ARR No. L4D27a

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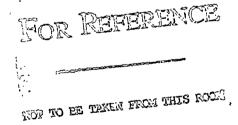
WARTIME REPORT

ORIGINALLY ISSUED

April 1944 as
Advance Restricted Report L4D27a

NUMERICAL EVALUATION OF THE ϵ-INTEGRAL OCCURRING IN

THE THEODORSEN ARBITRARY AIRFOIL POTENTIAL THEORY

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ADVANCE RESTRICTED REPORT

NUMERICAL EVALUATION OF THE ε-INTEGRAL OCCURRING IN

THE THEODORSEN ARBITRARY AIRFOIL POTENTIAL THEORY

By Irven Naiman

SUMMARY

A more precise method of evaluating the ε-integral occurring in the arbitrary airfoil theory of Theodorsen (NACA Reps. Nos. 411 and 452) is developed by retaining higher order terms in the Taylor expansion and by use of Simpson's rule. Formulas are given for routine calculation of the ε-integral and for the necessary computational coefficients. The computational coefficients are tabulated for a 40-point division of the range of integration from 0 to 2π. With no increase in computational work the systematic error in the numerical value of ε is reduced from the order of 1 percent to approximately 0.1 percent.

INTRODUCTION

The solution of the general problem by means of conformal transformation for the flow about an arbitrary airfoil (references 1 and 2), a symmetrical lattice (reference 3), and a biplane (reference 4) involves the determination of the imaginary part of a complex transformation function, given the real part. As shown in references 1 and 2 the real part may be expanded in a Fourier series and the imaginary part is the conjugate Fourier series. It is also shown in these references that the imaginary part ε may be obtained from the real part ψ by the following functional equation:

$$\epsilon(\varphi') = -\frac{1}{2\pi} \int_0^{2\pi} \psi(\varphi) \cot \frac{\varphi - \varphi'}{2} d\varphi \qquad (1)$$

This integral occurs frequently in conformal-
transformation problems involving the evaluation of the
functions on the circle. A procedure for the numerical
evaluation of this integral is given in references 1
and 2. This method, which is currently in use at LMAL,
gives an error of about 1.5 percent for a 40-point
division of the range of integration. An improvement
in the accuracy is therefore very desirable, particularly
if the labor involved is not increased. A revised method
given herein is found to involve a little less work than
that previously required and to give an error of only
about 0.1 percent. Constants for use in this more pre-
cise method have been computed and are presented in
table I.

EVALUATION OF THE ε-INTEGRAL

The evaluation of the ε-integral is complicated by
the discontinuity at $\varphi = \varphi'$. This difficulty may be
surmounted by a separate solution across the discon-
tinuity. When $s = \varphi - \varphi'$ is substituted in equation (1)

$$\epsilon(\varphi') = -\frac{1}{2\pi} \int_{-\varphi'}^{2\pi-\varphi'} \psi(\varphi' + s) \cot \frac{s}{2} \, ds$$

or, because of the periodicity of this function,

$$\epsilon(\varphi') = -\frac{1}{2\pi} \int_{-s}^{2\pi-s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \qquad (2)$$

The discontinuity now occurs at $s = 0$. For purposes of
numerical evaluation this integral may be broken up as
follows:

$$\epsilon(\varphi') = -\frac{1}{2\pi}\left[\int_{-s}^{s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds + \int_{s}^{2\pi-s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \right]$$

$$= \epsilon_1 + \epsilon_2 \qquad\qquad (3)$$

where

$$\epsilon_1 \equiv -\frac{1}{2\pi} \int_{-s}^{s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \qquad (4)$$

and

$$\epsilon_2 \equiv -\frac{1}{2\pi} \int_{s}^{2\pi-s} \psi(\varphi' + s) \cot \frac{s}{2} \, ds \qquad (5)$$

Evaluation of ϵ_1.- The first integral ϵ_1 includes the discontinuity and the limit s may be taken as some convenient small value. By a Taylor series expansion the integral is easily evaluated as follows:

$$\psi(\varphi' + s) = \psi(\varphi') + s\psi'(\varphi') + \frac{s^2}{2}\psi''(\varphi') + \frac{s}{3!}\psi'''(\varphi') + \ldots$$

