TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 925

A LEAST-SQUARES PROCEDURE FOR THE SOLUTION OF THE LIFTING-LINE INTEGRAL EQUATION

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Washington
February 1944

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SUMMARY

A least-squares procedure adapted to numerical calculation is presented for the approximate solution of the Prandtl lifting-line equation. Sufficient data are tabulated to permit a solution of the equation by purely numerical methods for an arbitrary symmetrical variation of the chord and the angle of attack. In addition, modified procedures are formulated for the analysis of wings in which the spanwise variation of the chord or angle of attack is discontinuous. The methods proposed are illustrated by explicit numerical analysis of rectangular wings without twist and with linear and quadratic twist, a tapered wing with rounded tips and partial-span flaps, and a rectangular wing with a cut-out. Comparisons are made in several cases with the results of other procedures.

The computation involved in the procedure is entirely mechanical and is conveniently carried out on a computing machine. The accuracy attained in a solution using the tabulated data should be comparable to the accuracy of the given wing data in all practical cases, while the time required is considerably less than that required by more elaborate procedures, such as those of Lotz and Betz, and only slightly greater than that required by less exact methods, such as those of Glauert and Tani. While the modified analysis applicable to a wing with a discontinuous angle of attack or chord requires a small amount of additional computation, it is probable that the resultant accuracy in such cases could be attained by the Lotz procedure only after a very lengthy series of calculations.

INTRODUCTION

The distribution of lift over the span of a wing in uniform motion is determined, according to the Prandtl theory of the lifting line, as the solution of a singular integro-differential equation the mathematical complexities of which are such that exact solutions have been obtained only in very special cases. While several methods have been devised for obtaining approximate solutions to this equation, it is felt that a new procedure based on a method of least squares which was presented in reference 6 may be of practical interest.

In the usual procedures an approximation to the lift function is assumed as the sum of a finite number of appropriate approximating functions with undetermined coefficients, after which the coefficients are determined in various ways so that the lifting-line equation is approximately satisfied. While it might be expected that the determination of these parameters would be most efficiently accomplished by a method of least squares, the only application of such a method known to the writer (reference 2) was not well adapted to numerical computation for arbitrarily varying chord and angle of attack. In addition, the single case treated was that of a wing with discontinuous angle of attack, for which the procedure as given in reference 2 failed to give satisfactory results.

The purpose of the present paper is to present a new least-squares procedure in which the major part of the numerical calculation can be readily carried out on a computing machine, and in which the amount of labor involved is not dependent upon the nature of the variation of the chord and the angle of attack. Since all the previous procedures are notably inadequate for the analysis of wings with discontinuous spanwise variation of angle of attack or chord, an explicit treatment of such cases is included.

This investigation, conducted at the Massachusetts Institute of Technology, was sponsored by, and conducted with financial assistance from, the National Advisory Committee for Aeronautics.

SYMBOLS

y	spanwise coordinate in units of the half span ($y = 0$ at the root, $y = \pm 1$ at the tips)
b	span
c	chord
c_R	root chord $(c(0))$
c^*	chord divided by root chord (c/c_R)
α	angle of attack
q	dynamic pressure
l	section lift (per unit length along span)
c_l	section lift coefficient (l/qc)
F	auxiliary lift function $(cc_l/mc_R,\ l/qmc_R)$
m	profile constant $(dc_l/d\alpha)$
μ	dimensionless constant $(mc_R/4b)$
S	projected wing area $\left(\frac{b}{2}\int_{-1}^{1} c(y)\, dy\right)$
L	total lift $\left(\frac{b}{2}\int_{-1}^{1} l(y)\, dy\right)$
C_L	coefficient of total lift (L/qS)
$\bar{\alpha}$	angle of attack corresponding to approximate lift distribution

MATRIX NOTATION

Representation of sets of linear equations.– In this paper a set of linear equations is represented in a condensed form. Thus, for example, the formal equation

$$\begin{matrix} A_0 & A_1 \end{matrix}$$
$$\left\| \begin{matrix} a_{00} & a_{10} \\ a_{01} & a_{11} \\ a_{02} & a_{12} \end{matrix} \right\| = \left\| \begin{matrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{matrix} \right\|$$

will be used to represent the coefficients of three linear equations in the two parameters A_0 and A_1, of which the first is

$$a_{00} A_0 + a_{10} A_1 = \alpha_0$$

Matrix multiplication.– A rectangular array of elements is called a matrix. For the present purposes it will be convenient to define two types of matrix products.

The first type, which will be called the star product, is illustrated by the following example:

$$\left\| \begin{matrix} c_0 \\ c_1 \\ c_2 \end{matrix} \right\| * \left\| \begin{matrix} a_{00} & a_{10} \\ a_{01} & a_{11} \\ a_{02} & a_{12} \end{matrix} \right\| = \left\| \begin{matrix} c_0 a_{00} & c_0 a_{10} \\ c_1 a_{01} & c_1 a_{11} \\ c_2 a_{02} & c_2 a_{12} \end{matrix} \right\|$$

In general, the star product of a one-column matrix of m elements into a matrix having m rows and n columns will be defined as a matrix having m rows and n columns, wherein each element is the product of the corresponding element in the original m by n matrix by the element of the one-column matrix which lies in its row.

The definition of the dot product of two matrices may be illustrated by the following example:

$$\begin{Vmatrix} b_{00} & b_{10} \\ b_{01} & b_{11} \\ b_{02} & b_{12} \end{Vmatrix} \cdot \begin{Vmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \\ a_{02} & a_{12} \end{Vmatrix} = \begin{Vmatrix} (b_{00}a_{00}+b_{01}a_{01}+b_{02}a_{02}) & (b_{00}a_{10}+b_{01}a_{11}+b_{02}a_{12}) \\ (b_{10}a_{00}+b_{11}a_{01}+b_{12}a_{02}) & (b_{10}a_{10}+b_{11}a_{11}+b_{12}a_{12}) \end{Vmatrix}$$

In general, the dot product of two rectangular matrices having an equal number of rows will be defined as a matrix wherein the element in the j-th row and the k-th column is the algebraic sum of all products of corresponding elements in the j-th column of the first factor matrix and the k-th column of the second factor matrix. (This product is equivalent to the conventional matrix product of the transpose of the first matrix into the second matrix.)

SOLUTION OF THE LIFTING-LINE EQUATION BY A METHOD OF LEAST SQUARES

According to the lifting-line theory, the lift $l(y)$ per unit span acting on an airfoil is determined in terms of the chord $c(y)$ and the angle of attack $\alpha(y)$ by the integro-differential equation

$$l(y) = c^*(y)\left\{q\, m\, c_R\, \alpha(y) - \frac{\mu}{\pi}\oint_{-1}^{1} \frac{dl}{d\eta}\, \frac{d\eta}{y-\eta}\right\} \tag{1a}$$

and the boundary conditions

$$l(\pm 1) = 0 \tag{1b}$$

where y is a spanwise coordinate measured from the root in units of the half span $b/2$, $c^*(y)$ is the ratio of the chord to the root chord,

$$c^*(y) = \frac{c(y)}{c_R} \tag{2}$$

q is the dynamic pressure, m is a profile constant $\left(\frac{dc_l}{d\alpha}\right)$, and μ is the dimensionless constant defined by the equation

$$\mu = \frac{m}{4}\, \frac{c_R}{b} \tag{3}$$

The integral appearing in equation (1) is not a proper integral, but is to be assigned its Cauchy principal value, according to the definition

$$\oint_{-1}^{1} = \lim_{\epsilon \to 0}\left\{\int_{-1}^{y-\epsilon} + \int_{y+\epsilon}^{1}\right\}$$

If the notation

$$F(y) = \frac{l(y)}{q\, m\, c_R} \tag{4}$$

is introduced, equation (1) can be written in the form

$$\left.\begin{aligned} \frac{F(y)}{c^*(y)} + \frac{\mu}{\pi}\oint_{-1}^{1}\frac{dF}{d\eta}\,\frac{d\eta}{y-\eta} &= \alpha(y) \\ F(\pm 1) &= 0 \end{aligned}\right\} \tag{5}$$

The usual method of obtaining an approximate solution to this equation consists in assuming that $F(y)$ can be approximated satisfactorily in the range $|y| \leqslant 1$ by a finite series of appropriate functions

$$F(y) \approx \sum_{n=1}^{N} A_n\, \varphi_n(y) \tag{6}$$

so that equation (5) becomes

$$\sum_{n=1}^{N} A_n\, \Phi_n(y) \approx \alpha(y) \tag{7}$$

where

$$\Phi_n(y) = \frac{\varphi_n(y)}{c^*(y)} + \frac{\mu}{\pi}\oint_{-1}^{1}\frac{d\varphi_n}{d\eta}\,\frac{d\eta}{y-\eta} \tag{8}$$

and in then determining the parameters A_n by requiring that equation (7) be a true equality at N arbitrarily chosen points (reference 1). This procedure thus leads to a set of N linear equations

$$\left.\begin{array}{c}\sum_{n=1}^{N} A_n \Phi_n(y_k) = \alpha(y_k) \\ k = 1, 2, \ldots N\end{array}\right\} \qquad (9)$$

in the N undetermined parameters A_n.

A more elaborate procedure consists in determining the parameters by requiring that the integral of the square of the difference between the two sides of equation (7), over the range $|y| \leq 1$, be a minimum (reference 2),

$$\int_{-1}^{1} \left[\sum_{n=1}^{N} A_n \Phi_n(y) - \alpha(y)\right]^2 dy = \text{minimum} \qquad (10)$$

Equating the partial derivatives to zero leads again to a set of N linear equations

$$\left.\begin{array}{c}\sum_{n=1}^{N} A_n \int_{-1}^{1} \Phi_p(y)\Phi_n(y)dy = \int_{-1}^{1} \Phi_p(y)\,\alpha(y)\,dy \\ p = 1, 2, \ldots N\end{array}\right\} \qquad (11)$$

which serves to determine the constants A_n.

In order to avoid the integrations involved in equations (11) and reduce the approximate solution of equation (5) to a purely numerical process, there is here presented a modification of the least square procedure which has been used previously in connection with other problems (reference 6). This method consists basically in approximating the integral in equation (10) by a sum of weighted values of the integrand, so that equation (10) is replaced by the condition

$$\sum_{k=1}^{M} D_k \left[\sum_{n=1}^{N} A_n \Phi_n(y_k) - \alpha(y_k)\right]^2 = \text{minimum} \qquad (12)$$

where D_k is the integration coefficient associated with the value of the integrand at the point y_k.

