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# SOME NOTES ON THE NUMERICAL SOLUTION OF SHEAR-LAG AND MATHEMATICALLY RELATED PROBLEMS

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# SOME NOTES ON THE NUMERICAL SOLUTION OF SHEAR-LAG AND MATHEMATICALLY RELATED PROBLEMS

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## SUMMARY

The analysis of box beams with shear deformation of the flanges can be reduced to the solution of a differential equation. The same equation is met in other problems of stress analysis. No analytical solutions of this equation can be given for practical cases, and numerical methods of evaluation must be used. Available methods are briefly discussed. Two numerical examples show the application of the step-by-step method of integration to shear-lag problems.

## INTRODUCTION

When a box beam is subjected to bending moments, the stress distribution differs somwehat from that given by the ordinary theory of bending. The reason for these differences, denoted by the term "shear-lag action," lies in the fact that the cover sheet suffers appreciable shear deformations, particularly after it buckles into diagonal-tension fields. Under certain simplifying assumptions discussed in reference 1, the analysis of this problem leads to a differential equation of the type

$$\frac{d^2y}{dx^2} - K_1^2 y + K_2 = 0 \qquad (1)$$

In this equation, $x$ is the distance along the span, and $y$ may denote, for instance, the bending stress in the flange, or the shear stress in the cover. The meaning of $K_1$ and $K_2$ depends on the meaning assigned to $y$.

When a box beam is subjected to torsion, there will

be in general a tendency of the cross sections to warp out of their plane. At the support, this warping is more or less completely prevented and bending stresses arise (reference 2). Under certain simplifying assumptions, the analysis of this problem also leads to an equation of type (1).

The same equation is obtained in a number of other cases. A quite well-known case is the beam under combined lateral loading and axial tension (reference 3).

Analytical solutions of equation (1) can be given when $K_1$ and $K_2$ are constants or simple functions of $x$ (reference 3). In many practical problems, however, $K_1$ and $K_2$ vary along the span in a manner that is difficult to define by simple mathematical functions, a difficulty occurring particularly in box-beam problems. It is therefore necessary to use numerical methods for the solution of given problems.

## NUMERICAL METHODS OF SOLUTION

All methods to be discussed in this paper depend on the assumption that it is permissible to divide the beam into a limited number of bays (5 to 10) and that the coefficients $K_1$ and $K_2$ may be assumed to be constant within each bay. This assumption, and the methods used for the solution of the differential equation, make the solutions inherently approximate. The state of affairs is comparable with the engineering usage of applying the formula $\sigma = \frac{Mc}{I}$ to tapered beams.

Trial-and-error method.- In reference 4, a somewhat unorthodox trial-and-error method was described for obtaining solutions of shear-lag problems. The method was found to be very rapid; under favorable circumstances, two cycles of the computation were sufficient to obtain an accuracy consistent with the conditions of the problem. The speed with which the analysis can be made, however, depends critically on the skill and the experience of the analyst. It therefore appears desirable to provide other methods of analysis that are less dependent on or entirely independent of the skill of the analyst. Another disadvantage of the trial-and-error method is that it becomes very cumbersome when applied to cases in which the bound-

ary conditions differ from those discussed in reference 4.

Fixed-point method.- A graphical method that requires no arbitrary trial assumptions whatever has been described by L. Kirste. In reference 5, the method was applied to the problem of bending stresses due to torsion; in reference 6, it was applied to the beam-column problem. Reference 6 being easily available, the details of the method need not be given here.

The outline of the procedure is as follows. From one end of the beam, the boundary condition given at this end being utilized, a sequence of "fixed points" to the other end of the beam is constructed. When the other end has been reached, the boundary condition given there is utilized to find the first point of the desired curve, which is then established by working back to the original starting point with the aid of the curve of fixed points.

For purposes of comparison with arithmetical methods, the drawing of the curve of fixed points may be counted as equivalent to one cycle of computation; the drawing of the desired curve, as equivalent to a second cycle. Furthermore, converting the given data into a graphical figure and scaling the final curve to obtain the numerical answers should be counted as equivalent to one cycle of computation. It may be said, therefore, that the graphical method is equivalent to a three-cycle arithmetical method.

