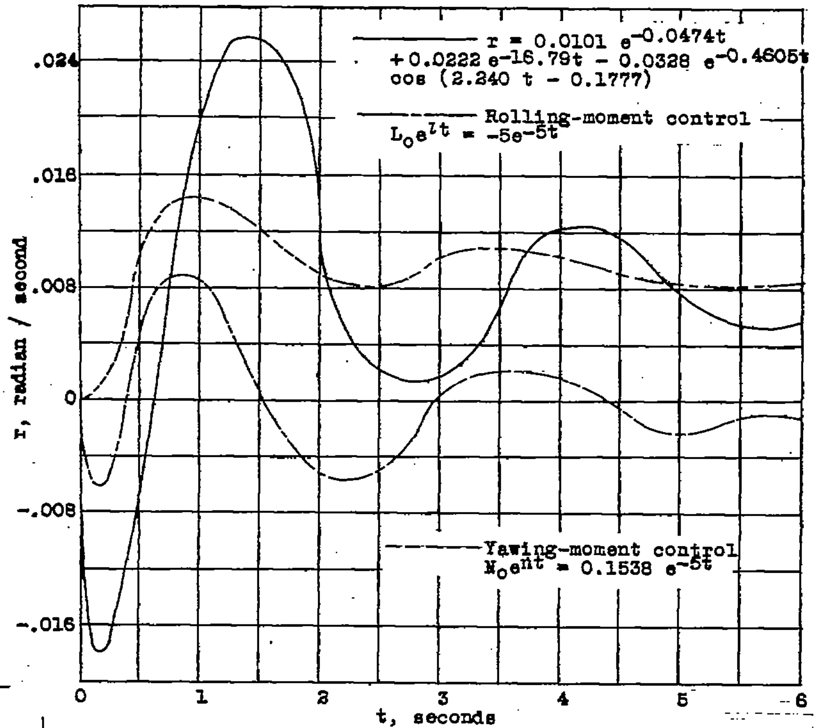


Figure 16.- Solution for motion in r under an initial disturbance of p_0 = 1 radian per second comparing free motion with motion resulting from rolling or yawing controls. Bristol Fighter at 0° angle of attack.