Equations (11) are now replaced by the following set of N linear equations:

$$\left.\begin{array}{c}\sum_{n=1}^{N} A_n \sum_{k=1}^{M} D_k \Phi_p(y_k)\, \Phi_n(y_k) = \sum_{k=1}^{M} D_k \Phi_p(y_k)\, \alpha(y_k) \\ p = 1, 2, \ldots N\end{array}\right\} \quad (13)$$

With the notation of the preceding section, the coefficients of this set of equations can be written in matrix form

$$\left\| D_k \Phi_n(y_k) \right\| \odot \left\| \Phi_n(y_k) \right\| = \left\| D_k \Phi_n(y_k) \right\| \odot \left\| \alpha(y_k) \right\| \quad (14)$$

where the row and column indices are k and n, respectively. It follows that the coefficients in equations (13) can be conveniently obtained by first writing the set of equations

$$\left.\begin{array}{c}\sum_{n=1}^{N} A_n \Phi_n(y_k) = \alpha(y_k) \\ k = 1, 2, \ldots M\end{array}\right\} \quad (15)$$

in matrix form, and by then forming the dot product of the auxiliary matrix

$$\left\| D_k \Phi_n(y_k) \right\|$$

into both sides of the resultant formal equation.

While equations (15) are of the same form as equations (9), the points y_k now correspond to the weighted ordinates in equation (12), so that the range of k is arbitrary. Thus, in place of satisfying equation (7) at a number of points, N, equal to the number of undetermined parameters, the present procedure satisfies this equation as nearly as possible at an arbitrary number of points, M.

Since equations (13) are homogeneous in the weighting coefficients D_k, these coefficients may be chosen as any convenient multiple of the actual integration coefficients. Also, it may be desirable in certain cases to interpret equation (12) otherwise than as an approximation to equation (10). That is, in place of requiring that the mean square of the difference between the two sides of the lifting-line equation be a minimum, it may be desirable to require that the equation be more nearly satisfied at some points than at others. The coefficients D_k are then of the nature of "influence coefficients" and may be determined by magnifying certain of the integration coefficients in proportion to the degree of satisfaction desired at corresponding intervals along the span.

If the computed values of the parameters are substituted in the left-hand sides of equations (15) the values of the angle of attack $\bar{\alpha}$ corresponding to the approximate lift distribution specified by equation (6) are determined at M points along the span. A comparison of this angle-of-attack distribution with the prescribed distribution α will give an indication of the degree of approximation attained in the solution.

OUTLINE OF THE PROCEDURE FOR NUMERICAL SOLUTION OF THE EQUATION

In this section a procedure involving a five-term approximation to $F(y)$ is explicitly developed for the case when $c(y)$ and $\alpha(y)$ are symmetrical with respect to the wing root. An analogous procedure can be developed for the treatment of the case of an anti-symmetrical angle of attack.

It is convenient to choose the approximating functions φ_n of equation (6) so that the functions defined by the integrals

$$\frac{\mu}{\pi}\int_{-1}^{1}\frac{d\varphi_n}{d\eta}\frac{d\eta}{y-\eta}$$

have a simple analytical expression and are regular in the neighborhood of the wing tips ($y = \pm 1$). Such functions are, in general, characterized by the property

$$\left.\begin{aligned} \varphi_n(y) &\sim (\text{const.})\,(1-y^2)^{\frac{2n-1}{2}} \\ y &\rightarrow \pm 1 \end{aligned}\right\} \qquad (16)$$

where n is a positive integer. In addition, the functions φ_n should be of a form readily adaptable to the approximation of the function F, the characteristic behavior of which usually is known.

In the present procedure an approximation to $F(y)$ is assumed in the form

$$F(y) \approx B\,y^2 \log\left(\frac{1+\sqrt{1-y^2}}{|y|}\right) + \sqrt{1-y^2}\sum_{n=0}^{3} A_{2n}\,y^{2n} \qquad (17)$$

With the exception of the first term, the approximating functions are conventional ones employed elsewhere. The coefficient of B, which is of the form required by equation (16), was originally chosen for use in cases when $\alpha(y)$ has a discontinuous first derivative at the root (e.g., in the case of a symmetrically linear angle of attack), since the contribution of this term to the integral representing the induced angle of attack,

$$\frac{\mu}{\pi}\int_{-1}^{1}\frac{d}{d\eta}\left[\eta^2 \log\left(\frac{1+\sqrt{1-\eta^2}}{|\eta|}\right)\right]\frac{d\eta}{y-\eta} = \mu(\pi|y|-1)$$

has a discontinuous derivative at the root $(y=0)$, while the function itself has a continuous derivative at this point. The function was, however, retained for use in the more general case since it complements the other approximating functions, being intermediate in behavior between the functions $\sqrt{1-y^2}$ and $y^2\sqrt{1-y^2}$. (See figure 1.)

With the approximation of equation (17) the lifting-line equation (5) becomes

$$\frac{1}{c^*(y)}\left\{B y^2 \log\left(\frac{1+\sqrt{1-y^2}}{[y]}\right) + \sqrt{1-y^2}\left[A_0+A_2y^2+A_4y^4+A_6y^6\right]\right\}$$
$$+\mu\left\{B\left(\pi[y]-1\right)+A_0+A_2\left(3y^2-\frac{1}{2}\right)+A_4\left(5y^4-\frac{3}{2}y^2-\frac{1}{8}\right)\right.$$
$$\left.+A_6\left(7y^6-\frac{5}{2}y^4-\frac{3}{8}y^2-\frac{1}{16}\right)\right\}$$
$$\approx \alpha(y) \qquad [y] \leqslant 1 \tag{18}$$

For the approximate integration indicated in equation (11) nine points y_k are chosen, equally spaced over the interval $0 \leqslant y \leqslant 1$, so that $y_k = k/8$, θ : $k = 0, 1, 2, \ldots 8$. If equation (18) is evaluated at these points, nine linear equations in the five parameters are obtained the coefficients of which are written in matrix form in equation (19).

	*	B	A_0	A_2	A_4	A_6
1		.00000	1.00000	.00000	.00000	.00000
$\frac{1}{c^*(1/8)}$		.04326	.99216	.01550	.00024	.00000
$\frac{1}{c^*(1/4)}$		.12897	.96825	.06052	.00378	.00024
$\frac{1}{c^*(3/8)}$		.23018	.92702	.13036	.01833	.00258
$\frac{1}{c^*(1/2)}$	*	.32924	.86602	.21651	.05413	.01353
$\frac{1}{c^*(5/8)}$		.40897	.78062	.30493	.11911	.04653
$\frac{1}{c^*(3/4)}$		.44739	.66144	.37206	.20928	.11772
$\frac{1}{c^*(7/8)}$		.40452	.48412	.37066	.28378	.21727
11		$\left[\frac{\sqrt{1-y^2}}{c^*(y)}\right]_{y=1}$	$\left[\frac{\sqrt{1-y^2}}{c^*(y)}\right]_{y=1}$	$\left[\frac{\sqrt{1-y^2}}{c^*(y)}\right]_{y=1}$	$\left[\frac{\sqrt{1-y^2}}{c^*(y)}\right]_{y=1}$	$\left[\frac{\sqrt{1-y^2}}{c^*(y)}\right]_{y=1}$

	B	A_0	A_2	A_4	A_6		
	−1.00000	1.00000	−.50000	−.12500	−.06250		$\alpha(0)$
	−.60730	1.00000	−.45312	−.14722	−.06894		$\alpha(1/8)$
	−.21460	1.00000	−.31250	−.19922	−.09309		$\alpha(1/4)$
	.17810	1.00000	−.07812	−.23706	−.14521		$\alpha(3/8)$
$+\mu$	.57080	1.00000	−.25000	−.18750	−.20312	=	$\alpha(1/2)$ (19)
	.96350	1.00000	.67188	.05200	−.17322		$\alpha(5/8)$
	1.35619	1.00000	1.18750	.61328	.18140		$\alpha(3/4)$
	1.74889	1.00000	1.79688	1.65747	1.34811		$\alpha(7/8)$
	2.14159	1.00000	2.50000	3.37500	4.06250		$\alpha(1)$

The elements in the last row of the second matrix are all zeros, unless the chord tapers to zero at the tips at least as rapidly as $\sqrt{1-y^2}$. In case the wing approximates an elliptical plan form near the tips, all these elements have an equal non-zero value which can be determined in a simple manner. If a constant λ is determined so that, in the immediate neighborhood of the tips,

$$\frac{c(y)}{c_R} \approx \lambda \sqrt{1-y^2} \tag{20}$$

then these elements are to be assigned the value $1/\lambda$. While cases in which the chord tapers to zero more rapidly than $\sqrt{1-y^2}$ involve certain mathematical difficulties, it is probable that for wings with rounded tips the error introduced by using a tip approximation of the form given in equation (20) would not be great.