The neatness and the straightforwardness of the fixed-point method are impressive. It is doubtful, however, whether it will be possible in some cases to achieve the necessary accuracy by graphical means. There are also practical objections to graphical methods on the grounds of ease of checking, filing, and transmitting computations. For such reasons, graphical computations are usually employed only when they save considerable time as compared with arithmetical methods. It is very questionable whether the fixed-point method has a very marked advantage over the step-by-step integration method to be next described.

Step-by-step integration method.- An orthodox step-by-step integration method may be used either in semi-graphical or in numerical form. Only the numerical form will be described here because it is believed to be preferable for all-around use.

It is impossible to start the final integration at either end of the beam because only one boundary condition is given at either end. It is therefore necessary to make an arbitrary trial assumption for the unknown boundary condition at one end before starting a first integration. When this first integration is completed, the boundary condition at the far end will not have been met (except by accident). A second integration is therefore made, starting with a different initial assumption for the unkown boundary condition and resulting in a different error of closure. If the errors of closure are plotted against the initial assumptions, it is possible to find what initial assumption must be made to reduce the error of closure to zero.

The graphical form of this method was described in reference 7 as applied to the beam-column problem. One quite important point, however, was apparently not noted or at least was not specifically mentioned in this paper. The curve of error of closure against initial assumptions is a straight line. Consequently, two trials are sufficient, in principle at least, to determine the desired initial assumption and it will not be necessary to make more than three cycles of computation. The third cycle will be the final one, a fact that may be used as a check against errors of computations.

In the appendix, detailed examples are given which show the application of the numerical procedure to shear-lag problems. The basic theory and the nomenclature is taken from reference 1.

Remarks on the accuracy of numerical solutions.- The errors of numerical solutions may be divided into three classes:

1. Errors arising from using a small number of significant figures.

2. Errors arising from using bays of finite length.

3. Errors arising from the assumption that the coefficients are constant in a bay.

Errors of the first class are important only when the calculation involves small differences between large figures. In most cases, this difficulty can be overcome by using a calculating machine; ordinarily a slide rule is sufficiently accurate.

A different method of overcoming the difficulty is to rearrange the computation so that no small differences are involved. In the case of shear-lag problems, this rearrangement can be accomplished by writing the equations not for the actual stresses but for the corrections that must be applied to the stresses given by the ordinary bending theory. (See appendix.) This procedure is probably more rational than the first one in several respects.

The magnitude of the errors of the second class can be estimated by making comparisons with known analytical solutions. For simple cases, it was found that five or six bays often give sufficient accuracy (about 1 percent). One rather extreme case investigated was a beam column with a compressive load near the buckling load ($L/j = 3$). In this case, the half-beam had to be divided into 10 bays. (Only the half-beam was used on account of symmetry.)

Errors of the third class can also be appraised by making comparisons with analytical solutions; they appear to be of the same order of magnitude as the errors of the first two classes.

The conclusion may be drawn, then, that the necessity for making "approximate" numerical solutions of the differential equation (1) is not, in general, the factor limiting the practical accuracy of the results. The factors limiting this accuracy are the simplifying assumptions, physical and mathematical, that must be made to reduce the problem to the solution of a simple differential equation.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., March 31, 1939.

# APPENDIX

## APPLICATION OF THE METHOD OF STEP-BY-STEP INTEGRATION TO SHEAR-LAG PROBLEMS

The differential equations of the shear-lag theory.- Two static equations and one elastic equation form the basic equations of the shear-lag theory for box beams with flat covers (reference 1).

$$\sigma_F' A_F = \frac{SW}{h} - \tau t \tag{1}$$

$$\sigma_L' A_L = \tau t \tag{2}$$

$$\tau' = - \frac{G_e}{Eb} (\sigma_F - \sigma_L) \tag{3}$$

The symbols used are those of reference 1. The prime denotes differentiation with respect to $x$. The $x$ origin will be taken at the tip of the beam for all cases.

The basic equations can be used to form the following differential equations.

$$\sigma_F'' - K^2 \sigma_F + K^2 \frac{M}{hA_T} = 0 \tag{4}$$

$$\sigma_L'' - K^2 \sigma_L + K^2 \frac{M}{hA_T} = 0 \tag{5}$$

$$\tau'' - K^2 \tau + K^2 \frac{S_W A_L}{ht A_T} = 0 \tag{6}$$

where

$$K^2 = \frac{G_e t}{E b} \left( \frac{1}{A_F} + \frac{1}{A_L} \right) \tag{7}$$

For convenience, the boundary conditions are summarized in table I.