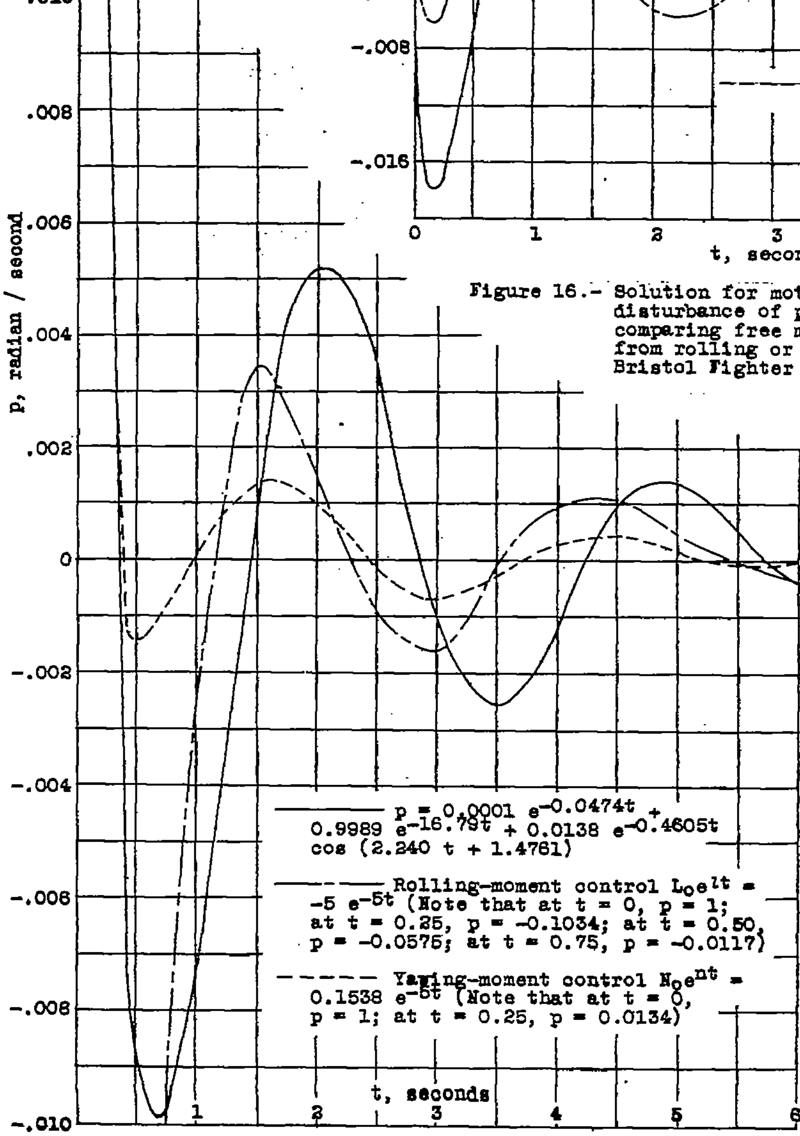


Figure 15.- Solution for motion in p under an initial disturbance of p_0 = 1 radian per second comparing free motion with motion resulting from rolling or yawing controls. Bristol Fighter at 0° angle of attack.

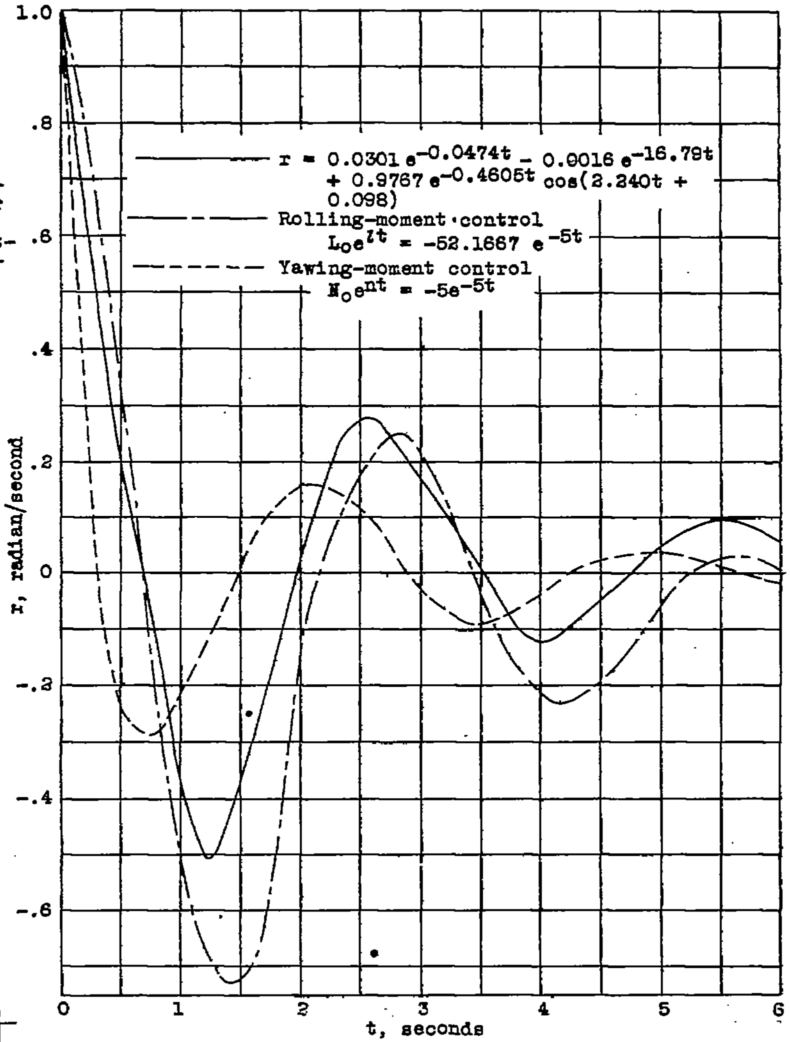


Figure 18.- Solution for motion in r under an initial disturbance of r_0 = 1 radian per second comparing free motion with motion resulting from rolling and yawing controls. Bristol Fighter at 0° angle of attack.