When this expansion is substituted in equation (4) the integrals containing the even-ordered derivatives are found to be identically zero. Equation (4) then becomes

$$\epsilon_1 = -\frac{1}{2\pi}\left[\psi' \int_{-s}^{s} s \cot \frac{s}{2} \, ds + \frac{\psi'''}{3!} \int_{-s}^{s} s^3 \cot \frac{s}{2} \, ds + \ldots\right] \qquad (6)$$

where the derivatives are evaluated at φ'. The Taylor expansion for $\cot \frac{s}{2}$ is

$$\cot \frac{s}{2} = \frac{2}{s} - \frac{s}{6} - \frac{s^3}{360} - \ldots$$

and equation (6) is thus obtained as

$$\epsilon_1 = -\frac{1}{2\pi}\left[4s\psi'\left(1 - \frac{s^2}{36} - \frac{s^4}{3600} - \dots\right) + \frac{2}{9}s^3\psi'''\left(1 - \frac{s^2}{20} - \frac{s^4}{1680} - \dots\right)\right.$$

$$\left. + \frac{s^5}{150}\psi^V\left(1 - \frac{5s^2}{84} - \dots\right) + \frac{s^7}{8820}\psi^{VII}\left(1 - \frac{7s^2}{108} - \dots\right) + \dots\right] \quad (7)$$

Evaluation of ϵ_2.- The second integral ϵ_2 (equation (5)) may be rearranged for convenience in numerical calculation as follows:

$$\epsilon_2 = -\frac{1}{2\pi}\left[\int_s^\pi \psi(\varphi' + s)\cot\frac{s}{2}\,ds + \int_\pi^{2\pi-s}\psi(\varphi' + s)\cot\frac{s}{2}\,ds\right]$$

$$= -\frac{1}{2\pi}\left[\int_s^\pi \psi(\varphi' + s)\cot\frac{s}{2}\,ds + \int_\pi^s \psi(\varphi' - s)\cot\frac{s}{2}\,ds\right]$$

$$= -\frac{1}{2\pi}\left\{\int_s^\pi\left[\psi(\varphi' + s) - \psi(\varphi' - s)\right]\cot\frac{s}{2}\,ds\right\} \quad (8)$$

where $-s$ has been substituted for s in the second integral and the limits have been rearranged accordingly.

NUMERICAL METHODS

Method of reference 1.- In reference 1 the interval 0 to 2π is divided into n equal parts of magnitude $2\pi/n$ (n is an even number). The values of ψ are designated $\psi_{-\frac{n}{2}}, \psi_{-\frac{n}{2}+1}, \dots, \psi_{-1}, \psi_0, \psi_1, \dots, \psi_{\frac{n}{2}}$ where ψ_0 is the value of ψ at $\varphi = \varphi'$ and $\psi_{-\frac{n}{2}} = \psi_{\frac{n}{2}}$

is the value at $\varphi = \varphi' \pm \pi$. The integrations are per-
formed over intervals of width $2\pi/n$ with the ψ values
at the midpoint of the interval. The range of integra-
tion for ϵ_1 is from $s = -\pi/n$ to $s = \pi/n$ and for
ϵ_2 from π/n to $2\pi - \frac{\pi}{n}$.

The first integral ϵ_1 is evaluated by retaining
only the first-order terms in s,

$$\epsilon_1 \cong -\frac{1}{2\pi} 4 s \psi_0' = -\frac{2}{n}\psi_0' \qquad (9)$$

where the slope ψ_0' is determined graphically at ψ_0,
that is, at $\varphi = \varphi'$ ($s = 0$).

The second integral ϵ_2 is composed of the sum of
integrals across each interval

$$\epsilon_2 = -\frac{1}{2\pi} \sum_{k=1}^{n-1} \int_{\frac{2k-1}{n}\pi}^{\frac{2k+1}{n}\pi} \psi(\varphi' + s) \cot \frac{s}{2} \, ds$$

The function ψ does not change much across the
interval and is therefore approximated by use of the
value at the midpoint $\psi_k \cong \psi\left(\varphi' + \frac{2k\pi}{n}\right)$. Then,

$$\epsilon_2 \cong -\frac{1}{2\pi} \sum_{k=1}^{n-1} \psi_k \int_{\frac{2k-1}{n}\pi}^{\frac{2k+1}{n}\pi} \cot \frac{s}{2} \, ds$$

$$\cong -\frac{1}{\pi} \sum_{k=1}^{n-1} \psi_k \log \frac{\sin \frac{2k+1}{2n}\pi}{\sin \frac{2k-1}{2n}\pi}$$

or, by equation (8), with $\psi_{n-k} = \psi_{-k}$

$$\epsilon_2 \cong -\frac{1}{\pi} \sum_{k=1}^{n/2} a_k(\psi_k - \psi_{-k}) \qquad (10)$$

where

$$\left.\begin{array}{l} a_k \equiv \log \dfrac{\sin \dfrac{2k + 1}{2n}\pi}{\sin \dfrac{2k - 1}{2n}\pi} \\[3em] a_{\frac{n}{2}} \equiv 0 \end{array}\right\} \qquad (11)$$