It will be noted that the equations for $\sigma_F$ and $\sigma_L$ are identical, so that $\sigma_F$ and $\sigma_L$ can be obtained by solving the same differential equation for different boundary conditions at the root. It will be found, however, that the stresses obtained in this manner satisfy the equation of static equilibrium with the external moment only at the root and at such stations along the span where the ratio $A_F/A_L$ is the same as at the root. If this ratio varies radically along the span, the solution will be reliable only near the root.

The numerical evaluation of the equations requires the computation of the second derivative. From equation (4), for example,

$$\sigma_F'' = K^2 \left(\sigma_F - \frac{M}{hA_T}\right) \tag{8}$$

This expression may be very small over a large part of the span, making it necessary to carry a large number of significant figures. It is therefore better to write the differential equation for the expression in parentheses, which is the correction that must be added to the stress calculated by the ordinary bending theory to account for shear lag, because $\sigma = M/hA_T$ is the stress calculated by the ordinary theory.

If this correction is denoted by

$$u = \sigma_F - \frac{M}{hA_T} \tag{9}$$

the differential equation becomes

$$u'' - K^2 u = 0 \tag{10}$$

where K has the same meaning as before. This equation can be solved by step-by-step integration, but the solution is somewhat easier than the solution of one of the equations (4), (5), or (6).

In analogy with equation (9), the correction for the stress $\sigma_L$ may be defined by

$$v = \frac{M}{hA_T} - \sigma_L \tag{9a}$$

giving the differential equation

$$v'' - K^2 v = 0 \tag{10a}$$

The solutions of (10) and (10a) for any given problem again fail to satisfy the equation of static equilibrium with the external moment except at the root and at those stations where the ratio $A_F/A_L$ is the same as at the root. The discrepancy was always found to be smaller than when solving equations (4) and (5). For general use, equation (10) therefore appears to be the best choice. The stresses $\tau$ and $\sigma_L$ can be found from statics after u, and therfore $\sigma_F$, have been found.

The equations discussed thus far apply to box beams with flat covers. For box beams with camber (fig. 2), equations corresponding to (4), (5), and (6) could be written. As in the case of beams without camber, it appears to be more convenient to use equation (10). The constant $K$ in equation (7) is then changed to

$$K^2 = \frac{G_e t}{Eb'}\left(\frac{1 + \frac{c}{h_W}}{A_F} + \frac{1}{A_L}\right) \tag{7a}$$

Example 1

Find the stress $\sigma_F$ in the axially loaded panel shown in figure 3. The following data are given:

$$\frac{P}{A_T} = 1.00 \qquad A_F = A_L = \frac{1}{2} A_T \qquad L = 2.00$$

$$K^2 = \frac{G_e t}{E b}\left(\frac{1}{A_F} + \frac{1}{A_L}\right) = 1.00$$

(These numerical values were chosen to emphasize method rather than arithmetic. They do not imply impractical sizes; they imply merely unconventional units for stress and length, which need not be specified here.)

The panel will be divided into five bays. Stresses and other functions at any station will be designated by a subscript denoting the values of $x/L$ in tenths; for instance, $\sigma_4$ designates the value of $\sigma_F$ at the station $x/L = 0.4$.

Two trial values must be assumed for $\sigma_1'$. If the shear stiffness were infinite, the stress would be constant, and the corresponding value $\sigma' = 0$ will be used as the first trial value. If the shear stiffness is large, but finite, the value of $\sigma$ at the root will be nearly

$$\sigma_{1o} = \frac{P}{A_T} = 1.00$$

while at the tip

$$\sigma_o = \frac{P}{A_F} = 2.00$$

Assuming that the stress decreases linearly between the tip and the root gives

$$\sigma' = \frac{\Delta\sigma}{\Delta x} = -\frac{1.00}{2.00} = -0.500$$

This value will be used for the second trial.