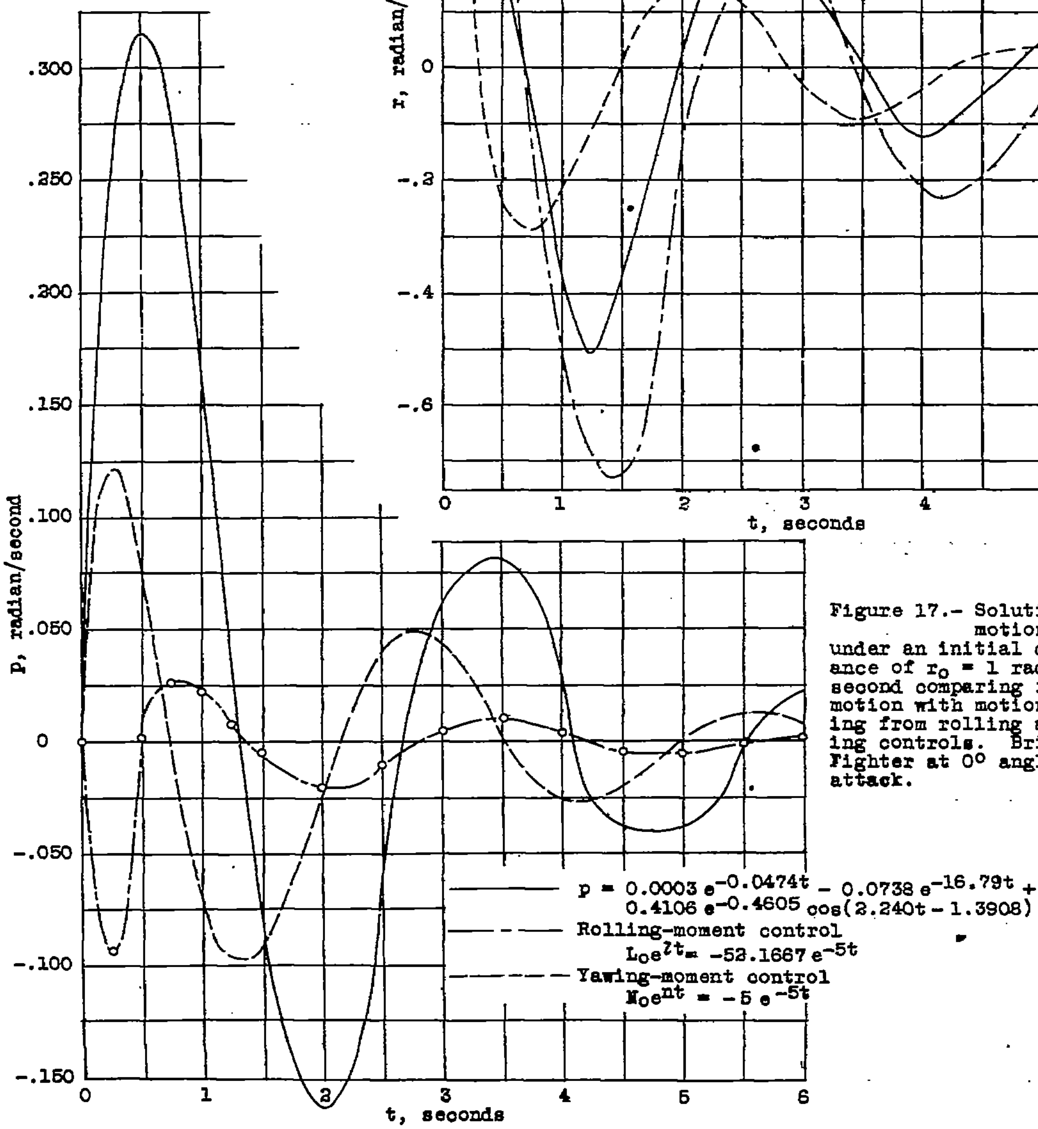


Figure 17.- Solution for motion in p under an initial disturbance of r_0 = 1 radian per second comparing free motion with motion resulting from rolling and yawing controls. Bristol Fighter at 0° angle of attack.