If now the values of the chord and angle of attack are known at the nine points along the span, and if μ is prescribed, equation (19) can be put in the form

$$\begin{matrix} B & A_0 & A_2 \ldots\ldots & A_6 \end{matrix}$$

$$\left\| \begin{matrix} b_0 & a_{00} & a_{20} \ldots\ldots & a_{60} \\ b_1 & a_{01} & a_{21} \ldots\ldots & a_{61} \\ b_2 & a_{02} & a_{22} \ldots\ldots & a_{62} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ b_8 & a_{08} & a_{28} & a_{68} \end{matrix} \right\| = \left\| \begin{matrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_8 \end{matrix} \right\| \tag{21}$$

where, for example,

$$\left. \begin{matrix} b_0 = -\mu \qquad b_1 = \dfrac{.04326}{c^*(1/8)} - .60730\mu \\ a_{22} = \dfrac{.06052}{c^*(1/4)} - .31250\mu \ldots\ldots a_{88} = \left[\dfrac{\sqrt{1-y^2}}{c^*(y)}\right]_{y=1} + 4.06250\mu \end{matrix} \right\} \tag{22}$$

and

$$\alpha_k = \alpha(k/8) \tag{23}$$

In accordance with the results of the preceding section, the coefficients of the linear equations (13) are obtained by forming the dot product of an auxiliary matrix into both sides of formal equation (21). This auxiliary matrix is formed by multiplying the elements of the k–th row of the left–hand matrix of equation (21) by the weighting coefficient D_k associated with point y_k in equation (12). If it is required that the mean square of the difference between the two sides of the lifting–line equation be as small as possible, so that equation (12) is to approximate equation (10), these coefficients are to be proportional to a set of integration coefficients. The best results have been obtained, in general, if coefficients proportional to those of Simpson's rule are used (for a detailed discussion, see reference 6, pp. 319–323), so that

$$\left.\begin{aligned} D_0 = D_8 &= 1/2 \\ D_1 = D_3 = D_5 = D_7 &= 2 \\ D_2 = D_4 = D_6 &= 1 \end{aligned}\right\} \qquad (24)$$

and the auxiliary matrix is of the form

$$\left\| \begin{matrix} \frac{1}{2}b_0 & \frac{1}{2}a_{00} & \frac{1}{2}a_{20} & \cdots\cdots\cdots & \frac{1}{2}a_{60} \\ 2b_1 & 2\,a_{01} & 2\,a_{21} & \cdots\cdots\cdots & 2\,a_{61} \\ b_2 & a_{02} & a_{22} & \cdots\cdots\cdots & a_{62} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{2}b_8 & \frac{1}{2}a_{08} & \frac{1}{2}a_{28} & \cdots\cdots\cdots & \frac{1}{2}a_{68} \end{matrix} \right\| \qquad (25)$$

If the dot product of matrix (25) into both sides of equation (21) is formed, the coefficients of a set of five linear equations in the five parameters $B, A_0, \ldots . A_6$ are obtained the solution of which, in connection with equation (17), determines the lift distribution over the span of the wing. The amount of numerical work involved is considerably reduced if use is made of the fact that, as can be shown, the matrix of coefficients in the final set of equations is symmetrical with respect to its principal diagonal.

The data required to compute the values of $F(y)$ at nine points are tabulated in the second matrix of equation (19) (if the elements of the last row are replaced by zeros). Thus, for example,

$$\left.\begin{aligned} F(0) &= A_0 \\ F(1/8) &= 0.04326B + 0.99216A_0 + 0.01550\ A_2 + 0.00024\ A_4 \\ &\cdots\cdots\cdots\cdots \\ F(1) &= 0 \end{aligned}\right\} \quad (26)$$

With the auxiliary function $F(y)$ known, the lift distribution $l(y)$ is determined from the equation

$$l(y) = m\, q\, c_R\, F(y)$$

and the section lift coefficient $c_l(y)$ follows from the equation

$$c_l(y) = m \frac{c_R}{c(y)} F(y)$$

If L is the total lift acting on the wing and S is the projected area of the wing, the coefficient of total lift, C_L, defined by the relationship

$$C_L = \frac{L}{q\,S}$$

is given by the formula

$$C_L = 2\mu \frac{b^2}{S} \int_{-1}^{1} F(y)\,dy$$

or, with the approximation of equation (17),

$$C_L = \pi \mu \frac{b^2}{S} \left[\frac{1}{3} B + A_0 + \frac{1}{4} A_2 + \frac{1}{8} A_4 + \frac{5}{64} A_6 \right] \qquad (27)$$

An indication as to the accuracy of the solution is obtained if the values of the angle of attack $\overline{\alpha}$ given by the left-hand sides of the equations represented by formal equation (21) are compared with the prescribed values of α; for example,

$$\overline{\alpha}_k = b_k B + a_{ok} A_o + \ldots + a_{6k} A_6 \approx \alpha_k \qquad (28)$$

It may be mentioned that in place of using the approximation of equation (17), the lifting-line equation may be first transformed by the substitution

$$y = -\cos\theta \qquad 0 \leqslant \theta \leqslant \pi$$

after which, if an approximation to the function $F(y)$ is assumed of the form

$$F(y) \approx B(\cos^2\theta) \log \left(\frac{1 + \sin\theta}{|\cos\theta|} \right) + \sum A_n \sin n\theta \qquad (17a)$$

equation (18) is replaced by

$$\frac{1}{c^*(\cos\theta)} \left\{ B\,(\cos^2\theta) \log \left(\frac{1 + \sin\theta}{|\cos\theta|} \right) + \sum A_n \sin n\theta \right\}$$

$$+ \mu \left\{ B(\pi\,|\cos\theta| - 1) + \sum n A_n \frac{\sin n\theta}{\sin\theta} \right\} \approx \alpha\,(\cos\theta) \qquad (18a)$$

and the present least-squares procedure is again applicable.

In the preceding developments it was assumed that the profile constant m does not vary along the wing span; that is, that parallel chord sections of the wing are geometrically similar. If this is not the case, the effect of varying m can be taken into account if in equation (5) and the subsequent equations the function $c^*(y)$ is replaced by the function

$$\frac{m(y)}{m_R} c^*(y)$$

and the definitions of the constant μ and the function $F(y)$ are modified so as to read

$$\mu = \frac{m_R}{4} \frac{c_R}{b} \tag{3a}$$

$$F(y) = \frac{l(y)}{q\, m_R\, c_R} \tag{4a}$$

where m_R is the value of m at the root.

EXPLICIT SOLUTION FOR A RECTANGULAR WING

In order to illustrate the procedure of the preceding section, a wing of rectangular plan form and aspect ratio $\frac{b}{c} = m$ (≈ 6) is considered. In this case, equations (2) and (3) become

$$c^*(y) = 1 \tag{29}$$

and

$$\mu = \frac{1}{4} \tag{30}$$

Each element of the left-hand matrix of equation (21) is thus the sum of the corresponding element of the second matrix of equation (19) and one-fourth the corresponding element of the third matrix of that equation, so that equation (21) becomes

B	A_0	A_2	A_4	A_6			
-.25000	1.25000	-.12500	-.03125	-.01562		α_0	
-.10856	1.24216	-.09778	-.03656	-.01723		α_1	
-.07531	1.21825	-.01761	-.04602	-.02304		α_2	
.27470	1.17702	-.11083	-.04093	-.03372		α_3	
.47194	1.11602	.27901	.00725	-.03725	=	α_4	(31)
.64985	1.03062	.47290	.13211	.00322		α_5	
.78644	.91144	.66893	.36260	.16307		α_6	
.84175	.73412	.81988	.69815	.55430		α_7	
.53540	.25000	.62500	.84375	1.01562		α_8	

The auxiliary matrix is now formed by multiplying the elements in the k-th row of the left-hand matrix by the weighting coefficient D_k, as given in equation (24). If the dot product of this matrix into both sides of equation (31) is taken, the following five linear equations are obtained:

$$\begin{array}{ccccc} B & A_0 & A_2 & A_4 & A_6 \end{array}$$

$$\begin{Vmatrix} 3.47075 & 4.19825 & 2.91636 & 1.84739 & 1.30533 \\ 4.19825 & 13.43179 & 3.09615 & 1.47864 & .89447 \\ 2.91636 & 3.09615 & 2.56410 & 1.77884 & 1.32531 \\ 1.84739 & 1.47864 & 1.77884 & 1.50586 & 1.26747 \\ 1.30533 & .89447 & 1.32531 & 1.26747 & 1.16176 \end{Vmatrix} = \begin{Vmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Vmatrix} \qquad (32)$$

where

$$\begin{Vmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Vmatrix} = \begin{Vmatrix} -.12500 & .62500 & -.06250 & -.01562 & -.00781 \\ -.21713 & 2.48431 & -.19556 & -.07312 & -.03446 \\ .07531 & 1.21825 & -.01761 & -.04602 & -.02304 \\ .54940 & 2.35405 & .22166 & -.08189 & -.06745 \\ .47194 & 1.11602 & .27901 & .00725 & -.03725 \\ 1.29969 & 2.06125 & .94580 & .26423 & .00645 \\ .78644 & .91144 & .66893 & .36260 & .16307 \\ 1.68349 & 1.46825 & 1.63975 & 1.39630 & 1.10860 \\ .26770 & .12500 & .31250 & .42188 & .50781 \end{Vmatrix} \odot \begin{Vmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Vmatrix} \qquad (33)$$

It may be remarked that the calculation of each element in equation (32) involves only a single continuous operation on a computing machine. For example, the coefficient of A_2 in the second row of equation (32) was obtained as the algebraic sum of products of corresponding elements of the second column of the auxiliary matrix (the first matrix on the right-hand side of equation (33)) and the third column of the left-hand matrix of equation (31):

$$-(.62500)(.12500) - (2.48431)(.09778) - (1.21825)(.01761) + \ldots$$
$$\ldots + (.12500)(.62500) = 3.09615$$

Since the matrix on the left-hand side of equation (32) is symmetrical, it was not necessary to compute the elements below its principal diagonal.

The solution of equations (32) is also conveniently carried out on a computing machine, the process being considerably shortened if account is taken of the symmetry in the matrix of coefficients. (See, for example, reference 5.)

In the case of a uniform angle of attack, for which $\alpha(y) = \alpha_T$ = constant, the right-hand matrix of equation (32) as determined from equation (33), is found to be

$$\alpha_T \begin{Vmatrix} 4.79185 \\ 12.36357 \\ 3.79199 \\ 2.23562 \\ 1.61592 \end{Vmatrix}$$

and the solution of equations (32) then gives

$$\left.\begin{aligned} B &= -0.25469\ \alpha_T \qquad A_0 = 0.82881\ \alpha_T \qquad A_2 = 0.98119\ \alpha_T \\ A_4 &= +1.32373\ \alpha_T \qquad A_6 = 1.36382\ \alpha_T \end{aligned}\right\} \quad (34)$$

The coefficient of total lift C_L is determined by equation (27),

$$C_L = 0.73065\ m\ \alpha_T \quad (35)$$

In table 1 and figure 2(a) the auxiliary lift function $F(y)$ (computed from equations (26)) is compared with the corresponding solution given by Glauert (reference 1). This solution was obtained by assuming a four-term approximation to $F(y)$ of the form given in equation (17a) (omitting the first term) and by then satisfying the lifting-line equation exactly at four points, not including the wing tip. The values of the left-hand side of the basic equation (5) (computed from equations (19)) are compared with the corresponding values determined from the Glauert solution in table 2. It is seen that the Glauert solution satisfies the lifting line equation extremely well except in the immediate vicinity of the wing tip where a large deviation occurs, while for the present solution the equation is reasonably well satisfied along the span. The two lift distributions agree very closely, however, except near the tip of the wing, and the Glauert lift coefficient, $C_L = 0.729\ m\ \alpha_T$, is nearly identical with the result of equation (35).