Table II shows the arrangement of the computation. The start of the first cycle of computation is as follows. At $x/L = 0$, the known boundary value $\sigma = 2.000$ is written down. The first trial value for $\sigma_1'$ is zero; therefore,

$$\Delta\sigma_{o-2} = \sigma'\Delta x = 0 \times 0.400 = 0$$

and consequently

$$\sigma_2 = \sigma_1 + \Delta\sigma = 2.000 + 0 = 2.000$$

Now, using equation (4) in the form

$$\sigma_n'' = K_n^2 \sigma_n - K_n^2 \left(\frac{P}{A_T}\right)_n$$

gives

$$\sigma_2'' = 2.000 - 1.000 = 1.000$$

(The subscript n denotes the station number and is added here to indicate that, in the general case, $K^2$ and $P/A_T$ vary along the span.)

Therefore

$$\Delta\sigma_{1-3}' = \sigma_2'' \Delta x = 1.000 \times 0.400 = 0.400$$

Adding this increment to the value at the preceding station gives

$$\sigma_3' = \sigma_1' + \Delta\sigma_{1-3}' = 0 + 0.400 = 0.400$$

From here on, the computation repeats itself in principle. From the slope $\sigma'$ just found, the increment $\Delta\sigma$ is calculated. Adding this increment to the stress at the preceding station gives the stress at the new station. Using the differential equation gives $\sigma''$, etc. In the general case, K and $P/A_T$ will be different for each station; the table would therefore include a row for $K^2\sigma$ and a row for $K^2 M/hA_T$.

Since an axially loaded panel can be considered as the cover of a beam in pure bending, the boundary condition that must be satisfied at the root is, according to table I,

$$\sigma_{10}' = \frac{S_W}{hA_F} = 0$$

In the first two trials, the values of $\sigma_{10}'$ obtained are 2.905 and 1.047, respectively. (Note that these values are obtained by adding $1/2\ \Delta x\sigma''$ to $\sigma_9'$.) These values of $\sigma'$ are obtained by assuming $\sigma_1' = 0$ and $\sigma_1' = -0.500$, respectively; by proportion, it is found that $\sigma_1'$ should be taken as -0.782 to satisfy the condition $\sigma_{10}' = 0$. This initial value of $\sigma'$ was used in the third cycle of the computation; the computation shows that $\sigma_{10}' = 0$ as required

(with a very small error). Comparison of the stresses with those found by formula (reference 1 or 2) shows that the error in the root stress is 0.3 percent.

## Example 2

As a second example, the analysis of a tapered beam with cambered cover will be given. The beam chosen is N.A.C.A. beam 4; the dimensions of this beam, as well as the results of strain-gage tests, are given in reference 1. The methods used in obtaining the basic data given in table III, rows 2 to 6, will be briefly outlined.

The compression flanges, the tension flanges, and the longitudinals are assumed to be concentrated at their respective centroids. The cover sheet is assumed to be fully effective in aiding the longitudinals, and the adjacent strips of sheet are added to the longitudinals or flanges. The web is also assumed to be fully effective in bending and is replaced by concentrated flanges of cross-sectional area $1/6\ A_W$. The area of the "substitute longitudinal" is calculated by

$$A_{LS} = A_L \frac{\sinh K_3 b}{K_3 b}$$

which is formula (4) of reference 1; the "substitute camber" is taken as $c_S = 1/2\ c$. The thickness of the cover sheet is 0.0114 inch, and the ratio $G/E$ is taken as 0.40. With the values given in rows 2 to 6, the values of $k^2$ can be computed by formula (7a).

If an arbitrary value of $u' = 1.00$ is assumed for the first bay, a step-by-step integration of equation (10) can then be made, as shown in rows 8 to 12.

According to table I, the boundary condition that must be fulfilled at the root is

$$u' = \frac{S_W A_L}{h A_F A_T}$$

Now at the root

$$S_W = P - \frac{M}{h} \tan \alpha = 0.473\ P$$

giving

$$u' = \frac{0.473\ P \times 0.463}{5.70 \times 0.177 \times 0.640}$$

For an applied load $P$ of 250 pounds, this equation becomes

$$u' = 85.0$$

The value obtained in the trial solution is $u' = 247.1$, so that the final values for $u$ are obtained by multiplying the $u$ values of the trial solution by $\frac{85.0}{247.1} = 0.344$. These final $u$ values are given in row 13. In row 14 are given the values of $\sigma = \frac{Mc}{I}$, that is, the flange stresses obtained by the ordinary bending theory, using the geometric properties of the beam given in reference 1. Adding the $u$ values to the $\sigma$ values gives the final solution as shown in row 15. Figure 4 gives a comparison between calculated and experimental stresses.