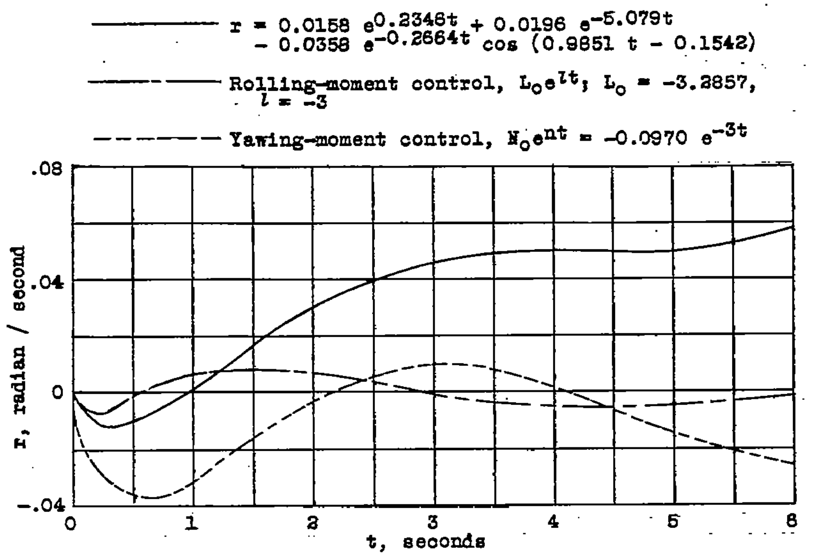


Figure 20.- Solution for motion in r under an initial disturbance of p_o = 1 radian per second, comparing free motion with motion resulting from rolling or yawing controls. Bristol Fighter at 16° angle of attack.

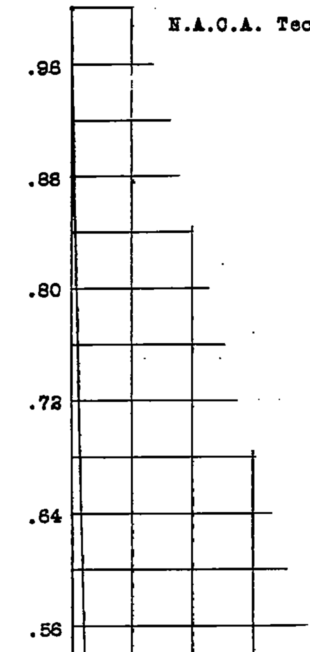
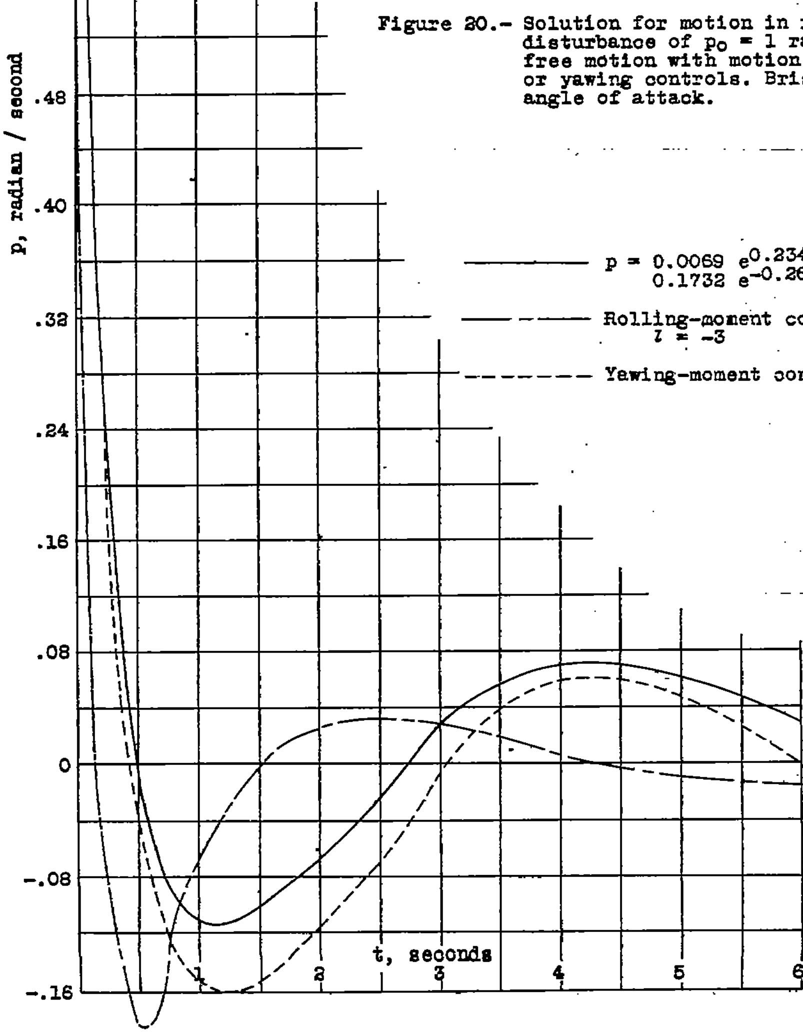



