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THE SQUARE ROOT METHOD

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ABSTRACT: Conditions under which solution of a set of linear equations can be effected by the square root method are given. While much literature exists on the technique of this method, we have been unable to find a discussion of conditions under which this technique succeeds. A method of decomposing a large class of hermitian matrices, (sometimes requiring quaternions), also appears to be new.

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This report contains the result of an investigation carried out in connection with a problem in least square fitting of a function which expresses complex yaw of a spin-stabilized projectile in terms of distance down range. This problem was resolved into a system of linear equations with complex coefficients, the solution of which is discussed here. The work was carried out under the sponsorship of ONR project NR-044-003, Numerical Analysis and BuOrd project NOL-Re3d-452-1-53, Free-Flight Aeroballistics of Spin-Stabilized Bodies.

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THE SQUARE ROOT METHOD

1. Introduction

Throughout this paper  $S = (s_{ij})$  will denote a triangular matrix, (i.e.,  $i > j$  implies  $s_{ij} = 0$ ),  ${}^tA$  will denote the transpose of the matrix  $A$ ;  $R, C, Q$  will denote respectively the field of real numbers, the field of complex numbers and the sfield of quaternions with real coefficients.  $M_n(F)$  will denote the ring of  $n \times n$  matrices with elements in a sfield  $F$ .

In the square root method one solves the system of linear equations

$$AX = G \text{ for } X,$$

( $A \in M_n(R)$ ,  $X$  an unknown column vector,  $G$  a given column vector), in the following manner (cf. ref. (a)):

One finds a matrix  $S$  such that  ${}^tS \cdot S = A$ . Assuming  $A$  non-singular, it follows  $S$  and  ${}^tS$  are non-singular so that

$${}^tS \cdot SX = G \text{ implies } SX = K$$

where  $K = {}^tS^{-1} \cdot G$ . Because of the simple forms of  $S$  and  ${}^tS$ , the calculation of  ${}^tS^{-1} \cdot G$  and of  $S^{-1} K$  is fairly simple. For another discussion of this method and further references see reference (b).

2. The Theorems

In what follows we are given a sfield  $F$  and an anti-automorphism of the sfield such that  $\sigma^2 = 1$ . If  $B = (b_{ij}) \in M_n(F)$  and  $\bar{B} = (\sigma b_{ij})$  then

$\sigma {}^tB = {}^t\bar{B}$ . We shall sometimes interpret  $B \in M_n(F)$  as a linear transformation on the left  $n$ -dimensional vector space of  $n$ -tuples of  $F$  over  $F$ . To say  $B$  is non-singular means  $B$  has an inverse which is equivalent to the requirement that the image of  $B$  be the whole space. This latter condition is easily seen to be equivalent to the requirement that the rows of the matrix  $B$  be left linearly independent.

Lemma a triangular matrix  $S \in M_n(F)$  is non-singular if and only if  $s_{ii} \neq 0$ , for  $i = 1, \dots, n$ .

(1) For  $n = 1$ , the lemma is trivial. Suppose  $\prod_{i=1}^n s_{ii} \neq 0$ , and  $n > 1$ . We may assume inductively that the matrix

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[1]

$$\begin{matrix} s_{22} & s_{23} & \dots & s_{2n} \\ 0 & s_{33} & \dots & s_{3n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & s_{nn} \end{matrix}$$

has left linearly independent rows. If the row vectors of S are dependent then there exists  $b \in F$  such that  $b s_{11} = 0$ . Since  $s_{11} \neq 0$ ,  $b = 0$ . This implies the dependence of the rows of the matrix [1] which contradicts the inductive assumption.

Now suppose  $\prod_{i=1}^n s_{ii} = 0$ . If  $s_{ii} = 0$  for some  $i > 1$  then by induction the rows of [1] are dependent and hence the rows of S. If  $\prod_{i=1}^n s_{ii} \neq 0$  then  $s_{11} = 0$ . The  $n$  vectors  $(s_{12}, s_{13}, \dots, s_{1n})$ ,  $1 \leq i \leq n$ , are dependent since they lie in an  $n-1$  dimensional space. If  $\sum_{i=1}^n b_i (s_{12}, \dots, s_{1n}) = 0$  then  $\sum_{i=1}^n b_i (s_{11}, s_{12}, \dots, s_{1n}) = 0$ .

If  $A \in M_n(F)$ , then for  $1 \leq k \leq n$ ,  $A_k \in M_k(F)$  denotes the matrix obtained from A by deleting all rows and columns except rows and columns 1 through k.