The complete integral is given by $\epsilon = \epsilon_1 + \epsilon_2$, or

$$\epsilon \cong -\left\{\frac{2}{n}\psi_0{}' + \frac{1}{\pi} \sum_{k=1}^{n/2} a_k(\psi_k - \psi_{-k})\right\} \qquad (12)$$

Values of the constants a_k were given in reference 1 for $n = 10$ and in reference 2 for $n = 20$. Revised values for these constants, together with those for $n = 40$, are given in table II.

Improved method.- The numerical accuracy of the evaluation of the ϵ-integral will be shown to be improved by the following method: The interval 0 to 2π is divided into n equal parts and the ψ values are designated as in the previous section. The second integral ϵ_2 is evaluated by Simpson's rule from ψ_1 to ψ_{n-1} $(\psi_{n-1} = \psi_{-1})$. The range of integration for ϵ_1 is therefore twice as large as that in the previous section, that is, $s = -2\pi/n$ to $2\pi/n$. The approximation in which only the first-order term of equation (7) is used is insufficient and the higher derivatives must be used. These derivatives are most conveniently obtained by numerical differentiation.

The Newton-Stirling formulas for derivatives (reference 5, p. 75) are

$$s\psi' = \delta\psi - \frac{1}{6}\delta^3\psi + \frac{1}{30}\delta^5\psi - \ldots$$

$$s^3\psi''' = \delta^3\psi - \frac{1}{4}\delta^5\psi + \ldots \left.\right\} \quad (13)$$

$$s^5\psi^V = \delta^5\psi - \ldots$$

where s = tabular interval $(2\pi/n)$. The mean central differences $\delta\psi$ can be expressed in terms of the tabular values as

$$2\delta\psi = \psi_1 - \psi_{-1}$$

$$2\delta^3\psi = \psi_2 - 2\psi_1 + 2\psi_{-1} - \psi_{-2} \left.\right\} \quad (14)$$

$$2\delta^5\psi = \psi_3 - 4\psi_2 + 5\psi_1 - 5\psi_{-1} + 4\psi_{-2} - \psi_{-3}$$

The substitution of relations (13) in equation (7) gives

$$\epsilon_1 = -\frac{1}{2\pi}\left\{2\delta\psi\left(2 - \frac{s^2}{18} - \frac{s^4}{1800} - \ldots\right) - 2\delta^3\psi\left(\frac{2}{9} - \frac{s^2}{270} - \ldots\right)\right.$$

$$\left. + 2\delta^5\psi\left(\frac{13}{450} - \frac{s^2}{1512} - \ldots\right) - \ldots\right\} \quad (15)$$

The further substitution of relations (14) in equation (15) gives

$$\epsilon_1 = -\frac{1}{2\pi}\left\{(\psi_1 - \psi_{-1})\left[\left(2 + \frac{4}{9} + \frac{19}{90} + \ldots\right)\right.\right.$$

$$\left.- \frac{s^2}{18}\left(1 + \frac{2}{15} + \frac{5}{84} + \ldots\right) - \ldots\right]$$

$$-(\psi_2 - \psi_{-2})\left[\left(\frac{2}{9} + \frac{38}{225} + \ldots\right) - s^2\left(\frac{1}{270} + \frac{1}{378} + \ldots\right) - \ldots\right]$$

$$\left.+(\psi_3 - \psi_{-3})\left[\left(\frac{19}{450} + \ldots\right) - s^2\left(\frac{1}{1512} + \ldots\right) - \ldots\right] - \ldots\right\}$$

or

$$\epsilon_1 \equiv b_1(\psi_{-1} - \psi_1) + b_2(\psi_{-2} - \psi_2) + b_3(\psi_{-3} + \psi_3) \quad (16)$$