In the case of a <u>symmetrically linear angle of attack</u>, for which $\alpha(y) = \alpha_T |y|$, the right-hand matrix of equation (32) is found to be

$$\alpha_T \begin{Vmatrix} 3.57658 \\ 5.43746 \\ 3.03389 \\ 2.03302 \\ 1.55018 \end{Vmatrix}$$

and equations (32) give

$$\left.\begin{aligned} B &= 0.11229\ \alpha_T \qquad A_0 = 0.16731\ \alpha_T \qquad A_2 = 1.25057\alpha_T \\ A_4 &= -1.67588\ \alpha_T \qquad A_6 = 1.48110\ \alpha_T \end{aligned}\right\} \quad (36)$$

The coefficient of total lift is then

$$C_L = 0.33270\ m\ \alpha_T$$

The variation of the function $F(y)$ is presented in table 1 and figure 2(b) in comparison with the Glauert solution, while in table 2 the left-hand side of equation (5) is evaluated for the two solutions and compared with the prescribed right-hand side. In this case the actual angle of attack $\bar{\alpha}$ corresponding to the Glauert solution deviates appreciably from the prescribed angle of attack α both near the root $(y = 0)$ and near the tip $(y = 1)$. The root deviation, which is due to the fact that the curve representing the function $\alpha(y)$ has a discontinuous derivative at $y = 0$, is decreased in the present solution by the presence of the first term in the approximation of equation (17), and the tip deviation for the present solution is also less pronounced. Corresponding differences occur between the two lift distributions, and the Glauert lift coefficient, $C_L = 0.320\ m\ \alpha_T$, differs from the result of equation (37) by about 4 percent.

In the case of a quadratic angle of attack, for which $\alpha(y) = \alpha_T y^2$, the right-hand matrix of equation (32) takes the form

$$\alpha_T \begin{Vmatrix} 2.70325 \\ 3.29199 \\ 2.41043 \\ 1.78438 \\ 1.43005 \end{Vmatrix}$$

The solution of equation (32) is found to be

$$\left.\begin{aligned} B &= -0.20524\ \alpha_T \qquad A_0 = 0.06665\ \alpha_T \qquad A_2 = 1.20613\ \alpha_T \\ A_4 &= -1.00759\ \alpha_T \qquad A_6 = 1.13356\ \alpha_T \end{aligned}\right\} \quad (37)$$

and equation (27) gives

$$C_L = 0.20607\ m\ \alpha_T$$

The lift distribution corresponding to this solution is presented in table 1 and figure 2(c) while in table 2 the functions $\bar{\alpha}$ and α are compared. This problem was not considered by Glauert. The lift distribution in this case is more nearly concentrated at the wing tip and the maximum lift value is only about 42 percent of the maximum value for the corresponding wing without twist. Also, a higher degree of approximation is indicated by the agreement of the two sides of the lifting-line equation.

MODIFIED PROCEDURE FOR WINGS WITH DISCONTINUOUS ANGLE OF ATTACK

It is known that at a point of discontinuity in the angle of attack the lift-distribution curve has an infinite derivative, so that an approximation of the type used in equation (17) could not be expected to give an accurate result. In such a case, in order that the right- and left-hand sides of equation (5) have the same discontinuity at a point, the function $F(y)$ must be continuous but must have the property that the integral

$$\frac{\mu}{\pi}\oint_{-1}^{1}\frac{dF}{d\eta}\,\frac{d\eta}{y-\eta} \tag{38}$$

has the same discontinuity as the angle of attack, α.

As can be verified in an elementary manner, the function

$$\Gamma_a(y) = (y-a)\log\left(\frac{1-ay+\sqrt{1-a^2}\,\sqrt{1-y^2}}{|y-a|}\right) \tag{39}$$

satisfies the equation

$$\frac{1}{\pi^2}\oint_{-1}^{1}\frac{d\Gamma_a}{d\eta}\,\frac{d\eta}{y-\eta} = \begin{cases} -\frac{1}{\pi}\cos^{-1}a & -1 \leq y < a \\ 1-\frac{1}{\pi}\cos^{-1}a & a < y \leq 1 \end{cases} \tag{40}$$

and has the correct behavior at the tips, since

$$\Gamma_a(y) \to \pm\sqrt{1-a^2}\,\sqrt{1-y^2} \qquad y \longrightarrow \pm 1$$

Thus the integral

$$\frac{\mu}{\pi}\oint_{-1}^{1}\frac{d\Gamma_a}{d\eta}\,\frac{d\eta}{y-\eta} \tag{41}$$

is piecewise constant in the interval $|y| \leq 1$ and has a jump of magnitude $\pi\mu$ at the point $y = a$. Suppose that $\alpha(y)$ is continuous except for finite jumps of magnitude J_n at the points $y = a_n$.

$$\alpha(a_n+) - \alpha(a_n-) = J_n \tag{42}$$

Then, if the function

$$P(y) = \frac{1}{\pi\mu}\sum_n J_n\,\Gamma_{a_n}(y) \tag{43}$$

is defined, it follows that the expression

$$\alpha(y) - \frac{\mu}{\pi} \oint_{-1}^{1} \frac{dP}{d\eta} \frac{d\eta}{y-\eta}, \tag{44}$$

is continuous.

This fact suggests the following procedure for solving the lifting-line equation (5) in such cases. If an approximation to the lift distribution function is assumed of the form

$$F(y) = P(y) - F_0(y) \tag{45}$$

where P is defined by equation (43) and F_0 is given by the right-hand side of equation (17), then equation (5) can be written

$$\frac{F_0(y)}{c^*(y)} + \frac{\mu}{\pi} \oint_{-1}^{1} \frac{dF_0}{d\eta} \frac{d\eta}{y-\eta} = \alpha^*(y) \tag{46}$$

where

$$\alpha^*(y) = \frac{P(y)}{c^*(y)} + \frac{\mu}{\pi} \oint_{-1}^{1} \frac{dP}{d\eta} \frac{d\eta}{y-\eta} - \alpha(y) \tag{47}$$

Equation (46) is formally equivalent to equation (5) if $\alpha(y)$ is replaced by $\alpha^*(y)$ and, since $\alpha^*(y)$ is continuous, the solution can be carried out as outlined in the preceding sections.

In calculating the total lift coefficient, C_L, the value of the integral

$$\int_{-1}^{1} \Gamma_a(y)\, dy = -\frac{\pi a}{2}\sqrt{1-a^2} \tag{48}$$

is needed.

As an example, suppose that the aileron deflection of a wing is such that

$$\alpha(y) = \begin{cases} 0 & -1 \leq y < -a \\ 1 & -a < y < a \\ 0 & a < y \leq 1 \end{cases} \tag{49}$$

In this case equation (43) becomes

$$P(y) = \frac{1}{\pi\mu}\left\{\Gamma_{-a}(y) - \Gamma_a(y)\right\}$$

or

$$P(y) = \frac{1}{\pi\mu}\left\{-(y-a)\log\left[\frac{1-ay+\sqrt{1-a^2}\sqrt{1-y^2}}{|y-a|}\right] + (y+a)\log\left[\frac{1+ay+\sqrt{1-a^2}\sqrt{1-y^2}}{|y+a|}\right]\right\} \tag{50}$$

Also, using equation (40), it is found that

$$\frac{\mu}{\pi}\oint_{-1}^{1}\frac{dP}{d\eta}\frac{d\eta}{y-\eta} = \frac{1}{\pi^2}\left\{-\oint_{-1}^{1}\frac{d\Gamma_a}{d\eta}\frac{d\eta}{y-\eta} + \oint_{-1}^{1}\frac{d\Gamma_{-a}}{d\eta}\frac{d\eta}{y-\eta}\right\}$$

$$= \begin{cases} -\left[-\frac{1}{\pi}\cos^{-1}a\right] + \left[-\frac{1}{\pi}(\pi-\cos^{-1}a)\right] & -1 \leqslant y < -a \\ -\left[-\frac{1}{\pi}\cos^{-1}a\right] + \left[1-\frac{1}{\pi}(\pi-\cos^{-1}a)\right] & -a < y < a \\ -\left[1-\frac{1}{\pi}\cos^{-1}a\right] + \left[1-\frac{1}{\pi}(\pi-\cos^{-1}a)\right] & a < y \leqslant 1 \end{cases}$$

$$\frac{\mu}{\pi}\oint_{-1}^{1}\frac{dP}{d\eta}\frac{d\eta}{y-\eta} =$$

$$= \begin{cases} \frac{2}{\pi}\cos^{-1}a - 1 & -1 \leqslant y < -a \\ \frac{2}{\pi}\cos^{-1}a & -a < y < a \\ \frac{2}{\pi}\cos^{-1}a - 1 & a < y \leqslant 1 \end{cases}$$

$$= \alpha(y) - \left(1 - \frac{2}{\pi}\cos^{-1}a\right) \tag{51}$$

Thus, finally, equation (47) gives

$$\alpha^*(y) = \frac{P(y)}{c^*(y)} - \left(1 - \frac{2}{\pi}\cos^{-1}a\right) \tag{52}$$

EXPLICIT SOLUTION FOR A TAPERED WING WITH ROUNDED TIPS AND PARTIAL-SPAN FLAPS

As an illustration of the modified procedure, a wing analyzed in reference 4 is now considered. The variation of the chord ratio $c(y)/c_R$ is represented in figure 3, while the following additional numerical data are given:

$$\mu = 0.1919 \qquad \frac{b^2}{S} = 10 \tag{53}$$

In the immediate neighborhood of the wing tips, it is found that, approximately,

$$c^*(y) \approx 1.64\sqrt{1 - y^2} \qquad y \longrightarrow \pm 1$$

and it follows from equation (20) that

$$\lambda = 1.64 \tag{54}$$

An angle of attack of 1 radian from 0 to 0.489 and 0 from 0.489 to 1 is prescribed, so that, in the notation of the example of the preceding section,

$$a = 0.489 \tag{55}$$

The equation to be solved is then

$$\frac{F_0(y)}{c^*(y)} + \frac{\mu}{\pi}\int_{-1}^{1} \frac{dF_0}{d\eta}\,\frac{d\eta}{y-\eta} = \alpha^*(y) \tag{56}$$

where

$$F_0(y) = P(y) - F(y) \tag{57}$$

and $P(y)$ and $\alpha^*(y)$ are defined in equations (50) and (52).