Theorem 1

Suppose F commutative. Let  $A \in M_n(F)$  be non-singular. If there exists  $S \in M_n(F)$  such that  $\sigma^t S, S = A$  then

- (a)  $|A_k| \neq 0, k = 1, \dots, n; |A_k|$  denotes the determinant of  $A_k$ .
- (b)  $\sigma x_k, x_k = |A_k|, k = 1, \dots, n$  has a solution in F.
- (c)  $\sigma^t A = S$ .

Proof

Since  $| \sigma^t S | = |S| = |A|$  and A is non-singular,  $|S| \neq 0$  and  $| \sigma^t S | \neq 0$ . Since  $|S| = \prod_{i=1}^n s_{ii}$ ,  $s_{ii} \neq 0$  and hence  $\sigma(s_{11}) \neq 0$ . It follows that  $|A_k| \neq 0, k = 1, \dots, n-1$ , since  $\sigma^t S_k \cdot S_k = A_k$ . Thus (a) is satisfied. For  $n \geq k-1$ , we have

$$0 \neq |A_k| \div |A_{k-1}| = \prod_{i=1}^k \sigma s_{ii} \cdot s_{ii} / \prod_{i=1}^{k-1} \sigma s_{ii} \cdot s_{ii} = \sigma s_{kk} \cdot s_{kk}$$

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For  $k = 1$  we have  $\sigma a_{11} \cdot a_{11} = a_{11}$ . Assuming  $\sigma x_0 \cdot x_0 = /A_{k-1}/$  we have  $\sigma x_0 \cdot x_0 \cdot \sigma a_{kk} \cdot a_{kk} = \sigma(x_0 a_{kk}) \cdot x_0 a_{kk} = /A_k/$ , (b) is demonstrated. To demonstrate (c):

$$\sigma A = \sigma(\sigma S \cdot S) = \sigma(\sigma S \cdot \sigma S) = \sigma S \cdot S = A$$

**Theorem 2**

Let  $F$  be a sfield. Let  $G$  denote the set of  $x \in F$  such that  $\sigma x = x$ . Suppose for every  $g \in G$  there exists  $x \in F$  such that  $\sigma x \cdot x = g$ . Given a non-singular matrix  $A \in M_n(F)$ . If

- (a)  $A_i, i = 1, \dots, n$  is non-singular.
- (b)  $\sigma A = A$  (i.e.,  $\sigma a_{ij} = a_{ji}$ ).

Then there exists  $S \in M_n(F)$  such that  $\sigma S \cdot S = A$ .

Proof

We observe first the existence of such an  $S$  is equivalent to the existence of  $s_{ij} \in F, 1 \leq i \leq j \leq n$ , such that

$$[2] \quad \sum_{r=1}^i \sigma s_{ri} \cdot s_{rj} = a_{ij}, \quad j \geq i \text{ since}$$

$$\sigma \sum_{r=1}^i \sigma s_{ri} \cdot s_{rj} = \sum_{r=1}^i \sigma s_{rj} \cdot s_{ri} = \sigma a_{ij} = a_{ji}$$

For  $n = 1$  the theorem is trivial. We note that conditions (a) and (b) hold for  $A_{n-1}$  (with  $(n-1)$  replacing  $n$ ). We assume inductively the existence of  $s_{ij}, 1 \leq i \leq j < n$  such that [2] is satisfied. We seek now  $s_{in}, i = 1, \dots, n$  satisfying

$$\sum_{r=1}^i \sigma s_{ri} \cdot s_{rn} = a_{in}$$

For  $i < n$ , we need solve a linear equation the coefficient of the unknown of which is  $\sigma s_{ii}$ . Since  $A_{n-1}$  is non-singular the lemma tells us  $s_{ii} \neq 0$  and so  $\sigma s_{ii} \neq 0$ . It remains only to find  $s_{nn}$  satisfying

$$\sum_{r=1}^n \sigma s_{rn} \cdot s_{rn} = a_{nn}$$

We observe that  $\sigma(\sigma s_{rn} \cdot s_{rn}) = (\sigma^2 s_{rn} \cdot \sigma^2 s_{rn}) = \sigma s_{rn} \cdot s_{rn}$ .