## REFERENCES

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3. Niles, Alfred S., and Newell, Joseph S.: Airplane Structures. Vol. II, 2d ed. John Wiley & Sons, Inc., 1938.

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5. Kirste, Léon: Sur le calcul des poutres en caisson. L'Aérotechnique, No. 169, bulletin "L'Aéronautique," No. 212, Jan. 1937, pp. 1-6.

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TABLE I

BOUNDARY CONDITIONS

| Tip ($x = 0$) | Root ($x = L$) | | |
|---|---|---|---|
| | Longitudinal built in | Longitudinal not built in | Elastically yielding support (fig. 1) |
| $\sigma_F = \frac{M}{hA_F}$ | $\sigma_F' = \frac{S_W}{hA_F}$ | $\sigma_F = \frac{M}{hA_F}$ | $\sigma_F' = \frac{S_W}{hA_F} - \frac{t}{A_F} \gamma G_e$ |
| $\sigma_L = 0$ | $\sigma_L' = 0$ | $\sigma_L = 0$ | $\sigma_L = \frac{t}{A_L} \gamma G_e$ |
| $\tau' = -\frac{G_e M}{EbhA_F}$ | $\tau = 0$ | $\tau' = \frac{G_e M}{EbhA_F}$ | $\tau = \gamma G_e$ |
| $u = \frac{M}{hA_T} \frac{A_L}{A_F}$ | $u' = \frac{S_W}{hA_F} \frac{A_L}{A_T}$ | $u = \frac{M}{hA_F} \frac{A_L}{A_T}$ | $u' = \frac{S_W}{hA_F} \frac{A_L}{A_T} - \frac{t}{A_F} \gamma G_e$ |
| $v = 0$ | $v' = \frac{S_W}{hA_T}$ | $v = \frac{M}{hA_T}$ | $v' = \frac{S_W}{hA_T} - \frac{t}{A_L} \gamma G_e$ |

TABLE II

SHEAR-LAG ANALYSIS OF AXIALLY LOADED PANELS

| x/L | 0 | | 0.2 | | 0.4 | | 0.6 | | 0.8 | | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **First trial solution** | | | | | | | | | | | |
| $\sigma'$ | | 0 | | 0.400 | | 0.864 | | 1.466 | | 2.303 | 2.905 |
| $\Delta\sigma$ | | 0 | | .160 | | .346 | | .586 | | .921 | |
| $\sigma$ | 2.000 | | 2.000 | | 2.160 | | 2.506 | | 3.092 | | 4.013 |
| $\sigma''$ | | | 1.000 | | 1.160 | | 1.506 | | 2.092 | | 3.013 |
| $\Delta\sigma'$ | | | .400 | | .464 | | .602 | | .837 | | 1.205 |
| **Second trial solution** | | | | | | | | | | | |
| $\sigma'$ | | -0.500 | | -0.180 | | 0.111 | | 0.420 | | 0.796 | 1.047 |
| $\Delta\sigma$ | | -.200 | | -.072 | | .044 | | .168 | | .318 | |
| $\sigma$ | 2.000 | | 1.800 | | 1.728 | | 1.772 | | 1.940 | | 2.258 |
| $\sigma''$ | | | .800 | | .728 | | .772 | | .940 | | 1.258 |
| $\Delta\sigma'$ | | | .320 | | .291 | | .309 | | .376 | | .503 |
| **Final solution** | | | | | | | | | | | |
| $\sigma'$ | | -0.782 | | -0.507 | | -0.313 | | -0.169 | | -0.053 | 0.001 |
| $\Delta\sigma$ | | -.313 | | -.203 | | -.125 | | -.068 | | -.021 | |
| $\sigma$ | 2.000 | | 1.687 | | 1.484 | | 1.359 | | 1.291 | | 1.270 |
| $\sigma''$ | | | .687 | | .484 | | .359 | | .291 | | .270 |
| $\Delta\sigma'$ | | | .275 | | .194 | | .144 | | .116 | | .108 |
| **Analytical solution** | | | | | | | | | | | |
| $\sigma$ | 2.000 | | 1.684 | | 1.482 | | 1.355 | | 1.287 | | 1.266 |