The functions $c^*(y)$, $P(y)$ and $\alpha^*(y)$ are tabulated at the nine points considered in table 3.

If the operations indicated in equation (19) (where α is replaced by α^*) are carried out, equation (21) takes the form

$$\begin{array}{ccccc} B & A_0 & A_2 & A_4 & A_6 \end{array}$$

$$\begin{Vmatrix} -.19190 & 1.19190 & -.09595 & -.02399 & -.01199 \\ -.07040 & 1.25021 & -.07042 & -.02800 & -.01323 \\ .10621 & 1.29847 & .00920 & -.03391 & -.01759 \\ .31748 & 1.33285 & .14545 & -.02293 & -.02469 \\ .54852 & 1.34659 & .33665 & -.03619 & -.02094 \\ .77976 & 1.32734 & .57247 & .18323 & .03444 \\ .97608 & 1.25020 & .82318 & .45254 & .22316 \\ 1.05476 & 1.05256 & 1.00377 & .82257 & .64496 \\ 1.02073 & .80166 & 1.08951 & 1.25742 & 1.38935 \end{Vmatrix} = \begin{Vmatrix} 1.85269 \\ 1.94784 \\ 1.94088 \\ 1.76657 \\ 1.09023 \\ .49025 \\ .09273 \\ -.17629 \\ -.32527 \end{Vmatrix} \tag{58}$$

The coefficients of the final set of linear equations, obtained by taking the dot product of the auxiliary matrix defined in equation (25) into both sides of equation (58), are then found,

$$\begin{array}{ccccc} B & A_0 & A_2 & A_4 & A_6 \end{array}$$

$$\begin{Vmatrix} 5.45683 & 7.35230 & 4.66690 & 3.11236 & 2.31514 \\ 7.35230 & 18.51243 & 5.71837 & 3.14707 & 2.12795 \\ 4.66690 & 5.71837 & 4.11195 & 2.92894 & 2.26282 \\ 3.11236 & 3.14707 & 2.92894 & 2.42110 & 2.05002 \\ 2.31514 & 2.12795 & 2.26282 & 2.05002 & 1.85166 \end{Vmatrix} = \begin{Vmatrix} 1.76261 \\ 15.54154 \\ .62091 \\ -.51871 \\ -.60723 \end{Vmatrix} \tag{59}$$

and the solution of the corresponding set of equations is

$$B = 4.20498 \qquad A_0 = 1.40506 \qquad A_2 = -9.94467$$
$$A_4 = 6.24691 \qquad A_6 = -1.96339 \tag{60}$$

In table 4 and figure 4 the auxiliary lift function $F(y)$, determined by the equation

$$F(y) = P(y) - F_0(y)$$

is presented in comparison with a solution given by Pearson (reference 4). This solution was obtained by using a ten-term series of the type given in equation (17a) (omitting the first term) and determining the parameters by a method given by Miss Lotz (reference 3). For further comparison there is included in figure 4 a solution obtained by the present least-squares procedure with an approximation of the type of equation (17), that is, without using the additional approximating function P(y). While the Pearson solution agrees closely with the present solution over a large part of the span, it appears that, even with a ten-term approximation, the correct behavior of the left curve cannot be satisfactorily approximated near the end of the flap except by the use of a term similar to the function P. The values of the angle of attack corresponding to the present solution and to the Pearson solution are compared with the prescribed values of α in table 5 and figure 5.

MODIFIED PROCEDURE FOR WINGS WITH DISCONTINUOUS CHORD VARIATION

Suppose that the angle of attack $\alpha(y)$ is continuous and that the chord $c(y)$ is continuous except for finite jumps at the points $y = a_n$. Then the function $F(y)/c^*(y)$ has corresponding discontinuities of magnitude $-\gamma_n F(a_n)$ where

$$\gamma_n = -\left[\frac{1}{c^*(a_n+)} - \frac{1}{c^*(a_n-)}\right] = \frac{c^*(a_n+) - c^*(a_n-)}{c^*(a_n+)c^*(a_n-)} \tag{61}$$

Hence, since α is continuous, the left-hand side of equation (5) must be continuous and the function

$$\frac{\mu}{\pi}\int_{-1}^{1}\frac{dF}{d\eta}\,\frac{d\eta}{y-\eta}$$

must have discontinuities of magnitude $+\gamma_n F(a_n)$ at the points $y = a_n$.

If the function $F(y)$ is written in the form

$$F(y) = F_1(y) - Q(y) \tag{62}$$

where

$$Q(y) = \frac{1}{\pi\mu} \sum_n K_n \Gamma_{a_n}(y) \qquad (63)$$

and $\Gamma_a(y)$ is defined by equation (39), the constants K_n can be determined so that the left-hand side of equation (5) is continuous and consequently the function $F_1(y)$ has a finite derivative inside the interval $|y| < 1$. For, according to equation (40), if equation (62) is introduced into equation (5) the left-hand side of the resulting equation has, at each point $y = a_m$, a discontinuity of magnitude.

$$-\gamma_m \left\{ F_1(a_m) - \frac{1}{\pi\mu} \sum_n K_n \Gamma_{a_n}(a_m) \right\} - K_m$$

It follows that the discontinuities will disappear if the constants K_n satisfy the equations

$$\sum_n K_n \left\{ \gamma_m \Gamma_{a_n}(a_m) - \pi \mu \delta_{mn} \right\} = \pi \mu \gamma_m F_1(a_m) \qquad (64)$$

$$m = 1, 2, 3, \ldots$$

where

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \qquad (65)$$

The constants K_n are thus determined from equation (64) as linear combinations of the values of $F_1(y)$ at the points of discontinuity.

Equation (5) can now be written in the form

$$\frac{F_1(y)}{c^*(y)} + \frac{\mu}{\pi} \oint_{-1}^{1} \frac{dF_1}{d\eta} \frac{d\eta}{y-\eta} - \left[\frac{Q(y)}{c^*(y)} + \frac{\mu}{\pi} \oint_{-1}^{1} \frac{dQ}{d\eta} \frac{d\eta}{y-\eta} \right] = \alpha(y) \qquad (66)$$

Since the function $F_1(y)$ has a finite derivative inside the interval $|y| < 1$, an approximation to $F_1(y)$ can be assumed in the form given by equation (17) and the term in brackets in equation (66) becomes a known linear combination of the parameters B and A_{2n}. If this term is evaluated at the points y_k with the help of equation (40) and if the set of coefficients of B and A_{2n} in the resultant expressions is written as a matrix, the matrix equation replacing equation (21) is obtained by subtracting this matrix from the left-hand side of equation (19). The least-squares procedure can then be applied as before.

As an example, suppose that the chord variation of a symmetrical wing has discontinuities of equal magnitude and opposite sign at the points $y = \pm a$ and write

$$\gamma = \frac{c^*(a+) - c^*(a-)}{c^*(a+)\ c^*(a-)} = -\frac{c^*(-a+) - c^*(-a-)}{c^*(-a+)\ c^*(-a-)} \qquad (67)$$

Then equation (63) becomes

$$Q(y) = \frac{1}{\pi\mu}\left[K_{-a}\Gamma_{-a}(y) + K_a\Gamma_a(y)\right] \qquad (68)$$

Also, since $F(y)$ is an even function of y and since

$$\left.\begin{aligned} \Gamma_a(a) &= \Gamma_{-a}(-a) = 0 \\ \Gamma_\alpha(-a) &= -\Gamma_{-a}(a) = 2a\log a \end{aligned}\right\} \qquad (69)$$

equation (64) becomes

$$\left.\begin{aligned} -\pi\mu K_{-a} - 2\gamma a\log a\ K_a &= -\pi\mu\gamma F_1(a) \\ -2\gamma a\log a\ K_{-a} - \pi\mu K_a &= \pi\mu\gamma F_1(a) \end{aligned}\right\} \qquad (70)$$

so that

$$K_a = -K_{-a} = \frac{\pi \mu \gamma}{2\gamma a \log a - \pi\mu} F_1(a) \qquad (71)$$

Thus equation (68) becomes

$$Q(y) = \frac{\gamma F_1(a)}{2\gamma a \log a - \pi\mu}\left\{\Gamma_{-a}(y) - \Gamma_a(y)\right\} = \frac{\pi\mu \gamma F_1(a)}{\pi\mu - 2\gamma a \log a} P(y) \qquad (72)$$

where $P(y)$ is defined by equation (50), and the integral equation (66) can be written in the form

$$\frac{F_1(y)}{c^*(y)} + \frac{\mu}{\pi}\oint_{-1}^{1} \frac{dF_1}{d\eta}\frac{d\eta}{y-\eta} - \beta(y) F_1(a) = \alpha(y) \qquad (73)$$

where

$$\beta(y) = \frac{\pi \mu \gamma}{\pi\mu - 2\gamma a \log a}\left\{\frac{P(y)}{c^*(y)} + \frac{\mu}{\pi}\oint_{-1}^{1} \frac{dP}{d\eta}\frac{d\eta}{y-\eta}\right\} \qquad (74)$$

EXPLICIT SOLUTION FOR A RECTANGULAR WING WITH A CUT-OUT

As an illustration of the procedure of the preceding section, the effect of a rectangular cut-out on the lift distribution over a rectangular wing with an original aspect ratio of magnitude $\frac{b}{c} = m$ without cut-out is analyzed. The length of the cut-out is taken to be one-quarter of the span and the width to be one-quarter of the chord (fig. 6). In this case there follows

$$\left.\begin{aligned} \mu &= \frac{3}{16} \\ a &= \frac{1}{4} \\ c^*(y) &= \begin{cases} 1, & |y| < \frac{1}{4} \\ \frac{4}{3}, & \frac{1}{4} < |y| \leq 1 \end{cases} \end{aligned}\right\} \qquad (75)$$

and equation (67) gives

$$\gamma = \frac{1}{4} \tag{76}$$

From equation (50),

$$P(y) = \frac{16}{3\pi}\left\{-\left(y-\frac{1}{4}\right)\log\left[\frac{1-\frac{1}{4}y+\frac{\sqrt{3}}{2}\sqrt{1-y^2}}{\left|y-\frac{1}{4}\right|}\right] + \left(y+\frac{1}{4}\right)\log\left[\frac{1+\frac{1}{4}y+\frac{\sqrt{3}}{2}\sqrt{1-y^2}}{\left|y+\frac{1}{4}\right|}\right]\right\} \tag{77}$$

and from equation (51),

$$\frac{\mu}{\pi}\int_{-1}^{1}\frac{dP}{d\eta}\frac{d\eta}{y-\eta} = \begin{cases} \frac{2}{\pi}\cos^{-1}\left(\frac{1}{4}\right), & |y| < \frac{1}{4} \\ \frac{2}{\pi}\cos^{-1}\left(\frac{1}{4}\right) - 1, & \frac{1}{4} < |y| \leq 1 \end{cases} \tag{78}$$

If $F_1(y)$ is assumed in the form given by equation (17), there follows

$$F_1(a) = F_1\left(\frac{1}{4}\right) = 0.12879\,B + 0.96825\,A_0 + 0.06052\,A_2 + 0.00378\,A_4 + 0.00024\,A_6 \tag{79}$$

The function $\beta(y)$ of equation (74) is evaluated at the nine points considered in table 6.