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Therefore  $(\sigma s_{rn} \cdot s_{rn}) \in G$ . Since  $\sigma a_{ij} = a_{ji}$ ,  $\sigma a_{nn} = a_{nn}$  so that  $a_{nn} \in G$ . One sees easily that  $G$  is an abelian group so that

$$a_{nn} - \sum_{r=1}^{n-1} \sigma s_{rn} \cdot s_{rn} \text{ is in } G$$

and the result follows.

3. The Special Cases

Corollary 1. Let  $A \in M_n(\mathbb{C})$  be non-singular and symmetric. A necessary and sufficient condition that there exist  $S \in M_n(\mathbb{C})$  such that  ${}^t S \cdot S = A$  is that  $|A_i|$ ,  $i = 1, \dots, n-1$ , be non-zero.

Proof

Let  $F = \mathbb{C}$  and  $\sigma = 1$ .

Corollary 2. Let  $A \in M_n(\mathbb{Q})$  be non-singular. Suppose  ${}^t \sigma A = A$  where

$$\sigma (z_1 + z_2 j) = \bar{z}_1 + z_2 j, (z_1 = a + bi, z_2 = c + di).$$

A necessary and sufficient condition that there exist  $S \in M_n(\mathbb{Q})$  such that  ${}^t \sigma S \cdot S = A$  is that  $A_i$ ,  $i = 1, \dots, n-1$  be non-singular.

Proof

One sees readily  $\sigma$  is an anti-automorphism of  $\mathbb{Q}$ .  $G$  consists of the elements of the form  $b + zj$  with  $b$  real. For  $b, z$  fixed we seek  $z_1$  and  $z_2$  such that  $(z_1 + z_2 j)(z_1 + z_2 j) = b + zj$ . If  $z = 0$ , this is easy. Assume  $z \neq 0$ .  $z_1, z_2$  must be non-zero and must satisfy  $z_1 \bar{z}_1 - z_2 \bar{z}_2 = b$  and  $2z_1 z_2 = z$ . This is equivalent to  $z_2 = \frac{z}{2z_1}$  and  $z_1 \bar{z}_1 - (z\bar{z}/4z_1 \bar{z}_1) = b$ . The latter is equivalent to  $4(z_1 \bar{z}_1)^2 - 4bz_1 \bar{z}_1 - z\bar{z} = 0$  which always has a solution,

$$\text{viz. } z_1 \bar{z}_1 = \frac{1}{2} (b + \sqrt{b^2 + z\bar{z}}) > 0.$$

We supplement corollary 2 with the remark that if  $A \in M_n(\mathbb{Q})$  is hermitian, and if one chooses the diagonal elements of  $S$  properly, (identifying  $z + 0j$  with  $z$ ),  $S$  contains elements of a particularly simple form. If we call  $z + 0j$  a complex quaternion and  $0 + zj$  a chaste quaternion then every element of  $S$  is either complex or chaste. Indeed the  $s_{ij}$  may always be chosen still more simply as either a positive real number or a positive real number times  $j$ . These remarks may be verified using the equations

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$$s_{ii} = (a_{ii} - \sum_{r=1}^{i-1} \sigma s_{ri} \cdot s_{ri})^{\frac{1}{2}} \text{ for } i > 1$$

$$s_{ij} = (a_{ij} - \sum_{r=1}^{i-1} \sigma s_{ri} \cdot s_{rj}) / s_{ii} \text{ ( } j > i > 1 \text{)}$$

$$s_{11} = (a_{11})^{\frac{1}{2}}, \quad s_{1j} = a_{1j} / s_{11}$$

and the facts: the inverse of a chaste quaternion is chaste, the product of a complex and chaste quaternion is chaste,  $\sigma s_{ri} \cdot s_{ri}$  is complex whether  $s_{ri}$  is complex or chaste.

Corollary 2 and 3 insure that a positive definite real symmetric matrix and a positive definite hermitian matrix are decomposable in the complex and quaternion domains respectively. However in these cases the decomposition is possible in the real and complex domains respectively. This is because  $s_{ii}^2 = (\text{positive real number})$  has a solution in the reals as does  $\sigma s_{ii} \cdot s_{ii} = (\text{positive real number})$ . For an interesting discussion of hermitian matrices see reference 3.

Utilizing theorem 2 and "The Principle of the Irrelevance of Algebraic Inequalities," (ref. 4, p. 4), one can prove that the necessary conditions associated in theorem 1 for the existence of a decomposition of a non-singular matrix, (in the case of a commutative F), is indeed sufficient. The results of the above paragraph are then obtainable directly from this theorem.



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