Figure 19.- Solution for motion in p under an initial disturbance of p_o = 1 radian per second, comparing free motion with motion resulting from rolling or yawing controls. Bristol Fighter at 16° angle of attack.

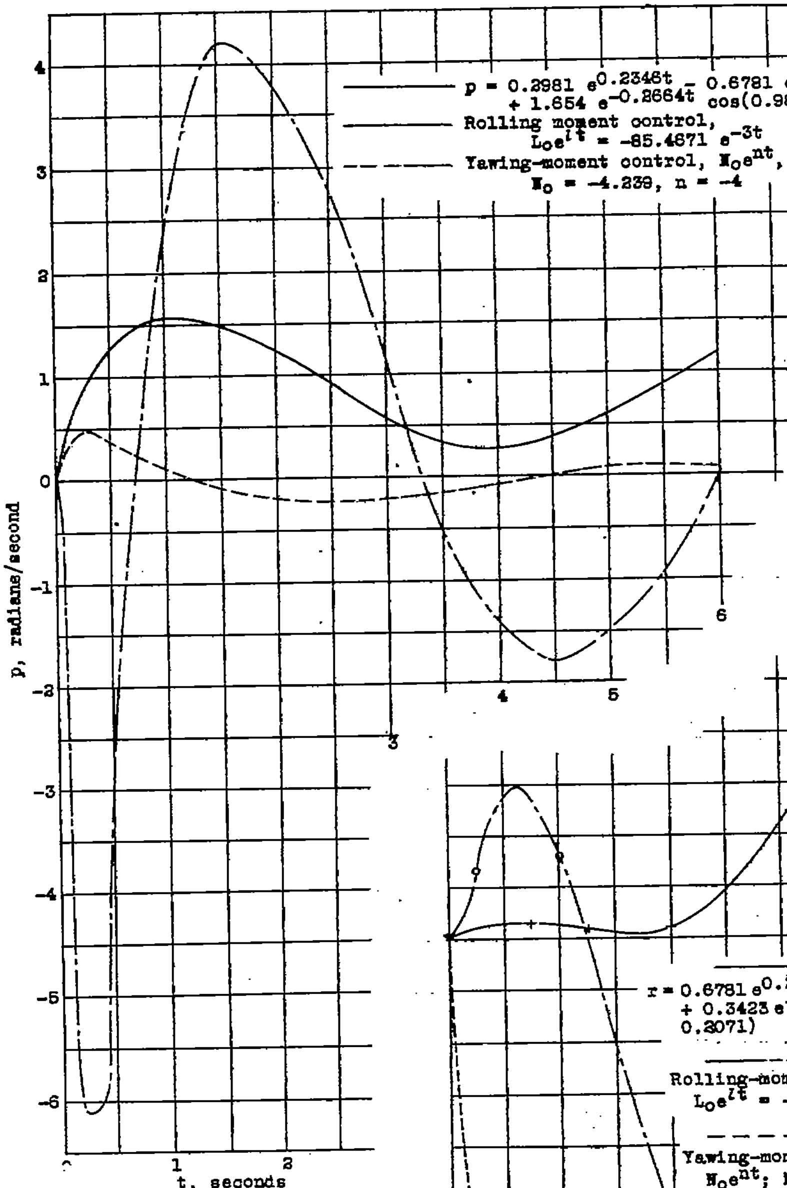


Figure 21.- Solution for motion in p under an initial disturbance of r_0 = 1 radian per second, comparing free motion with motion resulting from rolling or yawing controls. Bristol Fighter at 16° angle of attack.