TABLE III

SHEAR-LAG ANALYSIS OF TAPERED BEAM WITH CAMBERED COVER

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | x (in.) | 0 | | 19 | | 38 | | 57 | | 76 | | 95 |
| 2 | $h_w$ (in.) | 2.70 | | 3.30 | | 3.90 | | 4.50 | | 5.10 | | 5.70 |
| 3 | $A_F$ (sq.in.) | .143 | | .150 | | .157 | | .163 | | .170 | | .177 |
| 4 | $A_{LS}$(sq.in.) | .224 | | .272 | | .320 | | .367 | | .415 | | .463 |
| 5 | b' (in.) | 6.10 | | 7.32 | | 8.54 | | 9.76 | | 10.98 | | 12.20 |
| 6 | $c_s$ (in.) | .56 | | .67 | | .78 | | .90 | | 1.01 | | 1.13 |
| 7 | $K^2$ | .00967 | | .00729 | | .00575 | | .00472 | | .00394 | | .00334 |
| 8 | u' | | 1.00 | | 3.63 | | 13.23 | | 43.68 | | 131.1 | [247.1] |
| 9 | Δu | | 19.0 | | 68.9 | | 251.5 | | 830 | | 2,490 | |
| 10 | u(lb/sq.in.) | 0 | | 19.0 | | 87.9 | | 339.4 | | 1,169 | | 3,659 |
| 11 | u" | | | .1385 | | .505 | | 1.603 | | 4.60 | | 12.22 |
| 12 | Δu' | | | 2.63 | | 9.60 | | 30.45 | | 87.4 | | 232.0 |
| 13 | u(lb/sq.in.) | | | 7 | | 30 | | 117 | | 402 | | 1,260 |
| 14 | σ(lb/sq.in.) | 0 | | 1,795 | | 2,690 | | 2,920 | | 3,040 | | 3,015 |
| 15 | $\sigma_F$(lb/sq.in.) | 0 | | 1,802 | | 2,720 | | 3,037 | | 3,442 | | 4,275 |