where

$$b_1 \equiv \frac{1}{2\pi}\left[\left(2 + \frac{4}{9} + \frac{19}{90}\right) - s^2\left(\frac{1}{18} + \frac{1}{135} + \frac{5}{1512}\right) - \ldots\right]$$

$$= \frac{1}{2\pi}\left(\frac{239}{90} - \frac{167}{2520}s^2 - \ldots\right)$$

$$b_2 \equiv -\frac{1}{2\pi}\left[\left(\frac{2}{9} + \frac{38}{225}\right) - s^2\left(\frac{1}{270} + \frac{1}{378}\right) - \ldots\right] \qquad (17)$$

$$= -\frac{1}{2\pi}\left(\frac{88}{225} - \frac{2}{315}s^2 - \ldots\right)$$

$$b_3 \equiv \frac{1}{2\pi}\left(\frac{19}{450} - \frac{s^2}{1512} - \ldots\right)$$

The second integral ϵ_2 is evaluated by Simpson's rule from ψ_1 to ψ_{n-1}

$$\epsilon_2 \cong -\frac{1}{2\pi}\frac{2\pi}{3n}\left[\psi_1 \cot \frac{\pi}{n} + 4\psi_2 \cot \frac{2\pi}{n}\right.$$

$$\left. + 2\psi_3 \cot \frac{3\pi}{n} + \ldots + \psi_{n-1} \cot \frac{n-1}{n}\pi\right]$$

or, by equation (8), with $\psi_{n-k} = \psi_{-k}$

$$\epsilon_2 \cong \sum_{k=1}^{n/2} c_k(\psi_{-k} - \psi_k) \qquad (13)$$

where

$$\left.\begin{array}{l} c_1 \equiv \dfrac{1}{3n} \cot \dfrac{\pi}{n} \\[4pt] \cdot \\ \cdot \\ \cdot \\ c_k \equiv \dfrac{\alpha}{3n} \cot \dfrac{k\pi}{n} \\[4pt] \cdot \\ \cdot \\ \cdot \\ c_{\frac{n}{2}} \equiv 0 \end{array}\right\} \qquad (19)$$

where (except in the first term) $\alpha = 2$ for k odd and $\alpha = 4$ for k even. The complete integral is given by $\epsilon = \epsilon_1 + \epsilon_2$, or

$$\epsilon \cong \sum_{k=1}^{n/2} A_k(\psi_{-k} - \psi_k) \qquad (20)$$

where

$$A_k = b_k + c_k \qquad\qquad (21)$$

Values of A_k for $n = 40$ are given in table I.

ACCURACY OF EVALUATION

The accuracy of the two methods of evaluation described may be determined by integrating various harmonics. The results are presented as ratios of the integrated value to the correct value so that a value of unity is a correct evaluation. Values of this ratio for the harmonics are:

Harmonic	40-point method of reference 1	40-point present method
1	1.01434	0.99977
2	1.02662	1.00044
3	1.03700	.99940
4	1.04547	1.00060
5	1.05280	.99340
10	1.06112	.95420

Inasmuch as the higher harmonics generally enter in a much smaller proportion than the lower ones for such contours as are encountered for airfoil shapes, the error of the 40-point method of reference 1 is of the order of 1.5 percent, whereas that of the new method presented herein is approximately 0.1 percent.

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5. Davis, Harold T.: Tables of the Higher Mathematical
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TABLE I.- VALUES OF A_k FOR USE WITH EQUATION (20)

k	A_k n = 40	k	A_k n = 40
1	0.52827	11	0.01423
2	.14834	12	.02422
3	.07614	13	.01021
4	.10259	14	.01698
5	.04024	15	.00690
6	.06542	16	.01083
7	.02720	17	.00400
8	.04588	18	.00528
9	.01951	19	.00066
10	.03333	20	0

TABLE II.- VALUES OF a_k FOR USE WITH EQUATION (12);

METHOD OF REFERENCES 1 AND 2

k	a_k n = 10	n = 20	n = 40
1	1.06544	1.09037	1.09656
2	.44311	.49426	.50671
3	.23117	.31141	.33028
4	.10302	.21750	.24303
5	0	.15775	.19028
6		.11440	.15453
7		.08021	.12941
8		.05115	.10826
9		.02493	.09207
10		0	.07862
11			.06714
12			.05711
13			.04316
14			.04004
15			.03255
16			.02553
17			.01937
18			.01245
19			.00619
20			0