The nine linear equations involved in the least-squares procedure are to be obtained by evaluating (73) at the nine points $y_k = k/8$, $k = 0, 1, 2, \ldots 8$. With the approximation of equation (17), the matrix of coefficients of the unknown parameters B and A_{2n} in this set of equations, corresponding to the left-hand side of equation (21), is obtained as the difference between two matrices, the first of which is given directly by the left-hand side of equation (19). In the second matrix, which represents the coefficients of the values of the third term on the left-hand side of equation (73), the elements in the k-th row are the products of the coefficients of the parameters in equation (79) by the quantity $s(y_k)$.

The coefficients of the required nine equations are thus found in the form

$$
\begin{array}{ccccc} B & A_0 & A_2 & A_4 & A_6 \end{array}
$$

$$
\left\|\begin{array}{rrrrr}
-.24490 & .75651 & -.12069 & -.02512 & -.01182 \\
-.12560 & .76682 & -.09526 & -.02897 & -.01303 \\
.04424 & .82172 & -.01896 & -.03488 & -.01730 \\
.19912 & .83092 & .07988 & -.93090 & -.02530 \\
.35128 & .81698 & .20800 & .00536 & -.02794 \\
.48752 & .77392 & .35474 & .09908 & .00242 \\
.59186 & .69876 & .50264 & .27201 & .12231 \\
.63452 & .57466 & .61642 & .52371 & .41573 \\
.40518 & .21474 & .47045 & .63292 & .76172
\end{array}\right\| = \left\|\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right\| \qquad (80)
$$

The coefficients of the final set of linear equations are then found, by taking the dot product of the auxiliary matrix previously defined into both sides of equation (80),

$$
\begin{array}{ccccc} B & A_0 & A_2 & A_4 & A_6 \end{array}
$$

$$
\left\|\begin{array}{rrrrr}
1.97915 & 2.30991 & 1.66370 & 1.04882 & .74071 \\
2.30991 & 6.55641 & 1.75463 & .88374 & .54525 \\
1.66370 & 1.75463 & 1.45675 & 1.00542 & .74857 \\
1.04882 & .88374 & 1.00542 & .84761 & .71317 \\
.74070 & .54525 & .74857 & .71317 & .65352
\end{array}\right\| = \left\|\begin{array}{r} 3.45862 \\ 8.71573 \\ 2.77811 \\ 1.67224 \\ 1.21165 \end{array}\right\| \qquad (81)
$$

and the solution of this set of equations is determined,

$$
\left.\begin{array}{lll}
B = -1.45207 & A_0 = 1.29708 & A_2 = 2.59819 \\
 & A_4 = -2.38087 \qquad A_6 = 2.03971 &
\end{array}\right\} \qquad (82)
$$

In table 6 and figure 6 of the auxiliary lift function $F(y) = \frac{c}{mc_R} c_l$, given by the expression

$$
F(y) = F_1(y) - \frac{\pi \mu \gamma F_1(a)}{\pi \mu - 2\gamma a \log a} P(y) \qquad (83)
$$

and the function $\frac{F(y)}{c^*(y)} = \frac{1}{m} c_l$ are presented numerically and graphically. For the purpose of comparison there is included in figure 6 a curve representing the variation of the function $\frac{1}{m} c_l$ for the same wing without cut-out, as determined in a preceding section.

CONCLUSION

The explicit procedure given in this paper, using a five-term approximation and a nine-point weighting system, should give results with a degree of accuracy comparable to the accuracy of measurement of the physical data of the wing in all practical cases. More nearly exact solutions could be obtained by retaining a greater number of terms or by using a more elaborate weighting system.

While the present method is not a method of successive approximations, an indication of the accuracy attained can be found in the following two ways:

1. As mentioned in the text, if the computed values of the parameters specifying the lift function $F(y)$ are substituted in the left-hand sides of the nine linear equations represented by formal equation (21) the resultant values of the angle of attack corresponding to the function F can be compared with the corresponding prescribed values of the angle of attack.

2. It can be shown that if the last term in the approximation to the function $F(y)$ is not retained, the final set of four linear equations given by the least-squares procedure for the determination of the four remaining parameters is obtained from the corresponding set of five equations in the original five parameters by deleting the coefficients of A_8 and omitting the last equation. If the resultant set of equations is solved, a comparison of the lift distributions corresponding to the four-term and five-term approximations will afford an estimate of the effect of retaining additional terms.

The complete solution of a problem, including the tabulation of values of the functions $F(y)$ and $\overline{\alpha}(y)$, can be carried out in less than two hours by a competent computing-machine operator.

Massachusetts Institute of Technology,
Cambridge, Mass., February 1943.

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TABLE 1.— LIFT DISTRIBUTIONS FOR RECTANGULAR WING $\left(\frac{b}{c} = m\right)$

y	$\frac{1}{m\alpha_T} c_l(y)$				
	Present solution			Glauert solution	
	$\alpha = \alpha_T$	$\alpha = \alpha_T[y]$	$\alpha = \alpha_T y^2$	$\alpha = \alpha_T$	$\alpha = \alpha_T[y]$
0	0.8288	0.1673	0.0666	0.832	0.132
.125	.8262	.1898	.0757	.830	.157
.250	.8243	.2462	.1075	.822	.222
.375	.8169	.3171	.1562	.809	.307
.500	.7932	.3820	.2121	.786	.385
.625	.7478	.4272	.2686	.752	.435
.750	.6828	.4498	.3236	.695	.451
.875	.5826	.4362	.3567	.577	.421
.9325	.4736	.3760	.3235	.450	.355
1	0	0	0	0	0

TABLE 2.— COMPARISON OF ANGLE OF ATTACK $\bar{\alpha}$ CORRESPONDING TO THE LIFT DISTRIBUTIONS OF TABLE 1 WITH PRESCRIBED ANGLE OF ATTACK α

y	$\alpha = \alpha_T$			$\alpha = \alpha_T[y]$			$\alpha = \alpha_T y^2$	
	$\bar{\alpha}/\alpha_T$		α/α_T	$\bar{\alpha}/\alpha_T$		α/α_T	$\bar{\alpha}/\alpha_T$	α/α_T
	Present solution	Glauert solution		Present solution	Glauert solution		Present solution	
0	0.997	1.000	1.0	0.054	0.000	0.000	−0.002	0
.125	.986	1.000	1.0	.109	.050	.125	.004	.016
.250	1.003	1.000	1.0	.233	.196	.250	.065	.062
.375	1.022	1.003	1.0	.385	.376	.375	.159	.141
.500	1.018	.995	1.0	.521	.544	.500	.265	.250
.625	.982	.995	1.0	.620	.646	.625	.376	.391
.750	.954	1.005	1.0	.711	.729	.750	.526	.562
.875	1.030	1.020	1.0	.894	.851	.875	.790	.766
1	.952	.692	1.0	.974	.861	1.	.962	1.

TABLE 3.– AUXILIARY DATA FOR TAPERED WING

$\bar{y}$	$c^*(y)$	$\alpha(y)$	$P(\bar{y})$	$\alpha^*(y)$
0	1.	1.0	2.17796	1.85269
.125	.9375	1.0	2.13103	1.94784
.250	.8750	1.0	1.98288	1.94088
.375	.8125	1.0	1.69962	1.76657
.500	.7500	0	1.06538	1.09523
.625	.6875	0	.54692	.47025
.750	.6250	0	.26128	.09278
.875	.5625	0	.08380	–.17629
1.	0	0	0	–.32527

TABLE 4.– LIFT DISTRIBUTION FOR TAPERED WING

Present solution		Pearson solution	
$\bar{y}$	$\frac{c}{m\,c_R}\,c_l$	y	$\frac{c}{m\,c_R}\,c_l$
0.	0.7729	0.	0.7576
.1250	.7077	.1564	.7033
.2500	.6588	.3090	.6535
.3750	.6162	.4540	.4451
.4375	.5463	.4890	.3589
.5000	.3057	.5225	.2743
.5625	.1619	.5878	.1460
.6250	.1101	.7071	.0864
.7500	.0744	.8090	.0506
.8750	.0425	.8910	.0328
1.	0	.9511	.0224

TABLE 5.– COMPARISON OF ANGLE OF ATTACK $\overline{\alpha}$ CORRESPONDING TO THE LIFT DISTRIBUTIONS OF TABLE 4 WITH THE PRESCRIBED ANGLE OF ATTACK α

Present solution			Pearson solution		
y	$\overline{\alpha}(y)$	$\alpha(y)$	y	$\overline{\alpha}(y)$	$\alpha(y)$
0	1.157	1.0	0	1.043	1.0
.125	.936	1.0	.1564	.947	1.0
.250	1.008	1.0	.3090	1.058	1.0
.375	1.100	1.0	.4540	.724	1.0
.500	–.023	0	.4890	.505	———
.625	–.058	0	.5225	.288	0
.750	.029	0	.5878	–.010	0
.875	.020	0	.7071	.063	0
1.000	–.036	0	.8090	.021	0
———	———	———	.8910	.046	0
———	———	———	.9511	.044	0