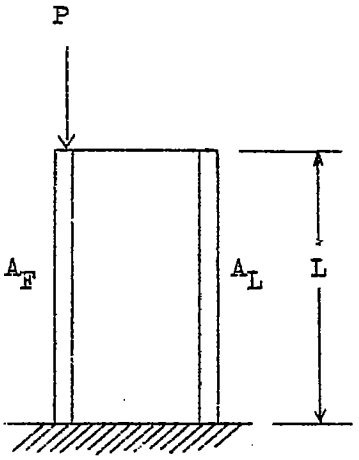


Figure 3

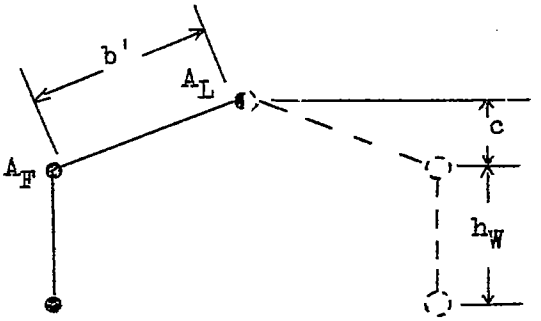


Figure 2

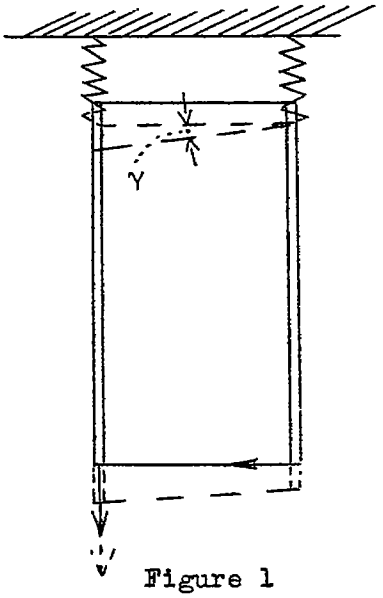


Figure 1

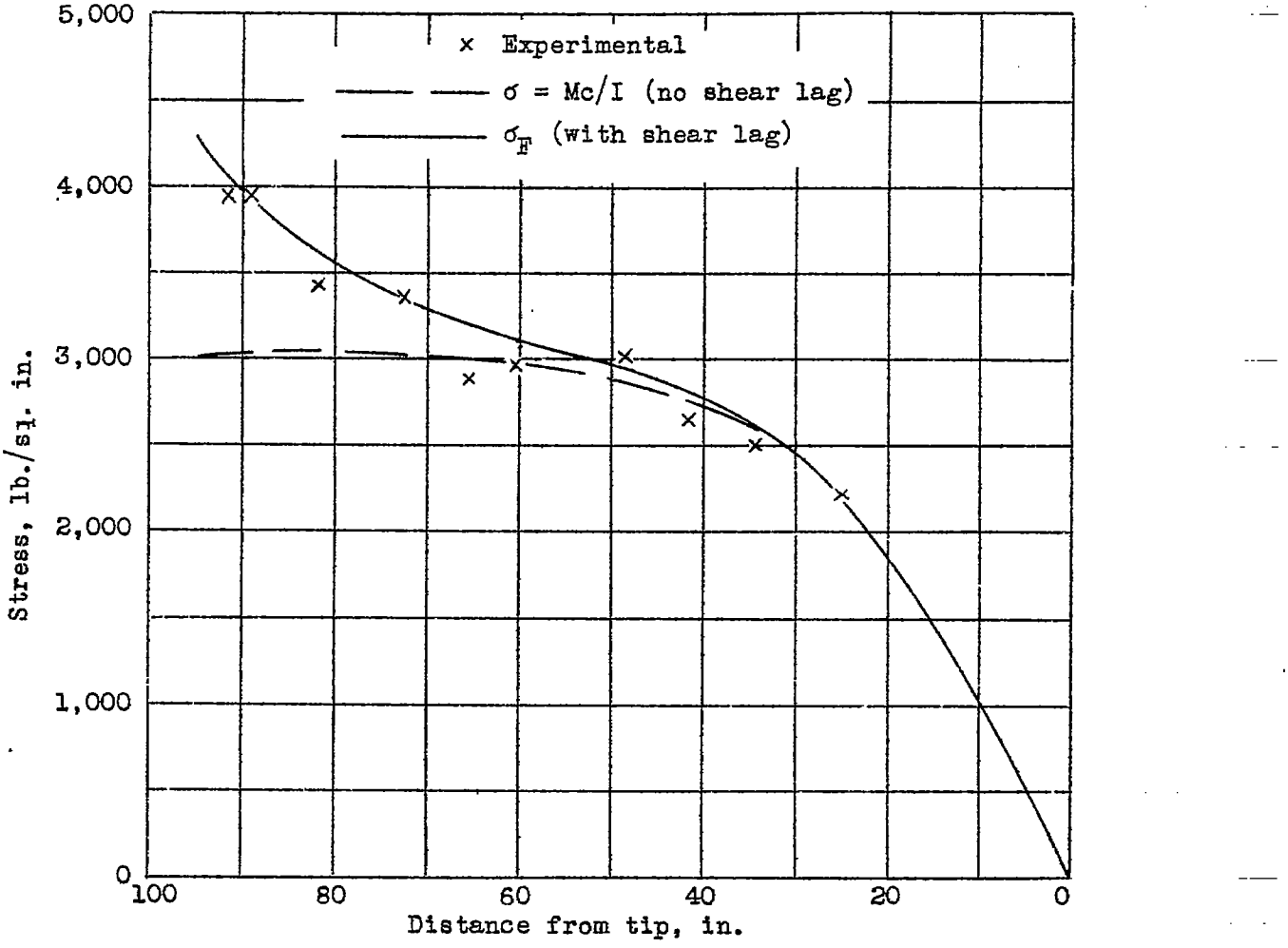


Figure 4.- Comparison between calculated and experimental stresses in N.A.C.A. beam 4. Data from reference 1.