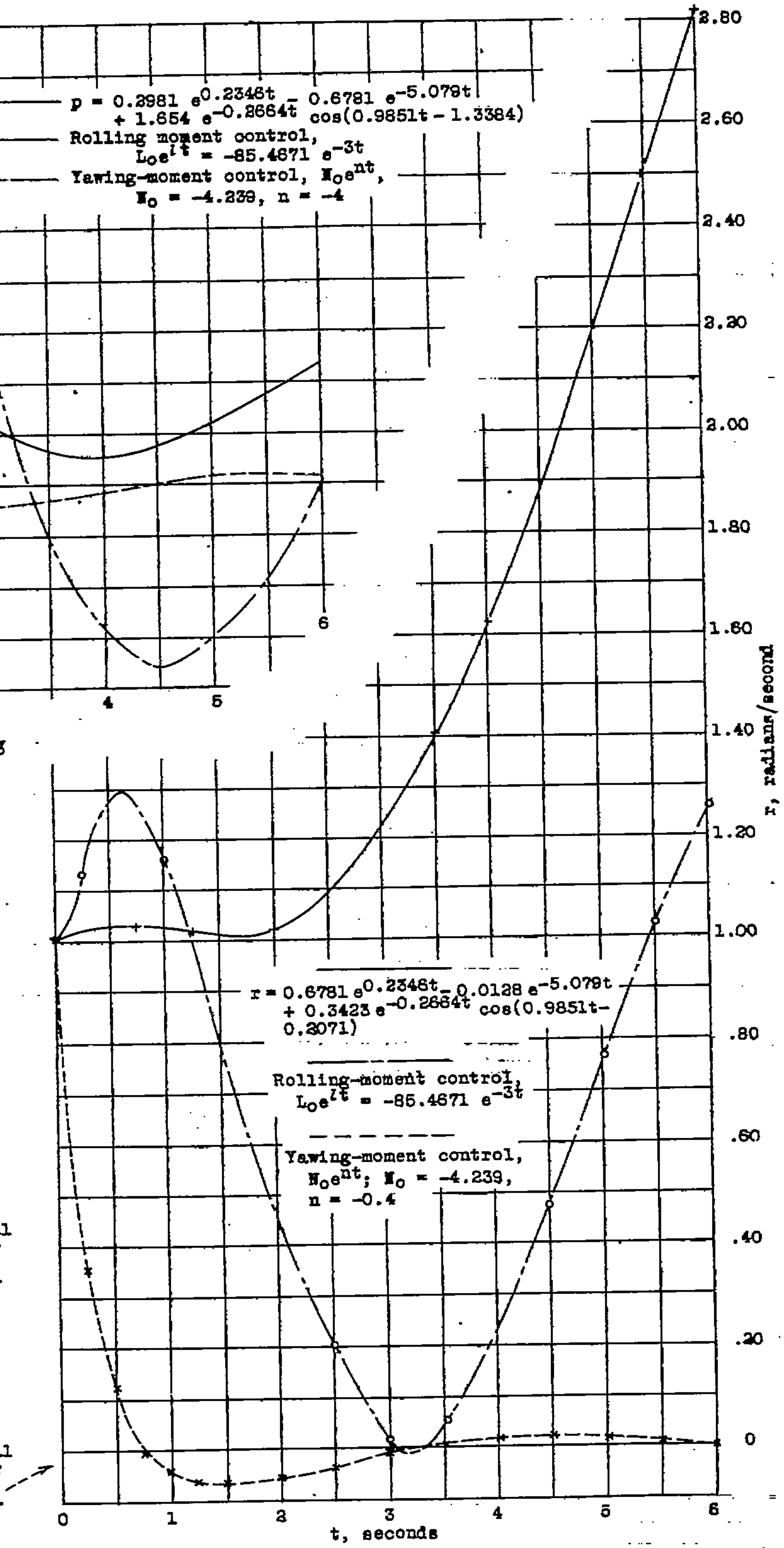


Figure 22.- Solution for motion in r under an initial disturbance of r_0 = 1 radian per second, comparing free motion with motion resulting from rolling or yawing controls. Bristol Fighter at 16° angle of attack.