TABLE 6.– NUMERICAL DATA FOR WING WITH CUT-OUT

y	P(y)	$\beta(y)$	$\frac{c}{m c_R} c_l$	$\frac{1}{m} c_l$
0	1.70627	0.44512	0.9339	0.9339
.125	1.59911	.42638	.9234	.9234
.250	1.13364	.34498	.9760	{.9760 {.7320
.375	.62280	.05355	1.0240	.7680
.500	.37225	.02069	1.0272	.7704
.625	.20701	–.00098	.9781	.7336
.750	.09497	–.01568	.8965	.6724
.875	.02492	–.02486	.7659	.5744
1	0	–.02813	0	0

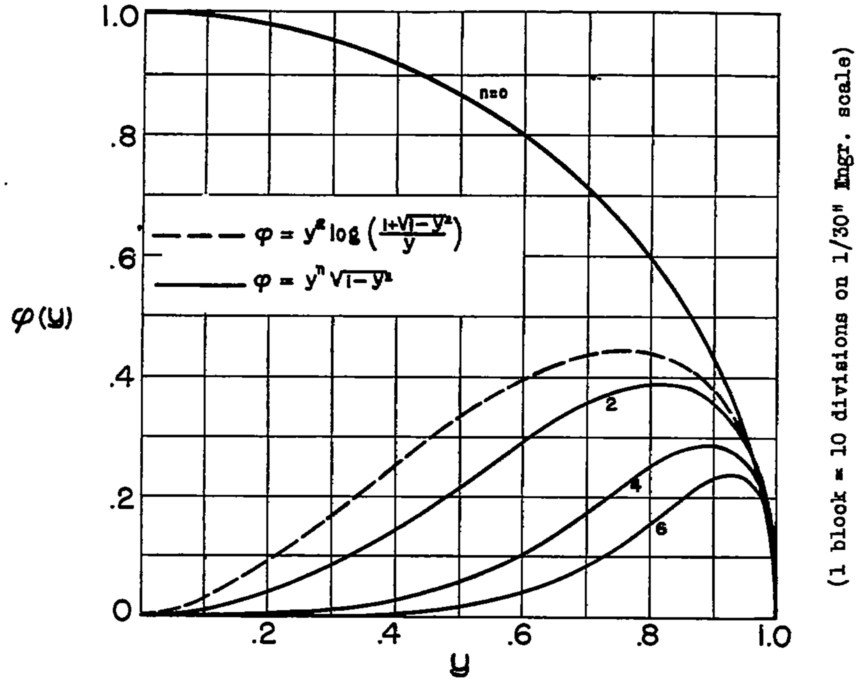
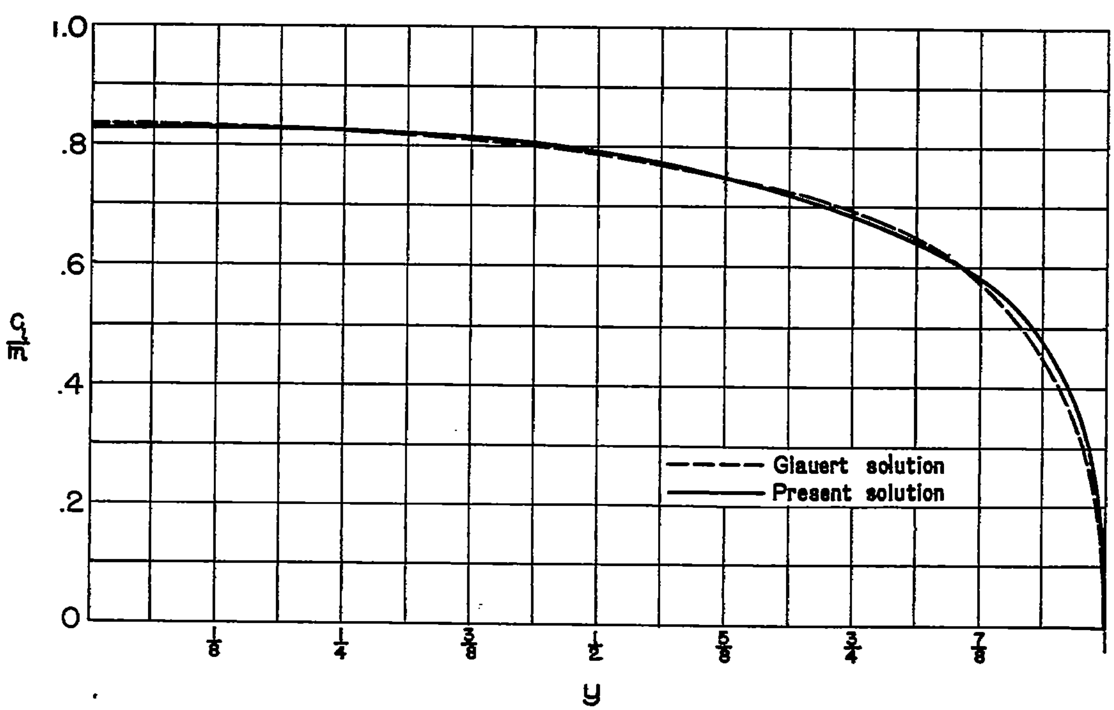
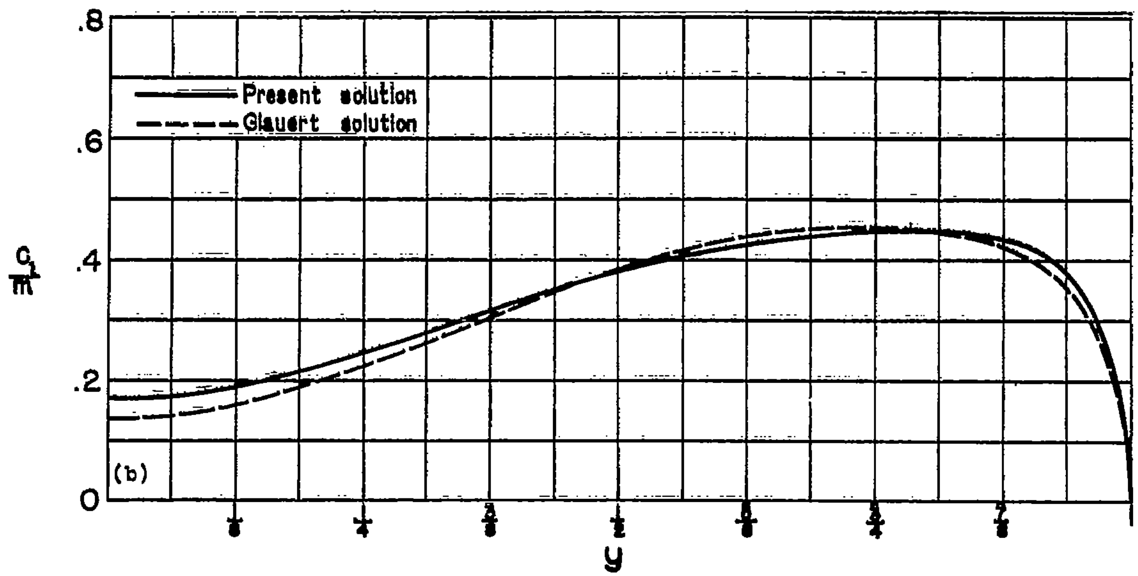


Figure 1.- The approximating functions $y^{2n}\sqrt{1-y^2}$ and $y^2\log(1+\sqrt{1-y^2}/|y|)$.



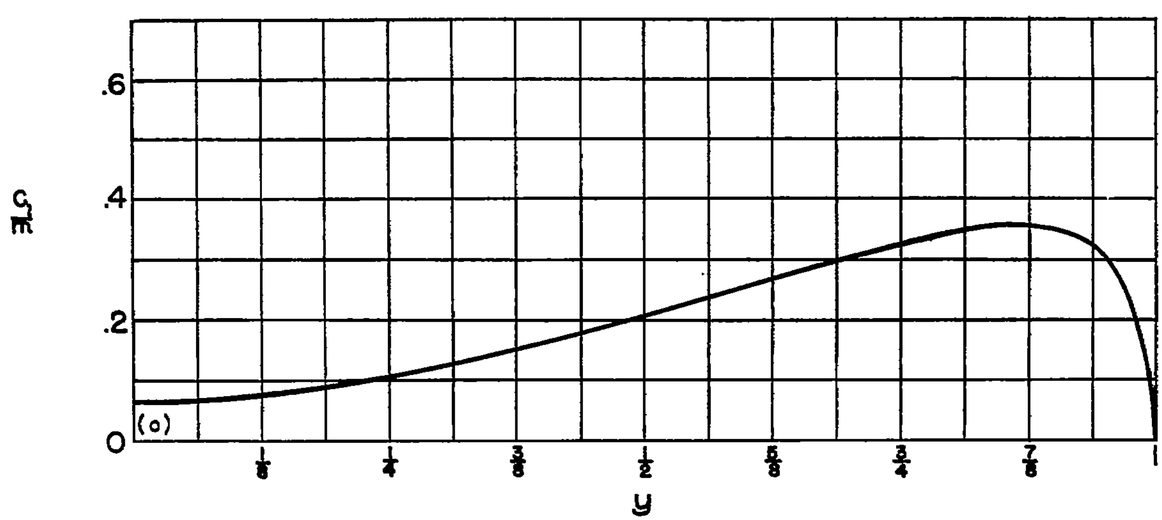
(a) Constant angle of attack ($\alpha = \alpha_T$).

Figure 2 (a to c).- Lift distributions for a rectangular wing (b/c = m).



(b) Symmetrically linear angle of attack ($\alpha = \alpha_T\ |y|$).

(1 block = 10/30")



(c) Quadratic angle of attack ($\alpha = \alpha_T y^2$).

Figure 3.- (Concluded).

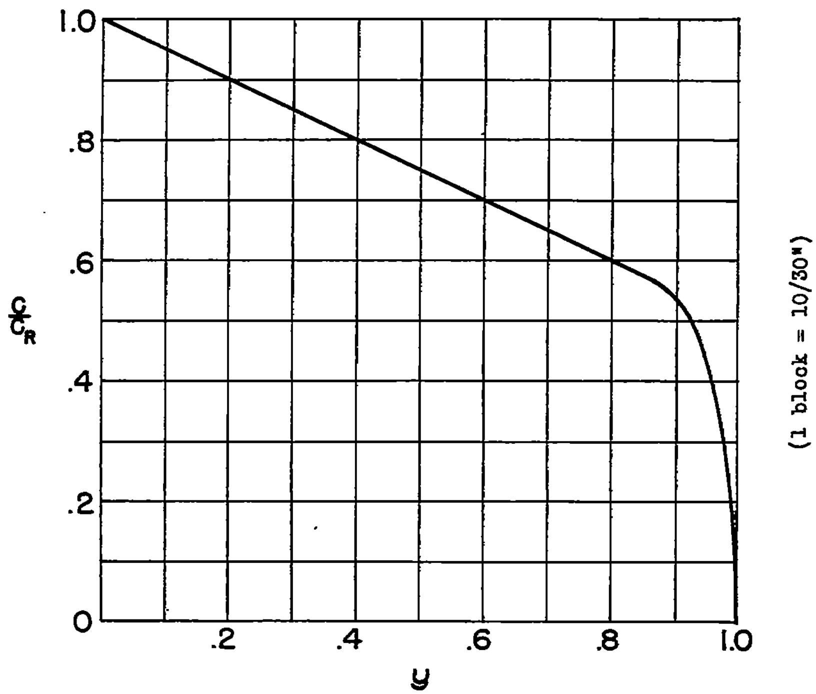


Figure 3.- Chord variation of tapered wing used in example.

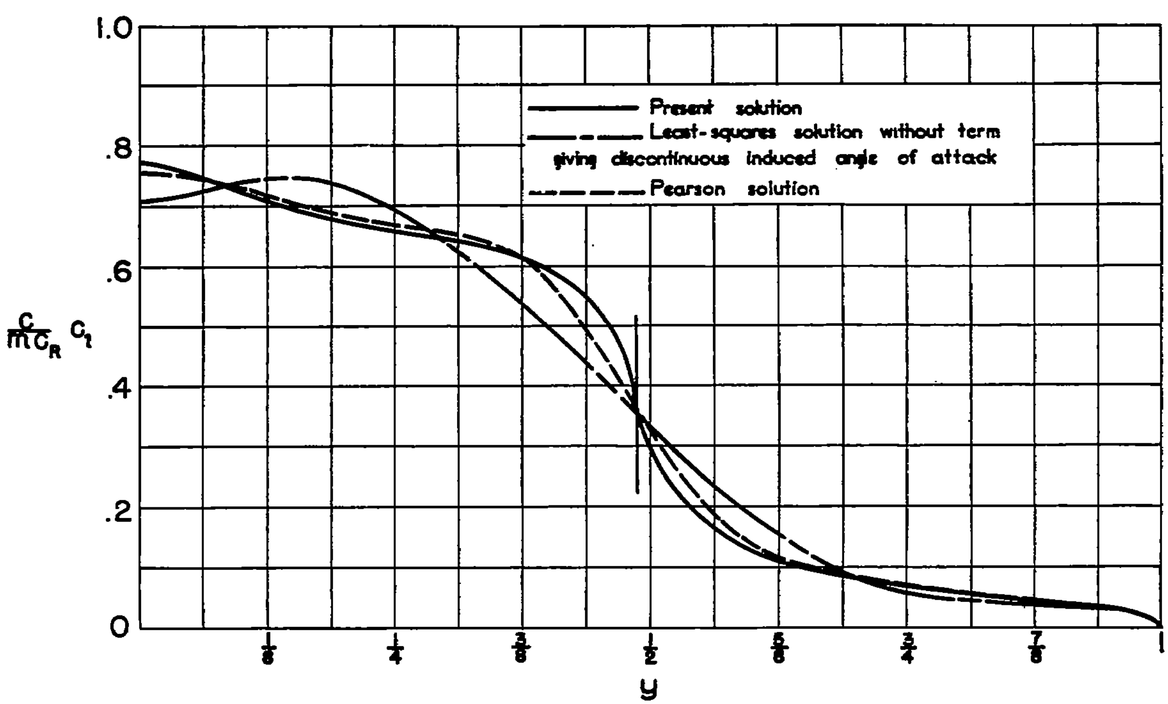


Figure 4.- Lift distribution for a tapered wing with rounded tips and partial-span flaps ($b^2/S = 10$).

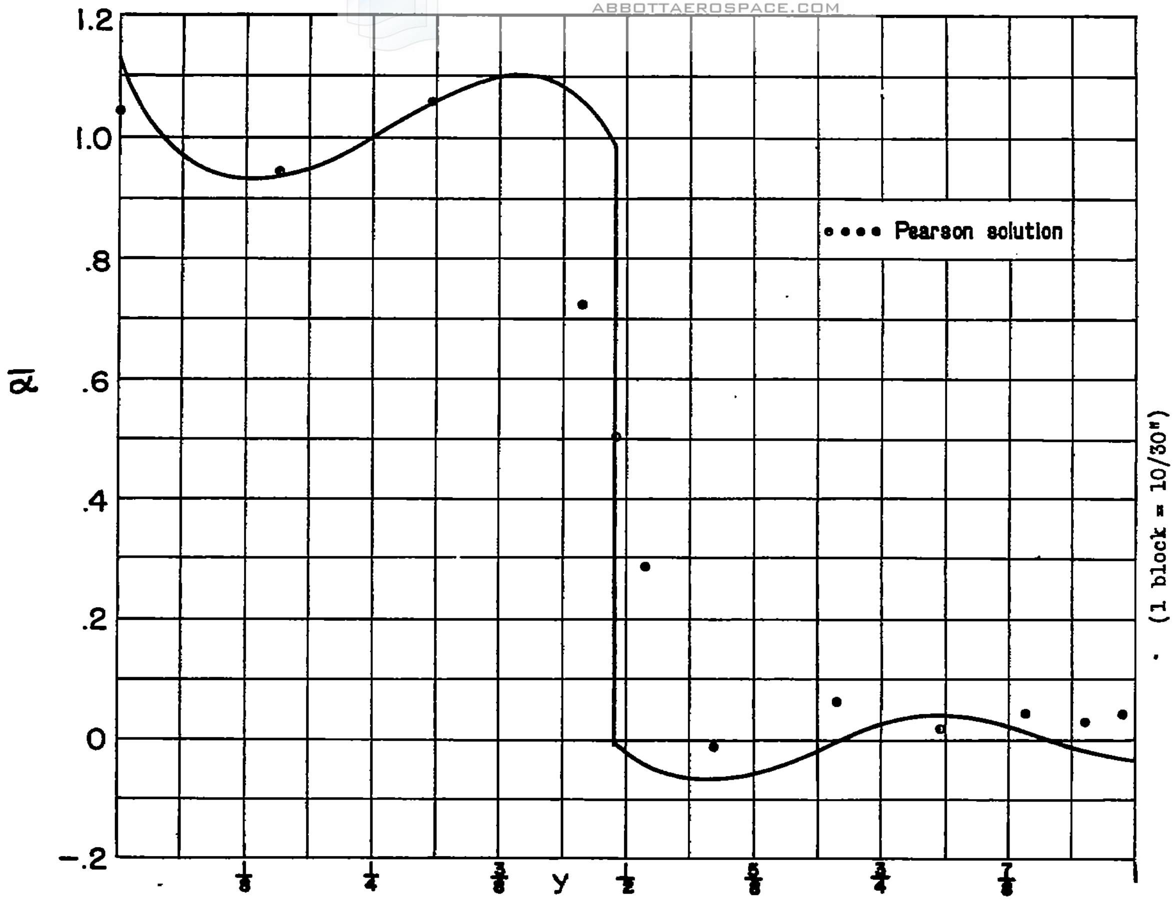


Figure 5.- Comparison of angle of attack corresponding to lift distributions of figure 4 with prescribed angle of attack.

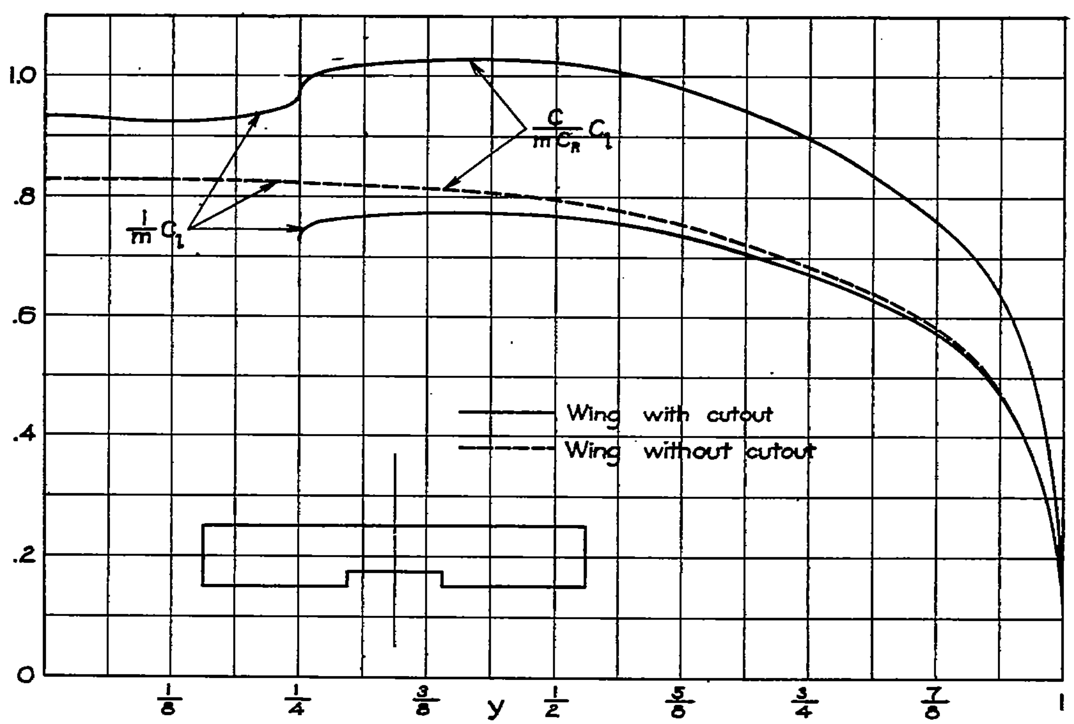


Figure 6.- Lift distribution for rectangular wing with cutout ($b/c_{max} = m$, $b^2/S = \frac{16}{15} m$)